Quantizations of Differential Equations

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Abstract

We discuss here the categorical approach to quantizations of monoidal categories and functors between them. We outline also some methods of finding quantizations and compare them with the known ones. Applying the scheme to differential equations we get that the quantized one is a differential equation which corresponds to quantized braiding.

Key words: Monoidal categories, quantizations, differential equations, Hopf algebras

1 Quantizations

1.1 Quantizations of Monoidal Categories

Let $\mathcal{C}$ be a strict monoidal category [[11],[14],[2]]equipped with a tensor product bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $\mathbf{k}$.

By a *quantization of category* $\mathcal{C}$ we mean ([7],[8]) a natural isomorphism of the bifunctor $\mathcal{Q} : \otimes \Longrightarrow \otimes$, $\mathcal{Q}_{X,Y} : X \otimes Y \to X \otimes Y$, $X, Y \in Ob(\mathcal{C})$, which preserves a unit, $\mathcal{Q}_{X,\mathbf{k}} = \mathcal{Q}_{\mathbf{k},X} = \mathbf{1}_X$ and the following diagram

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[r, "id_X \otimes \mathcal{Q}_{Y,Z}"] \arrow[d, "\mathcal{Q}_{X,Y} \otimes id_Z"] & X \otimes Y \otimes Z \arrow[d, "\mathcal{Q}_{X,Y \otimes Z}"] \\
X \otimes Y \otimes Z \arrow[r, "\mathcal{Q}_{X \otimes Y,Z}"] & X \otimes Y \otimes Z
\end{tikzcd}$$

commutes.

Natural isomorphisms act in the natural way on the totality of all quantizations. Namely, if $\mathcal{Q}$ is a quantization and λ is a unit preserving natural isomorphism of $\mathcal{C}$ then a new natural isomorphism $\lambda(\mathcal{Q}) : \otimes \Longrightarrow \otimes$ given by the formula

$$\lambda(\mathcal{Q})_{X,Y} = (\lambda_X \otimes \lambda_Y) \circ \mathcal{Q}_{X,Y} \circ \lambda^{-1}_{X\otimes Y}$$

is a quantization too.

Denote by $\mathcal{Q}(\mathcal{C})$ the totality of all quantizations of the category. The totality of quantizations is closed with respect to composition where $(\mathcal{Q}' \circ \mathcal{Q}'')_{X,Y} \stackrel{\mathrm{def}}{=} \mathcal{Q}'_{X,Y} \circ \mathcal{Q}''_{X,Y}$, and the composition determines a group structure. Denote by $H^2_q(\mathcal{C})$ the factor-set of all quantizations by the above action of the group of natural isomorphisms.

We call ([9]) $H^2_q(\mathcal{C})$ the *nonlinear 2nd cohomology group* of the monoidal category.

1.2 Quantizations of Functors

Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be a unit preserving functor between two monoidal categories. By a *quantization of* Φ we mean a natural isomorphism $\mathcal{Q} : \otimes \circ \Phi \times \Phi \Longrightarrow \Phi \circ \otimes$, $\mathcal{Q}_{X,Y} : \Phi(X) \otimes \Phi(Y) \to \Phi(X \otimes Y)$, $X, Y \in Ob(\mathcal{C})$, of the bifunctors which preserves units and satisfies the coherence condition. Namely, the following diagram

$$\begin{array}{ccc}
\Phi(X) \otimes \Phi(Y) \otimes \Phi(Z) & \xrightarrow{id_{\Phi(X)} \otimes \mathcal{Q}_{Y,Z}} & \Phi(X) \otimes \Phi(Y \otimes Z) \\
\Big\downarrow {\scriptstyle \mathcal{Q}_{X,Y} \otimes id_{\Phi(Z)}} & & \Big\downarrow {\scriptstyle \mathcal{Q}_{X,Y\otimes Z}} \\
\Phi(X \otimes Y) \otimes \Phi(Z) & \xrightarrow{\mathcal{Q}_{X\otimes Y,Z}} & \Phi(X \otimes Y \otimes Z)
\end{array}$$

commutes [cf.[3]].

The quantizations of the identity functor $id : \mathcal{C} \to \mathcal{C}$ are precisely quantizations of the category.

