# Exceptionally simple PDE

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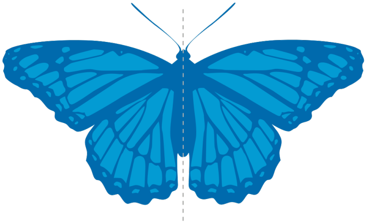
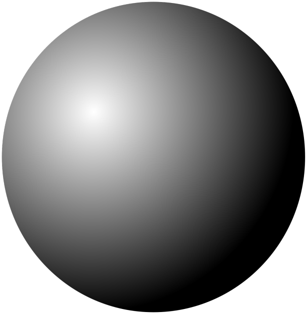
# Outline

1. Symmetry & various geometric realizations of $G_2$
2. New models: Exceptionally simple PDE
3. Geometry underlying the new models

Symmetry and $G_2$

# Symmetry





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- $\rightsquigarrow$ Lie group: group + manifold
- $\rightsquigarrow$ Lie algebra: vector space $\mathfrak{g}$ with a skew, bilinear $[\cdot,\cdot]$ s.t.

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0, \quad \forall a,b,c\in\mathfrak{g}.$$

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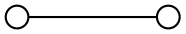
This is a beautiful piece of mathematics and is well-known. It is often presented with the aid of Dynkin diagrams, e.g.  .

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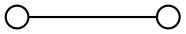
This is a beautiful piece of mathematics and is well-known. It is often presented with the aid of Dynkin diagrams, e.g.  .

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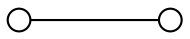
This is a beautiful piece of mathematics and is well-known. It is often presented with the aid of Dynkin diagrams, e.g.  .

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Classical cases: Easy. What about the exceptionals?

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(Over $\mathbb{R}$, $\exists$ 2 open orbits. Get the cpt and split real forms of $G_2$.)

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(Split form of $G_2$ arises from automorphisms of split-octonions.)

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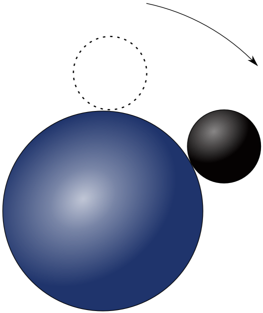
- Configuration space $M$ is 5-dimensional.
- No twisting or slipping $\Rightarrow$ constraints on velocity space $TM$. Get rank 2 distribution $D \subset TM$ of allowable directions.



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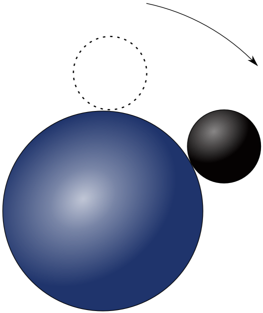

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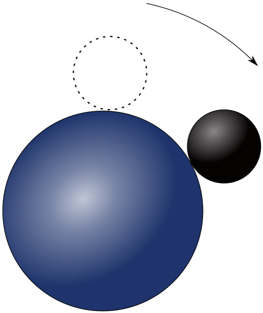

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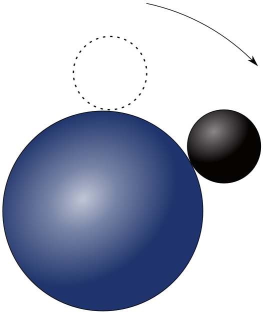

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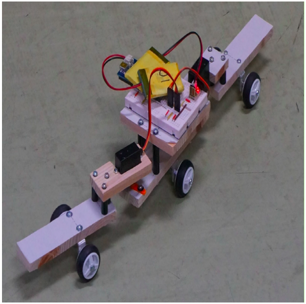
- $\rho \neq 3$: $\mathrm{SO}(3) \times \mathrm{SO}(3)$ symmetry
- $\rho = 3$: (split) $\mathfrak{g}_2$ symmetry *(Bryant, Zelenko, Bor–Montgomery, Baez–Huerta)*

# The hunt for a $G_2$-snake



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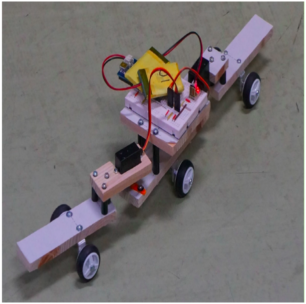

