LECTURE SERIES ON (CLASSICAL) BRST COHOMOLOGY

References:

- Notes on BRST Cohomology (2006) — Figueroa-O'Farrill
- Geometric BRST Quantization, I (CMP 1991) — FoF & Kimura
- Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras (AoP 1987) — Kostant & Sternberg
- BRST quantization ('70s) — Becchi, Rouet, Stora, Tyutin

Rough idea: BRST differential is Chevalley-Eilenberg differential and a contractible contribution.

COMPLEXES

$$C = \bigoplus_{p \in \mathbb{Z}} C^p \qquad \mathbb{Z}\text{-graded v.s.}$$

$$d : C \to C, \quad d(C^p) \subseteq C^{p+1}, \quad d^2 = 0$$

$(C^\bullet, d)$ is called "graded differential complex"

C "space of cochains"

d "differential"

$Z = \operatorname{Ker} d$ "space of cocycles"

$B = \operatorname{Im} d$ "space of coboundaries"

$(C^\bullet, d)$ is often depicted $\cdots \longrightarrow C^{p-1} \xrightarrow{d^{p-1}} C^p \xrightarrow{d^p} C^{p+1} \longrightarrow \cdots$

$$d^{p+1} \circ d^p = 0$$

$$H = H(C^\bullet, d) = Z/B \quad \text{"cohomology"}$$

$$H = \bigoplus_{p \in \mathbb{Z}} H^p, \quad H^p = Z^p / B^p = \ker d^p / \operatorname{Im} d^{p-1}.$$

Def A differential graded algebra (DGA) is $(C^\bullet; d)$ s.t.

(i) C is an algebra w/t $C^i \cdot C^j \subseteq C^{i+j}$;

(ii) d is a derivation of degree 1, i.e.,

$$d(\alpha\beta) = d\alpha \cdot \beta + (-1)^{|\alpha|} \alpha \cdot d\beta \qquad |\alpha| = \text{degree of } \alpha$$

Often tacitly assumed algebra is associative w/t unity $\mathbb{1} \in C^0$ and "commutative" in the sense that $\alpha \cdot \beta = (-1)^{|\alpha||\beta|} \beta \cdot \alpha$.

EX: M mnfd, de Rham complex $C^\bullet = \Omega^\bullet(M)$ is DGA (w.r.t. exterior derivative and wedge product)

EXERCISE: If $(C^\bullet; d)$ is DGA then $H(C; d)$ is (associative, commutative, graded) algebra.

CHEVALLEY-EILENBERG COHOMOLOGY

$\mathfrak{g}$ = Lie algebra $\qquad \rho : \mathfrak{g} \to \mathfrak{gl}(V)$ representation

$$C^p(\mathfrak{g}; V) := \Lambda^p \mathfrak{g}^* \otimes V \quad \text{"space of } p\text{-forms on } \mathfrak{g} \text{ w/t values in } V\text{"}$$

$$d : C^p(\mathfrak{g}; V) \longrightarrow C^{p+1}(\mathfrak{g}; V)$$

- $v \in V$: $dv(X) := \rho(X)v \quad \forall X \in \mathfrak{g}$
- $\alpha \in \mathfrak{g}^*$: $d\alpha(X,Y) := -\alpha[X,Y] \ \forall X,Y \in \mathfrak{g}$
- extend to $\Lambda^\cdot \mathfrak{g}^*$ by $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$
- extend to $\Lambda^\cdot \mathfrak{g}^* \otimes V$ by $d(\alpha \otimes v) = d\alpha \otimes v + (-1)^{|\alpha|} \alpha \wedge dv$

EXERCISES:

(i) Show $d^2 = 0$

(ii) Show $$d\omega(X_0, \dots, X_p) = \sum_{i=0}^{p} (-1)^i \rho(X_i)\, \omega(X_0, \dots, \hat{X}_i, \dots, X_p)$$
$$+ \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p)$$

For all $\omega \in \Lambda^p \mathfrak{g}^* \otimes V$.

