Classical BRST construction

- Symplectic Reduction $\widetilde{M} = M//G$
- Restriction from M to M_0 with cohomology
- Quotient from M_0 to $\widetilde{M} = M_0/G$ with cohomology
- BRST cohomology: directly from M to $\widetilde{M}$

Symplectic manifol (M, ω), i.e., closed $\omega \in \Omega^2(M)$ that is non-degenerate

$$\left(\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M) \text{ is isomorphism, with inverse } \# : \Omega^1(M) \longrightarrow \mathfrak{X}(M) \right)$$

$$\xi \longrightarrow \xi^\flat := i_\xi \omega$$

G conn. Lie group acting effectively on M preserving ω: infinitesimally

$$\mathfrak{g} \hookrightarrow \mathfrak{X}(M) \qquad \text{Lie alg. morphism}$$

$$X \longmapsto \xi_X \qquad \xi_X f(m) := \frac{d}{dt}\Big|_{t=0} f\left(e^{-tX} m\right)$$

$$0 = \mathcal{L}_{\xi_X} \omega = d i_{\xi_X} \omega + i_{\xi_X} \cancel{d\omega}^{=0} = d \xi_X^\flat$$

Def. $\xi \in \mathfrak{X}(M)$ is

- symplectic if $\xi^\flat$ is closed
- hamiltonian if $\xi^\flat$ is exact (i.e. $\exists\ \phi_\xi \in C^\infty(M)$ s.t. $\xi^\flat + d\phi_\xi = 0$)

$$\mathrm{sym}(M) = \{ \text{symplectic } \xi \} = \# \Omega^1_{\text{closed}}(M)$$

$$\mathrm{ham}(M) = \{ \text{hamiltonian } \xi \} = \# d C^\infty(M)$$

EXERCISE: $[\mathrm{sym}(M), \mathrm{sym}(M)] \subseteq \mathrm{ham}(M)$

Def. Action of G is hamiltonian if $\xi_X^\flat + d\phi_X = 0 \ \forall X \in \mathfrak{g}$. In this case we have a linear map $\mathfrak{g} \longrightarrow C^\infty(M)$, $X \longmapsto \phi_X$.

EX: $H^1_{dR}(M) = 0 \Longrightarrow$ action is hamiltonian

$H^1(\mathfrak{g}; \mathbb{R}) = 0 \Longrightarrow$ ________________

(M,ω) symplectic mnfd $\longrightarrow$ $C^\infty(M)$ is a Poisson algebra

$$\{f,g\} := \xi_f\, g = \omega(\xi_f, \xi_g)$$

where $\xi_f \in \mathrm{ham}(M)$ is given by $\xi_f^\flat + df = 0$. Main properties:

1. $\{f,g\} = -\{g,f\}$
2. $\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}$

It defines Lie algebra structure on $C^\infty(M)$

3. $\{f,gh\} = \{f,g\}h + g\{f,h\}$

REM: $\xi_{\phi_X} = -(d\phi_X)^\# = \xi_X$

EX: $M = T^*N$ (phase space of configuration space N)

$\omega = -d\theta$ for $\theta = \sum_i p^i dq^i$

coordinates on the fibers

coordinates on N

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i} \right)$$

Def. A symplectic action is Poisson if $\exists\ \mathfrak{g} \longrightarrow C^\infty(M)$ s.t.
$X \longmapsto \phi_X$

- $\xi_X^\flat + d\phi_X = 0$
- $\phi_{[X,Y]} = \{\phi_X, \phi_Y\}$

In this case the moment map $\Phi : M \longrightarrow \mathfrak{g}^*$ is G-equivariant
$m \longmapsto \phi_\cdot(m)$

and $M_0 := \Phi^{-1}(0)$ is closed embedded submnfd of M, provided $0 \in \mathfrak{g}^*$ is regular value of Φ. Consider space of leaves $\widetilde{M} := M_0/G$ and assume it is mnfd. (Also space of gauge orbits and denoted $M/\!/G$.)

THM (Marsden–Weinstein)

(M,ω) symplectic mnfd with Poisson action of G, assume $\widetilde{M}=M_0/G$ is mnfd

$\Longrightarrow \exists!$ symplectic form $\widetilde{\omega}$ on $\widetilde{M}$ s.t. $\pi^*\widetilde{\omega}=i^*\omega$, where $\pi: M_0 \to \widetilde{M}$ and

$i: M_0 \longrightarrow M$ are natural maps.

Idea of proof: M_0 is coisotropic submnfd, i.e., $(T_m M_0)^\perp \subseteq T_m M_0$ at any $m \in M_0$

$(v \in T_m M_0 \Longleftrightarrow d\Phi(v)=0 \Longleftrightarrow \underbrace{d\Phi(v)(X)}=0 \ \forall X \in \mathfrak{g}$ so $(T_m M_0)^\perp = T_m \mathcal{O}^G$

$v(\phi_X) = d\phi_X(v) =$ $\subseteq T_m M_0$

$-\xi_X^\flat(v) = \omega(v,\xi_X))$ $\hookrightarrow G$ preserves M_0)

Now $i^*\omega = \omega|_{TM_0}$ is degenerate with kernel $(TM_0)^\perp$ so it induces non-degenerate $\widetilde{\omega}$ on each $T_m M_0/(T_m M_0)^\perp = T_m M_0/T_m \mathcal{O}^G \cong T_{\pi(m)}\widetilde{M}$.

