Abstract

We demonstrate that the Betti numbers associated to an \( \mathbb{N}_0 \)-graded minimal free resolution of the Stanley-Reisner ring \( S/I_\Delta (d-1) \) of the \((d-1)\)-skeleton of a simplicial complex \( \Delta \) of dimension \( d \) can be expressed as a \( \mathbb{Z} \)-linear combination of the corresponding Betti numbers of \( \Delta \). An immediate implication of our main result is that the projective dimension of \( S/I_\Delta (d-1) \) is at most one greater than the projective dimension of \( S/I_\Delta \), and it thus provides a new and direct proof of this. Our result extends immediately to matroids and their truncations. A similar result for matroid elongations can not be hoped for, but we do obtain a weaker result for these.

1 Introduction

In this paper we investigate certain aspects of the relationship between an \( \mathbb{N}_0 \)-graded minimal free resolution of the Stanley-Reisner ring of a simplicial complex and those associated to its skeletons. Our main result is Theorem 3.1 which says that each of the Betti numbers associated to an \( \mathbb{N}_0 \)-graded minimal free resolution of \( S/I_\Delta (d-1) \), where \( I_\Delta (d-1) \) is the ideal generated by monomials corresponding to
nonfaces of the \((d - 1)\)-skeleton of a finite simplicial complex \(\Delta\), can be expressed as a \(\mathbb{Z}\)-linear sum of the Betti numbers associated to \(S/I_{\Delta}\).

Previous results on the Stanley-Reisner rings of skeletons include the classic [8, Corollary 2.6] which states that

\[
\text{depth } S/I_{\Delta} = \max \{ j : \Delta^{(j-1)} \text{ is Cohen-Macauley} \}. \tag{1}
\]

This result was later generalized to monomial ideals in [6, Corollary 2.5]. By the Auslander-Buchsbaum identity, it follows from (1) that

\[
p.d. \ I_{\Delta} \leq p.d. \ S/I_{\Delta^{(d-1)}} \leq 1 + p.d. \ S/I_{\Delta}.
\]

From the latter of these inequalities it is easily demonstrated, again by using the Auslander-Buchsbaum identity, that every skeleton of a Cohen-Macauley simplicial complex is Cohen-Macauley - a fact which was proved in [8, Corollary 2.5] as well.

That \(p.d. \ S/I_{\Delta^{(d-1)}} \leq 1 + p.d. \ S/I_{\Delta}\) can also be seen as an immediate consequence of our main result, and Theorem 3.2 thus provides a new and direct proof of this and therefore also of the fact that the Cohen-Macauley property is inherited by skeletons.

The projective dimension of Stanley-Reisner rings has seen recent research interest. Most notably, it was demonstrated in [12, Corollary 3.33] that

\[
p.d. \ S/I_{\Delta} \geq \max \{|C| : C \text{ is a circuit of the Alexander dual } \Delta^{*} \text{ of } \Delta\},
\]

with equality if \(S/I_{\Delta}\) is sequentially Cohen-Macauley.

Our main result extends immediately to a matroid \(M\) and its truncations. Such matroid truncations have themselves seen recent research interest. An example of this being [10], which contains the strengthening of a result by Brylawski [4, Proposition 7.4.10] concerning the representability of truncations.

Corresponding to our main result applied to matroid truncations, we give a considerably weaker result concerning matroid elongations. It says that the Betti table associated to the elongation of \(M\) to rank \(r\(M\) + 1 is equal to the Betti table obtained by removing the second column from the Betti table of \(S/I_{M}\) - but only in terms of zeros and nonzeros.

1.1 Structure of this paper

- In Section 2 we provide definitions and results used later on.
• In Section 3 we demonstrate that the Betti numbers associated to a \( \mathbb{N}_0 \)-graded minimal free resolution of the Stanley Reisner ring of a skeleton can be expressed as a \( \mathbb{Z} \)-linear combination of the corresponding Betti numbers of the original complex. This leads immediately to a new and direct proof that the property of being Cohen-Macauley is inherited from the original complex.

• In Section 4 we see how our main result applies to truncations of matroids. We also explore whether a similar result can be obtained for matroid elongations.

2 Preliminaries

2.1 Simplicial complexes

**Definition 2.1.** A simplicial complex \( \Delta \) on \( E = \{1, \ldots, n\} \) is a collection of subsets of \( E \) that is closed under inclusion.

We refer to the elements of \( \Delta \) as the faces of \( \Delta \). A facet of \( \Delta \) is a face that is not properly contained in another face, while a nonface is a subset of \( E \) that is not a face.

**Definition 2.2.** If \( X \subseteq E \), then \( \Delta|_X = \{ \sigma \subseteq X : \sigma \in \Delta \} \) is itself a simplicial complex. We refer to \( \Delta|_X \) as the restriction of \( \Delta \) to \( X \).

**Definition 2.3.** Let \( m \) be the cardinality of the largest face contained in \( X \subseteq E \). The dimension of \( X \) is \( \dim(X) = m - 1 \).

