Holomorphic structures in hydrodynamical models of nearly geostrophic flow

by

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Abstract

We study complex structures arising in Hamiltonian models of nearly geostrophic flows in hydrodynamics. This work is motivated by the observation that an elliptic Monge-Ampère equation arises in many of these models in connection with the relationship between a velocity field and the potential vorticity, and the Monge-Ampère operator defines an almost-complex structure on the model phase space. In this paper we show that a natural extension of the so-called geostrophic momentum transformation of semi-geostrophic theory, which has a special importance in both theoretical meteorology and numerical weather prediction, defines a special Kähler structure on the model phase space. Furthermore, analogues of the 'geostrophic momentum coordinates' can be shown to be special Lagrangian coordinates under certain conditions which relate to the physical approximations under consideration.

1 Introduction

The very complex flow of both the atmosphere and oceans is believed to be described accurately by the classical Navier-Stokes based equations of fluid motion. However, in the asymptotic regimes (parameterized by rotation and stratification) most relevant for weather and climate forecasting, it can be shown that the solutions of the fluid equations stay close to the solutions of much simpler dynamical systems. These approximate models seek to describe flows in which there is a dominant balance between the Coriolis, buoyancy and pressure-gradient forces on fluid particles. Such approximations to Newton's second law are commonly referred to as *balanced models*, and the so-called quasi-geostrophic and semi-geostrophic theories are examples of such models.

Salmon (1983, 1985, 1988) pioneered the systematic derivation of balanced models within the framework of Hamilton's principle. The rationale is to make approximations to the Lagrangian without disturbing the symmetry properties of the functional, thereby ensuring that the resulting

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model retains approximations to the conservation laws of the Euler equations. In particular, the particle relabelling symmetry corresponds, within the model approximations, to the fundamental meteorological-cum-oceanographic fluid-dynamical principle of conservation of 'potential vorticity'.

The semi-geostrophic model has a particular significance in dynamical meteorology. The reason is that semigeostrophy has certain useful and elegant mathematical properties, in particular Hamiltonian structure, Legendre duality, and the convexity of certain potential functions (Chynoweth and Sewell 1989, 1991; Roulstone and Norbury 1994; Roulstone and Sewell 1997). These permit both robust numerical integration — even in cases where frontal discontinuities form — together with a rather complete knowledge of mathematical properties such as existence and uniqueness of solutions (Cullen and Purser 1984, Purser and Cullen 1987, Cullen *et al.* 1991). Subsequently, quaternionic and Kähler structure has been identified within a class of balanced models in which the relationship between the potential vorticity and a stream function is governed by a Monge-Ampère equation (Roubtsov and Roulstone 1997). McIntyre and Roulstone (2000) address the issue as to whether higher-order corrections to semigeostrophy may be constructed while retaining some of the mathematical features that facilitate the integration of the equations both analytically and numerically. This issue is of practical importance in data assimilation too, given the recent development of schemes based on semi-geostrophic theory (Desroziers 1997). The purpose of this note is to demonstrate the existence of both special Lagrangian and special Kähler structures in models of nearly geostrophic flow. We shall use the shallow water equations as a paradigm for the equations of meteorology and oceanography.

2 Shallow water theory and balanced dynamics

The motion of a typical particle in shallow water theory can be described by expressing the current cartesian horizontal coordinates

$$x = x(a, b, t), \quad y = y(a, b, t) \tag{1}$$

as functions, on the right, of the particle labels a, b and the time t. The incompressibility hypothesis requires the current depth h to be a function $h(a, b, t)$ with the property

$$\frac{h(a, b, 0)}{h(a, b, t)} = \frac{\partial(x, y)}{\partial(a, b)}\ , \tag{2}$$

where the jacobian on the right is that of the mapping (1). The time derivative of (2) following the particle gives the differential equation of continuity.

The equations of horizontal momentum balance for flows over a bed which is rotating with position dependent Coriolis parameter $f(y)$ are

$$\ddot{x} + g\frac{\partial h}{\partial x} - \dot{y}f = 0, \;\; \ddot{y} + g\frac{\partial h}{\partial y} + \dot{x}f = 0\ . \tag{3}$$

Here g is a given constant, representing the combined effect of the acceleration due to gravity and a centrifugal component due to the Earth's rotation and the superposed dot denotes the time derivative following a particle.

