# Dispersionless integrable systems and their dispersive deformations

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Tromsø, February 2019

# General context

How to classify integrable PDEs of Kadomtsev-Petviashvili (KP) type,

$$(u_t - uu_x)_x - \varepsilon^2 u_{xxxx} = u_{yy}?$$

**Possible approach:**

- Classify 2+1D dispersionless systems which may (potentially) arise as dispersionless limits of integrable equations (method of hydrodynamic reductions).
- Reconstruct dispersive corrections (deformation of hydrodynamic reductions).

## Plan of talks:

- Different approaches to integrability in 1+1D. Example of KdV.
- Integrable systems of hydrodynamic type in 1+1D. The method of hydrodynamic reductions in 2+1D.
- Dispersive deformations of 2+1D dispersionless integrable systems.

# Different approaches to integrability. Example of KdV.

- The Korteweg de Vries (KdV) equation.
- Lax pair.
- Higher symmetries (commuting flows).
- Higher conservation laws.
- Bi-Hamiltonian structure, recursion operator.
- Soliton solutions.

## The Korteweg de Vries (KdV) equation

This equation models shallow water waves (it was first derived by Boussinesq in 1877 and rediscovered by Korteweg and de Vries in 1895):

$$u_t - 6uu_x + u_{xxx} = 0.$$

Figure 1: Diederik Korteweg (1848-1941) and Gustav de Vries (1866-1934), Dutch mathematicians

## Lax pair

A Lax pair for a nonlinear PDE is an overdetermined linear system (depending on auxiliary spectral parameter $\lambda$) whose compatibility condition is the given PDE.

Verify that KdV equation is the compatibility condition of the following linear system

$$\psi_{xx} = (u - \lambda)\psi,$$

$$\psi_t = (4\lambda + 2u)\phi_x - u_x\psi,$$

where $\lambda$ is a spectral parameter.

Integrability $\equiv$ existence of a Lax pair.

## Higher symmetries

We say that an evolutionary PDE

$$u_t = F(u, u_x, \dots)$$

possesses a 'higher' symmetry (commuting flow)

$$u_y = G(u, u_x, \dots)$$

if $u_{ty} = u_{yt}$.

Verify the commutativity of the following flows:

$$u_t = 6uu_x - u_{xxx},$$

$$u_y = 30u^2u_x - 20u_xu_{xx} - 10uu_{xxx} + u_{xxxxx}.$$

(KdV and its 5th-order commuting flow). Calculate 7th-order commuting flow of KdV.

Integrability $\equiv$ existence of infinitely many higher commuting flows.

## Higher conservation laws

Conservation laws of a nonlinear PDE are relations of the form

$$h(u, u_x, \dots)_t + f(u, u_x, \dots)_x = 0$$

which hold modulo the PDE ($h$ is the conserved density, $f$ is the flux). Verify that the following are conservation laws of the KdV equation:

$$u_t + (u_{xx} - 3u^2)_x = 0,$$

$$\left(\frac{u^2}{2}\right)_t + \left(uu_{xx} - \frac{1}{2}u_x^2 - 2u^3\right)_x = 0,$$

$$\left(u^3 + \frac{1}{2}u_x^2\right)_t + \left(-\frac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 + u_xu_{xxx} - \frac{1}{2}u_{xx}^2\right)_x = 0,$$

known as the Casimir, momentum and energy.

Find a general conservation law whose density $h$ depends on $u, u_x, u_{xx}$.

Integrability $\equiv$ existence of infinitely many higher conservation laws.

## Bi-Hamiltonian structure, recursion operator.

The KdV equation possesses bi-Hamiltonian structure

$$u_t = J_1 \frac{\delta I_1}{\delta u} = J_2 \frac{\delta I_2}{\delta u},$$

where

$$J_1 = \frac{\partial}{\partial x}, \qquad I_1 = \int\limits_{-\infty}^{\infty} (u^3 + \frac{1}{2} u_x^2) dx$$

and

$$J_2 = -\frac{\partial^3}{\partial x^3} + 4u \frac{\partial}{\partial x} + 2u_x, \qquad I_2 = \frac{1}{2} \int_{-\infty}^{\infty} u^2 dx.$$

Recursion operator $R$ is defined as

$$R = J_2 J_1^{-1} = -\frac{\partial^2}{\partial x^2} + 4u + 2u_x \left( \frac{\partial}{\partial x} \right)^{-1}.$$

Integrability $\equiv$ existence of a bi-Hamiltonian representation/recursion operator.

## Soliton solutions

N-soliton solutions are given by the explicit formula

$$u_n(x,t) = -2\frac{\partial^2}{\partial x^2}\ln\det \mathrm{A}(x,t)$$

where $\mathrm{A}$ is the $n \times n$ matrix

$$A_{rs} = \delta_{rs} + \frac{c_s}{k_s + k_r} e^{-(k_s+k_r)x+8k_s^3 t},$$

$c_s, k_s = const$. Verify that $u_n$ indeed solves the KdV equation for $n = 1, 2, 3$.

Asymptotically:

$$u_n(x,t) \sim -\sum_{s=1}^{n} 2k_s^2 \mathrm{sech}^2[k_s(x - 4k_s^2 t - x_s^{\pm})] \quad \text{as } t \to \pm\infty\,,$$

where $x_s$ are the phase shifts (given by a certain explicit formula).

Integrability $\equiv$ existence of exact n-soliton solutions.

## Further integrability cretiria

- Existence of a Bäcklund transformation
- Existence of a Wahlquist-Estabrook prolongation structure

The choice of definition of integrability depends on the class of PDEs under study