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Lecture 3 Homogeneous kinematical space(time)s (Monday 13 May 2019)

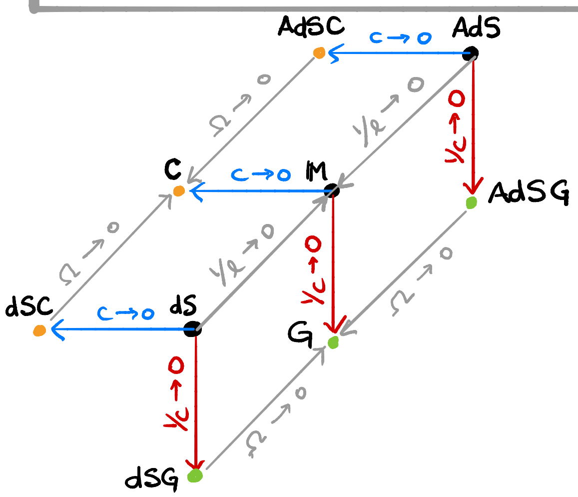
In Lecture 1, we classified the (simply-connected) maximally symmetric riemannian and lorentzian manifolds and identified their Lie algebras of Killing vectors. We saw in Lecture 2 how the zero-curvature limits $S \to \mathbb{E}$, $H \to \mathbb{E}$, $dS \to \mathbb{M}$ and $AdS \to \mathbb{M}$ induced a contraction of the Lie algebra of isometries. Similarly, the carrollian ($c \to 0$) and galilean ($1/c \to 0$) limits induced contractions of the Lie algebra of isometries. We remarked that all these Lie algebras are kinematical and we described the result of the classification of KLAs in generic dimension. For $n=3$ (manifold dimension $= n+1$) and $n=2$ there are other KLAs and we did not list them.
A KLA with n-dimensional space isotropy has a basis $(J_{ab} = -J_{ba}, B_a, P_a, H)$ where J_{ab} span a Lie subalgebra $\cong \mathfrak{so}(n)$, B_a, P_a are vectors of $\mathfrak{so}(n)$ and H is a scalar. We say that J_{ab} are (infinitesimal) rotations, B_a are boosts, P_a are spatial translations and H is time translation; but these identifications can only be made precise once we have a geometric model on which the KLA acts transitively.

Our aim in this lecture is to present the classification of (simply-connected) homogeneous space(time)s of kinematical Lie groups.

Def. A **kinematical Lie group** is a Lie group whose Lie algebra is kinematical. A **closed** subgroup $\mathcal{H}$ of a kinematical Lie group $\mathcal{K}$ is **admissible** if its Lie algebra $\mathfrak{h}$ is a subalgebra of $\mathfrak{k}$ consisting of rotations and boosts. That is, $\mathfrak{h} \supset \mathfrak{r} \cong \mathfrak{so}(n)$ and $\mathfrak{h} = \mathfrak{so}(n) \oplus V$ under $\operatorname{ad}_{\mathfrak{r}}$. We can always choose a basis for $\mathfrak{h}$ consisting of (J_{ab}, B_a).

Def. A **homogeneous kinematical space(time)** is a connected manifold admitting a transitive action of a kinematical Lie group with admissible stabilisers. In other words, $M \cong \mathcal{K}/\mathcal{H}$ where $\mathcal{K}$ is a kinematical Lie group and $\mathcal{H}$ is admissible. We may describe M (up to coverings) by the pair $(\mathfrak{k},\mathfrak{h})$ of Lie algebras: $\mathfrak{k} = \mathrm{Lie}(\mathcal{K})$ and $\mathfrak{h} = \mathrm{Lie}(\mathcal{H})$. Choosing a basis (J, B) for $\mathfrak{h}$, the pair $(\mathfrak{k},\mathfrak{h})$ is uniquely determined by the brackets of $\mathfrak{k}$ in that basis and, in particular, by the brackets which do not involve J.

	Examples	$[H,B]$	$[H,P]$	$[B,B]$	$[B,P]$	$[P,P]$	
	$\mathbb{E}$	P	0	$-J$	H	0	riemannian
	S	P	$-B$	$-J$	H	$-J$	
	H	P	B	$-J$	H	J	
	$\mathbb{M}$	$-P$	0	J	H	0	lorentzian
	dS	$-P$	$-B$	J	H	$-J$	
	AdS	$-P$	B	J	H	J	
	C	0	0	0	H	0	"carrollian"
$0 \uparrow c$	dSC	0	$-B$	0	H	$-J$	
	$AdSC$	0	B	0	H	J	
	G	$-P$	0	0	0	0	"galilean"
$0 \uparrow 1/c$	dSG	$-P$	$-B$	0	0	0	
	$AdSG$	$-P$	B	0	0	0	



Scholium $(\mathfrak{k},\mathfrak{h})$ is **reductive** if $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$ and it is **symmetric** if, in addition, $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$. In the above table, $\mathfrak{h} = \mathrm{Span}(J,B)$, $\mathfrak{m} = \mathrm{Span}(H,P)$ and all spaces are symmetric.