In particular, the dimension of a face \( \sigma \) is equal to \( |\sigma| - 1 \). We define \( \dim(\Delta) = \dim(E) \), and refer to this as the dimension of \( \Delta \).

**Definition 2.4 (The \( i \)-skeleton of \( \Delta \)).** For \( 0 \leq i \leq \dim(\Delta) \), let the \( i \)-skeleton \( \Delta^{(i)} \) be the simplicial complex

\[
\Delta^{(i)} = \{ \sigma \in \Delta : \dim(\sigma) \leq i \}.
\]

In particular, we have \( \Delta^{(d)} = \Delta \). The 1-skeleton \( \Delta^{(1)} \) is often referred to as the underlying graph of \( \Delta \).

**Remark.** Whenever \( \sigma \in \mathbb{N}_0^n \) the expression \( |\sigma| \) shall signify the sum of the coordinates of \( \sigma \). When, on the other hand, \( \sigma \subseteq \{1 \ldots n\} \), the expression \( |\sigma| \) denotes the cardinality of \( \sigma \).
2.2 Matroids

There are numerous equivalent ways of defining a matroid. It is most convenient here to give the definition in terms of independent sets. For an introduction to matroid theory in general, we recommend e.g. [13].

**Definition 2.5.** A matroid $M$ consists of a finite set $E$ and a non-empty set $I(M)$ of subsets of $E$ such that:

- $I(M)$ is a simplicial complex.
- If $I_1, I_2 \in I(M)$ and $|I_1| > |I_2|$, then there is an $x \in I_1 \setminus I_2$ such that $I_2 \cup x \in I(M)$.

The elements of $I(M)$ are referred to as the independent sets (of $M$). The bases of $M$ are the independent sets that are not contained in any other independent set; in other words, the facets of $I(M)$. Conversely, given the bases of a matroid, we find the independent sets to be those sets that are contained in a basis. We denote the bases of $M$ by $B(M)$. It is a fundamental result that all bases of a matroid have the same cardinality, which implies that $I(M)$ is a pure simplicial complex.

The dual matroid $\overline{M}$ is the matroid on $E$ whose bases are the complements of the bases of $M$. Thus

$$B(\overline{M}) = \{ E \setminus B : B \in B(M) \}.$$ 

**Definition 2.6.** For $X \subseteq E$, the rank function $r_M$ of $M$ is defined by

$$r_M(X) = \max \{|I| : I \in I(M), I \subseteq X \}.$$ 

Whenever the matroid $M$ is clear from the context, we omit the subscript and write simply $r(X)$. The rank $r(M)$ of $M$ itself is defined as $r(M) = r_M(E)$. Whenever $I(M)$ is considered as a simplicial complex we thus have $r(X) = \dim(X) + 1$ for all $X \subseteq E$, and $r(M) = \dim(I(M)) + 1$.

**Definition 2.7.** If $X \subseteq E$, then $\{ I \subseteq X : I \in I(M) \}$ form the set of independent sets of a matroid $M|_X$ on $X$. We refer to $M|_X$ as the restriction of $M$ to $X$.

**Definition 2.8 (Truncation).** The $i^{th}$ truncation $M^{(i)}$ of $M$ is the matroid on $E$ whose independent sets consist of the independent sets of $M$ that have rank less than or equal to $r(M) - i$. In other words

$$I(M^{(i)}) = \{ X \subseteq E : r(X) = |X|, r(X) \leq r(M) - i \}.$$
Observe that \( M^{(i)} = I(M)^{(r(M) - i - 1)} \), whenever \( I(M) \) is considered as a simplicial complex. That is, the \( i \)th truncation corresponds to the \( (d - i) \)-skeleton.

**Definition 2.9** (Elongation). For \( 0 \leq i \leq n - r(M) \), let \( M_{(i)} \) be the matroid whose independent sets are \( I(M_{(i)}) = \{ \sigma \in E : n(\sigma) \leq i \} \).

Since \( r(M_{(i)}) = r(M) + i \), the matroid \( M_{(i)} \) is commonly referred to as the elongation of \( M \) to rank \( r(M) + i \). It is straightforward to verify that for \( i \in [0, \ldots, n - r(M)] \) we have \( M_{(i)} = M^{(i)} \).

### 2.3 The Stanley-Reisner ideal, Betti numbers, and the reduced chain complex

Let \( \Delta \) be an abstract simplicial complex on \( E = \{1, \ldots, n\} \). Let \( \mathbb{K} \) be a field, and let \( S = \mathbb{K}[x_1, \ldots, x_n] \). By employing the standard abbreviated notation

\[
\lambda_1^{a_1} \lambda_2^{a_2} \cdots \lambda_n^{a_n} = x^a
\]

for monomials, we establish a 1–1 connection between monomials of \( S \) and vectors in \( \mathbb{N}_0^n \).

