EXAMPLES

EX 1

$\mathfrak{g} = \mathfrak{gl}(1|1)$

$$\underbrace{E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathfrak{sl}(1|1)} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$[E, F] = H, \quad [Z, E] = E, \quad [Z, F] = -F$$

$$E \mapsto \begin{pmatrix} 0 & \lambda_2 \\ 0 & 0 \end{pmatrix} \quad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H \mapsto \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad Z \mapsto \begin{pmatrix} \lambda_1 + \frac{1}{2} & 0 \\ 0 & \lambda_1 - \frac{1}{2} \end{pmatrix}$$

For any $\lambda_2 \neq 0$ it is irrep. of dim. 2 with 1-dimens. Schur algebra.

LIE THEOREM IS NOT TRUE FOR LIE SUPERALGEBRAS.

EX2 Heisenberg superalgebras in even and odd dimensions

- Heisenberg superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$

$$\mathfrak{g}_{\bar 0} = \langle H \rangle \,, \quad \mathfrak{g}_{\bar 1} = \langle p_1, \dots, p_m, q_1, \dots, q_m \rangle \,, \quad [p_i, q_i] = H$$

$(\lambda \neq 0)$

$$\alpha \in \Lambda^\bullet [z_1, \dots, z_m]$$

$$p_i \cdot \alpha = \frac{\partial \alpha}{\partial z_i} \,, \qquad q_i \cdot \alpha = \lambda\, z_i \wedge \alpha \,, \qquad H \cdot \alpha = \lambda\, \alpha$$

It is irrep. of dim. 2^m with 1-dimens. Schur algebra.

- Heisenberg superalgebra $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}R$ with $[\mathfrak{g}, R] = 0$, $[R, R] = H$

$$\alpha \otimes \theta \in \Lambda^\bullet \otimes \mathbb{C}[\varepsilon] \,, \qquad |\varepsilon| = \bar 1 \,, \qquad \varepsilon^2 = \lambda/2$$

$(\lambda \neq 0)$

$$X \cdot \alpha \otimes \theta = X \cdot \alpha \otimes \theta \quad \text{for all } X \in \mathfrak{g}$$

$$R \cdot \alpha \otimes \theta = (-1)^{|\alpha|} \alpha \otimes \varepsilon \cdot \theta$$

It is irrep. of dim. 2^{m+1} with 2-dimens. Schur alg. (consider odd morphism $\alpha \mapsto (-1)^{|\alpha|} \alpha \otimes \varepsilon$, $\alpha \otimes \varepsilon \mapsto -\frac{\lambda}{2} (-1)^{|\alpha|} \alpha$).

MAIN THM

$\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ solvable Lie superalgebra

$$L = \{ \lambda \in \mathfrak{g}^* \mid \lambda[\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}] = 0 \ \& \ \lambda \mathfrak{g}_{\bar 1} = 0 \} \quad \text{Character space of } \mathfrak{g}_{\bar 0}$$

Every $\lambda \in L$ determines a $(1|0)$-dim. rep. of $\mathfrak{g}_{\bar 0}$ and $\mathfrak{g}_{\bar 0}$-inv. symmetric bilinear form

$$F_\lambda : \bigodot^2 \mathfrak{g}_{\bar 1} \longrightarrow \mathbb{C}$$
$$(z_1, z_2) \longmapsto \lambda[z_1, z_2]$$

DEF. A subalgebra $h \leq \mathfrak{g}$ with $\mathfrak{g}_{\bar 0} = h_{\bar 0}$ is a polarization (for λ) if $h_{\bar 1}$ is maximal isotropic subspace for F_λ. (Clearly $h_{\bar 1} \supseteq n = \ker F_\lambda$.)

Given any polarization we may construct:

1. Since $\lambda([\mathfrak{h},\mathfrak{h}])=0$, λ determines $(1|0)$-dim. rep. $\mathbb{C}v$ of $\mathfrak{h}$

2. $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}v$ is $\mathfrak{g}$-rep.

<u>THM 1</u> (Kac '77, Sergeev '99)

(i) $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ is fin. dim. $\mathfrak{g}$-irrep.