Fact These are the simply-connected symmetric homogeneous kinematical space(time)s.

Symmetric spaces have a canonical torsion-free invariant affine connection: it is flat for $\mathbb{E}$, $\mathbb{M}$, C, G.

Remark The description of homogeneous spaces via pairs $(\mathfrak{k},\mathfrak{h})$ is analogous to describing a Lie group via its Lie algebra. Every (f.d. real) Lie algebra is isomorphic to the Lie algebra of a unique (up to iso.) simply-connected Lie group. Equivalently, Lie algebras (up to iso.) classify connected Lie groups (up to coverings). For pairs $(\mathfrak{k},\mathfrak{h})$ existence of a homogeneous space is not guaranteed and uniqueness is also in question.

Non-example $\mathfrak{k} = \underline{\mathfrak{su}}(3)$, $\mathfrak{h} = \left\{ \begin{pmatrix} i & & \\ & i\alpha & \\ & & -i(1+\alpha) \end{pmatrix} \middle| \alpha \in \mathbb{R}\setminus\mathbb{Q} \right\}$ ∄ homogeneous space described by $(\mathfrak{k},\mathfrak{h})$!

$\mathcal{K} = SU(3)$ or $PSU(3)$. The subgroup of $SU(3)$ gen'd by $\mathfrak{h}$ is contained in a maximal torus and is an irrational slope subgroup and hence not closed in $SU(3)$. (Neither is it closed in $PSU(3)$.)
(The irrational slope subgroup of the torus is not closed in the torus, but is closed in the universal cover.)

Theorem There is a bijection between (iso classes) of simply-connected homogeneous spaces and (iso classes) of effective, geometrically realisable pairs $(\mathfrak{k},\mathfrak{h})$. ($(\mathfrak{k}_1,\mathfrak{h}_1) \cong (\mathfrak{k}_2,\mathfrak{h}_2)$ if ∃ iso $\varphi: \mathfrak{k}_1 \xrightarrow{\cong} \mathfrak{k}_2$ sending $\mathfrak{h}_1$ to $\mathfrak{h}_2$.)

$(\mathfrak{k},\mathfrak{h})$ is effective if $\mathfrak{h}$ does not contain a nontrivial ideal of $\mathfrak{k}$

$(\mathfrak{k},\mathfrak{h})$ is geometrically realisable if there exists a Lie group $\mathcal{K}$ with Lie algebra (iso to) $\mathfrak{k}$ where the Lie subgroup $\mathcal{H}$ generated by $\mathfrak{h}$ is closed. Then $M := \mathcal{K}/\mathcal{H}$ is a geometric realisation of $(\mathfrak{k},\mathfrak{h})$.

Classification

One can classify simply-connected homogeneous kinematical space(time)s as follows:

0. Classify kinematical Lie algebras up to isomorphism.
1. For each KLA $\mathfrak{k}$, determine the possible admissible subalgebras $\mathfrak{h}$ up to the action of $\mathrm{Aut}(\mathfrak{k})$.
2. We discard the resulting pairs $(\mathfrak{k},\mathfrak{h})$ which are not effective. Because of the structure of $\mathfrak{k}$ and $\mathfrak{h}$ the only nontrivial ideal of $\mathfrak{k}$ contained in $\mathfrak{h}$ would be the ideal $\mathfrak{b}$ generated by the boosts and then $\mathfrak{h}/\mathfrak{b} \cong \mathfrak{r} \cong \underline{\mathfrak{so}}(n)$ and $\mathfrak{k}/\mathfrak{b}$ would be an aristotelian Lie algebra.

Scholium An aristotelian Lie algebra $\mathfrak{a}$ is an $\left(\frac{n(n+1)}{2}+1\right)$-dim'l real Lie algebra such that

