Def. An associative, commutative and with unity DGA $(K^\bullet, d)$ is called a <u>Koszul algebra</u> if $\exists\ \mathbb{Z}$-graded v.s. $E = \bigoplus_{k \in 2\mathbb{Z}+1} E^k$ s.t.

(i) $K = \Lambda^\bullet E \otimes R$ as tensor product of algebras, where R is a graded subalgebra in even degrees,

(ii) $K^p = (0)$ for all $p \lneq 0$ (equivalently $E^p = R^p = (0)$ for all $p \lneq 0$) and $R^0 = \mathbb{K}\mathbb{1}$

(iii) $dE \subseteq R$, $dR = 0$.

⚠ The index p in K^p is <u>not</u> the "exterior grading".

REM If $\dim E = \ell \lneq +\infty$ and an ordered basis $\{e_1, \dots, e_\ell\}$ of E is fixed then d is determined by $de_i = \phi_i \in R$, $1 \leq i \leq \ell$.

EX: For $W = \bigoplus_{k \gneq 0} W^k$, $W_{\bar 0} = \bigoplus_{k \in 2\mathbb{Z}} W^k$, $W_{\bar 1} = \bigoplus_{k \in 2\mathbb{Z}+1} W^k$ the Koszul algebra

$$K = \Lambda^\bullet W_{\bar 1} \otimes S^\bullet W_{\bar 0}$$

is called a <u>Cartan algebra</u>.

It is useful to consider "exterior grading" $K = \bigoplus_{q \geq 0} K_q$, where $K_q = \Lambda^q E \otimes R$. Clearly K is also a DGA w.r.t. this grading but with d a derivation of "exterior degree" -1. In particular $(K_\bullet, d)$ is a chain complex:

$$\underbrace{\begin{gathered}\cdots \to \Lambda^q E \otimes R \xrightarrow{d} \Lambda^{q-1} E \otimes R \xrightarrow{d} \cdots \cdots \longrightarrow E \otimes R \xrightarrow{d} R \xrightarrow{\phi} 0 \\ d(e_{i_1} \wedge \cdots \wedge e_{i_q} \otimes u) = \sum_{j=1}^{q} (-1)^{j+1} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_q} \otimes \phi_{i_j} u \text{ if } \dim E < +\infty\end{gathered}}_{\text{Koszul complex}}$$

$H(K_\bullet, d) = \bigoplus_{q \geq 0} H_q(K_\bullet, d)$ is associative, commutative and with unity bigraded algebra

Important fact: let $J = (dE)$ be the ideal of R generated by dE ($J = (\phi_1, \dots, \phi_\ell)$ if $\dim E < +\infty$). Then $H_0 = R/J$. We assume $\dim E \lneq +\infty$ from now on.

IDEA: Koszul complex tells us when a sequence $\{\phi_1,\dots,\phi_l\}\subseteq R$ is regular.

Def: $\{\phi_1,\dots,\phi_l\}$ is called regular if ϕ_j is not a zero-divisor in R/J_{j-1} where $J_{j-1}=(\phi_1,\dots,\phi_{j-1})$ for all $1\le j\le l$.

(Concretely if $u\in R$ is s.t. $u\phi_j\in(\phi_1,\dots,\phi_{j-1})$ then $u\in(\phi_1,\dots,\phi_{j-1})$.)

THM Assume $|\phi_j|=2k_j \gneq 0$ for all $1\le j\le l$. Then the following are equivalent:

1. $\{\phi_1,\dots,\phi_l\}\subseteq R$ is regular
2. $H_1(K_\bullet,d)=0$
3. $H_q(K_\bullet,d)=0$ for all $q\ge 1$

(i.e. $(K_\bullet,d)$ is free resolution by R-modules of R/J)

EX: $K=\Lambda^\bullet W_{\bar 1}\otimes S^\bullet W_{\bar 0}$ w/t $W_{\bar 0}=W_{\bar 1}$ as vector spaces and $de_i=e_i$ (for all $1\le j\le l$.
Then $(K_\bullet,d)$ is a resolution of $\mathbb{K}\cong S^\bullet W_{\bar 0}/S^+W_{\bar 0}$.

