Dispersive deformations of dispersionless integrable systems

- Deformations of hydrodynamic reductions of dKP.
- Classification of scalar integrable 2+1D soliton equations with simplest nonlocalities.

KP equation

$$(u_t - uu_x)_x - u_{xxxx} = u_{yy}.$$

Perturbative symmetry approach

$$(u_t - \varepsilon uu_x)_x - u_{xxxx} = u_{yy}.$$

Dispersive deformation

$$(u_t - uu_x)_x - \varepsilon^2 u_{xxxx} = u_{yy}.$$

Program of classification of 2+1D integrable systems:

- Classify 2+1D dispersionless systems which may (potentially) arise as dispersionless limits of integrable soliton equations (method of hydrodynamic reductions).
- Reconstruct dispersive corrections (deformation of hydrodynamic reductions).

Dispersive deformations of hydrodynamic reductions of dKP

E. V. Ferapontov and A. Moro, Dispersive deformations of hydrodynamic reductions of 2D dispersionless integrable systems, J. Phys. A: Math. Theor. 42 (2009) 035211, 15pp.

$$(u_t - uu_x)_x - \varepsilon^2 u_{xxxx} = u_{yy}.$$

Look for deformed n-phase solutions in the form

$$u = u(R^1, ..., R^n) + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots$$

where

$$R_t^i = \lambda^i(R)R_x^i + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots$$

$$R_y^i = \mu^i(R)R_x^i + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots$$

Here $(\dots)$ are required to be **polynomial** and **homogeneous** in the derivatives of R^i. Recall that $\lambda^i = u + (\mu^i)^2$ where $\mu^i,\ u$ satisfy the Gibbons-Tsarev system.

Deformations of one-phase reductions of dKP

$$(u_t - uu_x)_x - \varepsilon^2 u_{xxxx} = u_{yy}.$$

Deformed one-phase reductions (modulo the Miura group one can assume $u = R$):

$$\begin{aligned} R_t =&(\mu^2 + R)R_x \\ &+\varepsilon^2 \left((2\mu\mu' + 1)R_{xx} + (\mu\mu'' - \mu(\mu')^3 + (\mu')^2/2)R_x^2\right)_x + O(\varepsilon^4), \\ R_y =&\mu R_x \\ &+\varepsilon^2 \left(\mu' R_{xx} + \frac{1}{2}(\mu'' - (\mu')^3)R_x^2\right)_x + O(\varepsilon^4). \end{aligned}$$

Conjecture

For any soliton system in 2+1D, all hydrodynamic reductions of its dispersionless limit can be deformed into reductions of the dispersive counterpart (linear non-degeneracy of the dispersionless limit is required).

Generalised KP equation

$$u_t - uu_x + \varepsilon(A_1 u_{xx} + A_2 u_x^2) + \varepsilon^2(B_1 u_{xxx} + B_2 u_x u_{xx} + B_3 u_x^3) = w_y,$$

$$w_x = u_y.$$

Here $A_i(u),\ B_i(u)$ are certain functions of u. Let us require that all one-phase reductions can be deformed as

$$u = R, \qquad w = w(R) + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots,$$

where

$$R_t = (\mu^2 + R)R_x + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots,$$

$$R_y = \mu R_x + \varepsilon^2(\dots) + \varepsilon^4(\dots) + \dots,$$

$w' = \mu$. This gives $A_1 = A_2 = B_2 = B_3 = 0,\ B_1$=const, $\Longrightarrow$ KP.

Scalar third-order integrable 2+1 D soliton equations with simplest nonlocalities

E.V. Ferapontov, A. Moro and V.S. Novikov, Integrable equations in $2+1$ dimensions: deformations of dispersionless limits, J. Phys. A: Math. Theor. **42** *(2009) (18pp).*

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(...) + \epsilon^2(...), \quad w_x = u_y.$$

here $\varphi, \ \psi, \ \eta$ are functions of u and w, and $(\dots)$ denote terms which are polynomial in the derivatives of u and w with respect to x and y of orders 2 and 3, respectively. Here $w = D_x^{-1} D_y u$ is the nonlocality, no other non-local variables are allowed.

- Classify integrable dispersionless systems of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y.$$

- Reconstruct dispersive corrections which inherit all hydrodynamic reductions (sufficient to consider 1-component reductions up to the order ϵ^4).

Known examples

KP	$u_t = uu_x + w_y + \epsilon^2 u_{xxx}$
mKP	$u_t = (w - u^2/2)u_x + w_y + \epsilon^2 u_{xxx}$
Gardner	$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y + \epsilon^2 u_{xxx}$
VN	$u_t = (uw)_y + \epsilon^2 u_{yyy}$
mVN	$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u} \right)_y$
Harry Dym	$u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx}$

Here $w_x = u_y$. Notice that VN and mVN have coinciding dispersionless limits.