Another important kinematical concept is the so-called potential vorticity defined by

$$\xi = \frac{1}{h}\left(\frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{x}}{\partial y} + f\right), \tag{4}$$

which is conserved on particles. Common choices of the Coriolis parameter are $f = \mathit{constant}$ or βy, as approximations to $2\Omega \sin\phi$, depending on purpose. Here β and Ω are constants (related to the spin of the Earth), and ϕ is latitude. (Note, to avoid confusion, that Ω will be used to denote a symplectic structure later in the paper.)

The semi-geostrophic approximation to equations (3), in the case when f is a constant, is the replacement of the true acceleration by the time derivative of another vector

$$u_g = -\frac{g}{f}\frac{\partial h}{\partial y}, \;\; v_g = \frac{g}{f}\frac{\partial h}{\partial x}, \tag{5}$$

following the particle. This vector is a notional velocity, called the geostrophic velocity. The semi-geostrophic approximation seeks to find motions satisfying

$$\dot{u}_g + g\frac{\partial h}{\partial x} - \dot{y}f = 0, \quad \dot{v}_g + g\frac{\partial h}{\partial y} + \dot{x}f = 0. \tag{6}$$

Associated with these equations, the potential vorticity

$$\Xi = \frac{1}{h}\left[f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x,y)}\right], \tag{7}$$

is conserved.

We introduce a transformation of coordinates

$$X = x + \frac{g}{f^2}\frac{\partial h}{\partial x}, \quad Y = y + \frac{g}{f^2}\frac{\partial h}{\partial y}. \tag{8}$$

For a thorough discussion of this transformation, including its Legendre and contact properties, see Roulstone and Sewell (1997). The coordinates $X, Y)$ are called geostrophic coordinates because, when f is a constant, $\dot{X} = u_g$, $\dot{Y} = v_g$. Salmon (1985) studied certain *generalized* semi-geostrophic equations with pseudo-Hamiltonian form in X, Y space, namely

$$\dot{X} = -\frac{1}{f}\frac{\partial \Phi}{\partial Y}, \quad \dot{Y} = \frac{1}{f}\frac{\partial \Phi}{\partial X} \tag{9}$$

where

$$\Phi(X, Y, t) = \frac{1}{2}(u_g^2 + v_g^2) + gh. \tag{10}$$

In these coordinates, the potential vorticity takes the jacobian form

$$\Xi = \frac{f}{h}\frac{\partial(X,Y)}{\partial(x,y)} = \frac{1}{h}\left[f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x,y)}\right]. \tag{11}$$

The definition (11) can be rewritten as

$$\Xi = \frac{gf}{(H - \frac{1}{2}f(x^2+y^2))}\begin{vmatrix} \dfrac{\partial^2 H}{\partial x^2} & \dfrac{\partial^2 H}{\partial x \partial y} \\ \dfrac{\partial^2 H}{\partial y \partial x} & \dfrac{\partial^2 H}{\partial y^2} \end{vmatrix}, \tag{12}$$

where $H = \frac{1}{2}f(x^2+y^2) + gh$. This is the starting point for a Monge-Ampère type of equation in which $H(x, y, t)$ is the unknown, to be found in conjunction with suitable boundary conditions. Roubtsov and Roulstone (1997) show how the Monge-Ampère equation (12) can be expressed in terms of a 2-form restricted to the graph of h.

McIntyre and Roulstone (1996) studied a family of balanced models, all of which possess a 'Monge-Ampère/stream function' relationship, like the semi-geostrophic equations above, between the potential vorticity and the depth h. These balanced models are termed 'near-local' because they can be derived using the framework of constrained Hamiltonian systems (McIntyre and Roulstone *op. cit.*) in which the constraints depend upon the depth function and a finite number of its derivatives with respect to x and y. No non-local constraints are involved. Canonical coordinates, which should be considered as analogues of (8), were discovered together with an associated complex structure, and this was formulated in terms of quaternions by Roubtsov and Roulstone (1997).