Moreover, if $\Phi : \mathcal{C} \to \mathcal{C}'$ and $\Psi : \mathcal{C}' \to \mathcal{C}''$ are functors between monoidal categories and if $\mathcal{Q}^\Phi$ and $\mathcal{Q}^\Psi$ are corresponding quantizations then the natural isomorphism

$$\mathcal{Q}^{\Psi\circ\Phi}_{X,Y} \stackrel{\mathrm{def}}{=} \Psi\left(\mathcal{Q}^{\Phi}_{X,Y}\right) \circ \mathcal{Q}^{\Psi}_{\Phi(X),\Phi(Y)}$$

determines the quantization of the composition $\Psi \circ \Phi$.

We call *quantized functor* a pair $\left(\Phi, \mathcal{Q}^{\Phi}\right)$ consisting of a unit preserving functor and a quantization. The above statement means that the totality of quantized functors is closed with respect to the composition.

2 What do we quantize?

2.1 Algebras

Let $\mu : A \otimes A \to A$ be an algebra into category $\mathcal{C}$. We shall define a *quantization* A_q of the algebra given a quantization $\mathcal{Q}$ of the category as the same object A equipped with new product $\mu_q = \mu \circ \mathcal{Q}_{A,A} : A \otimes A \to A$.

Then (A, μ_q) is an algebra in $\mathcal{C}$.

2.2 Modules

Let (A, μ) be an algebra and $\mu_X : A \otimes X \to X$ be a left A-module in $\mathcal{C}$. By a *quantization* of the module we mean the same object X equipped with new multiplication $\mu_{X,q} = \mu_X \circ \mathcal{Q}_{A,X} : A \otimes X \to A$.

Then $(X, \mu_{X,q})$ is a left A-module in $\mathcal{C}$ too.

2.3 Coalgebras

Let $\Delta : A \to A \otimes A$ be a coalgebra in $\mathcal{C}$ with diagonal Δ. We define *quantization* of the coalgebra to be the same object A equipped with new diagonal Δ_q : $\mathcal{Q}^{-1}_{A,A} \circ \Delta : A \to A \otimes A$.

Then (A, Δ_q) is a coalgebra in $\mathcal{C}$.

2.4 Braidings

Let σ be a braiding in $\mathcal{C}$ and $\mathcal{Q}$ be a quantization. We define $\sigma^q_{X,Y} = \mathcal{Q}^{-1}_{Y,X} \circ \sigma_{X,Y} \circ \mathcal{Q}_{X,Y} : X \otimes Y \to Y \otimes X$. Then σ^q is a braiding into $\mathcal{C}$.

If (A, μ) is a σ-commutative algebra in $\mathcal{C}$ then (A, μ_q) is a σ^q-commutative algebra.

2.5 Bialgebras

In this subsection we fix a braiding σ in monoidal category $\mathcal{C}$.

Let (A, μ) be an algebra in $\mathcal{C}$. The tensor square $A^{\otimes 2} = A \otimes A$ can be considered as an algebra with respect to multiplication $\mu^{\otimes 2}_\sigma = (\mathbf{1}_A \otimes \sigma_{A,A} \otimes \mathbf{1}_A) \circ (\mu \otimes \mu) : A^{\otimes 2} \otimes A^{\otimes 2} \to A^{\otimes 2}$.

By *σ-bialgebra* (A, μ, Δ) in $\mathcal{C}$ we mean an algebra (A, μ) and a coalgebra (A, Δ) such that the diagonal Δ is an algebra morphism $\Delta : (A, \mu) \to (A^{\otimes 2}, \mu^{\otimes 2}_\sigma)$.

Then the quantization (A, μ_q, Δ_q) is a σ_q-bialgebra in $\mathcal{C}$.

2.6 Internal Homomorphisms

Assume that $\mathcal{C}$ is closed category. Then for any pair of objects X, Y one has the *internal homomorphisms* $\mathrm{Hom}\,(X, Y)$ object together with the composition $\mu^c : \mathrm{Hom}\,(Y, Z) \otimes \mathrm{Hom}\,(X, Y) \to \mathrm{Hom}\,(X, Z)\,,\ \mu^c : f \otimes g \longmapsto f * g$.

