Half-flat causal structures and integrable systems

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An outline

- Causal structures: definition, motivation, and history

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- The equivalence problem and fundamental invariants

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- Cayley structures

Definitions

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If the projective 2nd fund form of $\mathcal{C}_x \subset \mathbb{P}T_xM$, $\forall x \in M$ is non-degenerate everywhere one obtains a *causal structure.*

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Roughly speaking, if Σ_x not quadratic one has a (local) Finsler metric

$$TM \supset \Sigma^{2n+1} = \{v \in TM \mid F(v) = 1\}$$

assuming *radial transversality* and *non-deg of the 2nd fund form* of $\Sigma_x \subset T_xM$, $\forall x \in M$.

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- In the fibration $\pi \circ \iota : \mathcal{C} \to M$, the fibers $\mathcal{C}_x^{n-1} := (\pi \circ \iota)^{-1}(x)$ are mapped to immersed connected *tangentially non-degenerate* projective hypersurfaces via $\iota_x : \mathcal{C}_x \to \mathbb{P}T_xM$, i.e., they have non-deg projective 2nd fund form everywhere.

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if $\exists$ diffeo $\phi : U \to \tilde{U}$

where $x \in U \subset M, \tilde{x} \in \tilde{U} \subset \tilde{M}$

$$\begin{array}{ccc} \mathcal{C}|_U & \xrightarrow{\phi_*} & \tilde{\mathcal{C}}|_{\tilde{U}} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \tilde{\mu}} \\ U & \xrightarrow{\phi} & \tilde{U} \end{array}$$

$\tilde{x} = \phi(x)$

$\phi_*(\mathcal{C}_y) = \tilde{\mathcal{C}}_{\phi(y)}$

$\forall y \in U$

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- Note that $\mathcal{C}$ can be open and be immersed as an *open hypersurface* in $\mathbb{P}TM$.
- For the local aspects of causal geometry ι can be assumed to be an embedding in a sufficiently small neighborhood of $\mathcal{C}$.

Definitions and examples

Locally a causal structure can be expressed as

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Example : $L(x; y) = \frac{1}{3}(y^2)^3 + y^0 y^3 y^3 - y^1 y^2 y^3$:
Null cones are projectively equivalent to Cayley's cubic surface.

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Theorem (Hwang, 2013) Causal structures arising from smooth VMRTs are V-isotrivially flat.

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This picture can be generalized to higher dimensions.

The equivalence problem: causal geometry

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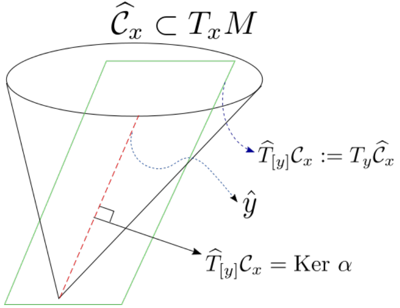

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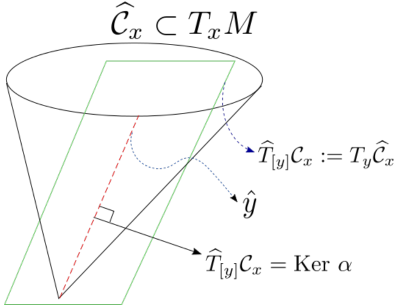

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$\mathrm{d}\eta^0 = -\zeta_1 \wedge \eta^1 - \cdots - \zeta_n \wedge \eta^n,$ $\eta^0 \wedge (\mathrm{d}\eta^0)^n \neq 0$	$\mathrm{d}\omega^0 = -\theta_1 \wedge \omega^1 - \cdots - \theta_{n-1} \wedge \omega^{n-1}$ $-2\phi_0 \wedge \omega^0, \quad \omega^0 \wedge (\mathrm{d}\omega^0)^{n-1} \neq 0$

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$\mathrm{d}\eta^0 = -\zeta_1 \wedge \eta^1 - \cdots - \zeta_n \wedge \eta^n,$ $\eta^0 \wedge (\mathrm{d}\eta^0)^n \neq 0$	$\mathrm{d}\omega^0 = -\theta_1 \wedge \omega^1 - \cdots - \theta_{n-1} \wedge \omega^{n-1}$ $-2\phi_0 \wedge \omega^0, \quad \omega^0 \wedge (\mathrm{d}\omega^0)^{n-1} \neq 0$
Geodesics: integral curves of the Reeb vector field $\eta^0(\mathbf{u}) = 1, \mathrm{d}\eta^0(\mathbf{u}, .) = 0$	Null geodesics: integral curves of the characteristic line field $\omega^0(\mathbf{v}) = 0, \mathrm{d}\omega^0(\mathbf{v}, .) = 0$

