Integrable systems of hydrodynamic type in 1+1D. The method of hydrodynamic reductions in 2+1D

- Systems of hydrodynamic type in 1+1D. Diagonalisability criterion. Semi-Hamiltonian property, commuting flows and conservation laws. The generalised hodograph method. Linear degeneracy.
- The method of hydrodynamic reductions in 2+1D. Example of dKP.
- The method of hydrodynamic reductions in higher dimensions.
- Dispersionless Lax pairs.

Based on

S.P. Tsarev, Geometry of Hamiltonian systems of hydrodynamic type. Generalized hodograph method. Izvestija AN USSR Math. **54***, N5 (1990) 1048-1068.*

E. V. Ferapontov and K. R. Khusnutdinova, On the integrability of (2+1)-dimensional quasilinear systems, Comm. Math. Phys. **248** *(2004) 187-206.*

Systems of hydrodynamic type in 1+1D. Diagonalisability criterion

Systems of hydrodynamic type are equations of the form

$$\mathbf{u}_t = v(\mathbf{u})\mathbf{u}_x, \quad \text{or} \quad u^i_t = v^i_j(\mathbf{u})u^j_x.$$

Example: Shallow water equations

$$h_t + (uh)_x = 0, \quad u_t + uu_x + h_x = 0.$$

We will be primarily dealing with systems that can be transformed into diagonal (Riemann invariant) form,

$$R^i_t = \lambda^i(R)R^i_x.$$

Diagonalisability criterion: the Haantjes tensor $H = 0$

$$N^i_{jk} = v^p_j \partial_{u^p} v^i_k - v^p_k \partial_{u^p} v^i_j - v^i_p(\partial_{u^j} v^p_k - \partial_{u^k} v^p_j),$$

$$H^i_{jk} = N^i_{pr} v^p_j v^r_k - N^p_{jr} v^i_p v^r_k - N^p_{rk} v^i_p v^r_j + N^p_{jk} v^i_r v^r_p.$$

Semi-Hamiltonian property

A diagonal system

$$R^i_t = \lambda^i(R)R^i_x, \qquad i = 1, \ldots, n, \tag{1}$$

is said to be semi-Hamiltonian if its characteristic speeds $\lambda^i(R)$ satisfy the relations

$$\partial_k \left(\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \right) = \partial_j \left(\frac{\partial_k \lambda^i}{\lambda^k - \lambda^i} \right).$$

$\partial_k = \partial_{R^k}$, $i \neq j \neq k$.

Verify that the following systems are semi-Hamiltonian:

$$R^i_t = (R^i \prod_k R^k)R^i_x, \qquad R^i_t = (R^i + \sum_k R^k)R^i_x.$$

We will see that the semi-Hamiltonian property implies integrability.

Conservation laws

Conservation laws of system (1) are relations of the form $h(R)_t = g(R)_x$ which must hold by virtue of (1).

Verify that this implies the following relations for the density h and the flux g:

$$\partial_i g = \lambda^i \partial_i h.$$

Verify that the elimination of g leads to the second-order system for the conserved density h:

$$\partial_i \partial_j h = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \partial_i h + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} \partial_j h, \qquad i \neq j.$$

Show that involutivity of this system for h is equivalent to the semi-Hamiltonian property.

This will prove the following results: system (1) is semi-Hamiltonian if and only if it possesses infinitely many conservation laws parametrised by n arbitrary functions of 1 variable.

Commuting flows

Systems

$$R^i_t = \lambda^i(R)R^i_x \qquad \text{and} \qquad R^i_y = \mu^i(R)R^i_x$$

are said to commute if $R^i_{ty} = R^i_{yt}$ (that is, they are compatible).

Verify that the commutativity condition implies the following relations:

$$\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}, \qquad i \neq j.$$

Show that involutivity of this system for μ^i is equivalent to the semi-Hamiltonian property.

This will prove the following results: system (1) is semi-Hamiltonian if and only if it possesses infinitely many commuting flows parametrised by n arbitrary functions of 1 variable.