We consider the totality of all internal homomorphisms as an "algebra" with partially determined operation $*$ and quantization of internal homomorphisms as the quantization of this algebra. Namely, we define a *quantization of internal homomorphisms* to be a natural isomorphism $\mathcal{Q}_h : \mathrm{Hom}\,(X, Y) \to \mathrm{Hom}\,(X, Y)$ such that $\mathcal{Q}_h(f) * x = f \underset{q}{*} x$ for all $f \in \mathrm{Hom}\,(X, Y)\,,\ x \in \mathrm{Hom}\,(\mathbf{1}, X)$ where $f \underset{q}{*} g = \mu^c \left(\mathcal{Q}_{\mathrm{Hom}(X,Y),\mathrm{Hom}(Y,Z)}\,(f \otimes g) \right)$.

Then $\mathcal{Q}_h\,(x) = x$ and $\mathcal{Q}_h\,(f * g) = \mathcal{Q}_h\,(f) * \mathcal{Q}_h\,(g)$.

3 Description of Quantizations

The description of quantizations is based on a some kind of non-abelian cohomology theory intrinsically connected to a monoidal category.

3.1 Quantizers and Non-Abelian Cohomologies

Let Ω be a totality of generators in $\mathcal{C}$. We shall assume that Ω contains an unit object $\mathbf{1}$ and closed with respect to the tensor product. In other words Ω is a monoid.

One can reformulate categorical constructions in terms of "geometry" of Ω. Namely, objects of the category we can view as "bundles" over Ω. If X is an object then the "fibre" of the corresponding bundle at point $\omega \in \Omega$ is $\mathrm{Mor}_{\mathcal{C}}(\omega, X)$. In this interpretation elements of objects correspond to sections the bundles and morphisms correspond to bundle morphisms.

To describe the quantizations we consider bundles $\mathbb{I}_n$ over $\Omega^n = \underbrace{\Omega \times \cdots \times \Omega}_{n \text{ times}}$ where the fibre $\mathbb{I}_n$ at $(\omega_1, ..., \omega_n)$ is $\mathbb{I}_n(\omega_1, ..., \omega_n) = \mathrm{Iso}(\omega_1 \otimes \cdots \otimes \omega_n)$.

Let Γ_n be the totality of all sections of $\mathbb{I}_n$. An element $s \in \Gamma_n$ establishes a correspondence $(\omega_1, ..., \omega_n) \in \Omega^n \longmapsto s(\omega_1, ..., \omega_n) \in \mathrm{Iso}(\omega_1 \otimes \cdots \otimes \omega_n)$. Note that Γ_n is a group with obvious group structure.

We define differentials $\delta_n : \Gamma_n \to \Gamma_{n+1}$, $n = 1, 2$ as follows:

✠ $\delta_1(s)(\omega_1, \omega_2) = (s(\omega_1) \otimes s(\omega_2)) \circ s^{-1}(\omega_1 \otimes \omega_2),$

✠✠ $\delta_2(s)(\omega_1, \omega_2, \omega_3) = (\mathbf{1}_{\omega_1} \otimes s(\omega_2, \omega_3)) \circ s(\omega_1, \omega_2 \otimes \omega_3) \circ s^{-1}(\omega_1 \otimes \omega_2, \omega_3) \circ (s^{-1}(\omega_1, \omega_2) \otimes \mathbf{1}_{\omega_3}).$

Then $\delta_2 \circ \delta_1 = 1$.

Tensor preserving natural transformations λ determine elements $s_\lambda \in \Gamma_1$ where $s_\lambda(\omega) = \lambda(\omega)$ and $s_\lambda \in \ker \delta_1$.

In the same manner quantizations $\mathcal{Q}$ determine elements $q \in \Gamma_2$ where $q(\omega_1, \omega_2) = \mathcal{Q}_{\omega_1, \omega_2}$ and $q \in \ker \delta_2$. We call these elements *quantizers.*

The correspondence between quantizations and quantizers can be extended to the map from $H_q^2(\mathcal{C})$ to the second non-abelian cohomology space $\ker \delta_2 \big/ \mathrm{im}\, \delta_1$.