Check that $\widetilde{\omega}$ is well-defined and closed. ■

EX: Assume action is hamiltonian. For every $X, Y \in \mathfrak{g}$ define function

$c(X,Y) = \phi_{[X,Y]} - \{\phi_X, \phi_Y\}$ and show that:

- it is closed $\left(d\phi_{[X,Y]} = -\xi^\flat_{[X,Y]} = -[\xi_X,\xi_Y]^\flat;\right.$

  $d\{\phi_X,\phi_Y\} = d(\omega(\xi_X,\xi_Y)) = -d\, \iota_{\xi_X}\xi_Y^\flat$

  $\left.= -\mathcal{L}_{\xi_X}\xi_Y^\flat = -(\mathcal{L}_{\xi_X}\xi_Y)^\flat.\right)$

- This defines $c: \Lambda^2\mathfrak{g} \longrightarrow H^0_{dR}(M)$, which is a CE cocycle

  $(c([X,Y],Z) + c([Z,X],Y) + c([Y,Z],X) =$ $\hookrightarrow [-,-]$ is Lie bracket

  $-\{\phi_{[X,Y]},\phi_Z\} - \{\phi_{[Z,X]},\phi_Y\} - \{\phi_{[Y,Z]},\phi_X\}$

  $= -\omega([\xi_X,\xi_Y],\xi_Z) - \cdots = -\{\phi_Z,\{\phi_Y,\phi_X\}\} - \cdots = 0)$

- Deduce that $[c] \in H^2(\mathfrak{g}; H^0_{dR}(M))$ is trivial iff $\exists\, \widetilde{\phi}_X$ s.t.

  $\xi_X^\flat + d\widetilde{\phi}_X = 0$ and $\widetilde{\phi}_{[X,Y]} = \{\widetilde{\phi}_X, \widetilde{\phi}_Y\}$

  ($[c]$ is trivial iff $\exists\, \alpha: \mathfrak{g} \longrightarrow H^0_{dR}(M)$ s.t. $c(X,Y) = -\alpha[X,Y]$

  If $[c]=0$ then set $\widetilde{\phi}_X := \phi_X + \alpha_X$. If such $\widetilde{\phi}_X$ exists,

  then set $\alpha_X := \widetilde{\phi}_X - \phi_X$.)

- $H^2(\mathfrak{g};\mathbb{R}) = 0 \Longrightarrow$ any hamiltonian action is Poisson. ■

例: $\mathbb{C}^n$ $\omega = \frac{1}{2i} \sum_k dz_k \wedge d\bar{z}_k$

$H_k = |z_k|^2$ $\{H_k, H_l\} = 0$ for all k, l

$G = \mathbb{R}^n$ or actually $\mathbb{T}^n$

Moment map $\Phi : \mathbb{C}^n \longrightarrow \mathbb{R}^n$

$$z \longmapsto (\underset{\overset{\vee\!|}{0}}{H_1(z)}, \dots, \underset{\overset{\vee\!|}{0}}{H_n(z)})$$

Φ^{-1} (pt generic) $= \mathbb{T}^n$

Action of G on $\mathbb{C}^n$ is $\zeta_{e_k}^{\flat} + dH_k = 0$

$$\longrightarrow \zeta_{e_k}^{\flat} + \bar{z}_k dz_k + z_k d\bar{z}_k = 0$$

$$\longrightarrow \zeta_{e_k} = -2i z_k \frac{\partial}{\partial z_k} + 2i \bar{z}_k \frac{\partial}{\partial \bar{z}_k}$$

In other words $\zeta_{e_k} = -2i(x_k + iy_k)\frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)$

$$+2i(x_k - iy_k)\frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)$$

$$= -2x_k\frac{\partial}{\partial y_k} + 2y_k\frac{\partial}{\partial x_k}$$

which restricted to Φ^{-1} (pt generic) are tangent to the S^1's.

So $M /\!/ G$ is a point.

Ex: $\mathbb{C}^{n+1}$ $\quad H = \sum_{k=1}^{n+1} |z_k|^2$

$G = \mathbb{R}$ or actually S^1

Moment map $\Phi : \mathbb{C}^{n+1} \longrightarrow \mathbb{R}$

$$z \longrightarrow H(z) \geq 0$$

Φ^{-1} (pt generic) $= S^{2n+1}$

$$\mathcal{J} = \sum_{k=1}^{n+1} \left(-2x_k \frac{\partial}{\partial y_k} + 2y_k \frac{\partial}{\partial x_k} \right)$$

So the action of G is $z \longrightarrow e^{i\theta} z$ $(\theta \in \mathbb{R})$ and

$M /\!/ G$ is $\mathbb{P}\mathbb{C}^n$.