3 Complex structures

Following McIntyre and Roulstone (1996), Roubtsov and Roulstone (1997) introduce the following notation for canonical coordinates for near-local balanced models

$$X = x + icq + ap, \quad Y = y - icp + aq, \tag{13}$$

where $a, c \in \mathbb{R}$, $i = \sqrt{-1}$ and $(p, q) = (\partial h/\partial x, \partial h/\partial y)$. McIntyre and Roulstone (*op. cit.*) show that the materially conserved potential vorticity, Ξ, can always be expressed in Jacobian form (cf. (11))

$$\Xi = \frac{f}{h}\frac{\partial(X, Y)}{\partial(x, y)} \ . \tag{14}$$

The choice of parameters a and c determines the form of the potential vorticity (and vice-versa). This choice is fixed by the physical considerations of the problem, but we shall find that the expressions for the coordinates in (13) is subject to certain restrictions if a holomorphic structure is desired.

One can readily show that

$$\frac{\partial(X, \overline{X}, Y, \overline{Y})}{\partial(x, y, p, q)} = 4c^2, \tag{15}$$

and we find

$$p = \frac{i}{2c}\left(Y - \overline{Y}\right) \ , \quad q = -\frac{i}{2c}\left(X - \overline{X}\right) \ , \tag{16}$$

$$x = \tfrac{1}{2}\left(X + \overline{X}\right) - \frac{ia}{2c}\left(Y - \overline{Y}\right) \ , \quad y = \tfrac{1}{2}\left(Y + \overline{Y}\right) + \frac{ia}{2c}\left(X - \overline{X}\right) \ . \tag{17}$$

It follows that

$$\frac{\partial}{\partial X} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + \frac{ia}{c}\frac{\partial}{\partial y} - \frac{i}{c}\frac{\partial}{\partial q}\right), \quad \frac{\partial}{\partial Y} = \tfrac{1}{2}\left(\frac{\partial}{\partial y} - \frac{ia}{c}\frac{\partial}{\partial x} + \frac{i}{c}\frac{\partial}{\partial p}\right). \tag{18}$$

In an attempt to describe the coordinates (13) in terms of a transformation similar to (8), Roubtsov and Roulstone (1997) made the following choice of coordinates on $\mathbb{C}^2$:

$$z = x + icq, w = y - icp. \tag{19}$$

This non-standard choice provides the following expression for the canonical symplectic 2-form on $\mathbb{R}^4$:

$$\Omega \equiv \mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q = \frac{i}{2c}(\mathrm{d}z \wedge \mathrm{d}w - \mathrm{d}\overline{z} \wedge \mathrm{d}\overline{w}), \tag{20}$$

which is not a (1,1)-form in the coordinates z, w, and hence is not a Kähler structure. This choice was motivated by the observation that the coordinates (13) result from the following transformation $\mathbb{C}^2 \mapsto \mathbb{C}^2$ (cf. (8)):

$$(z, w) \mapsto (X, Y). \tag{21}$$

The canonical 2-form is

$$\Omega = \frac{i}{2c}(\mathrm{d}X \wedge \mathrm{d}Y - \mathrm{d}\overline{X} \wedge \mathrm{d}\overline{Y}). \tag{22}$$

We argue, however, that the more natural choice of the coordinates on the phase space of near-local balanced models is

$$Z \equiv X, \quad W \equiv \overline{Y},$$

because there exists a Kähler structure in this representation. That is, we wish to place more emphasis on the complex structure than on the jacobian relationship (14) between Ξ and (X, Y).

The canonical structure in the coordinates (Z, W) is evidently a (1,1)-form

$$\Omega = \frac{i}{2c}(\mathrm{d}Z \wedge \mathrm{d}\overline{W} + \mathrm{d}W \wedge \mathrm{d}\overline{Z}) \tag{23}$$

and can be used to define a Kähler structure of the model. The second argument (see §4.1) for adopting the coordinates (Z, W) is that they can be considered as an example of *special Lagrangian* coordinates on the initial phase space (spanned by (z, w)) with *deformed* complex structure in accordance with the prescriptions for their existence given by Hitchin (1997) (see also Gross (1998)).

The holomorphic (2,0)-form $\mathrm{d}Z \wedge \mathrm{d}W$ is expressed in the original coordinates as

$$\mathrm{d}Z\wedge\mathrm{d}W = (\mathrm{d}z + a\mathrm{d}p)\wedge(\mathrm{d}\overline{w} + a\mathrm{d}q) = \mathrm{d}x\wedge\mathrm{d}y + a(\mathrm{d}x\wedge\mathrm{d}q + \mathrm{d}p\wedge\mathrm{d}y) + ic(\mathrm{d}x\wedge\mathrm{d}p + \mathrm{d}q\wedge\mathrm{d}y) + (a^2 + c^2)\mathrm{d}p\wedge\mathrm{d}q. \tag{24}$$

We will see below that there is essentially the only possible choice of the structure once we fix the elliptic Monge-Ampère operator for the balanced model.