Furthermore, identifying a subset of \( E \) with its indicator vector in \( \mathbb{N}_0^n \) (as is done in Definition 2.10 below) thus provides a 1–1 connection between squarefree monomials of \( S \) and subsets of \( E \).

**Definition 2.10.** Let \( I_\Delta \) be the ideal in \( S \) generated by monomials corresponding to nonfaces of \( \Delta \). That is, let

\[
I_\Delta = \langle x^\sigma : \sigma \notin \Delta \rangle.
\]

We refer to \( I_\Delta \) and \( S/I_\Delta \), respectively, as the **Stanley-Reisner ideal** and **Stanley-Reisner ring** of \( \Delta \).

Being a (squarefree) monomial ideal, the Stanley-Reisner ideal, and thus also the Stanley-Reisner ring, permits both the standard \( \mathbb{N}_0 \)-grading and the standard \( \mathbb{N}_0^n \)-grading. For \( b \in \mathbb{N}_0^n \) let \( S_b \) be the 1-dimensional \( \mathbb{K} \)-vector space generated by \( x^b \), and let \( S(a) \), \( S \) shifted by \( a \), be defined by \( S(a)_b = S_{a+b} \). Analogously, for \( j \in \mathbb{N}_0 \) let \( S_i \) be the \( \mathbb{K} \)-vector space generated by monomials of degree \( i \), and let \( S(j)_i \) be defined by \( S(j)_i = S_{i+j} \). For the remainder of this section let \( N \) be an \( \mathbb{N}_0^n \)-graded \( S \)-module.
Definition 2.11. An \((\mathbb{N}_0^n\text{- or } \mathbb{N}_0)\)-graded minimal free resolution of \(N\) is a left complex

\[
\begin{array}{c}
0 & \leftarrow F_0 & \phi_1 & F_1 & \phi_2 & F_2 & \cdots & \phi_l & F_l & \leftarrow 0
\end{array}
\]

with the following properties:

- \(F_i = \bigoplus_{a \in \mathbb{N}_0^n} S(-a)^{\beta_i,a}, \mathbb{N}_0^n\)-graded resolution
- \(\bigoplus_{j \in \mathbb{N}_0} S(-j)^{\beta_i,j}, \mathbb{N}_0\)-graded resolution
- \(\text{im} \phi_i = \ker \phi_{i-1}\) for all \(i \geq 2\), and \(F_0 / \text{im} \phi_1 \cong N\) (Exact)
- \(\text{im} \phi_i \subseteq mF_{i-1}\) (Minimal)
- \(\phi_i((F_i)_a) \subseteq (F_{i-1})_a\) (Degree preserving, \(\mathbb{N}_0^n\)-graded case)
- \(\phi_i((F_i)_j) \subseteq (F_{i-1})_j\) (Degree preserving, \(\mathbb{N}_0\)-graded case).

It follows from [7, Theorem A.2.2] that the Betti numbers associated to a \((\mathbb{N}_0\text{- or } \mathbb{N}_0^n)\)-graded minimal free resolution are unique, in that any other minimal free resolution must have the same Betti numbers. We may therefore without ambiguity refer to \(\{\beta_i,a(N;\mathbb{k})\}\) and \(\{\beta_i,j(N;\mathbb{k})\}\), respectively, as the \(\mathbb{N}_0^n\)-graded and \(\mathbb{N}_0\)-graded Betti numbers of \(N\) (over \(\mathbb{k}\)). Observe that

\[
\beta_{i,j}(N;\mathbb{k}) = \sum_{|a|=j} \beta_{i,a}(N;\mathbb{k})
\]

where \(|a| = a_1 + a_2 + \cdots + a_n\) (see Remark 2.11 above). Note also that for an \(\mathbb{N}_0^n\)-graded (that is, monomial) ideal \(I \subseteq S\), we have \(\beta_{i,\sigma}(S/I;\mathbb{k}) = \beta_{i-1,\sigma}(I;\mathbb{k})\) for all \(i \geq 1\), and \(\beta_{0,\sigma}(S/I;\mathbb{k}) = \begin{cases} 1, & \sigma = \emptyset \\ 0, & \sigma \neq \emptyset \end{cases} \).

The \(\mathbb{N}_0\)-graded Betti numbers of \(N\) may be compactly presented in a so-called Betti table:

\[
\begin{array}{l|llll}
& 0 & 1 & \cdots & l \\
\hline
j & \beta_{0,j}(N;\mathbb{k}) & \beta_{1,j+1}(N;\mathbb{k}) & \cdots & \beta_{l,j+l}(N;\mathbb{k}) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
k & \beta_{0,k}(N;\mathbb{k}) & \beta_{1,k+1}(N;\mathbb{k}) & \cdots & \beta_{l,k+l}(N;\mathbb{k})
\end{array}
\]
By the (graded) Hilbert Syzygy Theorem we have $F_i = 0$ for all $i \geq n$. If $F_l \neq 0$ but $F_i = 0$ for all $i > l$, we refer to $l$ as the length of the minimal free resolution. It can be seen from e.g. [5, Corollary 1.8] that the length of a minimal free resolution of $N$ equals its projective dimension (p. d. $N$).