Pf. I STEP

Set $R'=R/(\phi_1)$, $K'=K/(e_1,\phi_1)=\Lambda^\bullet\langle e_2,\dots,e_l\rangle\otimes R'$ and $\phi_i'=\phi_i+(\phi_1)\in R'$.
Define d' on K' by $d'e_i=\phi_i'$ for $i=2,\dots,l$ and $d'R'=0 \Rightarrow (K',d')$ is a Koszul algebra. Setting $I=(e_1,\phi_1)\subseteq K$ then we have exact sequence $0\to I\to K\to K'\to 0$ of differential complexes and therefore the "exact snake"

$$
\begin{array}{l}
\cdots\longrightarrow H_q(I)\longrightarrow H_q(K)\longrightarrow H_q(K')\longrightarrow \\
\longrightarrow H_{q-1}(I)\longrightarrow H_{q-1}(K)\longrightarrow H_{q-1}(K')\longrightarrow\cdots \qquad (*)\\
\longrightarrow \cdots \longrightarrow H_2(K')\longrightarrow \\
\longrightarrow H_1(I)\longrightarrow H_1(K)\longrightarrow H_1(K')\longrightarrow \overset{0}{\overset{\|}{H_0(I)}}
\end{array}
$$

$H_0(I)=0$ ($z\in I_0$ has the form $z=\phi_1 u$ for $u\in R$, so $z=d(e_1 u)$)

$H_1(I)=0 \Longleftrightarrow \phi_1$ is not a zero divisor

($z\in I_1$ has the form $z=e_1 u+v\phi_1$ where $v\in\langle e_2,\dots,e_l\rangle\otimes R$ and $u\in R$. Compute $dz=\phi_1 u+(dv)\phi_1=\phi_1(u+dv)$. If ϕ_1 is not a zero-divisor then $dz=0 \to u=-dv \to z=d(e_1 v)$.

Conversely, suppose $H_1(I)=0$ and that $\phi_1 x=0$ for some $x\in R^p$. We prove by induction on $|x|=p$ that $x=0$.

- $p=0$ obvious since $R^0=\mathbb{K}\cdot\mathbb{1}$
- $p>0$ By the induction hypothesis $\phi_1 w=0$ implies $w=0$ for $w\in K^{p-1}$.

  Setting $z=e_1 x\in I_1$ we have $dz=\phi_1 x=0 \to z=dy$ for some $y\in I_2$, $|y|=|e_1|+p-1$. We may write $y=e_1 u+v\phi_1$ for $u\in\langle e_2,\dots,e_l\rangle\otimes R$, $|u|=p-1$

  $v\in\Lambda^2\langle e_2,\dots,e_l\rangle\otimes R$, $|v|=p-2$

  Now $e_1 x=z=dy=\phi_1 u-e_1 du+\phi_1 dv \to$

  $e_1(x+du)=\phi_1\underbrace{(u+dv)}_{\text{Without } e_1} \to x+du=0 \to$

  $\phi_1(u+dv)=0$, $|u+dv|=p-1 \to u+dv=0$

  $\to x=-du=d(dv)=0$)

II STEP We show $H_1(I)=0 \to H_q(I)=0$ for all $q\geq 1$

Indeed $z\in I_q$ has the form $z=e_1 u+v\phi_1$, $u,v\in\Lambda^{\bullet}\langle e_2,\dots,e_l\rangle\otimes R$.

Then $dz=\phi_1 u-e_1 du+\phi_1 dv=\phi_1(u+dv)-e_1 du$ and $dz=0$ implies $\phi_1(u+dv)=0$. If $H_1(I)=0$, then ϕ_1 is not a zero divisor, so $u+dv=0$ and $z=d(e_1\cdot v)$.

III AND LAST STEP Essentially we proved the claimed equivalences for I and we may now use $(*)$ and induction on ℓ. I only show $1 \Rightarrow 2$ and leave $2 \Rightarrow 3$ and $2 \Rightarrow 1$ as exercises.

- $\ell = 1$ $K'_q = 0$ for all $q \geq 1 \Rightarrow H_1(K') = 0$
  If ϕ_1 is not a zero divisor then $H_1(I) = 0$ and $(*)$ gives

$$H_1(I) \longrightarrow H_1(K) \longrightarrow H_1(K') \longrightarrow 0$$
$$\begin{matrix} \| & & \| \\ 0 & & 0 \end{matrix}$$

  so that $H_1(K) = 0$.

- $\ell > 0$ By induction hypothesis, the implication is true for K'.
  Since $\{\phi'_2, \dots, \phi'_\ell\} \subseteq R'$ is regular, we have $H_1(K') = 0$ and since $H_1(I) = 0$ too, we have $H_1(K) = 0$ by $(*)$.

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