$M \rightsquigarrow R = C^\infty(M)$ commutative associative algebra over $\mathbb{R}$

$M_0 \rightsquigarrow$ vanishing ideal $J = \{ f \in R \mid f|_{M_0} = 0 \} \subseteq R$

$C^\infty(M_0) \cong R/J$ since M_0 is closed embedded submfd of M

Fix basis $\{X_1, \dots, X_\ell\}$ of $\mathfrak{g}$ and consider components $\{\phi_1, \dots, \phi_\ell\} \subseteq R$ of $\Phi : M \to \mathfrak{g}^*$.

<u>LEMMA</u> $J = (\phi_1, \dots, \phi_\ell)$ and the sequence $\{\phi_1, \dots, \phi_\ell\}$ is regular.

$\widetilde{M} \rightsquigarrow C^\infty(\widetilde{M}) \cong C^\infty(M_0)^{\mathfrak{g}} = H^0(\mathfrak{g}; C^\infty(M_0))$

This is not satisfactory since we would like to work directly with R: we will use free resolution by R-modules of R/J and "lift" CE differential to BRST chains.

<u>KOSZUL RESOLUTION</u> <u>$K^\bullet = \Lambda^\bullet \mathfrak{g} \otimes C^\infty(M)$ OF $C^\infty(M_0)$</u>

$$\cdots \longrightarrow \Lambda^q \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} \Lambda^{q-1} \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} C^\infty(M) \longrightarrow 0$$

$$\delta(X_{i_1} \wedge \cdots \wedge X_{i_q} \otimes F) = \sum_{j=1}^{q} (-1)^{j+1} X_{i_1} \wedge \cdots \wedge \widehat{X}_{i_j} \wedge \cdots \wedge X_{i_q} \otimes \phi_j F$$

$$H^q(K^\bullet, \delta) \cong \begin{cases} C^\infty(M_0) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

This is a complex of $\mathfrak{g}$-modules. E.g.:

$$X \cdot \left(\delta(X_1 \wedge X_2 \otimes f) \right) = X \cdot \left(X_2 \otimes \phi_1 f - X_1 \otimes \phi_2 f \right)$$

$$= [X, X_2] \otimes \phi_1 f + X_2 \otimes \underbrace{\mathfrak{z}_X(\phi_1)}_{} f + X_2 \otimes \phi_1 \mathfrak{z}_X(f) + \cdots$$

$$\mathfrak{z}_{\phi_X}(\phi_1) = \{\phi_X, \phi_1\} = \phi_{[X, X_1]}$$

$$\delta([X, X_1] \wedge X_2 \otimes f) = X_2 \otimes \phi_{[X, X_1]} f + \cdots$$

$$\delta(X_1 \wedge [X, X_2] \otimes f) = [X, X_2] \otimes \phi_1 f + \cdots$$

$$\delta(X_1 \wedge X_2 \otimes \mathfrak{z}_X f) = X_2 \otimes \phi_1 \mathfrak{z}_X f + \cdots$$

THE BRST COMPLEX

$$C^{p,q} := C^p(\mathfrak{g}; \Lambda^q \mathfrak{g} \otimes C^\infty(M)) = \Lambda^p \mathfrak{g}^* \otimes \Lambda^q \mathfrak{g} \otimes C^\infty(M) \quad , \quad C^n := \bigoplus_{\underbrace{p-q=n}_{\text{ghost number}}} C^{p,q}$$

$$D = d + (-1)^p \delta$$

THM $H^0(C^\cdot, D) \cong H^0(\mathfrak{g}; C^\infty(M_0)) \cong C^\infty(\tilde{M})$ as v.s.

POISSON ALGEBRA STRUCTURES

What about $H^0(C^\cdot; D)$? We will show that $C = \Lambda^\cdot(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes C^\infty(M)$ is graded Poisson superalgebra and $D = \{Q, -\}$ for some $Q \in C^1 = \bigoplus_{i \geq 0} C^{i+1,i}$.

This implies that $H(C, D)$ is graded Poisson superalgebra.

Def. A Poisson superalg. $P = P_{\bar{0}} \oplus P_{\bar{1}}$ has two bilinear operations preserving the parity:

① $P \otimes P \longrightarrow P$, $(a,b) \longrightarrow ab$ ② $P \otimes P \to P$, $(a,b) \to \{a,b\}$

s.t. $a(bc) = (ab)c$, $ab = (-1)^{|a||b|} ba$ } associative "commutative"

$\{a,b\} = -(-1)^{|a||b|}\{b,a\}$,

$\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{|a||b|}\{b,\{a,c\}\}$ } Lie superalgebra

$\{a,bc\} = \{a,b\}c + (-1)^{|a||b|} b\{a,c\}$ } (*)

If $P = \bigoplus_{n \in \mathbb{Z}} P^n$ with $P_{\bar{0}} = \bigoplus_{n \in 2\mathbb{Z}} P^n$, $P_{\bar{1}} = \bigoplus_{n \in 2\mathbb{Z}+1} P^n$ and ① and ② are compatible with $\mathbb{Z}$-degree then we speak of graded Poisson superalgebras.