(ii) Every fin. dim. irrep. of $\mathfrak{g}$ is isomorphic to some $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ (up to Π)

(iii) $\mathrm{rk}\,F_\lambda$ is even $\Leftrightarrow$ 1-dim. Schur algebra

$\mathrm{rk}\,F_\lambda$ is odd $\Leftrightarrow$ 2-dim. Schur algebra

LSA $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ w/t $\mathfrak{g}_{\bar 0}$ solvable **PREREQUISITES**

$\exists\, K \leq \mathfrak{g}$, codim $K = (0|1)$ (by LIE THM)

$\nu \in L$ character of adjoint rep. of $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}/K$

LEMMA 1 ν is character of $\mathfrak{g}$

Pf. Need to show $\nu\,[\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}] = 0$. Fix $z \in \mathfrak{g}_{\bar 1}$, $z \notin K_{\bar 1}$ so that

$\mathfrak{g} = K \oplus \mathbb{C}z$ as v.s. and $\mathfrak{g}/K = \mathbb{C}\bar{z}$

- $X, Y \in K_{\bar 1}$ $[X,Y]\cdot\bar{z} = X\cdot(\underbrace{Y\cdot\bar{z}}_{\in K_{\bar 0}}) - Y\cdot(\underbrace{X\cdot\bar{z}}_{\in K_{\bar 0}}) = 0$ since K acts on $\mathfrak{g}/K$

- $[z,z]\cdot\bar{z} = \overline{[[z,z],z]} = 0$

- $X \in K_{\bar 1}$ $[X,z]\cdot\bar{z} = \overline{[[X,z],z]} = \frac{1}{2}\overline{\underbrace{[X,[z,z]]}_{\in K_{\bar 1}}} = 0$

$\blacksquare$

<u>COROLLARY 1</u> A Lie superalg. $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ is solvable iff $\mathfrak{g}_{\bar 0}$ is solvable

<u>Pf.</u> We prove $\Leftarrow$ by induction on $\dim \mathfrak{g}_{\bar 1}$.

- $\dim \mathfrak{g}_{\bar 1} = 0$ ✓
- $\dim \mathfrak{g}_{\bar 1} > 0$ Since $[\mathfrak{g}, \mathfrak{g}] \subseteq \tilde{\mathfrak{g}} = [\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}] \oplus [\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}] \oplus \mathfrak{g}_{\bar 1}$

It is enough to show that $\tilde{\mathfrak{g}}$ is solvable. Take $K \subseteq \mathfrak{g}$ as before

$\Rightarrow$ $[\tilde{\mathfrak{g}}_{\bar 0}, \mathfrak{g}_{\bar 1}] \subseteq K_{\bar 1}$

(Lemma 1)

$\Rightarrow$ $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \subseteq \tilde{\mathfrak{g}}_{\bar 0} \oplus K_{\bar 1}$

By inductive hypothesis $\tilde{\mathfrak{g}}_{\bar 0} \oplus K_{\bar 1}$ is solvable, hence $\mathfrak{g}$ is solvable.

Implication $\Rightarrow$ is obvious. ∎

Using LIE THM:

LEMMA 2 Let W be fin. dim. $\mathfrak{g}_{\bar{0}}$-module, $f : \odot^2 W \to \mathbb{C}$ $\mathfrak{g}_{\bar{0}}$-inv., V $\mathfrak{g}_{\bar{0}}$-inv. isotropic subspace. Then $\exists$ $\mathfrak{g}_{\bar{0}}$-inv. maximal isotropic subspace $\supseteq V$.

COROLLARY 2 For every $\lambda \in L$, there exists a polarization $\mathfrak{h}$.

DESCRIPTION OF IRREDUCIBLE MODULES

- $\lambda \in L$, $\mathfrak{h} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{h}_{\bar 1}$ polarization $\longrightarrow$ $(1|0)$-dim rep. $\mathbb{C}v$ of $\mathfrak{h}$

- $\mathfrak{n} = \ker f_\lambda$, $\mathfrak{h}_{\bar 1}^{\perp} = \mathfrak{h}_{\bar 1} \oplus \mathbb{C}R$ as v.s. (If $\mathrm{rk}\, f_\lambda$ is even set $R = 0$.)