A sequence $f_1, \ldots, f_r \in \langle x_1, x_2, \ldots, x_n \rangle$ is said to be a regular $N$-sequence if $f_{i+1}$ is not a zero-divisor on $N/(f_1N + \cdots + f_iN)$.

**Definition 2.12.** The depth of $N$ is the common length of a longest regular $N$-sequence. Whenever $N$ is $\mathbb{N}_0$-graded the polynomials may be assumed to be homogeneous.

In general we have depth $N \leq \dim N$, where $\dim N$ denotes the Krull dimension of $N$. The following is a particular case of the famous Auslander-Buchsbaum Theorem.

**Theorem 2.1** (Auslander-Buchsbaum).

\[ \text{p. d. } N + \text{depth } N = n. \]

**Proof.** See e.g. [7, Corollary A.4.3].

Note that the Krull dimension $\dim S/I_\Delta$ of $S/I_\Delta$ is one more than the dimension of $\Delta$ (see [7, Corollary 6.2.2]). The simplicial complex $\Delta$ is said to be Cohen-Macauley if depth $S/I_\Delta = \dim S/I_\Delta$. That is, if $S/I_\Delta$ is Cohen-Macauley as an $S$-module.

**Definition 2.13.** Let $\mathcal{F}_i(\Delta)$ denote the set of $i$-dimensional faces of $\Delta$. That is,

\[ \mathcal{F}_i(\Delta) = \{ \sigma \in \Delta : |\sigma| = i + 1 \}. \]

Let $\mathbb{k}\mathcal{F}_i(\Delta)$ be the free $\mathbb{k}$-vector space on $\mathcal{F}_i(\Delta)$. The (reduced) chain complex of $M$ over $\mathbb{k}$ is the complex

\[ 0 \leftarrow \mathbb{k}\mathcal{F}_{-1}(\Delta) \xleftarrow{\delta_0} \cdots \xleftarrow{\delta_1} \mathbb{k}\mathcal{F}_{i-1}(\Delta) \xleftarrow{\delta_i} \mathbb{k}\mathcal{F}_i(\Delta) \xleftarrow{\delta_{i+1}} \cdots \xleftarrow{\delta_{\dim(\Delta)}} \mathbb{k}\mathcal{F}_{\dim(\Delta)} \leftarrow 0, \]

where the boundary maps $\delta_i$ are defined as follows: With the natural ordering on $E$, set $\text{sign}(j, \sigma) = (-1)^{r-j}$ if $j$ is the $r$th element of $\sigma \subseteq E$, and let

\[ \delta_i(\sigma) = \sum_{j \in \sigma} \text{sign}(j, \sigma) \sigma \setminus j. \]

Extending $\delta_i$ $\mathbb{k}$-linearly, we obtain a $\mathbb{k}$-linear map from $\mathbb{k}\mathcal{F}_i(\Delta)$ to $\mathbb{k}\mathcal{F}_{i-1}(\Delta)$. 

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**Definition 2.14.** The \(i\)th reduced homology of \(\Delta\) over \(\mathbb{k}\) is the vector space
\[
\check{H}_i(\Delta; \mathbb{k}) = \ker(\delta_i) / \im(\delta_{i+1}).
\]

The following is one of the most celebrated results in the intersection between algebra and combinatorics.

**Theorem 2.2** (Hochster’s formula).
\[
\beta_{i, \sigma}(S/I_\Delta; \mathbb{k}) = \beta_{i-1, \sigma}(I_\Delta; \mathbb{k}) = \dim_{\mathbb{k}} \check{H}_{|\sigma|-i-1}(\Delta|\sigma; \mathbb{k}).
\]

**Proof.** See [11, Corollary 5.12] and [7, p. 81].

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### 3 Betti numbers of \(i\)-skeletons

Let \(\Delta\) be a \(d\)-dimensional simplicial complex on \(\{1, \ldots, n\}\), and let \(\mathbb{k}\) be a field. In this section we shall demonstrate how each of the Betti numbers of \(S/I_{\Delta(d-1)}\) can be expressed as a \(\mathbb{Z}\)-linear combination of the Betti numbers of \(S/I_{\Delta}\).

#### 3.1 The first rows of the Betti table

**Lemma 3.1.**
\[
\check{H}_i(\Delta|\sigma; \mathbb{k}) = \check{H}_i(\Delta^{(d-1)}|\sigma; \mathbb{k})
\]
for all \(0 \leq i \leq d - 2\).

**Proof.** By the definition of a skeleton we have \(\mathcal{F}_i(\Delta|\sigma) = \mathcal{F}_i(\Delta^{(d-1)}|\sigma)\) and thus also \(\mathbb{k}\mathcal{F}_i(\Delta|\sigma) = \mathbb{k}\mathcal{F}_i(\Delta^{(d-1)}|\sigma)\), for all \(-1 \leq i \leq d - 1\). In other words, the reduced chain complexes of \(\Delta|\sigma\) and \(\Delta^{(d-1)}|\sigma\) are identical except for in homological degree \(d\). The result follows.