EX: $C^\infty(M)$ (with trivial odd part)

$\Lambda^\cdot(\mathfrak{g} \oplus \mathfrak{g}^*)$ ① wedge product ② $X, Y \in \mathfrak{g}$ $\alpha, \beta \in \mathfrak{g}^*$

$$\{\alpha, X\} = \{X, \alpha\} = \alpha(X)$$

$$\{X, Y\} = \{\alpha, \beta\} = 0$$

and then extend using (*)

EX: P_1, P_2 Poisson superalgebras $\Rightarrow$ $P_1 \otimes P_2$ is Poisson superalgebra

$$(a\otimes u)(b\otimes v) = (-1)^{|u||b|} ab \otimes uv$$

$$\{a\otimes u, b\otimes v\} = (-1)^{|u||b|}\left(\{a,b\}\otimes uv + ab\otimes\{u,v\}\right)$$

For all $a,b\in P_1$, $u,v\in P_2$.

Then $\Lambda^\bullet(\mathfrak{g}\oplus\mathfrak{g}^*)\otimes C^\infty(M)$ is Poisson superalgebra and it is graded by the ghost number since $\{C^{p,q}, C^{r,s}\}\subseteq C^{p+r-1,q+s-1}\oplus C^{p+r,q+s}$.

PROP: $D=\{Q,-\}$, where $Q = \underbrace{\alpha^i\phi_i}_{\mathfrak{g}^*\otimes C^\infty(M)} - \frac{1}{2}F_{ij}^k \underbrace{\alpha^i\wedge\alpha^j\wedge X_k}_{\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}}$.

Pf. Since d is an odd derivation w.r.t. wedge on $\Lambda^\bullet\mathfrak{g}^*$ and similarly δ w.r.t. wedge on $\Lambda^\bullet\mathfrak{g}$, it is enough to check the claim on $F\in C^\infty(M)$, $Y\in\mathfrak{g}$ and $\beta\in\mathfrak{g}^*$:

(i) $\{Q,F\} = \{\alpha^i\phi_i, F\} = \alpha^i\{\phi_i,F\} = \alpha^i \mathcal{L}_{X_i}F \in \mathfrak{g}^*\otimes C^\infty(M)$
agrees with CE differential dF

(ii) $\{Q,\beta\} = -\frac{1}{2}F_{ij}^k\alpha^i\wedge\alpha^j\{X_k,\beta\} = -\frac{1}{2}f_{ij}^k\beta_k\alpha^i\wedge\alpha^j\in\Lambda^2\mathfrak{g}^*$
agrees with $d\beta$

(iii) $\{Q,Y\} = \alpha^i(Y)\phi_i + f_{ij}^k\alpha^j(Y)\alpha^i\wedge X_k \in C^\infty(M)\oplus\mathfrak{g}^*\otimes\mathfrak{g}$
agrees with $\delta Y + dY$ ∎

REM: Q is called classical BRST operator (of Batalin & Vilkoviskii) and usually denoted by $Q = c^i\phi_i - \frac{1}{2}f_{ij}^k c^i c^j b_k$, where c^i are ghosts and b_i antighosts (classical ones.)

THM $H^0(C^\bullet, D) \cong C^\infty(\hat{M})$ as Poisson algebras.

Proof is omitted but let me note an important ingredient. Since $\{\phi_i, \phi_j\} = F_{ij}^{\,k}\phi_k$ ($f_{ij}^{\,k}$ = struct. const. of $\mathfrak{g}$), then $\{J, J\} \subseteq J$ and we may also write

$$
\begin{aligned}
C^\infty(\tilde{M}) &= \{ f \in C^\infty(M_0) \mid \mathfrak{g} \cdot f = 0 \} \\
&= \{ f \in C^\infty(M_0) \mid \{\phi_i, f\} = 0 \text{ on } M_0 \} \\
&= \{ f \in C^\infty(M) \mid \{\phi_i, f\} \subseteq J \} / J \\
&= N(J)/J
\end{aligned}
$$

which gives an alternative description of the structure of a Poisson algebra of $C^\infty(\tilde{M})$.