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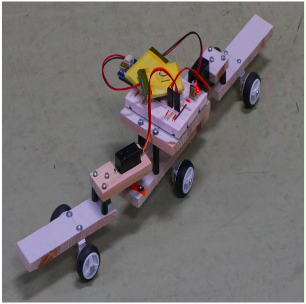

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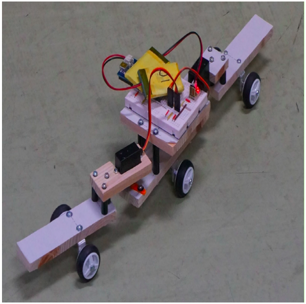

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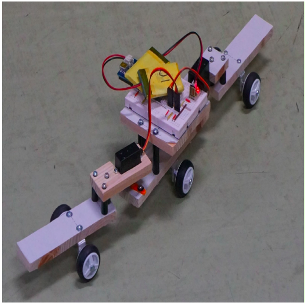

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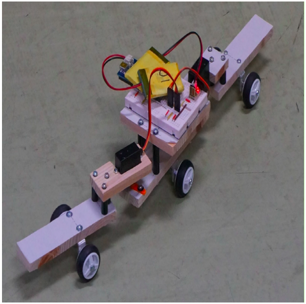

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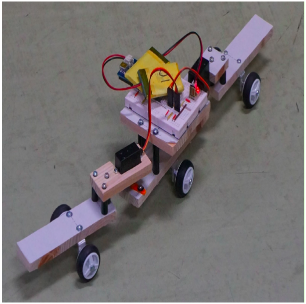

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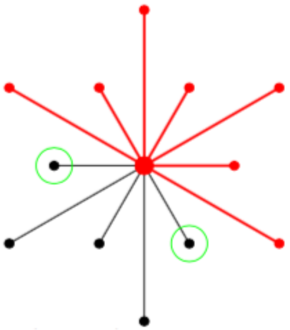
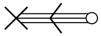
(He doubts it: "A $G_2$-snake may be as mythical as a unicorn or yeti.")

# $(2,3,5)$ from the $G_2$ root diagram

$G_2/P_1$





| Dim | Geometric structure | Model |
|---|---|---|
| 7 | Parabolic Goursat PDE $\mathcal{F}$ | $9(u_{xx})^2 + 12(u_{yy})^2(u_{xx}u_{yy} - (u_{xy})^2) + 32(u_{xy})^3 - 36u_{xx}u_{xy}u_{yy} = 0$ |
| 6 | Involutive pair of PDE $\mathcal{E}$ | $u_{xx} = \frac{1}{3}(u_{yy})^3, \quad u_{xy} = \frac{1}{2}(u_{yy})^2$ |
| 5 | $(2,3,5)$-distrib. $\overline{\mathcal{E}}$ | $dU - PdX$, $dP - QdX$, $dZ - Q^2dX$ (a.k.a. Hilbert–Cartan: $Z' = (U'')^2$) |
| 5 | $G_2$-contact structure (contact twisted cubic field) | $dz + x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2 = 0$, $dx_2^2 + \sqrt{3}dy_1dy_2 = 0$, $dx_2dy_2 - 3dx_1dy_1 = 0$, $dy_2^2 + \sqrt{3}dx_1dx_2 = 0$ |

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Yamaguchi (1999): Generalized Cartan's reduction thms (1910, 1911). (PDE are non-explicit).

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NOTE: The Jordan algebra structure plays no role in this talk.

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Let $n-1 := \dim(W)$. Basis $\{\mathrm{w}_a\}$ on $W$, dual basis $\{\mathrm{w}^a\}$ on $W^*$.
Let $\{x^i\}_{i=0}^{n-1}$ lin.coords on $\mathbb{C} \oplus W$.