Classification of integrable dispersionless limits

Integrability conditions:

E.V. Ferapontov and K.R. Khusnutdinova, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen. **37***, no. 8 (2004) 2949–2963.*

$$\varphi_{uu} = -\frac{\varphi_w^2 + \psi_u\varphi_w - 2\psi_w\varphi_u}{\eta}, \quad \varphi_{uw} = \frac{\eta_w\varphi_u}{\eta}, \quad \varphi_{ww} = \frac{\eta_w\varphi_w}{\eta},$$

$$\psi_{uu} = \frac{-\varphi_w\psi_w + \psi_u\psi_w - 2\varphi_w\eta_u + 2\eta_w\varphi_u}{\eta}, \quad \psi_{uw} = \frac{\eta_w\psi_u}{\eta}, \quad \psi_{ww} = \frac{\eta_w\psi_w}{\eta},$$

$$\eta_{uu} = -\frac{\eta_w\left(\varphi_w - \psi_u\right)}{\eta}, \quad \eta_{uw} = \frac{\eta_w\eta_u}{\eta}, \quad \eta_{ww} = \frac{\eta_w^2}{\eta}.$$

In involution, straightforward to solve: three main cases corresponding to $\eta = 1,\ \eta = u,\ \eta = e^w h(u)$.

New integrable examples

Example 1.

$$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2[B^3(u) - \beta B^2(u)u_x],$$

$B = \beta u D_x - D_y$.

Example 2.

$$u_t = \frac{4}{27}\gamma^2 u^3 u_x + (w + \gamma u^2)u_y + u w_y + \varepsilon^2[B^3(u) - \frac{1}{3}\gamma u_x B^2(u)],$$

$B = \frac{1}{3}\gamma u D_x + D_y$.

Example 3.

$$u_t = \frac{\delta}{u^3} u_x - 2w u_y + u w_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx},$$

$\delta = 0$ gives the Harry Dym equation.

Discrete equations as dispersive deformations

Consider a discrete wave-type equation,

$$\triangle_{t\bar{t}} u - \triangle_{x\bar{x}} f(u) - \triangle_{y\bar{y}} g(u) = 0,$$

equivalently,

$$u_{tt} - f(u)_{xx} - g(u)_{yy} + \frac{\epsilon^2}{12}(u_{tttt} - f(u)_{xxxx} - g(u)_{yyyy}) + \ldots = 0.$$

The corresponding dispersionless limit is

$$u_{tt} - f(u)_{xx} - g(u)_{yy} = 0.$$

Dispersionless limit possesses solutions of the form $u = R(x, y, t)$ where

$$R_t = \lambda(R) R_x, \quad R_y = \mu(R) R_x,$$

(one-phase reductions), here $\lambda^2 = f' + g' \mu^2$.

Method of dispersive deformations

Let us require that all one-phase reductions of the dispersionless PDE are 'inherited' by the discrete equation:

$$R_t = \lambda(R)R_x + \epsilon(b_1 R_{xx} + b_2 R_x^2) + \epsilon^2(b_3 R_{xxx} + b_4 R_x R_{xx} + b_5 R_x^3) + O(\epsilon^3),$$

$$R_y = \mu(R)R_x + \epsilon(a_1 R_{xx} + a_2 R_x^2) + \epsilon^2(a_3 R_{xxx} + a_4 R_x R_{xx} + a_5 R_x^3) + O(\epsilon^3).$$

This requirement allows us to reconstruct the coefficients $a_i(R), b_i(R)$ in terms of λ, μ. It also leads to strong constraints on $f(u), g(u)$ (integrability conditions):

$$f'' + g'' = 0, \;\; g''(1 + f') - g'f'' = 0, \;\; f''^2(1 + 2f') - f'(f' + 1)f''' = 0.$$

Setting $f(u) = u - \ln(e^u + 1), \; g(u) = \ln(e^u + 1)$, we obtain the discrete equation

$$\triangle_{t\bar{t}}u - \triangle_{x\bar{x}}[u - \ln(e^u + 1)] - \triangle_{y\bar{y}}[\ln(e^u + 1)] = 0,$$

known as 'gauge-invariant form' of the Hirota equation.

Comparison of 1+1 and 2+1 deformation schemes

1+1D:

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + \varepsilon^2(\dots) + \dots$$

- Dispersionless integrable systems form infinite dimensional parameter spaces.
- Terms at ε^2 contain extra functional freedom (central invariants).
- Any integrable system of hydrodynamic type possesses integrable dispersive deformations (not proved in full generality).

2+1D:

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + \varepsilon^2(\dots) + \dots$$

- Dispersionless integrable systems form finite dimensional parameter spaces.
- Terms at ε^2 contain no functional freedom.
- It is still unclear whether any dispersionless integrable system possesses a nontrivial dispersive deformation.