Jet-determination of symmetries of parabolic geometries*

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At $x \in M$,

- $\mathbf{X} \in \mathfrak{X}(M)$ *is k-jet determined if* $j^k_x(\mathbf{X}) \neq 0$.
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Example (Conformal structures)

If $g = \sum_{i=1}^n (dx^i)^2$, then here are all CKV $\mathbf{X} = X^i \partial_{x^i}$ for $(M, [g])$:

$$X^i = s^i + m^i{}_j x^j + \lambda x^i + r^j x_j x^i - \frac{1}{2} r^i x_j x^j$$

Here, $\mathcal{S}$ is 2-jet determined (everywhere).

Parabolic geometries

- G: semisimple Lie group, P: parabolic subgroup; $\mathfrak{g} = \mathfrak{g}_{-\nu} \oplus \ldots \oplus \mathfrak{g}_{\nu}$ with $\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$ and $\mathfrak{p} = \mathfrak{g}^0$, $\mathfrak{p}_+ = \mathfrak{g}^1$.

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Notation:

- Fix $x \in M$ and fix $u \in \pi^{-1}(x)$.
- $\xi \in \mathfrak{inf}(\mathcal{G}, \omega)$ corresponds $\mathbf{X} \in \mathcal{S} := \pi_*(\mathfrak{inf}(\mathcal{G}, \omega))$.

Main results: Jet-det of syms of parabolic geometries

Theorem

$\mathcal{S}$ is 2-jet determined everywhere. If G is simple, then at any $x \in M$ where $\kappa_H(x) \neq 0$, $\mathcal{S}$ is 1-jet determined at x.

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This is connected to the following:

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Given $0 \neq \omega_u(\xi) \in \mathfrak{g}^i \backslash \mathfrak{g}^{i+1}$. Then:

$$\begin{array}{lll} i < 0 : & j^0_x(\mathbf{X}) \neq 0 & \text{(0-jet determined)} \\ 0 \leq i < \nu : & j^0_x(\mathbf{X}) = 0,\ j^1_x(\mathbf{X}) \neq 0 & \text{(1-jet determined)} \\ i = \nu : & j^1_x(\mathbf{X}) = 0,\ j^2_x(\mathbf{X}) \neq 0 & \text{(2-jet determined)} \end{array}$$

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IDEA: For 1-jet det, want to show that a certain Tanaka prolongation does not reach the top-slot $\mathfrak{g}_\nu$.

Key technical advance: improved Tanaka prolongation result

Main results: Rigidity

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Theorem (Torsion-free parabolic geometries)

If $0 \neq \mathbf{X} \in \mathcal{S}$ and $j^1_x(\mathbf{X}) = 0$, i.e. $\omega_u(\xi) \in \mathfrak{g}_\nu$, then the geometry is flat on open set $U \subset M$ with $x \in \overline{U}$.

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Theorem (General parabolic geometries)

Suppose that:

(i) *$\omega_u(\xi)$ lies in the* **open** *G_0-orbit of $\mathfrak{g}_\nu$.*

(ii) *G/P is not $A_\ell/P_{s,s+1}$, $2 \leq s < \frac{\ell}{2}$ or B_ℓ/P_ℓ, $\ell \geq 5$ odd.*

Then the geometry is flat on an open set $U \subset M$ with $x \in \overline{U}$.

Part 1: Symmetry and Tanaka prolongation

Symmetry as a graded object

Čap–Neusser (2009): Fix $u \in \mathcal{G}$.

- $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is injective on $\{\xi_u : \xi \in \mathfrak{inf}(\mathcal{G}, \omega)\}$.

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Regularity $\Rightarrow \kappa_H(u) \in H^2_+(\mathfrak{g}_-,\mathfrak{g})$.

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Let $\mathfrak{a}_0 \subset \mathfrak{g}_0$ *be a subalg. Define* $\mathfrak{a} \subset \mathfrak{g}$ *by* $\mathfrak{a}_{\leq 0} = \mathfrak{g}_{\leq 0}$ *and* $\mathfrak{a}_k = \{X \in \mathfrak{g}_k \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{k-1}\}$ *for* $k > 0$. *Write* $\mathfrak{a} = \mathrm{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0)$.

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If $\mathfrak{g}$ *is simple and* $0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})$, *then* $\mathfrak{a}^\phi_\nu = 0$.