The matrix $\mathcal{B}$ (see (33) *et seq.*) of the 'momentum' part of the transformation

$$\mathcal{P} \equiv ap + icq, \quad \mathcal{Q} \equiv icp + aq, \tag{25}$$

has a diagonal real part and symmetric imaginary part, and hence the coordinates (Z, W) obey, trivially, the prescriptions to be special Lagrangian. We recall that this property means that there is an integrable almost-complex structure on the phase space of the model which induces a flat metric such that the cotangent components $\mathcal{P}, \mathcal{Q}$ (which are evidently Lagrangian) are also of "minimal volume" with respect to this metric. These conditions are discussed in detail in Theorem 5.2 of Gross (1998).

This observation shows that the phase space of the near-local balanced models with the coordinates (Z, W) can be considered as a Lagrangian fibration over the base manifold with the local coordinates (x, y). We will discuss below a possible extension of this point of view.

4 Special Kähler structure

There exists a (complex-valued) generating function

$$S = icpq + \tfrac{1}{2}a(p^2 + q^2) \tag{26}$$

such that

$$Z - x = \frac{\partial S}{\partial p} = icq + ap, \quad W - y = \frac{\partial S}{\partial q} = icp + aq. \tag{27}$$

Let us consider a potential function $\Phi = \phi + S$. Direct verification shows that the function Φ is holomorphic in the coordinates (Z, W):

$$\frac{\partial \Phi}{\partial \overline{Z}} = 0, \quad \frac{\partial \Phi}{\partial \overline{W}} = 0, \tag{28}$$

and

$$\frac{\partial \Phi}{\partial Z} = p, \quad \frac{\partial \Phi}{\partial W} = q. \tag{29}$$

Now we will show that this function can be used as a *holomorphic prepotential* of special Kähler structure (Freed (1999)) on the phase space of our models. Indeed, from the derivatives of the potential function Φ we can construct the matrix

$$||\tau_{ij}|| = \begin{pmatrix} \dfrac{\partial^2 \Phi}{\partial Z^2} & \dfrac{\partial^2 \Phi}{\partial Z \partial W} \\ \dfrac{\partial^2 \Phi}{\partial W \partial Z} & \dfrac{\partial^2 \Phi}{\partial W^2} \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{i}{2c} \\ -\dfrac{i}{2c} & 0 \end{pmatrix}$$

and a real function

$$K = \frac{1}{2} Im\left(\frac{\partial \Phi}{\partial Z}\overline{Z} + \frac{\partial \Phi}{\partial W}\overline{W}\right) = -cpq.$$

So we see that the Kähler potential is (up to the -1) an imaginary part of our holomorphic potential Φ and as usual our canonical form is

$$\Omega = i\partial\partial K = \frac{i}{2} Im(\tau_{ij}) \mathrm{d}u_i \wedge \mathrm{d}\overline{u}_j,$$

where $u_1 = Z, u_2 = W$. This simple example is a special Kähler geometry on the cotangent bundle to $\mathbb{R}^2$ slightly deformed by the transformation (21).

An important consequence of this observation is that the potential $K = -cpq$ leads to a flat *pseudo-metric* on the Lagrangian fibres of the inital cotangent bundle complimentary to $\mathbb{R}^2$. This fact is relevant to an observation (Roubtsov and Roulstone (2000)) that we can associate a (2,2)-pseudometric with any generic Monge-Ampère operator on a 4-dimensional symplectic manifold which admits a Lagrangian fibration (in particular for $T^*M, dim_{\mathbb{R}} M = 2$).

4.1 A link to Monge-Ampère operators

Let Ω be the canonical symplectic form and ω be a nondegenerate 2-form on $\mathbb{R}^4$ representing an elliptic Monge-Ampère operator (Lychagin (1979), Roubtsov and Roulstone (1997)). This means that we have an almost-complex structure on $\mathbb{R}^4$ whose integrability conditions (vanishing of the Nijenhuis tensor) is equivalent to the condition that the form ω, by a (conformal) symplectomorphism of $\mathbb{R}^4$, is reduced to a form with constant coefficients and its Pffafian is equal to 1 (Lychagin *et al.* 1993).