  ↳ Vector of norm 1

- $\mathfrak{h}_{\bar 1} = \mathfrak{n} \oplus F$ v.s.

PROP. 1 If $u \in \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}v$ satisfies $\underbrace{F \cdot u = 0}_{\text{Equivalently } \mathfrak{h}_{\bar 1}}$

$\longrightarrow$ $u \in \underbrace{\langle v, R\cdot v \rangle}_{\text{This does not depend on choice of } R}$

"CHARACTERIZATION OF SINGULAR VECTORS"

Pf. Weight of $R \cdot v$ w.r.t. $\mathfrak{h}$ is also λ (exercise). Induction on $\mathrm{rk}\, f_\lambda$:

- $\mathrm{rk}\, f_\lambda = 0 \longrightarrow \mathfrak{g} = \mathfrak{h}$ and $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) = \mathbb{C}v$

- $\mathrm{rk}\, f_\lambda > 0$ Select subalg. $K \supseteq \mathfrak{h}$ with $\dim \mathfrak{g}_{\bar 1}/K_{\bar 1} = 1$ (by LIE THM) and write $\mathfrak{g}_{\bar 1} = K_{\bar 1} \oplus \mathbb{C}z$.

Furthermore by PBW THM:

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}v = \mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}v \oplus z \cdot \mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}v$$

$$u = u_0 + z \cdot u_1$$

$\left(\text{since } 2z^2 = z \cdot z + z \cdot z = [z, z] \in \mathfrak{g}_{\bar{0}} = \mathfrak{k}_{\bar{0}} \text{ in } \mathcal{U}(\mathfrak{g})\right)$.

We split into two cases: $\mathfrak{k}_{\bar{1}}^{\perp} \not\subseteq \mathfrak{k}_{\bar{1}}$ & $\mathfrak{k}_{\bar{1}}^{\perp} \subseteq \mathfrak{k}_{\bar{1}}$.

$K_{\bar{1}}^{\perp} \not\subseteq K_{\bar{1}}$

i) We may assume $z \in K_{\bar{1}}^{\perp} \Longrightarrow z \in h_{\bar{1}}^{\perp}$ and $z \notin h_{\bar{1}}$ (since $K_{\bar{1}} \supseteq h_{\bar{1}}$)

Hence $\operatorname{rk}(f_\lambda)$ is odd and we may take $z = R$. Clearly $\operatorname{rk}(f_\lambda|_{K_{\bar{1}}})$ is even and h is still a polarization for $\lambda|_{K_{\bar{1}}}$.

Assume $f \cdot u = 0$ for all $f \in F \Longrightarrow 0 = f \cdot u = f \cdot u_0 + f \cdot z \cdot u_1$
$= f \cdot u_0 - z \cdot f \cdot u_1 + [f, z] \cdot u_1 \Longrightarrow z \cdot f \cdot u_1 = 0$
$\Longrightarrow f \cdot u_1 = 0$

By induction hypothesis $u_1 \in \mathbb{C} v$. Therefore

$$\underbrace{[f, z]}_{\in \mathfrak{g}_{\bar{0}}} u_1 = \lambda [f, z] u_1 = f_\lambda(\underset{h_{\bar{1}}}{\underset{\cap}{f}}, \underset{h_{\bar{1}}^{\perp}}{\underset{\cap}{z}}) u_1 = 0$$

and $f \cdot u_0 = 0$. Hence $u_0 \in \mathbb{C} \cdot v \Longrightarrow u \in \langle v, z \cdot v \rangle$.