(i) $\mathfrak{so}(n) \cong \mathfrak{r} < \mathfrak{a}$

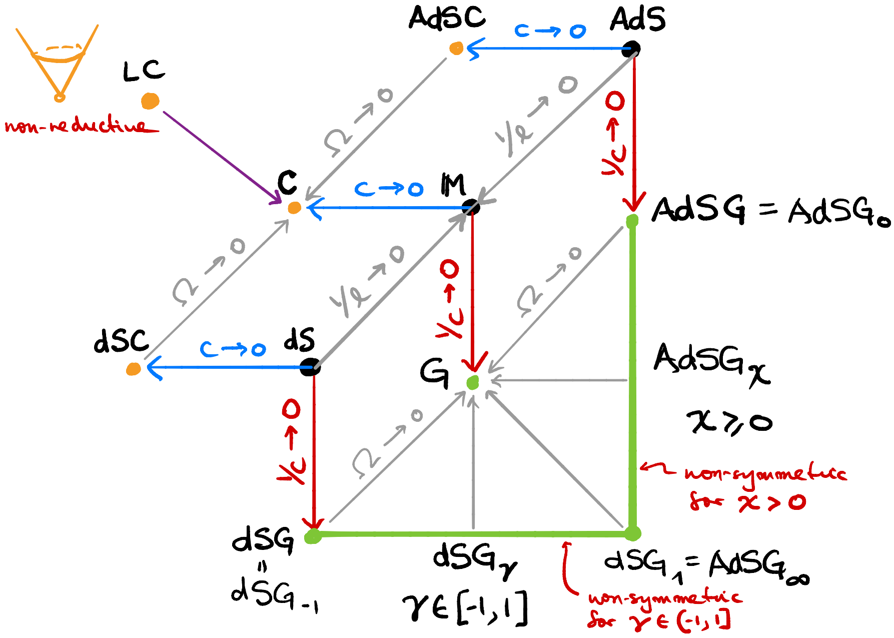
and

(ii) under $\mathrm{ad}_{\mathfrak{r}}$, $\mathfrak{a} = \mathfrak{r} \oplus V \oplus S$
$\qquad\qquad\qquad\qquad\qquad J \quad P \quad H$

determined by $[H,P]=\alpha P$ and $[P,P]=\beta J$

There are generically 4 aristotelian LAs up to iso:
$(\alpha,\beta) \in \{(0,0),(1,0),(0,1),(0,-1)\}$.
For $n=2$ there is an additional aristotelian LA: Heisenberg!

8. We show that the remaining $(\mathfrak{k},\mathfrak{h})$ are geometrically realisable. This is the most painful part of the process. Here is part of the end result:



We recognise M, AdS, dS as the maximally symmetric lorentzian spacetimes. Their riemannian cousins are not shown.

We have also their carrollian limits, together with the (future) lightcone LC.

And we also have their galilean limits which are points in two one-parameter families of galilean spacetimes.

	$[H,B]$	$[H,P]$	$[B,B]$	$[B,P]$	$[P,P]$
LC	B	$-P$	0	$H+J$	0
dSG$_\gamma$	$-P$	$\gamma B+(1+\gamma)P$	0	0	0
AdSG$_\chi$	$-P$	$(1+\chi^2)B+2\chi P$	0	0	0

Question What invariant structures do the carrollian and galilean spacetimes possess?

Let $M = \mathcal{K}/\mathcal{H}$ be a homogeneous space with $\mathcal{H}$ connected and $\mathcal{K}$ simply-connected. Let $(\mathfrak{k},\mathfrak{h})$ denote the corresponding pair and let $o \in M$ be a point with stabiliser $\mathcal{H}$.

$\mathcal{H}$ (and hence $\mathfrak{h}$) act on T_oM. The adjoint action of $\mathfrak{h}$ on $\mathfrak{k}$ has $\mathfrak{h}$ as a submodule and hence $\mathfrak{k}/\mathfrak{h}$ is an $\mathfrak{h}$-module and $T_oM \cong \mathfrak{k}/\mathfrak{h}$ as $\mathfrak{h}$-modules. If $(\mathfrak{k},\mathfrak{h})$ is reductive, then $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ $\exists$ $\mathfrak{h}$-module $\mathfrak{m}$, and $T_oM \cong \mathfrak{m}$. The action of $\mathfrak{h}$ on T_oM is called the linear isotropy representation.

Theorem There is a bijection between invariants of the linear isotropy representation and $\mathcal{K}$-invariant tensor fields on M.

For example, $\xi_o \in (\mathfrak{k}/\mathfrak{h})^{\mathfrak{h}}$ is the value at o of a $\mathcal{K}$-invariant vector field $\xi \in \mathcal{X}(M)$. Similarly, $\tau_o \in ((\mathfrak{k}/\mathfrak{h})^*)^{\mathfrak{h}}$ is the value at $o \in M$ of a $\mathcal{K}$-invariant one-form $\tau \in \Omega^1(M)$. Et cetera.

To determine the invariant structure on the homogeneous kinematical spacetimes, it is enough to study the linear isotropy representation. Remark All the homogeneous kinematical space(time)s are reductive with the exception of the carrollian future lightcone LC

Results The lorentzian HKSs admit an $\mathfrak{h}$-invariant $\eta_o \in S^2(\mathfrak{k}/\mathfrak{h})^*$ which is lorentzian.

Similarly, the riemannian HKSs admit an $\mathfrak{h}$-invariant $\eta_o \in S^2(\mathfrak{k}/\mathfrak{h})^*$ which is positive-definite.

For the carrollian HKSs, there is an invariant $\kappa_o \in \mathfrak{k}/\mathfrak{h}$ and an invariant corank-1 $\eta_o \in S^2(\mathfrak{k}/\mathfrak{h})^*$ such that $\eta_o(\kappa_o,-)=0$.

For the galilean HKSs, there is an invariant $\tau_o \in (\mathfrak{k}/\mathfrak{h})^*$ and an invariant corank-1 $h_o \in S^2(\mathfrak{k}/\mathfrak{h})$ s.t. $h_o(\tau_o,-)=0$.

A null hypersurface in a lorentzian manifold is carrollian. Dually, if a null Killing vector acts freely on a lorentzian manifold, the quotient is galilean.