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Let $\{x^i\}_{i=0}^{n-1}$ lin.coords on $\mathbb{C} \oplus W$. Param. submfld of $J^2(\mathbb{C}^n, \mathbb{C})$:

$$\mathcal{E}: \ (u_{ij}) = \begin{pmatrix} u_{00} & u_{0b} \\ u_{a0} & u_{ab} \end{pmatrix} = \begin{pmatrix} \mathfrak{C}(t^3) & \frac{3}{2}\mathfrak{C}_b(t^2) \\ \frac{3}{2}\mathfrak{C}_a(t^2) & 3\mathfrak{C}_{ab}(t) \end{pmatrix}, \quad t \in W.$$

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## Theorem (T. 2017, Contact symmetries of $\mathcal{E}$)

| $W$ | $\mathcal{JS}_{2\ell-5}$ ($\ell \geqslant 3$) | $\mathcal{JS}_{2\ell-6}$ ($\ell \geqslant 5$) | $\mathbb{C}$ | $\mathcal{J}_3(\underline{0})$ | $\mathcal{J}_3(\mathbb{R}_\mathbb{C})$ | $\mathcal{J}_3(\mathbb{C}_\mathbb{C})$ | $\mathcal{J}_3(\mathbb{H}_\mathbb{C})$ | $\mathcal{J}_3(\mathbb{O}_\mathbb{C})$ |
|---|---|---|---|---|---|---|---|---|
| $n$ | $2\ell-3$ | $2\ell-4$ | 2 | 4 | 7 | 10 | 16 | 28 |
| $sym(\mathcal{E})$ | $B_\ell$ | $D_\ell$ | $G_2$ | $D_4$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |

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## Theorem (Degenerate cases)

- $u_{ij} = 0$, $1 \leqslant i,j \leqslant n$: *point sym* $= A_{n+1}$. (*NOTE:* $\mathfrak{sl}_2$ *excluded!*)
- $u_{ijk} = 0$, $1 \leqslant i,j,k \leqslant n$: *contact sym* $= C_{n+1}$.

# Other exceptionally simple models

$$\mathcal{F}: \quad \begin{cases} u_{00} = t^a t^b u_{ab} - 2\mathfrak{C}(t^3), \\ u_{0a} = t^b u_{ab} - \frac{3}{2}\mathfrak{C}_a(t^2) \end{cases} \qquad (t \in W).$$

$$\mathcal{V} = \{[\mathbf{V}(\lambda,t)] : [\lambda,t] \in \mathbb{P}(\mathbb{C} \oplus W)\} \subset \mathbb{P}(\mathcal{C}), \text{ where}$$
$$\mathbf{V}(\lambda,t) = \lambda^3 \mathbf{X}_0 - \lambda^2 t^a \mathbf{X}_a - \tfrac{1}{2}\mathfrak{C}(t^3)\mathbf{U}^0 - \tfrac{3}{2}\lambda\mathfrak{C}_a(t^2)\mathbf{U}^a,$$
$$\text{with } \mathbf{X}_i = \partial_{x^i} + u_i \partial_u, \quad \mathbf{U}^i = \partial_{u_i}.$$

$$\tau(\mathcal{V}) = \{\mathrm{Q} = 0\} \subset \mathbb{P}(\mathcal{C}), \text{ where}$$
$$\mathrm{Q} = (\omega^i\theta_i)^2 + 2\theta_0\mathfrak{C}(\Omega^3) - 2\omega^0\mathfrak{C}^*(\Theta^3) - 9\mathfrak{C}_a(\Omega^2)(\mathfrak{C}^*)^a(\Theta^2),$$
$$\text{with } \omega^i = dx^i,\ \theta_i = du_i,\ \Omega = \omega^a \otimes w_a,\ \Theta = \theta_a \otimes w^a.$$

$$\overline{\mathcal{E}}: \quad Z_a = \tfrac{3}{2}\mathfrak{C}_a(T^2),\ U_{ab} = 3\mathfrak{C}_{ab}(T) \quad (T \in W).$$

# Envelopes

Consider the family of inhom. linear PDE param. by $t \in W$:

$$\boxed{\mathcal{M}_t := u_{00} - 2t^a u_{a0} + t^a t^b u_{ab} - \mathfrak{C}(t^3) = 0} \qquad (*)$$

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- first-order envelope: $\{\mathcal{M}_t = 0, \partial_{t^a}\mathcal{M}_t = 0\}$ yields $\mathcal{F}$.
- 2nd-order envelope: $\{\mathcal{M}_t = 0, \partial_{t^a}\mathcal{M}_t = 0, \partial_{t^a}\partial_{t^b}\mathcal{M}_t = 0\}$ yields $\mathcal{E}$.

NOTE: (*) generalizes the classical "Goursat parametrization".