$K_i^\perp \subseteq K_i$ $\quad \mathrm{rk}\left(f_\lambda\big|_{K_i}\right) = \mathrm{rk}(f_\lambda) - 2$ (so we may apply induction)

First note that $K_i^\perp \subseteq h_i$. (Indeed, if this were not true then $\exists X \in K_i^\perp$, $X \notin h_i \Rightarrow h_i \oplus \mathbb{C}X$ would be isotropic since $K_i^\perp \subseteq K_i$, but this contradicts maximality.)

n

n_1

Now $\dim K_i + \dim K_i^\perp = \dim g_i + \dim n \Rightarrow \dim K_i^\perp = \dim n + 1$

Therefore $\exists\, \eta \in K_i^\perp$, $\eta \notin n$ and we may shift it by elements of n so that $\eta \in F$ as well. Clearly $f_\lambda(\eta, z) \neq 0$ (otherwise $\eta \in n$) $\Rightarrow$

$$F = \tilde{F} \oplus \mathbb{C}\eta$$

where $\tilde{F} = F \cap (\mathbb{C}z)^\perp$.

As before $u = u_0 + z \cdot u_1$, with $u_0, u_1 \in \mathcal{U}(k) \otimes_{\mathcal{U}(h)} \mathbb{C}_Y$ and $F \cdot u = 0$ implies $F \cdot u_1 = 0$ and by induction $u_1 \in \langle Y, R \cdot Y \rangle$. Now

$$\underbrace{[f, z]}_{\in \mathfrak{n}_0} \cdot u_1 = f_\lambda(f, z)\, u_1$$

For all $f \in F \Longrightarrow [f, z] \cdot u_1 = 0$ for all $f \in \tilde{F} \Longrightarrow f \cdot u_0 = 0$ for all $f \in \tilde{F}$. By induction $u_0 \in \langle Y, R \cdot Y \rangle \Longrightarrow F \cdot u_0 = 0$. Finally

$$0 \underset{\eta \in F}{=} \eta \cdot u = \eta \cdot u_0 + \eta \cdot z\, u_1 \underset{\eta \in F}{=} [\eta, z]\, u_1 = \underbrace{f_\lambda(\eta, z)}_{\neq 0}\, u_1$$

$$\Longrightarrow u_1 = 0 \Longrightarrow u \in \langle Y, R \cdot Y \rangle. \qquad \blacksquare$$

COROLLARY 3 $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$ is fin. dim. irreducible.

Proof. $\mathfrak{g}_{\bar{1}} = F^* \oplus \underbrace{F \oplus \mathfrak{n}}_{\mathfrak{h}_{\bar{1}}} \oplus \mathbb{C}\cdot R$ (no $\mathbb{C}\cdot R$ if $\mathrm{rk} f_\lambda$ is even)

Dual isotropic subspaces

$R \perp F^* \oplus F \oplus \mathfrak{n}$

$F = \langle p_1, \dots, p_n \rangle \qquad F^* = \langle q_1, \dots, q_n \rangle$

$$\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathbb{C}v \underset{v.s.}{\simeq} \Lambda^\bullet F^* \otimes_{\mathbb{C}} \langle v, R\cdot v \rangle \text{ by PBW}$$

It has natural grading given by degree in $\Lambda^\bullet F^*$. The action of $\mathfrak{h}_{\bar{1}}$ lowers the degree, e.g., for any $X \in \mathfrak{h}_{\bar{1}}$ we have:

- $\deg = 0$ $\qquad X \cdot (\mathbb{1} \otimes \langle v, R\cdot v \rangle) = X \otimes \langle v, R\cdot v \rangle =$

$$= \lambda(X) \langle v, R\cdot v \rangle = 0$$

• $\underline{\deg} = 1$

$$X \cdot \left(q_i \otimes \langle v, R\cdot v\rangle\right) = X q_i \otimes \langle v, R\cdot v\rangle$$

$$= -q_i X \otimes \langle v, R\cdot v\rangle + \underbrace{[X, q_i]}_{\in \mathfrak{g}_{\bar 0} = \mathfrak{h}_{\bar 0}} \otimes \langle v, R\cdot v\rangle$$

$$= -q_i \overset{=0}{\cancel{\lambda(X)}} \langle v, R\cdot v\rangle + \underbrace{\lambda[X, q_i] \langle v, R\cdot v\rangle}_{\in \Lambda^0 F^*}$$