Causal vs. Finsler

Finsler	**Causal**
Indicatix bdle $\Sigma^{2n+1} \to M^{n+1}$ Loc. expressed as $F = 1$	(Proj.) null cone bdle $\mathcal{C}^{2n} \to M^{n+1}$ Loc. expressed as $L = 0$
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$g = (\eta^0)^2 + \delta_{ij}\eta^i\eta^j$ is well-def on Σ (osc. quadric)	$[g] = [2\omega^0\omega^n + \varepsilon_{ab}\omega^a\omega^b]$ is well-def on $\mathcal{C}$ (osc. quadric)

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Half-flatnesss in 4D conformal geometry

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Let us look at the structure equations

$$\begin{aligned}
\mathrm{d}\omega^0 &= \psi_0 \wedge \omega^0 - \theta^1 \wedge \omega^2 - \theta^2 \wedge \omega^1, \\
\mathrm{d}\omega^1 &= -\gamma^1 \wedge \omega^0 - \psi_1 \wedge \omega^1 - \theta^1 \wedge \omega^3 + F_2 \theta^2 \wedge \omega^2 + F_1 \theta^2 \wedge \omega^0 \\
\mathrm{d}\omega^2 &= -\gamma^2 \wedge \omega^0) - (\psi_0 + \psi_2) \wedge \omega^1 - \theta^2 \wedge \omega^3 + E_2 \theta^1 \wedge \omega^1 + E_1 \theta^1 \wedge \omega^0 \\
\mathrm{d}\omega^3 &= -\gamma^1 \wedge \omega^2 - \gamma^2 \wedge \omega^1 - (\psi_1 + \psi_2) \wedge \omega^3 + F_0 \theta^2 \wedge \omega^0 + E_0 \theta^1 \wedge \omega^0, \\
\mathrm{d}\theta^1 &= -\pi_1 \wedge \omega^0 - \pi_3 \wedge \omega^1 - \psi_2 \wedge \theta^1 \\
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If $E_2 = F_2 = 0$, then W_4 and V_4 generate W^{sd} and W^{asd}.

Double fibrations

For 4D indefinite self-dual causal structure:

$$T^3 \xleftarrow[E_2,\,V_4=0]{} \mathcal{C}^6 \xrightarrow[E_2,\,F_2=0]{} M^4$$

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Theorem

$$\text{indefinite half-flat causal on } M^4 \Longleftrightarrow \text{path geom. on } T^3$$

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Thus quartic polynomial in conformal geometry is replaced by three polynomials in the causal setting

Submaximal indefinite 4D causal str

Theorem : The submaximal 4D causal str of indefinite signature that does not descend to a conformal structure is V-isotrivially flat where V is the Cayley cubic and its infinitesimal symmetry algebra is 8-dimensional and solvable.

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Note that classically the Cayley cubic is associated to pair of ODEs:

$$z_1'' = z_2, \quad z_2'' = 0.$$

It appears that this pair is point equivalent to Egorov projective structure.

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The 4D structure group preserving $[\rho]$ is given by

$$\begin{pmatrix} \mathbf{f_0} & \mathbf{u} & \mathbf{v} & \frac{1}{\mathbf{f_0}}\mathbf{u}\mathbf{v} - \frac{1}{3\mathbf{f_0}^2}\mathbf{u}^3 \\ 0 & \mathbf{f_0}\,\mathbf{f_1} & \mathbf{u}\,\mathbf{f_1} & \mathbf{v}\,\mathbf{f_1} \\ 0 & 0 & \mathbf{f_0}\,\mathbf{f_1}^2 & \mathbf{u}\,\mathbf{f_1}^2 \\ 0 & 0 & 0 & \mathbf{f_0}\,\mathbf{f_1}^3 \end{pmatrix}$$

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A *Cayley structure* is a V-isotrivial causal structure where V is projectively equivalent to the Cayley cubic.
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$$\rho = \tfrac{1}{3}(\omega^2)^3 + \omega^0\omega^3\omega^3 - \omega^1\omega^2\omega^3$$

The 4D structure group preserving $[\rho]$ is given by

$$\begin{pmatrix} \mathbf{f_0} & \mathbf{u} & \mathbf{v} & \frac{1}{\mathbf{f_0}}\mathbf{u}\mathbf{v} - \frac{1}{3\mathbf{f_0}^2}\mathbf{u}^3 \\ 0 & \mathbf{f_0}\,\mathbf{f_1} & \mathbf{u}\,\mathbf{f_1} & \mathbf{v}\,\mathbf{f_1} \\ 0 & 0 & \mathbf{f_0}\,\mathbf{f_1}^2 & \mathbf{u}\,\mathbf{f_1}^2 \\ 0 & 0 & 0 & \mathbf{f_0}\,\mathbf{f_1}^3 \end{pmatrix}$$

Thus, any Cayley structure is equipped with an invariant flag given by

$$P_3 = \{\omega^3 = 0\} \supset P_2 = \{\omega^3 = \omega^2 = 0\} \supset P_1 = \{\omega^3 = \omega^2 = \omega^1 = 0\}$$