AN OPERATOR EXPRESSION FOR d (used in physical applications of CE cohomology)

$\{X_i\}$ basis of $\mathfrak{g}$ w/t $[X_i, X_j] = f_{ij}^k X_k$, $\{\alpha^i\}$ dual basis of $\mathfrak{g}^*$

$b_i = i_{X_i} : \Lambda^p \mathfrak{g}^* \longrightarrow \Lambda^{p-1} \mathfrak{g}^*$ "ANTIGHOST"

$c^i = \alpha^i \wedge \cdot : \Lambda^p \mathfrak{g}^* \longrightarrow \Lambda^{p+1} \mathfrak{g}^*$ "GHOST"

In terms of these operations and the representation ρ, which we abstract, we may write BRST operator (not to be confused w/t complete BRST operator)

$d = c^i X_i - \frac{1}{2} f_{ij}^k c^i c^j b_k$, which increases "ghost number" by 1.

(Indeed $dv = c^i X_i \cdot v = \alpha^i \otimes X_i \cdot v$, $d\alpha^k = -\frac{1}{2} f_{ij}^k \alpha^i \wedge \alpha^j$.)

REM: In context of quantized gauge theories, fields of unnatural statistics (or ghosts) were introduced to compensate propagation of unphysical degrees of freedom. The Lagrangian of Faddeev-Popov is d-invariant

$$\underbrace{\cdots \to C^{r-1}(\mathfrak{g};V) \to C^r(\mathfrak{g};V) \to C^{r+1}(\mathfrak{g};V) \to \cdots}_{\text{CE complex}}$$

$$\underbrace{H^{\bullet}(\mathfrak{g};V) = \bigoplus_{r\geq 0} H^r(\mathfrak{g};V)}_{\text{CE cohomology}}$$

EX:

(i) $H^0(\mathfrak{g};V) = V^{\mathfrak{g}}$ "invariants of V"

(ii) $H^1(\mathfrak{g};\mathbb{R}) = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$

(iii) $H^2(\mathfrak{g};\mathbb{R})$ = equiv. classes of central extensions $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}\cdot Z$ central element

$$[c] \longmapsto [X+\lambda Z, Y+\mu Z]_{\hat{\mathfrak{g}}} := [X,Y]_{\mathfrak{g}} + c(X,Y)Z$$

(iv) $H^1(\mathfrak{g};\mathfrak{g}) = \{\text{derivations of } \mathfrak{g}\} / \{\text{inner derivations}\}$

(v) $H^2(\mathfrak{g};\mathfrak{g})$ = equiv. classes of infinitesimal deformations of $\mathfrak{g}$

$$\mu_t : \Lambda^2\mathfrak{g} \to \mathfrak{g} \quad \forall\, t \in (-\varepsilon,\varepsilon) \text{ s.t. } \begin{cases} \mu_t \text{ is Lie bracket} \\ \mu_0(X,Y) = [X,Y] \end{cases}$$

μ_t and μ_t' are called equivalent if $\exists\, u_t : \mathfrak{g} \to \mathfrak{g}$ s.t. $\begin{cases} \mu_t' = u_t^* \mu_t \\ u_0 = id \end{cases}$

Check that $\left.\frac{\partial}{\partial t}\right|_{t=0} \mu_t \in Z^2(\mathfrak{g};\mathfrak{g})$ and if μ_t and μ_t' are equivalent then

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \mu_t' = \left.\frac{\partial}{\partial t}\right|_{t=0} \mu_t + d\underbrace{\left(\left.\frac{\partial}{\partial t}\right|_{t=0} u\right)}_{\in C^1(\mathfrak{g};\mathfrak{g})}.$$

RESOLUTIONS

Left resolution : graded differential complex $(K^\cdot, \delta)$ of shows
(or projective)

$$\cdots \to K^q \xrightarrow{\delta} K^{q-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} K^1 \xrightarrow{\delta} K^0 \xrightarrow{\delta} 0$$

such that its homology is in zero degree

$$H^q(K^\cdot, \delta) \simeq \begin{cases} V & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

Right resolution : the same but w/t $0 \xrightarrow{\delta} K^0 \xrightarrow{\delta} K^1 \to \cdots$
(acinjective)

REM: The (left) resolution can be "augmented" to an exact sequence

$$\cdots \xrightarrow{\delta} K^1 \xrightarrow{\delta} K^0 \xrightarrow{\varepsilon} V \to 0$$

where the "augmentation map" $\varepsilon : K^0 \to K^0/\delta K^1 \simeq V$ is canonical projection. Sometimes it is written $K^\cdot \xrightarrow{\varepsilon} V \to 0$.