**Proposition 3.1.** For all \(i\) and \(j \leq d + i - 1\) we have
\[
\beta_{i,j}(S/I_\Delta; \mathbb{k}) = \beta_{i,j}(S/I_{\Delta(d-1)}; \mathbb{k}).
\]
Proof. If \( j \leq d + i - 1 \) then \( j - i - 1 \leq d - 2 \). By Theorem 2.2 and Lemma 3.1 then, we have

\[
\beta_{i,j}(S/I_{\Delta}; \mathbb{k}) = \sum_{|\sigma| = j} \beta_{i,\sigma}(S/I_{\Delta}; \mathbb{k}) = \sum_{|\sigma| = j} \dim_{\mathbb{k}} \tilde{H}_{|\sigma|-i-1}(\Delta_{|\sigma|}; \mathbb{k}) = \sum_{|\sigma| = j} \dim_{\mathbb{k}} \tilde{H}_{|\sigma|-i-1}(\Delta^{(d-1)}_{|\sigma|}; \mathbb{k}) = \sum_{|\sigma| = j} \beta_{i,\sigma}(S/I_{\Delta^{(d-1)}}; \mathbb{k}) = \beta_{i,j}(S/I_{\Delta^{(d-1)}}; \mathbb{k}).
\]

\( \square \)

3.2 The final row of the Betti table

The Hilbert series of \( S/I_{\Delta} \) over \( \mathbb{k} \) is \( H(S/I_{\Delta}) = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{k}}(S/I_{\Delta})_i t^i \). Let \( f_i(\Delta) = |\mathcal{F}_i(\Delta)| \). By [7, Section 6.1.3, Equation (6.3)] we have

\[
H(S/I_{\Delta}) = \frac{\sum_{i=0}^{n} (-1)^i \sum_{j} \beta_{i,j}(S/I_{\Delta}; \mathbb{k})}{(1 - t)^n}.
\]

On the other hand, we see from [7, Proposition 6.2.1] that

\[
H(S/I_{\Delta}) = \frac{\sum_{i=0}^{d} f_{i-1}(\Delta) t^i (1 - t)^{d+1-i}}{(1 - t)^{d+1}}.
\]

Combined, these two equations imply

\[
\sum_{i=0}^{d+1} f_{i-1}(\Delta) t^i (1 - t)^{n-i} = \sum_{i=0}^{n} (-1)^i \sum_{j} \beta_{i,j}(S/I_{\Delta}; \mathbb{k}) t^i,
\] (2)

and

\[
\sum_{i=0}^{d} f_{i-1}(\Delta^{(d-1)}) t^i (1 - t)^{n-i} = \sum_{i=0}^{n} (-1)^i \sum_{j} \beta_{i,j}(S/I_{\Delta^{(d-1)}}; \mathbb{k}) t^i.
\] (3)

Remark. From here on we shall employ the convention that \( i! = 0 \) for \( i < 0 \), and that \( \binom{j}{k} = 0 \) if one or both of \( j \) and \( k \) is negative.
Differentiating both sides of equation (2) \( n - d - 1 \) times, we get

\[
\sum_{i=0}^{d+1} f_{i-1}(\Delta) \sum_{l=0}^{n-d-1} (-1)^l \binom{n-d-1}{l} \frac{i!(n-i)!}{(i-n+d+1+l)! (n-i-l)!} t^{i-n+d+1+l} (1-t)^{n-i-l} = \sum_{i=0}^{n} (-1)^i \sum_{j \geq n-d-1} \beta_{i,j}(S/I_\Delta; \mathbb{k}) \frac{j!}{(j-(n-d-1))!} t^{j-n+d+1}.
\]

When evaluated at \( t = 1 \), the left side of the above equation is 0 except when \( i = d+1 \) and \( l = n - d - 1 \). Thus, we have

\[
(-1)^{n-d-1} (n - d - 1)! f_d(\Delta) = \sum_{i=0}^{n} (-1)^i \sum_{j \geq n-d-1} \beta_{i,j}(S/I_\Delta; \mathbb{k}) \frac{j!}{(j-(n-d-1))!},
\]

and

\[
f_d(\Delta) = \sum_{i=0}^{n} (-1)^{n+d+i+1} \sum_{j \geq n-d-1} \binom{j}{n-d-1} \beta_{i,j}(S/I_\Delta; \mathbb{k}).
\]

**Lemma 3.2.** For all \( i \) and \( j \geq d+i+2 \) we have

\[
\beta_{i,j}(S/I_\Delta; \mathbb{k}) = 0.
\]

**Proof.** If \(|\sigma| \geq d+i+2\), then \(|\sigma| - i - 1 \geq \dim(\Delta) + 1\), which implies

\[
\dim_{\mathbb{k}} \bar{H}_{|\sigma|-i-1}(\Delta; \mathbb{k}) = 0.
\]

