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Lecture 1 — Maximally symmetric space(time)s

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M : C^∞, connected, $\partial M = \emptyset$.

$C^\infty(M)$, $\mathfrak{X}(M)$, $\Omega^\cdot(M)$

functions, vector fields, forms

A metric g on M is a non-degenerate section of S^2T^*M. We do not demand that g be positive-definite. A pair (M,g) is a riemannian manifold. The Fundamental Lemma of Riemannian geometry says that (M,g) admits a unique affine connection ∇ (the Levi-Civita connection) which is

- metric : $\nabla g = 0$
- torsion-free : $\nabla_X Y - \nabla_Y X = [X,Y] \quad \forall\, X,Y \in \mathfrak{X}(M)$

Exercise Derive the Koszul formula:

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z g(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y])$$

Every mathematical structure admits a notion of automorphisms. In the case of a riemannian (M,g) these are the isometries : diffeomorphisms $\varphi : M \to M$ s.t. $\varphi^* g = g$. At least when g is positive-definite or lorentzian (signature $(n-1,1)$, $n = \dim M$), isometries define a finite-dimensional Lie group. Its Lie algebra consists of the Killing vector fields : $\xi \in \mathfrak{X}(M)$ s.t.

$$\mathcal{L}_\xi g = 0 \quad \Rightarrow \text{ Killing vector fields define a Lie subalgebra of } \mathfrak{X}(M).$$

We can rewrite this equation in a more useful form.

Lemma

$\xi \in \mathfrak{X}(M)$ is Killing iff $A_\xi : TM \to TM$, defined by $A_\xi(X) = -\nabla_X \xi$ is skew-symmetric : $g(A_\xi X, Y) = -g(X, A_\xi Y)$.

Proof
$$0 = (\mathcal{L}_\xi g)(X,Y) = \xi(g(X,Y)) - g([\xi,X],Y) - g(X,[\xi,Y])$$

$$\overset{\nabla g = 0}{=} g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y) - g([\xi,X],Y) - g(X,[\xi,Y])$$

$$= g(\nabla_\xi X - [\xi,X], Y) + g(X, \nabla_\xi Y - [\xi,Y])$$

$$\overset{\nabla \text{ is torsion-free}}{=} g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \qquad \blacksquare$$

This still does not show that the Lie algebra of Killing vectors is finite-dimensional. That will follow from Killing's identity. Recall, ξ Killing $\Leftrightarrow$ $A_\xi \in \mathfrak{so}(TM)$.
Let's differentiate A_ξ: $\nabla_X A_\xi \in \mathfrak{so}(TM)$.

$$(\nabla_X A_\xi)(Y) = \nabla_X(A_\xi(Y)) - A_\xi(\nabla_X Y) = -\nabla_X\nabla_Y \xi + \nabla_{\nabla_X Y}\xi$$

$\therefore$

$$\begin{aligned}(\nabla_X A_\xi)(Y) - (\nabla_Y A_\xi)(X) &= -\nabla_X\nabla_Y\xi + \nabla_Y\nabla_X\xi + \nabla_{\nabla_X Y}\xi - \nabla_{\nabla_Y X}\xi \\ &\overset{\nabla \text{ is torsion-free}}{=} -[\nabla_X,\nabla_Y]\xi + \nabla_{[X,Y]}\xi \\ &= R(X,Y)\xi \qquad \text{(definition of } R) \\ &\overset{R(X,Y)=-R(Y,X)\ \&\ \text{algebraic Bianchi identity}}{=} -R(Y,\xi)X + R(X,\xi)Y\end{aligned}$$

$\nabla_X A_\xi,\ R(X,\xi) \in \mathfrak{so}(TM)$

Define $S(X,Y,Z) := g((\nabla_X A_\xi - R(X,\xi))Y, Z)$. Then $S(X,Y,Z) = -S(X,Z,Y)$
but also $S(X,Y,Z) = S(Y,X,Z)$. $\Rightarrow$ $S \equiv 0$ $\Leftrightarrow$ $\boxed{\nabla_X A_\xi = R(X,\xi)}$ Killing's identity

Let $E := TM \oplus \mathfrak{so}(TM)$, a rank-$\frac{n(n+1)}{2}$ real VB over (M^n, g). Define a connection D on E by

$$D_X\begin{pmatrix}\xi \\ A\end{pmatrix} = \begin{pmatrix}\nabla_X\xi + A(X) \\ \nabla_X A + R(\xi,X)\end{pmatrix} = \begin{pmatrix}\nabla_X & \iota_X \\ R(-,X) & \nabla_X\end{pmatrix}\begin{pmatrix}\xi \\ A\end{pmatrix}$$

Proposition (Killing transport)

$\xi \in \mathfrak{X}(M)$ is Killing $\Leftrightarrow$ $\begin{pmatrix}\xi \\ A_\xi\end{pmatrix} \in \Gamma(E)$ is D-parallel ∎

$\dim\{\sigma \in \Gamma(E) \mid D\sigma = 0\} \leq \operatorname{rank} E$ since parallel sections are uniquely determined by their value at any given point. (M connected!)