REM Usually we want V to be a module and by a result of Jacobson there always exists a projective resolution $(K^\cdot, \delta)$ which is a differential complex of modules (idea: K^0 = free module generated by elements of V
K^1 = " " in the Kernel of $K^0 \xrightarrow{\varepsilon} V$
and so on ... This is the complex of syzygies)

A resolution of V is an homological description of V in terms of simpler objects.

RESOLUTIONS OF A $\mathfrak{g}$-MODULE V

$\mathfrak{g}$ Lie algebra $\quad \rho: \mathfrak{g} \to \mathfrak{gl}(V)$ rep.

$$\cdots \xrightarrow{\delta} K^q \xrightarrow{\delta} K^{q-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} K^1 \xrightarrow{\delta} K^0 \xrightarrow{\delta} 0$$

resolution of V, i.e., complex $(K^\bullet, \delta)$ of $\mathfrak{g}$-modules w/t only non-trivial homology V in degree zero.

Idea: tensor the resolution w/t $\Lambda^\bullet \mathfrak{g}^*$ to get

$$\begin{tikzcd}
\cdots \arrow[r] & C^{p-1}(\mathfrak{g}; K^{q+1}) \arrow[r, "d"] \arrow[d, "\delta"] & C^{p}(\mathfrak{g}; K^{q+1}) \arrow[r, "d"] \arrow[d, "\delta"] & C^{p+1}(\mathfrak{g}; K^{q+1}) \arrow[r] \arrow[d, "\delta"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(\mathfrak{g}; K^{q}) \arrow[r, "d"] \arrow[d, "\delta"] & C^{p}(\mathfrak{g}; K^{q}) \arrow[r, "d"] \arrow[d, "\delta"] & C^{p+1}(\mathfrak{g}; K^{q}) \arrow[r] \arrow[d, "\delta"] & \cdots \\
\cdots \arrow[r] & C^{p-1}(\mathfrak{g}; K^{q-1}) \arrow[r, "d"] & C^{p}(\mathfrak{g}; K^{q-1}) \arrow[r, "d"] & C^{p+1}(\mathfrak{g}; K^{q-1}) \arrow[r] & \cdots
\end{tikzcd}$$

EXERCISE: Use that δ is a map of $\mathfrak{g}$-modules to show that the diagram is commutative.

We have a "bigraded" complex $C^{p,q} := C^p(\mathfrak{g}; K^q)$ w/t anticommuting differentials:

- CE differential $D' = d : C^{p,q} \to C^{p+1,q}$
- resolution differential $D'' = (-1)^p \delta : C^{p,q} \to C^{p,q-1}$

$\Longrightarrow$ $D = D' + D''$ satisfies $D^2 = 0$ and it sends C^n into C^{n+1}, where $C^n := \bigoplus_{p-q=n} C^{p,q}$.

$(C^\bullet, D)$ "total complex" or "BRST complex"

D "total differential" or "BRST differential"

$n = p - q$ "total degree" or "ghost number"

THM $H^n(C^\bullet, D) \underset{\text{v.s.}}{\cong} H^n(\mathfrak{g}; V)$ for all $n \in \mathbb{Z}$.

Pf: I STEP (TIC-TAC-TOE TECNIQUE) $V \cong K^0/\delta K^1 \Rightarrow C^n(\mathfrak{g}; V) \cong \dfrac{C^n(\mathfrak{g}; K^0)}{C^n(\mathfrak{g}; \delta K^1)} = \dfrac{C^{n,0}}{\delta C^{n,1}}$