• $\underline{\deg} = 2$

$$X \cdot \left(q_i q_j \otimes \langle v, R\cdot v\rangle\right) = X q_i q_j \otimes \langle v, R\cdot v\rangle$$

$$= -q_i X q_j \otimes \langle v, R\cdot v\rangle + [X, q_i] q_j \otimes \langle v, R\cdot v\rangle$$

$$= \overset{=0}{\cancel{q_i q_j X \otimes \langle v, R\cdot v\rangle}} - q_i \underbrace{[X, q_j]}_{\in \mathfrak{g}_{\bar 0}} \otimes \langle v, R\cdot v\rangle$$

$$+ q_j \underbrace{[X, q_i]}_{\in \mathfrak{g}_{\bar 0}} \otimes \langle v, R\cdot v\rangle + \underbrace{[[X, q_i], q_j]}_{\in \mathfrak{g}_{\bar 1}} \otimes \langle v, R\cdot v\rangle$$

which is in $\Lambda^1 F^* \oplus \Lambda^0 F^*$

Let $W \subseteq \mathrm{Ind}_{\mathfrak h}^{\mathfrak g}(\lambda)$ be a non-trivial submodule. Pick non-zero $w \in W$. Repeatedly applying h_i to w will lead to a non-zero element of W that is annihilated by h_i (a "singular vector"). By PROP I, this is either v or $\mathbb{R}\cdot v \Longrightarrow W = \mathrm{Ind}_{\mathfrak h}^{\mathfrak g}(\lambda)$. ■

GENERAL IMPORTANT FACT: V $\mathfrak h$-rep., $\mathrm{Ind}_{\mathfrak h}^{\mathfrak g}(V) := \mathcal U(\mathfrak g) \otimes_{\mathcal U(\mathfrak h)} V$

Frobenius reciprocity:

$$\mathrm{Hom}_{\mathfrak g}\big(\mathrm{Ind}_{\mathfrak h}^{\mathfrak g}(V), W\big) \underset{\text{v.s.}}{\cong} \mathrm{Hom}_{\mathfrak h}\big(V, \mathrm{Res}_{\mathfrak h}^{\mathfrak g}(W)\big)$$

$\mathfrak g$-rep.

given by restriction to $V \subseteq \mathrm{Ind}_{\mathfrak h}^{\mathfrak g}(V)$.

COROLLARY 4 $z(f_\lambda)$ is even $\Leftrightarrow$ 1-dim. Schur algebra

$z(f_\lambda)$ is odd $\Leftrightarrow$ 2-dim. Schur algebra

Pf: Focus $V = \mathbb{C}\cdot v \Rightarrow$

$$\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda), \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)) \cong \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}\cdot v, \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)).$$

Take $\mathfrak{h}$-equiv. $\varphi : \mathbb{C}v \longrightarrow \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)$, then $\varphi(v)$ is a singular vector. $\blacksquare$

Two technical results. Here $\mathfrak{g}$ is not necessarily solvable and $\mathfrak{K} \leq \mathfrak{g}$ with $\dim \mathfrak{g}/\mathfrak{K} = 1$ (0/1). Write $\mathfrak{g} = \mathfrak{K} \oplus \mathbb{C}_z$ as v.s.

<u>LEMMA 3</u> $\rho : \mathfrak{K} \to \mathfrak{gl}(U)$ irrep., then $\mathrm{Ind}_{\mathfrak{K}}^{\mathfrak{g}}(U)$ is reducible iff U has $\mathfrak{g}$-module struct. extending ρ.

<u>LEMMA 4</u> If V is $\mathfrak{g}$-irrep. and $U \subseteq V$ irreducible non-zero proper $\mathfrak{K}$-submodule, then $V \simeq \mathrm{Ind}_{\mathfrak{K}}^{\mathfrak{g}}(U)$.

$$\mathrm{Ind}_{\mathfrak{K}}^{\mathfrak{g}}(U) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{K})} U$$

irrep. of $\mathfrak{K}$

v.s. $\parallel\!\wr$

$$U \oplus zU$$

We construct $\mathfrak{K}$-composition series for $\mathrm{Ind}_{\mathfrak{K}}^{\mathfrak{g}}(U)$.