The generalised hodograph formula

Given 2 commuting flows

$$R_t^i = \lambda^i(R) R_x^i \qquad \text{and} \qquad R_y^i = \mu^i(R) R_x^i,$$

verify that the implicit relations (the generalised hodograph formula of Tsarev)

$$\mu^i(R) = x + \lambda^i(R) t, \qquad i = 1, \ldots, n,$$

provides a solution of system (1). Since, for semi-Hamiltonian systems, commuting flows μ^i depend on n arbitrary functions of 1 variable, the generalised hodograph formula provides a **generic** solution of system (1).

Linear degeneracy

Linearly degenerate systems are characterised by the condition

$$\partial_i \lambda^i = 0, \qquad 1 = 1, \ldots, n.$$

Linear degeneracy is known to prevent breakdown of smooth initial data $R^i(x, 0)$.

The method of hydrodynamic reductions in 2+1D

Goes back to Yanenko, Meleshko, Burnat, Peradzyński, Grundland, Kodama, Gibbons, Tsarev. Applies to quasilinear equations

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0.$$

Consists of seeking n-phase solutions

$$\mathbf{u} = \mathbf{u}(R^1, ..., R^n).$$

The phases $R^i(x, y, t)$ are required to satisfy a pair of commuting equations

$$R^i_t = \lambda^i(R) R^i_x, \qquad R^i_y = \mu^i(R) R^i_x,$$

(called hydrodynamic reductions). Commutativity conditions: $\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}$.

Definition A 2+1 quasilinear system is said to be integrable if, for any n, it possesses infinitely many n-component reductions parametrized by n arbitrary functions of one variable (n=3 is sufficient).

One-phase solutions are of the form $\mathbf{u} = \mathbf{u}(R)$ where $R(x, y, t)$ satisfies the equations

$$R_t = \lambda(R)R_x, \quad R_y = \mu(R)R_x.$$

Generic one-phase solution is constant along a one-parameter family of planes:

$$w(R) = x + \lambda(R)t + \mu(R)y.$$

Two-phase solutions are of the form $\mathbf{u} = \mathbf{u}(R^1, R^2)$ where $R^1(x, y, t), R^2(x, y, t)$ satisfy the equations

$$R^i_t = \lambda^i(R)R^i_x, \quad R^i_y = \mu^i(R)R^i_x.$$

Generic two-phase solution is constant along a two-parameter family of lines:

$$w^1(R^1, R^2) = x + \lambda^1(R^1, R^2)t + \mu^1(R^1, R^2)y,$$
$$w^2(R^1, R^2) = x + \lambda^2(R^1, R^2)t + \mu^2(R^1, R^2)y.$$

where $\frac{\partial_j w^i}{w^j - w^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}$.

‘Decomposition’ point of view

Hydrodynamic reductions can be viewed as decompositions of a given 2+1 PDE into a pair of consistent 1+1 PDEs. Further examples of this construction are as follows:

Example 1 The linearly degenerate system

$$u_t = vu_x, \quad v_t = uv_x$$

can be decoupled into a pair of consistent ODEs,

$$u_x = \frac{f(u)}{u-v}, \quad v_x = \frac{g(v)}{v-u}$$

and

$$u_t = \frac{vf(u)}{u-v}, \quad v_t = \frac{ug(v)}{v-u}.$$

Here f and g are arbitrary functions.

Example 2 The KdV equation,

$$4u_t + u_{xxx} - 6uu_x = 0,$$

can be decoupled into a pair of consistent ODEs (Dubrovin's equations) via the ansatz $u = -2(R^1 + ... + R^n)$:

$$R^i_x = \frac{\sqrt{P(R^i)}}{\prod_{k\neq i}(R^i - R^k)}, \qquad R^i_t = \frac{\sqrt{P(R^i)}(R^i - \sum R^s)}{\prod_{k\neq i}(R^i - R^k)}.$$

Here $P(R) = -4\prod_1^{2n+1}(R - E_s)$, $\sum E_s = 0$. This construction gives n-gap solutions of KdV. Note that R^i satisfy the linearly degenerate system

$$R^i_t = (R^i - \sum R^s)R^i_x.$$

Example of dKP

$$(u_t - uu_x)_x = u_{yy}.$$

First-order (hydrodynamic) form:

$$u_t - uu_x = w_y, \quad u_y = w_x.$$

N-phase solutions: $u = u(R^1, ..., R^n),\ w = w(R^1, ..., R^n)$ where

$$R^i_t = \lambda^i(R)R^i_x, \quad R^i_y = \mu^i(R)R^i_x.$$

Verify that the substitution of u, w into the above first-order system imply

$$\partial_i w = \mu^i \partial_i u, \qquad \lambda^i = u + (\mu^i)^2.$$

Derive the following equations for $u(R)$ and $\mu^i(R)$ (Gibbons-Tsarev system):

$$\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \qquad \partial_i \partial_j u = 2\frac{\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}.$$

In involution! General solution depends on n arbitrary functions of one variable.