Upshot

$KV(M,g) := \{\xi \in \mathfrak{X}(M) \mid \mathcal{L}_\xi g = 0\}$ is a real Lie algebra and $\dim KV(M^n,g) \leq \frac{n(n+1)}{2}$.

We say that (M^n, g) is maximally symmetric if $\dim KV(M^n,g) = \frac{n(n+1)}{2}$.

Example (Euclidean space) Take $M^n = \mathbb{A}^n$ with affine coordinates $(x^1, \dots, x^n)$ and $g_0 = \sum_i (dx^i)^2$.
Then $\nabla_{\frac{\partial}{\partial x^i}} = \frac{\partial}{\partial x^i}$ and $R = 0$, so if $\xi = \sum_i \xi^i \frac{\partial}{\partial x^i}$ is Killing, then Killing's identity says

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\xi^k = 0 \Rightarrow \xi^k = \underbrace{a^k}_{n} + \sum_\ell \underbrace{\lambda^k{}_\ell}_{n(n-1)/2} x^\ell \quad \text{where} \quad \lambda^k{}_\ell = -\lambda^\ell{}_k$$

$\therefore$ $(\mathbb{A}^n, g_0)$ is maximally symmetric. The Lie algebra $KV(\mathbb{A}^n, g_0)$ is the euclidean Lie algebra.

Let (M,g) and $\xi \in KV(M,g)$. Then $[\xi,\zeta] \in KV(M,g)$. What is the Killing transport data associated with $[\xi,\zeta]$? $[\xi,\zeta] = \nabla_\xi \zeta - \nabla_\zeta \xi = A_\xi \zeta - A_\zeta \xi$, and hence

$$A_{[\xi,\eta]}(X) = -\nabla_X [\xi,\eta] = -\nabla_X (A_\xi \zeta - A_\zeta \xi)$$
$$= -(\nabla_X A_\xi)(\zeta) - A_\xi(\nabla_X \zeta) + (\nabla_X A_\zeta)(\xi) + A_\zeta(\nabla_X \xi)$$
$$= -R(X,\xi)\zeta + A_\xi A_\zeta X + R(X,\zeta)\xi - A_\zeta A_\xi X$$
$$= R(\xi,\zeta)X + [A_\xi, A_\zeta]X$$

$$\therefore \left[\begin{pmatrix} \xi \\ A_\xi \end{pmatrix}, \begin{pmatrix} \zeta \\ A_\zeta \end{pmatrix} \right] = \begin{pmatrix} A_\xi \zeta - A_\zeta \xi \\ [A_\xi, A_\zeta] + R(\xi,\eta) \end{pmatrix}$$

$\therefore$ For $(\mathbb{A}^n, g_0)$, $R=0$, hence $KV(\mathbb{A}^n, g_0) \overset{\text{v.s.}}{=} \mathbb{R}^n \oplus \underline{so}(n)$ and for all $A,B \in \underline{so}(n)$, $x,y \in \mathbb{R}^n$,

$$\left.\begin{aligned} [A,B] &= A\cdot B - B\cdot A \\ [A,x] &= Ax \\ [x,y] &= 0 \end{aligned}\right\}$$

euclidean Lie algebra

$\mathbb{R}^n \rtimes \underline{so}(n)$ or

$0 \to \mathbb{R}^n \to e_n \to \underline{so}(n) \to 0$

Remarks ① R is the obstruction to $KV(M,g)$ to be a Lie subalgebra of e_n. In general, $KV(M,g)$ is a filtered Lie algebra $\mathfrak{h} \oplus V$, $\mathfrak{h} < \underline{so}(n)$, $V \subset \mathbb{R}^n$ with $\text{fdeg}\, \mathfrak{h} = 0$, $\text{fdeg}\, V = -1$, and the associated graded Lie algebra is a Lie subalgebra of e_n. (← KV of the flat model!)

② Maximally symmetric (M,g) have $\dim KV(M,g) = \frac{n(n+1)}{2}$. What are about sub-maximally symmetric? Can one have (M,g) with $\dim KV(M,g) = \frac{n(n+1)}{2} - 1$? There is a gap! (Ask Boris & Dennis!)