$\mathrm{Ind}_{\mathfrak{K}}^{\mathfrak{g}}(U)/U \cong \Pi U$ as $\mathfrak{K}$-modules (twisted by 1-dim. rep. of $\mathfrak{K}$)

Indeed for any $X \in \mathfrak{K}$ we have $[X, \mathfrak{z}] = \lambda(X)\mathfrak{z} + X'$, where $X' \in \mathfrak{K}$ and $\lambda : \mathfrak{K} \longrightarrow \mathbb{C}$ is s.t. $\lambda|_{\mathfrak{K}_{\bar{1}}} = 0$ & $\lambda[\mathfrak{K},\mathfrak{K}] = 0$.

Therefore
$$X \cdot (\mathfrak{z} \cdot u) = (-1)^{|X|}\mathfrak{z} \cdot (X \cdot u) + \lambda(X)\mathfrak{z} \cdot u + X' \cdot u$$

$\in U$

For every $u \in U$

$$\Longrightarrow \quad X \cdot \overline{\mathfrak{z} \cdot u} = (-1)^{|X|}\overline{\mathfrak{z} \cdot (X \cdot u)} + \lambda(X)\overline{\mathfrak{z} \cdot u}$$

In particular $0 \subseteq U \subseteq \mathrm{Ind}_{\mathfrak{K}}^{\mathfrak{g}}(U)$ is $\mathfrak{K}$-composition series, for which Jordan-Hölder applies: length and irred. factors, up to permutation, are invariants of a composition series.

In particular a non-zero $\mathfrak{k}$-submodule of $\mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(U)$ is either $\mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(U)$ or it is irreducible and $\cong U$ or twisted ΠU.

<u>Proof of Lemma 3</u>

Assume U has $\mathfrak{g}$-module structure extending ρ, then $\exists$ $\mathfrak{g}$-equiv. map $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} U \twoheadrightarrow U$ with non-trivial kernel.

Conversely, if $\mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(U)$ has non-zero proper $\mathfrak{g}$-submodule W then $W \cong U$ or twisted ΠU as $\mathfrak{k}$-module. In the first case there is nothing to prove, in the second case $U \cong \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(U)/W$ as $\mathfrak{k}$-module and the claim holds.

<u>Proof of Lemma 4</u> Kernel of $\mathfrak{g}$-equiv. $\mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(U) \longrightarrow V$ is either zero or it has dimension $\dim U$. In the second case, $\dim V = \dim U \Longrightarrow U \simeq V$, a contradiction ∎

PROP 2 Let U be an irred. fin. dim. $\mathfrak{g}$-rep. Then $U \cong \mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}}(\lambda)$ for some $\lambda \in L$ and a polarization (up to Π).

Pf: Induction on $\dim \mathfrak{g}_{\bar{1}}$

- $\dim \mathfrak{g}_{\bar{1}} = 0$ ✓ by LIE THM
- $\dim \mathfrak{g}_{\bar{1}} > 0$ Select subalg. K with $K_{\bar{0}} = \mathfrak{g}_{\bar{0}}$ and $\dim \mathfrak{g}_{\bar{1}}/K_{\bar{1}} = 1$.

  Two cases: U K-irred. & U K-red.

U K-irred.

By involution $\exists\ \lambda \in \perp$ & polarization $\hat{h} \subseteq K$ s.t.

$$U = \mathrm{Ind}_{\hat{h}}^{K}(\lambda) = \mathcal{U}(K) \otimes_{\mathcal{U}(\hat{h})} \mathbb{C}v \quad (\text{up to } \Pi)$$

If $\hat{h}$ were a polarization in $\mathfrak{g}$ too, then $\mathrm{Ind}_{\hat{h}}^{\mathfrak{g}}(\lambda) \cong \mathrm{Ind}_{K}^{\mathfrak{g}}(U)$ would be irred. by COROLLARY 3 and U would not admit a $\mathfrak{g}$-module structure extending action of K by LEMMA 3, a contradiction.

Let $h = \hat{h} \oplus \mathbb{C}z$ be a polarization in $\mathfrak{g}$ ($z \notin K$ by maximality).