So by Hochster’s formula we have that if \( j \geq d+i+2 \) then

\[
\beta_{i,j}(S/I_\Delta; \mathbb{k}) = \sum_{|\sigma|=j} \beta_{i,\sigma}(S/I_\Delta; \mathbb{k}) = \sum_{|\sigma|=j} \dim_{\mathbb{k}} \bar{H}_{|\sigma|-i-1}(\Delta; \mathbb{k}) = 0.
\]

\( \square \)

According to Proposition 3.1 and Lemma 3.2 and because \( f_i(\Delta) = f_i(\Delta^{(d-1)}) \) for all \( i \neq d \), subtracting equation (3) from equation (2) yields

\[
f_d(\Delta) t^{d+1} (1-t)^{n-d-1} = \sum_{i=0}^{n} (-1)^i (\beta_{i,d+i}(S/I_\Delta; \mathbb{k}) - \beta_{i,d+i}(S/I_{\Delta(d-1)}; \mathbb{k})) t^{d+i}
\]

\[
+ \sum_{i=0}^{n} (-1)^i \beta_{i,d+i+1}(S/I_\Delta; \mathbb{k}) t^{d+i+1}.
\]
Let $1 \leq u \leq n$. Differentiating both sides of the above equation $d + u$ times yields

$$
\frac{d}{d\Delta} \sum_{i=0}^{d+u} (-1)^i \binom{d+u}{i} \frac{(d+1)!}{(l-u+1)!(n-d-1-l)!} t^{l-u+1} (1-t)^{n-d-1-l}
$$

$$
= \sum_{i=u}^{n} (-1)^i \left( \beta_{d+i}(S/I_{\Delta}; \mathbb{K}) - \beta_{d+i+1}(S/I_{\Delta(d-1)}; \mathbb{K}) \right) \frac{(d+i)!}{(i-u)!} t^{i-u}
$$

$$
+ \sum_{i=u-1}^{n} (-1)^i \beta_{d+i+1}(S/I_{\Delta}; \mathbb{K}) \frac{(d+i+1)!}{(i-u+1)!} t^{i-u+1}.
$$

Evaluating at $t = 0$, we get

$$
\delta' * \left( (-1)^{u-1} \frac{d}{d\Delta} \frac{(d+u)!}{(u-1)!(n-d-u)!} \right)
$$

$$
= (-1)^u \text{big} \left( \beta_{u,d+u}(S/I_{\Delta}; \mathbb{K}) - \beta_{u,d+u}(S/I_{\Delta(u-1)}; \mathbb{K}) \right) (d+u)! + (-1)^{u-1} \beta_{u-1,d+u}(S/I_{\Delta}; \mathbb{K}) (d+u)!,
$$

where

$$
\delta' = \begin{cases} 
1, & 1 \leq u \leq n-d \\
0, & u > n-d
\end{cases}
$$

Summarizing the above:

**Proposition 3.2.** For $1 \leq u \leq n$, we have

$$
\beta_{u,d+u}(S/I_{\Delta(d-1)}; \mathbb{K}) = \beta_{u,d+u}(S/I_{\Delta}; \mathbb{K}) - \beta_{u-1,d+u}(S/I_{\Delta}; \mathbb{K}) + \binom{n-d-1}{u-1} \delta,
$$

where

$$
\delta = \begin{cases} 
\frac{d}{d\Delta} = \sum_{i=0}^{n} (-1)^{n+d+i} \sum_{j=0}^{n-d-1} \binom{j}{n-d-1} \beta_{i,j}(S/I_{\Delta}; \mathbb{K}), & 1 \leq u \leq n-d \\
0, & u > n-d
\end{cases}
$$

Bringing together Propositions 3.1 and 3.2, we get

**Theorem 3.1.** For all $i \geq 1$, we have

$$
\beta_{i,j}(S/I_{\Delta(d-1)}; \mathbb{K}) = \begin{cases} 
\beta_{i,j}(S/I_{\Delta}; \mathbb{K}), & j \leq d + i - 1 \\
\beta_{i,d+i}(S/I_{\Delta}; \mathbb{K}) - \beta_{i-1,d+i}(S/I_{\Delta}; \mathbb{K}) + \binom{n-d-1}{i-1} \delta, & j = d + i \\
0, & j \geq d + i - 1
\end{cases}
$$
where
\[
\delta = \begin{cases} 
  f_d(\Delta) = \sum_{k=0}^{n} (-1)^{n+d+k+1} \sum_{j \geq n-d-1}^{j} \binom{j}{n-d-1} \beta_{k,j}(S/I_{\Delta}; F_3), & 1 \leq i \leq n-d \\
  0, & i > n-d. 
\end{cases}
\]

