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Lecture 2 — Kinematical Lie algebras (Wednesday 8 May 2019)

Last time, we met the maximally symmetric riemannian and lorentzian manifolds: (M^{n+1}, g) with $\dim KV(M^{n+1}, g) = \frac{(n+1)(n+2)}{2}$.

$$KV(M, g) = \{ \xi \in \mathfrak{X}(M) \mid \mathcal{L}_\xi g = 0 \}$$

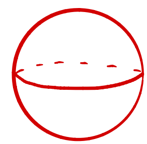
They have constant sectional curvature:

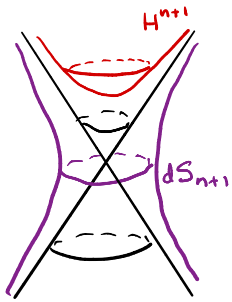
$$R(X,Y)Z = \lambda \left(g(Y,Z)X - g(X,Z)Y \right) \quad \exists\, \lambda \in \mathbb{R}$$

		$+$	0	$-$
riemannian	R	S^{n+1}	$\mathbb{E}^{n+1}$	H^{n+1}
lorentzian	L	dS_{n+1}	$\mathbb{M}^{n+1}$	AdS_{n+1}

$S^{n+1} \subset \mathbb{E}^{n+2}$

$\underline{\mathfrak{so}}(n+2)$





$H^{n+1} \subset \mathbb{M}^{n+2}$

$dS_{n+1} \subset \mathbb{M}^{n+2}$

$\underline{\mathfrak{so}}(1, n+1)$

$AdS_{n+1} \subset \mathbb{R}^{2,n}$

$(x^1)^2 + \cdots + (x^n)^2 - (x^{n+1})^2 - (x^{n+2})^2 = -\ell^2$

$\underline{\mathfrak{so}}(2,n)$

The Lie algebras of isometries of S^{n+1}, H^{n+1}, dS_{n+1} and AdS_{n+1} are (semi) simple.

For the flat examples $\mathbb{E}^{n+1}$ and $\mathbb{M}^{n+1}$, the Lie algebra of isometries is not semisimple: the translations span an ideal. Indeed,

$$\mathbb{E}^{n+1}: \ \mathfrak{e} = \underline{\mathfrak{so}}(n+1) \ltimes \mathbb{R}^{n+1} \qquad \mathbb{M}^{n+1}: \ \mathfrak{p} = \underline{\mathfrak{so}}(1,n) \ltimes \mathbb{R}^{n+1}$$

There is an obvious geometric limit (zero curvature limit) relating S^{n+1} or H^{n+1} to $\mathbb{E}^{n+1}$ and, similarly, AdS_{n+1} or dS_{n+1} to $\mathbb{M}^{n+1}$. Such geometric limits induce contractions of the Lie algebras of isometries.

A (real, f.d.) Lie algebra consists of a vector space $\mathfrak{g}$ and a bracket $[-,-] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ obeying the Jacobi identity. If $\varphi \in GL(\mathfrak{g})$, then

$$[X,Y]_\varphi := \varphi[\varphi^{-1}X, \varphi^{-1}Y]$$

defines another Lie algebra structure on $\mathfrak{g}$ which is isomorphic to $(\mathfrak{g},[,])$. Let $\mathcal{O}(\mathfrak{g},[-,-]) = \{ (\mathfrak{g},[-,-]_\varphi) \mid \varphi \in GL(\mathfrak{g})\}$. A Lie algebra $(\mathfrak{g},[,]_0)$ is said to be a contraction of $(\mathfrak{g},[,])$ if $(\mathfrak{g},[,]_0) \not\cong (\mathfrak{g},[,])$ but $(\mathfrak{g},[-,-]_0) \in \overline{\mathcal{O}(\mathfrak{g},[-,-])}$ (¯: the closure of the orbit.) A more concrete way to describe contractions is this. Let $\varphi_\varepsilon \in GL(G)$ for $\varepsilon \in (0,1]$ with $\varphi_1 = \mathrm{id}$. Then the limit $\lim_{\varepsilon\to 0} \varphi_\varepsilon$ may not exist, but if $[-,-]_0 := \lim_{\varepsilon \to 0} [-,-]_{\varphi_\varepsilon}$ does, then by continuity, $(\mathfrak{g},[-,-]_0)$ is a Lie algebra.