If $z \cdot v = 0$ we have well-defined $\mathfrak{g}$-equiv. map

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(h)} \mathbb{C}w \longrightarrow U$$

$$u \otimes w \longmapsto u \cdot v$$

which is an isomorphism by irreducibility. If $z \cdot v \neq 0$, then for any $p \in \hat{h}_{\bar 1}$ we have

Since U is $\mathfrak{g}$-module

$$p\cdot z\cdot v = [p, z]v = f_\lambda(p, z)v = 0$$

$$z\cdot z\cdot v = \frac{1}{2}[z, z]v = \frac{1}{2}f_\lambda(z, z)v = 0$$

$h_{\bar 1}$ is isotropic in $\mathfrak{g}_{\bar 1}$

In particular $z\cdot v$ is a singular vector in U, therefore its weight w.r.t. $\mathfrak{g}_0$ is also λ. In this case $\exists$ odd $\mathfrak{g}$-equiv. map

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(h)} \mathbb{C}\cdot w \longmapsto U$$

$$u \otimes w \longmapsto u\cdot z\cdot v$$

U K-red. By LEMMA 4 $U \cong \mathrm{Ind}_K^{\mathfrak g}(W)$ for K-irrep. $W \subseteq U$.

Then $W \cong \mathrm{Ind}_{\hat{\mathfrak h}}^{K}(\lambda)$ for a polarization $\hat{\mathfrak h} \subseteq K$ & $\lambda \in \perp$

by induction $\Longrightarrow$ $U \cong \mathrm{Ind}_{\hat{\mathfrak h}}^{\mathfrak g}(\lambda)$.

Let $\mathfrak h \supseteq \hat{\mathfrak h}$ be a polarization in $\mathfrak g$ and $\mathrm{Ind}_{\hat{\mathfrak h}}^{\mathfrak g}(\lambda) \longrightarrow \mathrm{Ind}_{\mathfrak h}^{\mathfrak g}(\lambda)$ the natural $\mathfrak g$-module morphism. By irreducibility, it is an isom. and by dimensional reasons $\hat{\mathfrak h} = \mathfrak h$ and $U \cong \mathrm{Ind}_{\mathfrak h}^{\mathfrak g}(\lambda)$.

$\blacksquare$

SOME INTERESTING CONSEQUENCES

$\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ solvable $\quad U \cong \mathrm{Ind}_{h}^{\mathfrak{g}}(\lambda)$ fin. dim. $\mathfrak{g}$-irrep.

(i) $\dim U = \dim \Lambda^{\bullet} \mathfrak{g}_{\bar 1}/h_{\bar 1} = 2^{s}$, $s \leq \frac{1}{2} \dim \mathfrak{g}_{\bar 1} + 1$

(ii) $\dim U = 1 \iff \mathfrak{g}_{\bar 1}$ is isotropic for f_λ

Otherwise $s > 0$ and in that case $\dim U_{\bar 0} = \dim U_{\bar 1}$

(iii) All $\mathfrak{g}$-irrep. are 1-dim. $\iff [\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}] \subseteq [\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}]$

Put it differently, in this case

$\{ \lambda \in \mathfrak{g}^* \mid \lambda[\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}] = 0 \ \& \ \lambda \mathfrak{g}_{\bar 1} = 0 \}$ Character space of $\mathfrak{g}_{\bar 0}$

$\{ \lambda \in \mathfrak{g}^* \mid \lambda[\mathfrak{g}, \mathfrak{g}] = 0 \ \& \ \lambda \mathfrak{g}_{\bar 1} = 0 \}$ Character space of $\mathfrak{g}$

are equal!