**Example 3.1.** Let $T$ be one of the two irreducible triangulations of the real projective plane (see [1]) – namely the one corresponding to an embedding of the complete graph on 6 vertices. Clearly then, we have $n = 6$ and $d = 2$. The Betti table of $S/I_T$ over $F_3$ is

\[
\beta[S/I_T](F_3) = \begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
3 & 0 & 10 & 15 & 6
\end{array}
\]

In this case $f_d(\Delta) = \binom{4}{3} \beta_{1,4}(S/I_T; F_3) - \binom{3}{3} \beta_{2,5}(S/I_T; F_3) + \binom{6}{3} \beta_{3,6}(S/I_T; F_3) = 10$. By Theorem 3.1 the Betti numbers of $S/I_{T(1)}$ are

\[
\begin{align*}
\beta_{1,4}(S/I_{T(1)}; F_3) &= \beta_{1,4}(S/I_T; F_3) + \binom{3}{0} \delta = 10 + 10. \\
\beta_{2,5}(S/I_{T(1)}; F_3) &= \beta_{2,5}(S/I_T; F_3) - \beta_{1,5}(S/I_T; F_3) + \binom{3}{1} \delta = 15 + 30. \\
\beta_{3,6}(S/I_{T(1)}; F_3) &= \beta_{3,6}(S/I_T; F_3) - \beta_{2,6}(S/I_T; F_3) + \binom{3}{2} \delta = 6 - 0 + 30. \\
\beta_{4,7}(S/I_{T(1)}; F_3) &= \beta_{4,7}(S/I_T; F_3) - \beta_{3,7}(S/I_T; F_3) + \binom{3}{3} \delta = 0 - 0 + 10.
\end{align*}
\]

\[
\beta[S/I_{T(1)}](F_3) = \begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 20 & 45 & 36 & 10
\end{array}
\]
Remark. Observe that as
\[
\beta_\ell[S/I_T](\mathbb{F}_2) =
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
3 & 0 & 10 & 15 & 6 & 1 \\
4 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]
the simplicial complex \(T\) of Example 3.1 is an example of a pure simplicial complex whose Betti numbers depend upon the field \(\mathbb{K}\) – as opposed to what is the case for matroids.

### 3.3 The projective dimension of skeletons

Let \(p.d. S/I_\Delta\) denote the projective dimension of \(S/I_\Delta\). By Auslander-Buchsbaum Theorem we have
\[
p.d. S/I_\Delta = n - \text{depth } S/I_\Delta \\
\geq n - \dim S/I_\Delta \\
= n - (d + 1),
\]
so \(n - d - 1 \leq p.d. S/I_\Delta \leq n\).

As for the skeletons, we have

**Corollary 3.1.**
\[
\]

**Proof.** Let \(p = p.d. S/I_\Delta\). By Proposition 3.1 it suffices to show that
\[
\beta_{p+2,d+p+2}(S/I_{\Delta(d-1)}; \mathbb{K}) = 0.
\]

But by Theorem 3.2 we have
\[
\beta_{p+2,d+p+2}(S/I_{\Delta(d-1)}; \mathbb{K}) = \beta_{p+2,d+p+2}(S/I_\Delta; \mathbb{K}) - \beta_{p+1,d+p+2}(S/I_\Delta; \mathbb{K}) + \delta \\
= 0 - 0 - \delta = 0,
\]
where the last equality is due to \(p + 2 > n - d\). \(\square\)

**Corollary 3.2.** If \(\Delta\) is Cohen-Macauley, then so is \(\Delta^{(d-1)}\).
Proof. Let $\Delta$ be a simplicial complex with $\dim(\Delta) = d$ and $\text{depth } S/I_\Delta = \dim S/I_\Delta$. As $\dim S/I_{\Delta(d-1)} = d$, we only need to prove that $\text{depth } S/I_{\Delta(d-1)} = d$ as well.

Since $\dim S/I_{\Delta(d-1)} \leq \dim S/I_{\Delta(d-1)} = d$, we have by the Auslander-Buchsbaum Theorem that $\text{p.d. } S/I_{\Delta(d-1)} \geq n-d$. On the other hand, since $\text{p.d. } S/I_{\Delta(d-1)} = n-d$, we see from Corollary 3.1 that $\text{p.d. } S/I_{\Delta(d-1)} \leq n-d$. We conclude that $\text{p.d. } S/I_{\Delta(d-1)} = n-d$ and, by Auslander-Buchsbaum again, that $\text{depth } S/I_{\Delta(d-1)} = d$. 

4 Betti numbers of truncations and elongations of matroids

Let $M$ be a matroid on $\{1, \ldots, n\}$, with $r(M) = k$. As was established in [3], the dimension of $\tilde{H}_i(M; \mathbb{F})$ is in fact independent of the field $\mathbb{F}$. Thus for matroids, the ($\mathbb{N}_0$- or $\mathbb{N}_0^n$-graded) Betti numbers are not only unique, but independent of the choice of field. We shall therefore omit referring to or specifying a particular field $\mathbb{F}$ throughout this section. By a slight abuse of notation we shall denote the Stanley-Reisner ideal associated to the set of independent sets $I(M)$ of $M$ simply by $I_M$.