Geometry behind the new models

# What is a 2nd order PDE?

| Global | Local |
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| Contact mfld<br>$(M^{2n+1}, \mathcal{C})$ | $(x^i, u, u_i)$, $\sigma := du - u_i dx^i$<br>$\mathcal{C} = \{\sigma = 0\} = \mathrm{span}\{\partial_{x^i} + u_i \partial_u, \partial_{u_i}\}$ |

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| $\mathcal{C}$ is a field of conformal symplectic spaces | $d\sigma\|_{\mathcal{C}} = dx^i \wedge du_i\|_{\mathcal{C}} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$<br>$\partial_{x^i} + u_i \partial_u, \partial_{u_i}$ is a CS-basis |

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A 2nd order PDE $\Sigma$ is a submanifold of $M^{(1)}$. A contact sym is a sym of $(M^{(1)}, \mathcal{C}^{(1)})$ that preserves $\Sigma$.

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IDEA: contact mfld + additional structure.

# Twisted cubic $\mathcal{V} \subset \mathbb{P}(V)$

Let $\mathcal{V} = \{[v^3] : [v] \in \mathbb{P}^1\}$, $V := S^3\mathbb{C}^2$, and $[\eta]$ CS-form on $V$:

$$\eta(f,g) := \frac{1}{3!}(f_{xxx}g_{yyy} - 3f_{xxy}g_{yyx} + 3f_{xyy}g_{yxx} - f_{yyy}g_{xxx}),$$

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Wrt CS-basis, i.e. $\eta = \begin{pmatrix} 0 & \mathrm{id}_2 \\ -id_2 & 0 \end{pmatrix}$, have coords $\begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}$ on $\mathrm{LG}(V)$.

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# Canonical objects associated to the twisted cubic

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These all inherit $G_0 \cong GL_2$ invariance from $\mathcal{V}$.
(and $\mathfrak{g}_0 \subsetneq \mathfrak{csp}_4$ is a maximal subalgebra.)

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| $G_2/P_2$ | $A_1/P_1$ | twisted cubic |
| $D_4/P_2$ | $(A_1/P_1)^3$ | $\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ |
| $F_4/P_1$ | $C_3/P_3$ | $\mathrm{LG}(3,6)$ |
| $E_6/P_2$ | $A_5/P_3$ | $\mathrm{Gr}(3,6)$ |
| $E_7/P_1$ | $D_6/P_6$ | $D_6$-spinor variety |
| $E_8/P_8$ | $E_7/P_7$ | Freudenthal variety |

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~~Type A~~: $\mathcal{V} = \mathbb{P}^{n-1} \dot\sqcup \mathbb{P}^{n-1}$ (reducible); ~~Type C~~: $\mathcal{V} = \mathbb{P}(V)$ (not proper).

# $G$-contact structures

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**Definition ($G \neq A_\ell, C_\ell$)**

*A G-contact structure is a contact mfld $(M^{2n+1}, \mathcal{C})$ with any of:*

- *a field of sub-adjoint varieties $\mathcal{V} \subset \mathbb{P}(\mathcal{C})$;*

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- Have unique max sym model: flat model, has sym $\mathfrak{g}$.
- Can efficiently compute syms of $\mathcal{E}, \mathcal{F}$ by using $\mathcal{V}$ instead. (In the flat cases, can do this uniformly and by-hand!)

# Where does $\mathfrak{C}$ come from?

| | | | | | |
|---|---|---|---|---|---|
| | $A_1$ | $A_2$ | $C_3$ | $F_4$ | Hyperplane section of Severi |
| $\mathfrak{f}_0^{ss}$ | $A_2$ | $A_2 \times A_2$ | $A_5$ | $E_6$ | Severi varieties |
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**Lemma**

*From L.-M., $\exists$ CS-basis adapted to $V = \mathbb{C} \oplus W \oplus \mathbb{C} \oplus W^*$ s.t. $\mathcal{V} \subset \mathbb{P}(V)$ is given by* $[\lambda, t^a] \to \left[\lambda^3, -\lambda^2 t^a, -\frac{\mathfrak{C}(t^3)}{2}, -\frac{3\lambda \mathfrak{C}_a(t^2)}{2}\right]$.

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- Uniform descriptions via a cubic form on a Jordan algebra.
- Moral of the story: Sometimes, complicated questions have exceptionally simple answers.