CLASSIFICATION OF MODULES $\mathrm{Ind}_{h}^{\mathfrak{g}}(\lambda)$

$\lambda \in L$, h polarization (for λ)

$\Theta_h \in L$ by $\Theta_h(X) = \begin{cases} -\frac{1}{2} \,\mathrm{Str}_{\mathfrak{g}/h}(\mathrm{ad}\, X) & \text{if } X \in \mathfrak{g}_{\bar 0} \\ 0 & \text{if } X \in \mathfrak{g}_{\bar 1} \end{cases}$

Note that $\Theta_h[h,h] = 0$, so Θ_h determines $(1|0)$-dim. rep. of h

Def. $(\lambda, h) \sim (\mu, t) \iff \lambda - \Theta_h = \mu - \Theta_t$

PROP3

(i) $\mathrm{Ind}_{h}^{\mathfrak{g}}(\lambda) \cong \mathrm{Ind}_{t}^{\mathfrak{g}}(\mu)$ up to Π iff $(\lambda, h) \sim (\mu, t)$

(ii) h is also a polarization for any $\lambda + \alpha \Theta_h$ $(\alpha \in \mathbb{C})$

In particular $I(\lambda) := \mathrm{Ind}_{h}^{\mathfrak{g}}(\lambda + \Theta_h)$ is irreducible and it depends up to Π only on $\lambda \in L$, not on the choice of polarization.

THM 2 (Sergeev '99)

The assignment $\lambda \mapsto I(\lambda)$ is a bijection between $\perp$ and the isomorphism classes of $\mathfrak{g}$-irrep. up to Π.

EX1 $\mathfrak{gl}(1|1)$, $[\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}] = 0$

$$\lambda : \mathfrak{g}_{\bar 0} \longrightarrow \mathbb{C}, \quad \lambda(Z) = \lambda_1, \quad \lambda(H) = \lambda_2$$

$$F_\lambda(E,E) = F_\lambda(F,F) = 0, \quad F_\lambda(E,F) = \lambda_2$$

- $\lambda_2 = 0 \Longrightarrow \mathfrak{h}_{\bar 1} = \mathfrak{g}_{\bar 1}$ & $\Theta_{\mathfrak{h}} = 0$ (dim irrep $= 1$)
- $\lambda_2 \neq 0 \Longrightarrow \mathfrak{h}_{\bar 1} = \mathbb{C} \cdot E$ & $\Theta_{\mathfrak{h}} = +\frac{1}{2} Z^*$

$\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} (\lambda + \Theta_{\mathfrak{h}})$ with $\lambda_2 \neq 0$ are all irrep. dim $\underset{*}{>} 1$ (up to Π)

These are the irrep. of first slide!

EX2 Exercise: do irrep. of Heisenberg superalgebras

EX3 (Sergeev example)

$\Lambda^\bullet[\xi_1,\xi_2]$

$\mathfrak{g}$ with basis

$$\left\{\begin{array}{l} x=\xi_1\partial_{\xi_1},\quad y=\xi_2\partial_{\xi_1},\quad z=\xi_1\xi_2,\quad u=1 \\ \eta_1=\partial_{\xi_1},\quad \eta_2=\partial_{\xi_2}-\xi_1\xi_2\partial_{\xi_1} \\ \eta_{-1}=\xi_1,\quad \eta_{-2}=\xi_2 \end{array}\right. \quad \begin{array}{l} \| \ \mathfrak{g}_{\bar 0} \\ \| \ \mathfrak{g}_{\bar 1} \end{array}$$

$[\mathfrak{g}_{\bar 0},\mathfrak{g}_{\bar 0}]=\langle y,z\rangle,\quad [\mathfrak{g}_{\bar 1},\mathfrak{g}_{\bar 1}]=\mathfrak{g}_{\bar 0}$

$\lambda:\mathfrak{g}_{\bar 0}\longrightarrow\mathbb{C}$ ~~dual element~~ u^* to u

$F_\lambda: S^2\mathfrak{g}_{\bar 1}\longrightarrow\mathbb{C},\quad F_\lambda(\eta_1,\eta_{-1})=F_\lambda(\eta_2,\eta_{-2})=1$

$\mathfrak{h}=\mathfrak{g}_{\bar 0}\oplus\langle\eta_{-1},\eta_{-2}\rangle$

$\Theta_{\mathfrak{h}}:\mathfrak{g}_{\bar 0}\longrightarrow\mathbb{C},\quad \Theta_{\mathfrak{h}}=\frac{1}{2}x^*$