We can choose a holomorphic (2,0)-form corresponding to this complex structure in such a way that

$$\mathrm{d}\tilde{Z} \wedge \mathrm{d}\tilde{W} = \omega - i\Omega. \tag{30}$$

Now we can address the following question: what freedom of choice is there in the special Lagrangian coordinates $(\tilde{Z}, \tilde{W})$ on the phase space such that the underlying almost-complex structure is given by an elliptic Monge-Ampère operator as above? We shall show that the answer places a restriction on the choice of parameters of the model (a and c in expression (13)) which are defined by the asymptotic regime (balanced approximation to (4)) to be modelled.

For an elliptic Monge-Ampère operator with real coefficients

$$\omega = \alpha \mathrm{d}p \wedge \mathrm{d}y + \beta(\mathrm{d}q \wedge \mathrm{d}y - \mathrm{d}p \wedge \mathrm{d}x) + \gamma \mathrm{d}x \wedge \mathrm{d}q + \delta \mathrm{d}x \wedge \mathrm{d}y + \phi \mathrm{d}p \wedge \mathrm{d}q, \tag{31}$$

the Pfaffian

$$Pf(\omega) = \frac{\omega \wedge \omega}{\Omega \wedge \Omega} = \alpha\gamma + \beta^2 - \delta\phi > 0. \tag{32}$$

The coordinates $(\tilde{Z}, \tilde{W})$ can be re-expressed using the coefficients of (31)

$$\tilde{Z} = \alpha_1 x + \beta_{11} p + \beta_{12} q, \quad \tilde{W} = \alpha_2 y + \beta_{21} p + \beta_{22} q, \tag{33}$$

where

$$\beta_{11} = \alpha; \beta_{22} = \gamma, \beta_{12} = \beta + i, \beta_{21} = \beta - i, \alpha_1 = \alpha_2 = \delta. \tag{34}$$

We compute

$$\det \mathcal{B} = \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix} = \alpha\gamma - (\beta^2 + 1)$$

and comparing with (31), we obtain $\det \mathcal{B} = \phi$. Then, using a conformal rescaling by the Pfaffian we can suppose that, up to a symplectomorphism of $\mathbb{R}^4$, either δ or ϕ can be chosen equal to 1. It is then easy to see, from (30), that in the choice we have for the coordinates $\tilde{Z}, \tilde{W}$, after imposing ellipticity and the conformal choice of $\phi = 1$, there remains only two degrees of freedom — the dependence on δ and α (or on δ and γ).

As the example of this ideology let us consider the Monge-Ampère operators associated with the transformations (13) and (21). We have the 2-form

$$\mathrm{d}X \wedge \mathrm{d}Y = \mathrm{d}x \wedge \mathrm{d}y + (a^2 - c^2)\mathrm{d}p \wedge \mathrm{d}q + a(\mathrm{d}x \wedge \mathrm{d}q + \mathrm{d}p \wedge \mathrm{d}y) - ic(\mathrm{d}x \wedge \mathrm{d}p + \mathrm{d}y \wedge \mathrm{d}q).$$

The *effective* part of this form which corresponds to the Monge-Ampère operator (Lychagin (1979)) is the real part, $\Re(dX \wedge dY)$, of $\mathrm{d}X \wedge \mathrm{d}Y$. In our notation, (31), $\alpha = \gamma = a, \phi = a^2 - c^2, \delta = 1$. Hence, the Pfaffian $Pf(\Re(dX \wedge dY)) = c^2$ and we obtain an elliptic Monge-Ampère operator together with a *new* (conformal) almost complex structure on $\mathbb{R}^4$ which was implicit in the McIntyre and Roulstone (1996) description of balanced models:

$$\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial y} + \frac{\partial}{\partial q}, \qquad \frac{\partial}{\partial y} \mapsto -a\frac{\partial}{\partial x} - \frac{\partial}{\partial p},$$
$$\frac{\partial}{\partial p} \mapsto -(a^2 - c^2)\frac{\partial}{\partial y} - a\frac{\partial}{\partial q}, \qquad \frac{\partial}{\partial q} \mapsto (a^2 - c^2)\frac{\partial}{\partial x} + a\frac{\partial}{\partial p}. \tag{35}$$

This structure is obviously integrable in the constant coefficient case.