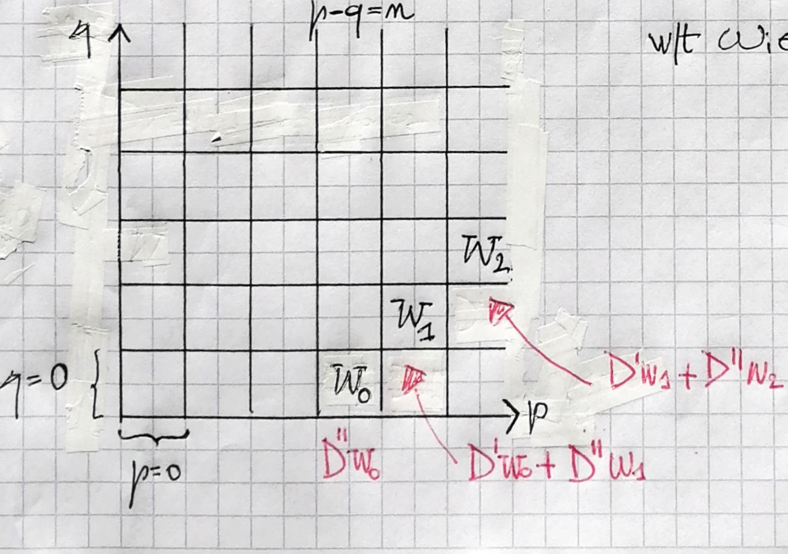
$d : C^n(\mathfrak{g}; V) \to C^{n+1}(\mathfrak{g}; V)$ is induced from $d : C^{n,0} \to C^{n+1,0}$ and projecting modulo $\delta C^{n+1,1}$.

Hence $Z^n(\mathfrak{g}; V) \cong \{ w \in C^{n,0} \mid dw \in \delta C^{n+1,1} \} / \delta C^{n,1}$

$$B^n(\mathfrak{g}; V) \cong dC^{n-1,0} + \delta C^{n,1} / \delta C^{n,1}$$

II STEP $\omega \in C^n = \bigoplus_{p-q=n} C^{p,q}$ $\quad \omega = \omega_0 + \omega_1 + \omega_2 + \cdots$ w/t $\omega_i \in C^{n+i,i}$

EX: $n=3$



$$D\omega=0 \Longleftrightarrow \begin{cases} D''\omega_0=0 \quad (\text{automatic}) \\ D'\omega_0+D''\omega_1=0 \\ D'\omega_1+D''\omega_2=0 \\ \vdots \\ D'\omega_{top}=0 \quad (\text{automatic}) \end{cases}$$

We claim that $\omega \mapsto \omega_0$ gives rise to a map in cohomology $H^n(C^\bullet;D) \to H^n(\mathfrak{g};V)$. Indeed $d\omega_0 \in \delta C^{n+1,1}$, hence it defines a cocycle in $Z^n(\mathfrak{g};V)$ and if $\omega=D\varphi$ for some $\varphi=\varphi_0+\varphi_1+\dots \in C^{n-1}$, $\varphi_i \in C^{n-1+i,i}$, then in particular $\omega_0=D'\varphi_0+D''\varphi_1 \in dC^{n-1,0}+\delta C^{n,1}$, which defines a coboundary in $B^n(\mathfrak{g};V)$.

III STEP

$\omega_0 \in C^{n,0}$ s.t. $D'\omega_0 \in D''C^{n+1,1}$ (this defines cocycle in $Z^n(\mathfrak{g};V)$)

$\Rightarrow D'\omega_0+D''\omega_1=0 \Rightarrow 0=D'D'\omega_0=D''D'\omega_1$

$\Rightarrow D'\omega_1 \in C^{n+2,1}$ is D''-cocycle

But D'' has no cohomology there, hence $D'\omega_1+D''\omega_2=0$ for some $\omega_2 \in C^{n+2,2}$. Continuing in this way we get $\omega \in C^n$ s.t. $D\omega=0$. (If $\omega \in B^n(\mathfrak{g};V)$ then $\omega_0=D'\varphi_0+D''\varphi_1$ for some $\varphi_0 \in C^{n-1,0}$ and $\varphi_1 \in C^{n,1}$. Hence $D''\omega_1=-D'\omega_0=D''D'\varphi_1$ and again $\omega_1=D'\varphi_1+D''\varphi_2$ for some $\varphi_2 \in C^{n+1,2}$ and so on until $\omega=D\varphi$.

So we have a map $H^n(\mathfrak{g};V) \longrightarrow H^n(C^\bullet;D)$.

IV STEP The maps are inverses one to the other. ∎