③ We shall be interested in Lorentzian manifolds (M^n, g) where g has signature $(n-1,1)$. The flat model is Minkowski spacetime $(\mathbb{A}^n, \eta)$ where relative to affine coordinates $(x^0, x^1, \dots, x^{n-1})$ the metric is given by $\eta = -c^2(dx^0)^2 + (dx^1)^2 + \dots + (dx^{n-1})^2$

speed of light (often $c \equiv 1$)

Examples ① The round spheres $S^n = q^{-1}(\ell) \subset \mathbb{R}^{n+1}$ (radius >0), $q : \mathbb{R}^{n+1} \to \mathbb{R}$, $x \mapsto \|x\| := \sqrt{\sum_i (x^i)^2}$

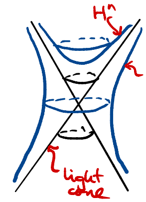
Any orthogonal linear transformation $x \mapsto Ax$, $A \in O(n+1)$ is an isometry. The Lie algebra of $O(n+1)$ is $\frac{n(n+1)}{2}$-dimensional $\Rightarrow S^n$ is maximally symmetric.

② **Hyperbolic space** $H^n \subset M^{n+1}$ (Minkowski spacetime) as $\{(x^0, x^1, \ldots, x^n) \in M^{n+1} \mid \eta(x,x) = -\ell^2\}$

$-(x^0)^2 + \sum_i (x^i)^2$; radius of curvature

Every **Lorentz transformation** $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $\eta(Ax, Ay) = \eta(x,y)$ is an isometry. The Lie algebra $\mathfrak{so}(n,1)$ of Lorentz transformations has $\dim = \frac{n(n+1)}{2}$ & hence H^n is maximally symmetric.

③ **de Sitter spacetime** $dS_n \subset M^{n+1}$ as $\{x \in M^{n+1} \mid \eta(x,x) = \ell^2\}$



maximally symmetric ← $KV(dS_n) \cong \mathfrak{so}(n,1)$ as for H^n. The **light cone** $\eta(x,x) = 0$ is singular at $x = 0$. The **future** and **past lightcones** are not singular, but they do not admit (invariant) metrics: they admit a **carrollian structure**.

④ **Anti de Sitter spacetime** $AdS_n \subset \mathbb{R}^{n+1}$, $\{x \in \mathbb{R}^{n+1} \mid (x^1)^2 + \cdots + (x^{n-1})^2 - (x^n)^2 - (x^{n+1})^2 = -\ell^2\}$

radius of curvature

maximally symmetric ← $KV(AdS_n) \cong \mathfrak{so}(n-1,2)$

Remark Typically, we define (A)dS spacetimes as the universal covers of the above quadrics.

e.g., $AdS_3 \cong \widetilde{SL(2,\mathbb{R})}$ whereas the quadric is $\approx SL(2,\mathbb{R})$.

Theorem A simply-connected, maximally symmetric riemannian (resp. lorentzian) manifold is isometric to euclidean space, $S(\ell)$ or $H(\ell)$ (resp., Minkowski, $dS(\ell)$ or $AdS(\ell)$ spacetimes) $\exists!\, \ell > 0$.

Proof If $\dim KV(M^n, g) = \frac{n(n+1)}{2}$, then the Killing transport connection D on E is flat. We calculate its curvature:

$$R^D(X,Y)\begin{pmatrix}\xi \\ A\end{pmatrix} = D_{[X,Y]}\begin{pmatrix}\xi \\ A\end{pmatrix} - [D_X, D_Y]\begin{pmatrix}\xi \\ A\end{pmatrix}$$

Exercise!

$$\vdots$$

$$= \begin{pmatrix} 0 \\ (\nabla_\xi R + [A,R])(X,Y) \end{pmatrix}$$

← algebraic Bianchi and ∇ torsion-free

← differential Bianchi

But for maximal symmetry, we can choose ξ, A independently and arbitrarily at any point. Therefore, $\nabla_\xi R = 0$ for all $\xi \in T_pM$ and all $p \in M$ $\Rightarrow$ (M,g) is locally symmetric

and $[A,R] = 0$ for all $A \in \mathfrak{so}(T_pM)$ and all $p \in M$ $\Rightarrow$ $R_p : \Lambda^2 T_pM \to \mathfrak{so}(T_pM)$ is $\mathfrak{so}(T_pM)$-equivariant. This means that

$$R(X,Y)Z = \lambda\big(g(X,Z)Y - g(Y,Z)X\big) \quad \exists \lambda \in C^\infty(M) \quad \text{but } \nabla R = 0 \Rightarrow \lambda \text{ is constant.}$$

So (M,g) has constant sectional curvature. One calculates the Riemann curvature of $\mathbb{E}^n, M^n, S^n, H^n, dS_n, AdS_n$ and one sees they have constant sectional curvature with $\lambda \propto 1/\ell^2$. So (M,g) is locally isometric to one of them & one passes to their universal covers. ∎