We can associate the form $\Re(\mathrm{d}X \wedge \mathrm{d}Y)$ (and the associated Monge-Ampère operator) with its "dual" counterpart, $\mathcal{J}$, by using the above mentioned (2,2)-pseudo-metric associated with $\Re(\mathrm{d}X \wedge \mathrm{d}Y)$:

$$\mathcal{J} = -2\mathrm{d}p \wedge \mathrm{d}q + \frac{2(a^2 + c^2)}{(a^2 - c^2)^2}\mathrm{d}x \wedge \mathrm{d}y - \frac{2a}{a^2 - c^2}(\mathrm{d}x \wedge \mathrm{d}q + \mathrm{d}p \wedge \mathrm{d}y)$$

with the Pfaffian

$$Pf(\mathcal{J}) = -\frac{4c^2}{(a^2-c^2)^2}.$$

It is clear that this "dual" form exists iff $a^2 \neq c^2$. The singular values of the parameters correspond to a degeneration of the pseudo-metric.

On the other hand, for the "deformed" coordinates Z, W, the analogous calculations give us two *hyperbolic* Monge-Ampère operators, which correspond to two effective 2-forms ω_1, ω_2, such that

$$\mathrm{d}Z \wedge \mathrm{d}W = \omega_1 + i\omega_2.$$

Their Pfaffians are equal: $Pf(\omega_1) = Pf(\omega_2) = -c^2$, and hence they define *symplectically equivalent* Monge-Ampère operators. The choice of the parameters $a = 1, c = \sqrt{3}$ in the McIntyre-Roulstone model implies that the "dual" hyperbolic operator which corresponds to the form $\mathcal{J}$ is symplectically equivalent to both operators defined by the pair ω_1, ω_2. Such "duality" is the subject of on-going research.

We should make a further remark about these special Lagrangian coordinates. As discussed by Hitchin (1997) and Gross (1998), the salient feature of their existence is the question of the integrability to the corresponding almost-complex structure. This question looks trivial in the "linear" approximation we have discussed here, though the theorem of S. Lie in the form of Lychagin and Roubtsov (1983) gives the integrability conditions in terms of

$$\mathrm{d}\omega = \tfrac{1}{2}\mathrm{d}\log|Pf(\omega)| \wedge \omega \tag{36}$$

for the corresponding effective Monge-Ampère form (another version of this statement is known also as "Hitchin's lemma" (see Hitchin (1987)). The foregoing special Lagrangian structure has relied on ω, in (31), having constant coefficients. Some interesting cases when this is not so are the subject of a further study.

4.2 A contact transformation

Finally, we show how another property of the semi-geostrophic equations, namely a contact structure (Purser 1993; Roulstone and Sewell 1997; Sewell and Roulstone 1994), generalizes to other near-local balanced models.

We can introduce a "semi-holomorphic" contact bundle corresponding to the coordinates (Z, W) with local coordinates $(Z, W; \Phi; P, Q)$, where $\Phi = \phi + S$ and

$$P = \frac{\partial \Phi}{\partial Z} = p, \; Q = \frac{\partial \Phi}{\partial W} = q. \tag{37}$$

One can show that

$$\mathrm{d}\Phi = \mathrm{d}\phi + \mathrm{d}S = \mathrm{d}\phi + ic(p\mathrm{d}q + q\mathrm{d}p) + a(p\mathrm{d}p + q\mathrm{d}q), \tag{38}$$

and furthermore

$$\mathrm{d}\Phi - \frac{\partial \Phi}{\partial Z}\mathrm{d}Z - \frac{\partial \Phi}{\partial W}\mathrm{d}W = \mathrm{d}\phi - p\mathrm{d}x - q\mathrm{d}y. \tag{39}$$

This is a "contactification" of the conformal symplectomorphism (23).

5 Summary

This work is a contribution to the ongoing research into balanced models in geophysical fluid dynamics, in which the objective is to understand the properties of the equations that assist solution strategies. In particular, the Kähler structure associated with a Monge-Ampère equation facilitates an understanding as to why complex coordinates arise in a natural way in these models — something that would be difficult to deduce from the physics alone. We have shown that a natural generalization (13) of the so-called geostrophic momentum transformation (8) of semi-geostrophic theory, discovered by McIntyre and Roulstone (1996), possesses a special Lagrangian and special

Kähler structure. These features may be of practical importance too, if one considers the possibility of using higher-order balance conditions in data assimilation schemes (e.g. Desroziers 1997).