Half-flat Cayley structures

The first order structure equations for half-flat Cayley structures is

$$\begin{aligned}
\mathrm{d}\omega^0 &= -\phi_0 \wedge \omega^0 - \theta^2 \wedge \omega^1 - \theta^1 \wedge \omega^2,\\
\mathrm{d}\omega^1 &= -(\phi_0 + \phi_1) \wedge \omega^1 - \theta^2 \wedge \omega^2 - \theta^1 \wedge \omega^3,\\
\mathrm{d}\omega^2 &= -(\phi_0 + 2\phi_1) \wedge \omega^2 - \theta^2 \wedge \omega^3 + c\, a_6 \omega^0 \wedge \omega^1 + a_2 \omega^0 \wedge \omega^3,\\
\mathrm{d}\omega^3 &= -(\phi_0 + 3\phi_1) \wedge \omega^3 + a_6 \omega^0 \wedge \omega^2 + a_4 \omega^0 \wedge \omega^3\\
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If $a_1 = 0$, one obtains a class of path geometries depending on 6 functions of 2 variables.
If $W_4 = 0$, one obtains a class of projective structures depending on 2 constants.

Half-flat Cayley str: A zoo of geometric structures

Vanishing of a_i implies integrability of P_2, P_3 in

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5. $a_6 = 0, a_5 \neq 0$ + another diff cond $\to$ Association of a point equivalence class of 3rd order scalar ODE.

DFK construction of a Lax pair

A *torsion-free* pair of ODEs $z_i'' = F_i(t, z, z')$, defines a half-flat conformal structure on its solution space. Dunajski, Ferapontov and Kruglikov gave the following construction of a Lax pair for them

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The 4-fold J has a conformal structure with α-planes given by

$$V_1 = \lambda \tfrac{\partial}{\partial \omega^0} + \tfrac{\partial}{\partial \omega^1}, V_2 = \lambda \tfrac{\partial}{\partial \omega^2} + \tfrac{\partial}{\partial \omega^3}$$

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The lift of $\{V_1, V_2\}$ to N is ambiguous up to $\frac{\partial}{\partial\lambda}$.

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Define $Z_1 = V_1 + m\frac{\partial}{\partial\lambda}, Z_2 = V_2 + n\frac{\partial}{\partial\lambda}$.

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$[Z_1, Z_2] \equiv 0$ modulo $\frac{\partial}{\partial\omega^0}, \frac{\partial}{\partial\omega^2}$ gives two PDEs of order 3 in four variables which is involutive. The solutions depend on 6 functions of 3 variables.

Half-flat Cayley str: characterizing Fel's torsion

Given a pair $(z^i)'' = F^i(t, z, z')$, let $\mathrm{D}_t = \partial_t + p^i\partial_{z^i} + F^i\partial_{p^i}$, and $\mathcal{D} = \mathrm{span}\{\partial_{p^1}, \partial_{p^2}\}$.

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Given $X \in \mathrm{span}\{\mathrm{D}_t\}$, a frame $\mathbf{V} = (V_1, V_2)$ for $\mathcal{D}$ is called normal if

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Theorem : There is a one to one correspondence between 3-dimensional path geometries arising from half-flat Cayley structures and point equivalence classes of pairs of second order ODEs satisfying

$$\mathrm{rank}\,\mathbf{T}^X = 1, \qquad \nabla_X\mathbf{T}^X = \phi\mathbf{T}^X, \qquad \hat{\mathbb{S}}(\mathbf{T}^X) = 0,$$

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As a result $\mathbf{T}^X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in some normal frame. The above ODE gives

$$V_1 = A_1 + \lambda B_1, \qquad V_2 = A_2 + \lambda B_2 + \frac{\lambda^2}{2}A_1 + \frac{\lambda^3}{6}B_1,$$

for some vector fields A_1, A_2, B_1, B_2 such that $\mathrm{ad}_X A_i = \mathrm{ad}_X B_i = 0$.

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$$V_1 = Y_2 + EY_1 + \lambda(Z_2 + EZ_1) \mod \mathrm{D}_t$$

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Following the conformal case, we find the functions m, n and finally they result in 7 equations of order 3 in 3 functions E, F^1, F^2 which is not involutive!

Future directions

- Integrating structure equations to give explicit examples of Cayley structures
- The moduli space of rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(2)$ in a 3-fold is equipped with a cubic form which gives Cayley cubic when restricted to certain hyperplane.
- Examples of uniruled manifolds for which the VMRTs give a Cayley structure.
- An alternative description of Cayley structures in terms of a pair of PDEs of finite type by passing to its space of null geodesics.
- Many of our discussion extends to a larger class of half-flat V-isotrivial causal structures.

Thank you for your attention!