Ferapontov — Lecture 2.   8.2.2019

Finite dim Hamiltonian formalism:

Phase space M^n. $(u^1, \ldots, u^n)$ $f_i = \frac{\partial f}{\partial u^i}$, $g_i = \frac{\partial g}{\partial u^i}$

Functions $f(u)$, $g(u)$

Poisson bracket $\{f,g\} = f_i \omega^{ij} g_j$, ω^{ij} is a $(2,0)$ tensor

- skew-sym: $\{f,g\} = -\{g,f\}$ $(\Longleftrightarrow \omega^{ij} = -\omega^{ji})$
- Jacobi: $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$

$$(\Longleftrightarrow \omega^{ij}{}_{,k}\omega^{kl} + \omega^{jl}{}_{,k}\omega^{ki} + \omega^{li}{}_{,k}\omega^{kj} = 0)$$

Darboux thm: If $\det(\omega^{ij}) \neq 0$, then ω^{ij} can be made constant coeff.

Hamiltonian system: $u^i_t = \omega^{ij} h_j$, where h is the Hamiltonian

eg. $\omega^{ij} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \Rightarrow$ Hamiltonian systems are $p^i_t = \frac{\partial h}{\partial q_i}$

$$q^i_t = \frac{-\partial h}{\partial p_i}$$

∞-dim Hamiltonian formalism:

Phase space: $(u^1(x), \ldots, u^n(x))$ on $M^n(u^1, \ldots, u^n)$

Functionals: $F = \int f(x, u, u_x, \ldots)\,dx$, $G = \int g(x, u, u_x, \ldots)\,dx$

Poisson bracket: $\{F,G\} = \int \frac{\delta F}{\delta u^i} A^{ij} \frac{\delta G}{\delta u^j}\,dx$

Where $\frac{\delta F}{\delta u^i} = \frac{\partial f}{\partial u^i} - \left(\frac{\partial f}{\partial u^i_x}\right)_x + \left(\frac{\partial f}{\partial u^i_{xx}}\right)_{xx} - \ldots$, etc.

& A^{ij} is a matrix differential operator in $\frac{d}{dx}$, with coeffs depending on u's and their x-derivatives.

- Skew-sym + Jacobi identities $\Longleftrightarrow$ complicated constraints for the coeffs of A^{ij}.

Darboux thm: If A^{ij} is "non-degenerate", then it can be transformed to u-independent form.

Hamiltonian systems:

$$u^i_t = A^{ij} \frac{\delta H}{\delta u^j} \quad , \quad H = \int h(x, u, u_x, \dots)\, dx$$

eg $A^{ij} = \eta^{ij} \frac{d}{dx} \quad , \eta^T = \eta.$

$\Rightarrow$ Ham. sys: $u^i_t = \eta^{ij} \left(\frac{\delta H}{\delta u^j}\right)_x.$

Shallow water eqns:

eg. $\begin{cases} h_t + (uh)_x = 0 \\ u_t + uu_x + h_x = 0 \end{cases} \Leftrightarrow \begin{pmatrix} h \\ u \end{pmatrix}_t = \underbrace{-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx}}_{A^{ij}} \begin{pmatrix} u^2/2 + h \\ uh \end{pmatrix}$

$$H = \int \left(\frac{u^2 h}{2} + \frac{h^2}{2}\right) dx \quad , \quad \frac{\delta H}{\delta u} = uh, \quad \frac{\delta H}{\delta h} = \frac{u^2}{2} + h$$

eg. (KdV)

$$u_t = 6uu_x - u_{xxx} = \underbrace{\frac{d}{dx}}_{A} \left(\frac{\delta H}{\delta u}\right) \quad , \quad H = \int \left(u^3 + \frac{u_x^2}{2}\right) dx$$

$$\frac{\delta H}{\delta u} = 3u^2 - u_{xx}$$

Dubrovin–Novikov

$$A^{ij} = g^{ij}(u) \frac{d}{dx} + b^{ij}_k(u)\, u^k_x$$

Suppose $\det(g^{ij}) \neq 0$. Then set $b^{ij}_k = -g^{is} \Gamma^j_{sk}$.

Thm: A^{ij} is Hamiltonian $\Longleftrightarrow$ g^{ij} is flat metric & Γ^{ij}_{sk} its Levi-Civita connection.

$$A^{ij} = g^{ij}(u)\frac{d}{dx} + b^{ij}_k(u)\, u^k_x + c\, u^i_x \left(\frac{d}{dx}\right)^{-1} u^j_x$$

Note: $u^i_x \left(\frac{d}{dx}\right)^{-1} u^j_x h_j = u^i_x \left(\frac{d}{dx}\right)^{-1} h_x = u^i_x h$

Thm: A^{ij} Hamiltonian $\Longleftrightarrow$ g^{ij} metric of constant curvature c.

$$A^{ij} = g^{ij}(u)\frac{d}{dx} + b^{ij}_k(u)\, u^k_x + w^i_k(u)\, u^k_x \left(\frac{d}{dx}\right)^{-1} w^j_l(u)\, u^l_x$$

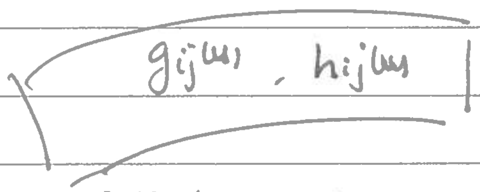
Thm: Skew-sym & Jacobi id.

$$\Longleftrightarrow \begin{cases} g_{is} w^s_k = g_{ks} w^s_i \\ \nabla_k w^i_j = \nabla_j w^i_k \\ R^{ij}_{kl} = w^i_k w^j_l - w^i_l w^j_k \end{cases}$$

(Gauss-Peterson-Codazzi eqns)

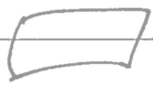
$\mathbb{E}^{n+1}$

$$w^i_k = g^{is} h_{sk}$$



hypersurface M^n $(u^1, \dots, u^n)$

Dirac bracket: $\tilde{\omega}^{ij} = \omega^{ij} - \{u^i, f^\alpha\}\{f^\alpha, f^\beta\}^{-1}\{f_\beta, u^j\}$

 $f_\alpha(u) = 0$.