Generalized dKP

$$(u_t - f(u)u_x)_x = u_{yy}.$$

First-order (hydrodynamic) form:

$$u_t - f(u)u_x = w_y, \quad u_y = w_x.$$

N-phase solutions: $u = u(R^1, ..., R^n), \; w = w(R^1, ..., R^n)$ where

$$R^i_t = \lambda^i(R)R^i_x, \quad R^i_y = \mu^i(R)R^i_x.$$

Verify that

$$\partial_i w = \mu^i \partial_i u, \qquad \lambda^i = f(u) + (\mu^i)^2.$$

Derive equations for $u(R)$ and $\mu^i(R)$ (generalized Gibbons-Tsarev system):

$$\partial_j \mu^i = f'(u)\frac{\partial_j u}{\mu^j - \mu^i}, \qquad \partial_i\partial_j u = 2f'(u)\frac{\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}.$$

Verify that involutivity of this system implies $f'' = 0$ (integrability condition).

The method of hydrodynamic reductions in higher dimensions

Consider the 4D second heavenly equation,

$$\theta_{tx} + \theta_{zy} + \theta_{xx}\theta_{yy} - \theta_{xy}^2 = 0.$$

Its first-order quasilinear form is obtained by setting $\theta_{xx} = u$, $\theta_{xy} = v$, $\theta_{yy} = w$, $\theta_{tx} = p$:

$$u_y = v_x, \;\; u_t = p_x, \;\; v_y = w_x, \;\; v_t = p_y, \;\; v_z = (v^2 - uw - p)_x.$$

Hydrodynamic reductions express u, v, w, p as functions of Riemann invariants $R^1, ..., R^n$ that solve a triple of commuting hydrodynamic type systems

$$R_t^i = \lambda^i(R)\, R_x^i, \quad R_y^i = \mu^i(R)\, R_x^i, \quad R_z^i = \eta^i(R)\, R_x^i.$$

Verify that this leads to the following generalised Gibbons-Tsarev-type system:

$$\partial_j \mu^i = \frac{(\mu^j - \mu^i)^2}{\eta^j - \eta^i + u(\mu^j - \mu^i)}\, \partial_j u, \quad \partial_j \eta^i = \frac{(\mu^j - \mu^i)(\eta^j - \eta^i)}{\eta^j - \eta^i + u(\mu^j - \mu^i)}\, \partial_j u,$$

$$\partial_i \partial_j u = 2 \frac{\mu^j - \mu^i}{\eta^j - \eta^i + u(\mu^j - \mu^i)}\, \partial_i u \partial_j u.$$

Verify that this system is in involution, so that its general solution depends on 2n arbitrary functions of one variable.

Definition A d-dimensional quasilinear system is said to be integrable if, for any n, it possesses infinitely many n-component reductions parametrized by (d-2)n arbitrary functions of one variable.

Based on

E. V. Ferapontov and K. R. Khusnutdinova, Hydrodynamic reductions of multi-dimensional dispersionless PDEs: the test for integrability, J. Math. Phys. **45***, no. 6 (2004) 2365-2377.*

Dispersionless Lax pairs

The dKP equation,

$$u_t - uu_x = w_y, \quad u_y = w_x,$$

possesses the Lax pair (Zakharov):

$$\psi_t = \frac{1}{3}\psi_x^3 + u\psi_x + w, \quad \psi_y = \frac{1}{2}\psi_x^2 + u.$$

Observation Integrability by the method of hydrodynamic reductions is equivalent to the existence of a dispersionless Lax pair of the form

$$\psi_t = F(\mathbf{u}, \psi_x), \quad \psi_y = G(\mathbf{u}, \psi_x).$$

(Proved for various classes of integrable models).