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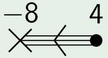
$(2,3,5)$-geometry		$\mathfrak{a}^\phi_+ = 0$

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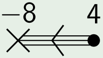
$(2,3,5)$-geometry		$\mathfrak{a}^\phi_+ = 0$
torsion-free pairs of 2nd order ODE		$\mathfrak{a}^\phi_{\geq 2} = 0$

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- $\mathfrak{S} \leq \mathfrak{U}$ is immediate.
- If $\mathfrak{g}$ is simple and $\kappa_H(u) \neq 0$, then $\mathfrak{s}_\nu(u) = 0$ $\rightsquigarrow \quad j^1_x(\mathbf{X}) \neq 0, \quad \forall \mathbf{X} \in \mathcal{S}$.

Reformulating the Tanaka prolongation result

Let $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$. On $\Gamma(\mathcal{A}M) \cong \mathfrak{X}(\mathcal{G})^P$, have algebraic bracket $\{\cdot,\cdot\}$ and geometric bracket $[\cdot,\cdot]$.

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$$\{t_n, \{..., \{t_1, s\}...\}\}(u) \cdot \kappa_H(u) = 0.$$

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Reformulating the Tanaka prolongation result

Let $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$. On $\Gamma(\mathcal{A}M) \cong \mathfrak{X}(\mathcal{G})^P$, have algebraic bracket $\{\cdot,\cdot\}$ and geometric bracket $[\cdot,\cdot]$.

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Fix $u \in \mathcal{G}$, sym s, $s(u) \in \mathfrak{g}^k \subset \mathfrak{p}_+$, and $t_j \in \Gamma(\mathcal{A}^{-i_j}M)$ with $k - i_1 - \ldots - i_n \geq 0$. Then

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- Thus, if s sym with $s(u) \in \mathfrak{p}$, then $[s,t](u) = -\{s,t\}(u)$.

Proof ideas

Recall: $\kappa_H \in \Gamma\left(\mathcal{G} \times_P \frac{\ker(\partial^*)}{\mathrm{im}(\partial^*)}\right)$, $\mathfrak{p}_+$-action on $\frac{\ker(\partial^*)}{\mathrm{im}(\partial^*)}$ is trivial.

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$|1|$-graded case: ✓. General case is a complicated induction. Have to contend with identities like

$$[t_2,[t_1,s]] = D_{t_2}[t_1,s] - D_{[t_1,s]} t_2 - \kappa(\Pi(t_2), \Pi([t_1,s])) + \{t_2,[t_1,s]\}.$$

Sample of general case

Let $s(u) \in \mathfrak{g}^2$ and $t_1, t_2 \in \Gamma(\mathcal{A}^{-1}M)$. From $0 = D_{t_2} D_{t_1} D_s \kappa_H$,

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- $[t_2,[t_1,s]](u) - \{t_2,\{t_1,s\}\}(u) = 0 \mod \mathfrak{ann}_{\mathfrak{p}}(\kappa_H(u))$.

Part 2: Structure of $\mathfrak{g}_\nu$ and rigidity

Čap–Melnick criteria

$\mathbf{X} \in \mathcal{S}$ has higher-order fixed point at x if $0 \neq E := \omega_u(\xi) \in \mathfrak{p}_+$.
Jacobson–Morozov $\Rightarrow$ std $\mathfrak{sl}_2$-triple $\{F, H, E\}$.

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(CM.1) $H \in \mathfrak{g}_0$;

(CM.2) Eigenvalues of H on $\mathfrak{g}_-$ are ≤ 0, the gen. eigenspace for eigenv. with real part zero is $C^-_{\mathfrak{g}}(E) = \{X \in \mathfrak{g}_- \mid [X, E] = 0\}$, and $\mathrm{ad}_H|_{C^-_{\mathfrak{g}}(E)} = 0$;

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Theorem (Čap–Melnick (2013))

Suppose (CM.1-3) hold, $\kappa_H(x) = 0$, and $v = \pi_*(\omega_u^{-1}(F)) \in T_x M$. Then $\exists \gamma : (-\epsilon, +\epsilon) \to M$, $\gamma(0) = x$, $\gamma'(0) = v$, preserved by flow of $\mathbf{X}$ & on which it acts by proj. transf. Let $\gamma^+ = \gamma((0, +\epsilon)) \subset M$, $\exists$nbd U of γ^+, $\bar{U} \ni x$, on which the geometry is flat.

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Our study: General grading, but suppose $E \in \mathfrak{g}_\nu$ ("top slot").

Structure theory for the top slot $\mathfrak{g}_\nu$

Definition

R: reductive, V : R-irrep, $\mathcal{V} \subset \mathbb{P}(V)$ closed orbit. If the only R-orbits are $Sec_k(\mathcal{V}) \backslash Sec_{k-1}(\mathcal{V})$, then V is sub-cominuscule.