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References

Chynoweth, S. and Sewell, M.J. (1989) Dual variables in semi-geostrophic theory. *Proc. R. Soc. Lond.* A **424**, 155-186.
Chynoweth, S. and Sewell, M.J. (1991) A concise derivation of the semi-geostrophic equations. *Q. J. R. Meteorol. Soc.*, **117**, 1109-1128.
Cullen, M.J.P., Douglas, R.J., Roulstone, I. and Sewell, M.J. Generalized semi-geostrophic theory on a sphere. *In preparation.*
Cullen, M.J.P., Norbury, J. and Purser, R.J. (1991) Generalised Lagrangian solutions for atmospheric and oceanic flows. *Siam J. Appl. Math.*, **51**, 20-31.
Cullen, M.J.P. and Purser, R.J. (1984) An extended Lagrangian theory of semi-geostrophic frontogenesis. *J. Atmos. Sci.*, **41**, 1477-1497.
Desroziers, G. (1997) A coordinate change for data assimilation in spherical geometry of frontal structures. *Mon. Wea. Rev.*, **125**, 3030-3038.
Freed, D.S. (1999) Special Kähler manifolds. *Commun. Math. Phys.*, **203**, 31-52.
Gross, M. (1998) Special Lagrangian fibrations II: geometry. To appear in *Surv. in Differential Geom.* (mathAG/9809072)
Hitchin, N.J. (1987) Monopoles, minimal surfaces and algebraic curves. *N.A.T.O. Advanced Study Institute*, **105**, L'Université de Montréal.
Hitchin, N.J. (1997) The moduli space of special Lagrangian submanifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **25**, 503-515.
Lychagin, V.V. (1979) Non-linear differential equations and contact geometry. *Russian Math. Surveys*, **34:1**, 137-165.
Lychagin, V.V. and Roubtsov, V.N. (1983) On Sophus Lie theorems for Monge-Ampère equations (in Russian). *Doklady Bielorussian Academy of Science*, **27:5**, 396-398.
Lychagin, V.V., Roubtsov, V.N. and Chekalov, I.V. (1993) A classification of Monge-Ampère equations. *Ann. Scient. Ec. Norm. Sup.*, 4^e série, **26**, 281-308.
McIntyre, M.E. and Roulstone, I. (1996) Hamiltonian balanced models: constraints, slow manifolds and velocity splitting. *Forecasting Research Scientific Paper 41, Meteorological Office, U.K.* submitted to *J. Fluid Mech.*
McIntyre, M.E. and Roulstone, I. (2000) Are there higher-accuracy analogues of semi-geostrophic theory? To appear in *Mathematical Modelling of Atmosphere and Ocean Dynamics*, I. Roulstone and J. Norbury (eds.), C.U.P.
Purser, R.J. (1993) Contact transformations and Hamiltonian dynamics in generalized semi-geostrophic theories. *J. Atmos. Sci.*, **50**, 1449-1468.
Purser, R.J. and Cullen, M.J.P. (1987) A duality principle in semi-geostrophic theory. *J. Atmos. Sci.* **44**, 3449-3468.
Roubtsov, V.N. and Roulstone, I. (1997) Examples of quaternionic and Kähler structures in Hamiltonian models of nearly geostrophic flow. *J. Phys. A.* **30**, L63-L68.
Roulstone, I. and Norbury, J. (1994). A Hamiltonian structure with contact geometry for the semi-geostrophic equations. *J. Fluid Mech.*, **272**, 211-233.
Roulstone, I. and Sewell, M.J. (1997) The mathematical structure of theories of semi-geostrophic type. *Phil. Trans. R. Soc. Lond.* **355**, 2489-2517.
Salmon, R. (1983) Practical use of Hamilton's principle. *J. Fluid Mech.*, **132**, 431–444.
Salmon, R. (1985) New equations for nearly geostrophic flow. *J. Fluid Mech.*, **153**, 461-477.
Salmon, R. (1988) Semi-geostrophic theory as a Dirac bracket projection. *J. Fluid Mech.*, **196**, 345-358.

Sewell, M.J. and Roulstone, I. (1994) Families of lift and contact transformations. *Proc. R. Soc. Lond.*, A **447**, 493-512.