4.1 Truncations

Note that the $i$th truncation of $M$ corresponds to the $(k-i-1)$-skeleton of $I(M)$; a fact which enables us to invoke Theorem 3.1. In addition, it follows from [9, Corollary 3(b)] that the minimal free resolutions of $S/I_M$ have length $n-k$. We thus have

Proposition 4.1. For all $i$, we have

$$
\beta_{i,j}(S/I_M^{(1)}) = \begin{cases} 
\beta_{i,j}(S/I_M), & j \leq k+i-2, \\
\beta_{i,k+i-1}(S/I_M) - \beta_{i-1,k+i-1}(S/I_M) \\
+ \sum_{n-k}^n (-1)^{n-k} \sum_{v \geq n-k} \binom{v}{n-k} \beta_{u,v}(S/I_M), & j = k+i-1, \\
0, & j \geq k+i.
\end{cases}
$$
4.2 Elongations

When it comes to elongations, the Betti numbers of $M$ provide far less information about the Betti numbers of $M(1)$ than what was the case with truncations. We do however have the following.

**Proposition 4.2.** For $i \geq 1$,

$$\beta_{i,j}(I_{M(l)}) \neq 0 \iff \beta_{i-1,j}(I_{M(l+1)}) \neq 0.$$  

**Proof.** According to [9, Theorem 1], we have that $$\beta_{i,\sigma}(I_M) \neq 0 \iff \sigma$$ is minimal with the property that $n_{M}(\sigma) = i + 1$.

Since $\beta_{i,j} = \sum_{|\sigma| = j} \beta_{i,\sigma}$, we see that

$$\beta_{i,j}(I_{M(l)}) \neq 0 \iff$$

There is a $\sigma$ such that $|\sigma| = j$ and $\sigma$ is minimal with the property that $n_{M(l)}(\sigma) = i + 1$

$$\iff$$

There is a $\sigma$ such that $|\sigma| = j$ and $\sigma$ is minimal with the property that $n_{M(l+1)}(\sigma) = i$

$$\iff$$

$$\beta_{i-1,j}(I_{M(l+1)}) \neq 0.$$  

\square

In terms of Betti tables, this implies that when it comes to zeros and nonzeros the Betti table of $I_{M(l)}$ is equal to the table you get by deleting the first column from the table of $I_{M}$. As the following counterexample (computed using MAGMA [2]) demonstrates, there can be no result for elongations analogous to Theorem 3.1.

Let $M$ and $N$ be the matroids on $\{1, \ldots, 8\}$ with bases

$$B(M) = \{ \{1,3,4,6,7\}, \{1,2,3,6,8\}, \{1,2,3,4,8\}, \{1,2,3,5,8\}, \{1,2,5,6,8\},$$

$$\{1,2,3,4,7\}, \{1,2,3,5,7\}, \{1,2,5,6,7\}, \{1,3,4,5,7\}, \{1,3,4,6,8\},$$

$$\{1,2,4,6,8\}, \{1,2,4,6,7\}, \{1,3,4,5,8\}, \{1,2,4,5,7\}, \{1,4,5,6,7\},$$

$$\{1,2,3,6,7\}, \{1,3,5,6,7\}, \{1,4,5,6,8\}, \{1,3,5,6,8\}, \{1,2,4,5,8\} \}$$
and

\[ B(N) = \{ \{1,3,4,6,7\}, \{1,2,3,4,8\}, \{1,2,3,5,8\}, \{1,2,5,6,8\}, \{1,2,3,4,7\}, \{1,2,3,5,7\}, \{1,2,5,6,7\}, \{1,3,4,5,7\}, \{1,3,4,6,8\}, \{1,2,4,6,8\}, \{1,2,4,6,7\}, \{1,3,4,5,8\}, \{1,2,4,5,7\}, \{1,3,4,5,6\}, \{1,2,4,5,6\}, \{1,3,5,6,7\}, \{1,2,3,5,6\}, \{1,2,3,4,6\}, \{1,3,5,6,8\}, \{1,2,4,5,8\}\} \].

Both \( I_M \) and \( I_N \) have Betti table

\[
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
2 & 1 & 0 & 0 \\
3 & 0 & 0 & 0 \\
4 & 1 & 4 & 0 \\
5 & 0 & 5 & 4 \\
\end{array}
\]

but while \( I_{M(1)} \) has Betti table

\[
\begin{array}{c|c}
 & 1 & 2 \\
\hline
5 & 1 & 0 \\
6 & 5 & 5 \\
\end{array}
\]

the ideal \( I_{N(1)} \) has Betti table

\[
\begin{array}{c|c}
 & 1 & 2 \\
\hline
5 & 2 & 0 \\
6 & 3 & 4 \\
\end{array}
\]

This shows that the Betti numbers associated to a matroid do not determine those associated to its elongation.

**References**