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G/P	G_0^{ss}	Sub-cominuscule variety $\mathcal{V} \subset \mathbb{P}(\mathfrak{g}_1)$
A_ℓ/P_k	$A_{k-1} \times A_{\ell-k}$	$\mathrm{Seg}(\mathbb{P}^{k-1} \times \mathbb{P}^{\ell-k}) \hookrightarrow \mathbb{P}(\mathbb{C}^k \boxtimes \mathbb{C}^{\ell+1-k})$
B_ℓ/P_1 D_ℓ/P_1	$B_{\ell-1}$ $D_{\ell-1}$	quadrics $Q^{2\ell-3} \hookrightarrow \mathbb{P}^{2\ell-2}$ $Q^{2\ell-4} \hookrightarrow \mathbb{P}^{2\ell-3}$
C_ℓ/P_ℓ	$A_{\ell-1}$	$\mathbb{P}^{\ell-1} \hookrightarrow \mathbb{P}(S^2\mathbb{C}^\ell)$
D_ℓ/P_ℓ	$A_{\ell-1}$	$\mathrm{Gr}(2,\ell) \hookrightarrow \mathbb{P}(\bigwedge^2 \mathbb{C}^\ell)$
E_6/P_6	D_5	$\mathbb{S}_5 = D_5/P_5 \hookrightarrow \mathbb{P}^{15}$
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Proposition

The top-slot $\mathfrak{g}_\nu$ is a sub-cominuscule G_0-module.

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Definition

Let G be complex simple. The TSOC is an ordered sequence $\{\beta_1, \beta_2, ...\} \subset \Delta(\mathfrak{g}_\nu)$, where $\beta_1 = \lambda$ is the highest root of $\mathfrak{g}$, and

$$\beta_j = \max\{\alpha \in \Delta(\mathfrak{g}_\nu) \,|\, \alpha \in \{\beta_1, ..., \beta_{j-1}\}^\perp\}, \quad j \geq 2.$$

(Remark: This max is *unique*.) Let e_γ be a root vector for γ.

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Definition

Let G be complex simple. The TSOC is an ordered sequence $\{\beta_1, \beta_2, ...\} \subset \Delta(\mathfrak{g}_\nu)$, where $\beta_1 = \lambda$ is the highest root of $\mathfrak{g}$, and

$$\beta_j = \max\{\alpha \in \Delta(\mathfrak{g}_\nu) \,|\, \alpha \in \{\beta_1, ..., \beta_{j-1}\}^\perp\}, \quad j \geq 2.$$

(Remark: This max is *unique*.) Let e_γ be a root vector for γ.

Theorem

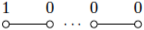
The TSOC parametrizes all G_0-orbits in $\mathbb{P}(\mathfrak{g}_\nu)$ via

$$[e_{\beta_1}], \quad [e_{\beta_1} + e_{\beta_2}], \quad [e_{\beta_1} + e_{\beta_2} + e_{\beta_3}], \quad ...$$

with $\langle \beta_i, \beta_i \rangle = \langle \lambda, \lambda \rangle$ for all i.

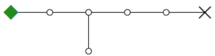
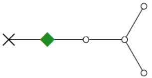
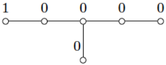
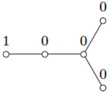
A Dynkin diagram recipe

Let $\mathfrak{T}^0(\mathfrak{g}_0^{ss}, \lambda)$ = effective $\mathfrak{g}_0^{ss}$-action on $\mathfrak{g}_\nu$. Iterative algorithm:

- Termination condition: $\mathfrak{T}^0(\mathfrak{g}_0^{ss}, \lambda) = \emptyset$ or 

- From $\mathfrak{D}(\mathfrak{g}, \mathfrak{p})$, remove contact node(s) (diamond), then remove cross-free connected components.