Example (Zero-curvature limit)

The $\ell \to \infty$ limits of $H^{n+1}(\ell)$, $S^{n+1}(\ell)$, $dS_{n+1}(\ell)$, $AdS_{n+1}(\ell)$ give either $\mathbb{E}^{n+1}$ (in the riemannian cases) or $\mathbb{M}^{n+1}$ (in the lorentzian cases).

H^{n+1} $\quad \underline{so}(1,n+1) \quad \varphi_{1/\ell}(J_{ab}) = J_{ab} \quad \varphi_{1/\ell}(J_{0a}) = \ell J_{0a}$

$$[J_{0a},J_{0b}]_\varphi := \varphi_{1/\ell}[\varphi^{-1}_{1/\ell}J_{0a}, \varphi^{-1}_{1/\ell}J_{0b}] = \frac{1}{\ell^2}J_{ab} \longrightarrow 0$$

S^{n+1} $\quad \underline{so}(n+2) \quad \varphi_{1/\ell}(J_{ab}) = J_{ab} \quad a,b \le n+1 \quad \varphi_{1/\ell}(J_{a,n+2}) = \ell J_{a,n+2}$

$$[J_{a,n+1}, J_{b,n+1}]_\varphi = -\frac{1}{\ell^2}J_{ab} \to 0$$

$\Big\}\ \underline{so}(n+1) \ltimes \mathbb{R}^{n+1}$

dS_{n+1} $\quad \underline{so}(1,n+1) \quad \varphi_{1/\ell}(J_{\mu\nu}) = J_{\mu\nu}, \quad 0 \le \mu,\nu \le n \quad \varphi_{1/\ell}(J_{\mu,n+1}) = \ell J_{\mu,n+1}$

$$[J_{\mu,n+1}, J_{\nu,n+1}]_\varphi = -\frac{1}{\ell^2}J_{\mu\nu} \longrightarrow 0$$

AdS_{n+1} $\quad \underline{so}(2,n) \quad \varphi_{1/\ell}(J_{\mu\nu}) = J_{\mu\nu} \quad 0 \le \mu,\nu \le n \quad \varphi_{1/\ell}(J_{\mu,n+1}) = \ell J_{\mu,n+1}$

$$[J_{\mu,n+1}, J_{\nu,n+1}]_\varphi = +\frac{1}{\ell^2}J_{\mu\nu} \to 0$$

$\Big\}\ \underline{so}(1,n) \ltimes \mathbb{R}^{n+1}$

All these Lie algebras $\left(\underline{so}(n+2), \underline{so}(1,n+1), \underline{so}(2,n), \mathfrak{e}_{n+1}, \mathfrak{p}_{n+1}\right)$ are examples of kinematical Lie algebras (with space isotropy).

Definition A kinematical Lie algebra (with n-dimensional space isotropy) is a real Lie algebra $\mathfrak{k}$ of dimension $\frac{(n+1)(n+2)}{2}$ satisfying the following:

(1) $\exists\ \underline{so}(n) \cong \mathfrak{r} < \mathfrak{k}$

(2) Under $\mathrm{ad}_{\mathfrak{r}}$, $\mathfrak{k} \cong \mathfrak{r} \oplus 2V \oplus S$ ← 1-dim'l scalar rep of $\underline{so}(n)$

V: n-dim'l vector rep of $\underline{so}(n)$

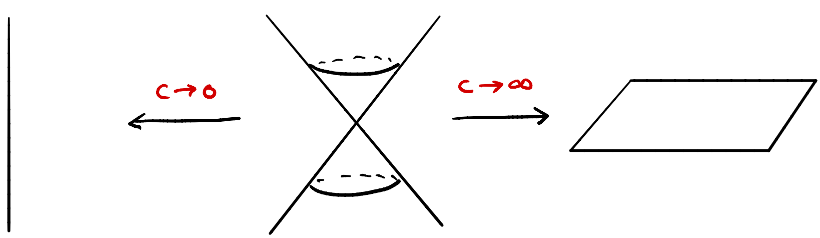
We normally write the basis of a KLA $\mathfrak{k}$ as $\langle \underbrace{J_{ab}}_{\mathfrak{so}(n)}, \underbrace{B_a, P_a}_{2V}, \underbrace{H}_{S} \rangle$ but we abbreviate the kinematical Lie brackets as:

$$[J,J] = J \qquad [J,B] = B \qquad [J,P] = P \qquad [J,H] = 0$$

Different KLAs are characterised by the additional Lie brackets:

(M,g)	$[B,B]$	$[B,P]$	$[P,P]$	$[H,B]$	$[H,P]$
S	$-J$	H	$-J$	P	$-B$
H	$-J$	H	J	P	B
$\mathbb{E}$	$-J$	H	0	P	0
dS	J	H	$-J$	$-P$	$-B$
AdS	J	H	J	$-P$	B
$\mathbb{M}$	J	H	0	$-P$	0

Different limits to the zero-curvature limit are obtained by taking the speed of light c to either 0 or ∞. The $c \to 0$ limit is called the carrollian (or ultra-relativistic) limit. The lightcone $c^2t^2 = \|x\|^2$ collapses to the time axis in the $c \to 0$ limit. The name "carrollian" was coined by Lévy-Leblond in reference of Lewis Carroll's 'Alice in Wonderland'. (cf. Red Queen) The limit $c \to \infty$ is called the galilean (or non-relativistic) limit: the light cone collapses to the plane $t = 0$.



At the level of the KLA, the carrollian limit is obtained by

$$\varphi_c(J) = J \qquad \varphi_c(B) = \tfrac{1}{c}B \qquad \varphi_c(P) = P \qquad \varphi_c(H) = \tfrac{1}{c}H \qquad (\text{and } c \to 0)$$

whereas the galilean limit is obtained by

$$\varphi_{1/c}(J) = J \qquad \varphi_{1/c}(B) = cB \qquad \varphi_{\frac{1}{c}}(P) = cP \qquad \varphi_{\frac{1}{c}}(H) = H \qquad (\text{and } \tfrac{1}{c} \to 0).$$

Under the carrollian limit:

(M,g)	$[B,B]$	$[B,P]$	$[P,P]$	$[H,B]$	$[H,P]$	$B \leftrightarrow P$
S	$-J c^2$	H	$-J$	$P c^2$	$-B$	$\cong \mathfrak{e}$
H	$-J c^2$	H	J	$P c^2$	B	$\cong \mathfrak{p}$
$\mathbb{E}$	$-J c^2$	H	0	$P c^2$	0	carroll
dS	$J c^2$	H	$-J$	$-P c^2$	$-B$	$\cong \mathfrak{e}$
AdS	$J c^2$	H	J	$-P c^2$	B	$\cong \mathfrak{p}$
$\mathbb{M}$	$J c^2$	H	0	$-P c^2$	0	Caroll

Under the galilean limit

(M,g)	$[B,B]$	$[B,P]$	$[P,P]$	$[H,B]$	$[H,P]$	
S	$-J \frac{1}{c^2}$	$H \frac{1}{c^2}$	$-J \frac{1}{c^2}$	P	$-B$	} Newton–Hooke
H	$-J \frac{1}{c^2}$	$H \frac{1}{c^2}$	$J \frac{1}{c^2}$	P	B	
$\mathbb{E}$	$-J \frac{1}{c^2}$	$H \frac{1}{c^2}$	$0 \frac{1}{c^2}$	P	0	Galilean
dS	$J \frac{1}{c^2}$	$H \frac{1}{c^2}$	$-J \frac{1}{c^2}$	$-P$	$-B$	} Newton–Hooke
AdS	$J \frac{1}{c^2}$	$H \frac{1}{c^2}$	$J \frac{1}{c^2}$	$-P$	B	
$\mathbb{M}$	$J \frac{1}{c^2}$	$H \frac{1}{c^2}$	$0 \frac{1}{c^2}$	$-P$	0	Galilean

In general, there are several other isomorphism classes of KLAs:

$$[H,B] = \gamma B \qquad [H,P] = P \qquad \gamma \in (-1,1]$$

and

$$[H,B] = \chi B + P \qquad [H,P] = \chi P - B \qquad \chi > 0$$

$\gamma = -1$ & $\chi = 0$ are $\cong$ to N-H KLAs

For $n=3$, the $\times$ product $V \times V \to V$ gives more KLAs
For $n=2$, the symplectic structure $V \times V \to S$ does too.
For $n=1$, any 3d LA is kinematical (Bianchi 1898)
For $n=0$, $\exists!$ 1-dim'l LA.

The classifications of KLAs follow using deformation theory of the static KLA where all non-kinematical Lie brackets vanish.