Example (E_7/P_7 : 3 G_0-orbits in $\mathbb{P}(\mathfrak{g}_\nu)$, $\nu = 1$)

Dynkin diagram sequence	 $\rightsquigarrow$	 $\rightsquigarrow$	$\times$
$\mathfrak{T}^0(\mathfrak{g}_0^{ss}, \lambda)$			$\emptyset$
TSOC in weight notation	$\beta_1 = \lambda_1$	$\beta_2 = -\lambda_1 + \lambda_6$	$\beta_3 = -\lambda_6 + 2\lambda_7$

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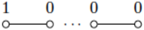
- Termination condition: $\mathfrak{T}^0(\mathfrak{g}_0^{ss}, \lambda) = \emptyset$ or 

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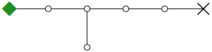
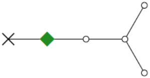
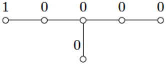
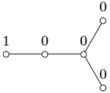
Dynkin diagram sequence	 $\rightsquigarrow$	 $\rightsquigarrow$	$\times$
$\mathfrak{T}^0(\mathfrak{g}_0^{ss}, \lambda)$			$\emptyset$
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β's are determined from contact nodes (in the original labelling). Note that $\sum_{i=1}^{j} \beta_i$ is dominant.

$\alpha(H) = B(H, H_\alpha)$, $h_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} H_\alpha$. Find std $\mathfrak{sl}_2$-triple $\{e_\alpha, h_\alpha, e_{-\alpha}\}$.

Adapted $\mathfrak{sl}_2$-triples

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Lemma

Have std $\mathfrak{sl}_2$-triples $\{E_j, H_j, F_j\}$ given by

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Express h_α via dual basis $\{Z_i\} \subset \mathfrak{h}$ to simple roots $\{\alpha_i\} \subset \mathfrak{h}^*$:

$$\alpha = \sum_i r_i \lambda_i \quad \Rightarrow \quad h_\alpha = \sum_i r_i \frac{\langle \alpha_i, \alpha_i \rangle}{\langle \alpha, \alpha \rangle} Z_i.$$

Rmk: $\langle \beta_i, \beta_i \rangle = \langle \lambda, \lambda \rangle$. Also, coeffs of all H_j wrt Z_i are ≥ 0.

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Example (E_7/P_7)

$H_1 = Z_1$, $H_2 = Z_6$ and $H_3 = 2Z_7$.

Specializing the Čap–Melnick criteria

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(CM.2) If $\alpha \in \Delta(\mathfrak{g}_+)$, then $\beta_i + \alpha \notin \Delta$ (since β_i are in the top-slot), so $\langle \beta_i, \alpha \rangle \geq 0$, and $\beta_i - \alpha \in \Delta$ iff $\langle \beta_i, \alpha \rangle > 0$. Have

$$[H_j, e_{-\alpha}] = \sum_{i=1}^{j} -\alpha(h_{\beta_i}) e_{-\alpha} = -\sum_{i=1}^{j} \underbrace{\langle \alpha, \beta_i^\vee \rangle}_{\in \mathbb{Z}_{\geq 0}} e_{-\alpha}.$$

Zero-eigenspace: sum of root spaces for $-\alpha \in \{\beta_1, ..., \beta_j\}^\perp$ (same as $C_{\mathfrak{g}}^-(E_j)$). Thus, (CM.2) ✓.

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(CM.3) H_j acts s.s. on $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ ✓. Wrt Z_i, coeffs of H_j are ≥ 0, so it suffices to check:

> (CM.3'): $H_j(\mu) \geq 0$, μ any lowest weight of $H^2_+(\mathfrak{g}_-, \mathfrak{g})$.

By Kostant, $\mu = -w \cdot \lambda$, where $w \in W^{\mathfrak{p}}(2)$.

Top-slot open orbits

Example (E_7/P_7; $\lambda = \lambda_1$, $w = (76)$)

$\mu = -w \cdot \lambda = [-2, -2, -3, -4, -3, -1, +1]$ (root notation).

- $H_1 = Z_1$: $H_1(\mu) = -2$;
- $H_2 = Z_6$: $H_2(\mu) = -1$;
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Theorem (General parabolic geometries)

Suppose that:

(i) $\omega_u(\xi)$ *lies in the* **open** G_0*-orbit of* $\mathfrak{g}_\nu$.

(ii) G/P *is not* $A_\ell/P_{s,s+1}$, $2 \leq s < \frac{\ell}{2}$ *or* B_ℓ/P_ℓ, $\ell \geq 5$ *odd.*

Then the geometry is flat on an open set $U \subset M$ *with* $x \in \overline{U}$.

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$y'' = (xy' - y)^3$ has $\mathfrak{sl}_2$ symmetry $x\partial_y + \partial_p$, $x\partial_x - y\partial_y - 2p\partial_p$, $y\partial_x - p^2\partial_p$. The isotropy dim at the origin is 2.

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Example (2-dim projective structures; A_2/P_1)

Above ODE example comes from a projective str. with syms $x\partial_y$, $x\partial_x - y\partial_y$, $y\partial_x$. The isotropy dim at the origin is 3.