3.2 Examples

3.2.1 Graded Categories

Let G be a group and let $\mathcal{C}$ be the category of G-graded $\mathbf{k}$-modules equipped with the standard monoidal structure. Then $\Omega = G$ and $H_q^2(\mathcal{C}) = H^2(G, U(\mathbf{k}))$ where $U(\mathbf{k})$ is the group of invertible elements of the ring $\mathbf{k}$. Therefore, quantizations of the category are in one-to-one correspondence with the second cohomology group.

3.2.2 Harmonic Oscillator

One of the most interesting application of graded quantizations we can find in dynamical systems. If such system possesses a non-trivial group of symmetries then the corresponding algebra of functions inherits a grading .

Let us consider, for example, 1-dimensional oscillator with Hamiltonian $H = p^2 + q^2$. The symmetry algebra of conformal symplectic transformations contains two-dimensional abelian algebra generated by the rotation and the scale transformation. The algebra of polynomials $A = \mathbb{C}[p,q]$ has the canonical $G = \mathbb{Z} \oplus \mathbb{Z}$ grading with respect to the symmetry algebra. Namely, components $A_{(a,b)}$, $a, b \in \mathbb{N}$, are generated by the polynomials $z^a \overline{z}^b$ where $z = p + \imath q$. Therefore, this algebra can be viewed as algebra in the monoidal category of $\mathbb{Z} \oplus \mathbb{Z}$ - graded $\mathbb{C}$-vector spaces. The cohomologies $H^2(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{C}^*)$ represent by 2-cocycles of the form $q(x,y) = \exp(\imath \langle Qx, y \rangle)$ where $x, y \in \mathbb{Z} \oplus \mathbb{Z}$ and

$$Q = \begin{Vmatrix} 0 & \nu \\ -\nu & 0 \end{Vmatrix}.$$

The quantization of the algebra corresponding to the quantizer q gives the following multiplication table for the quantized algebra

$$z \underset{q}{\cdot} z = z \cdot z, \quad z \underset{q}{\cdot} \overline{z} = e^{\imath\nu} z \cdot \overline{z}, \; \overline{z} \underset{q}{\cdot} z = e^{-\imath\nu} z \cdot \overline{z}, \; \overline{z} \underset{q}{\cdot} \overline{z} = \overline{z} \cdot \overline{z}.$$

Denote by $H_c = p^2 + q^2$ the classical Hamiltonian and by $H_q = p * p + q * q$ the quantum one.

Then $H_q = H_c \cos \nu$, and

$$p * q = pq - \frac{\sin\nu}{2} H_c, \;\; q * p = pq + \frac{\sin\nu}{2} H_c,$$
$$p * q - q * p = H_c \sin\nu.$$

Note that the algebra is the quantum plane [[12]].

3.2.3 Modules over Hopf Algebras

Let H be a Hopf algebra over ring $\mathbf{k}$ and let $\mathcal{C} = H$-mod be the monoidal category of left H-modules. In this case one has a single generator H and the quantizer is represented by an element $q \in H \underset{\mathbf{k}}{\otimes} H$ satisfying the following conditions

$$(\Delta \otimes \mathbf{id})(q) \cdot (q \otimes 1) = (\mathbf{id} \otimes \Delta)(q) \cdot (1 \otimes q),$$
$$q \cdot \Delta(h) = \Delta(h) \cdot q, \;\; \forall h \in H,$$
$$(\varepsilon \otimes \mathbf{id})(q) = (\mathbf{id} \otimes \varepsilon)(q) = 1,$$

where $\varepsilon : H \to \mathbf{k}$ is the counit.

The quantization $\mathcal{Q}$ corresponding to quantizer q has the form $\mathcal{Q}_{X,Y} : x \otimes y \in X \otimes Y \longmapsto q \cdot (x \otimes y) \in X \otimes Y$.

Note that quantizers q which satisfy the first and the third coherence conditions produce quantizations of the forgetful functor.

The unit preserving natural isomorphisms correspond to invertible elements $h \in H$. The action the group of units $U(H)$ on quantizers given by the formula $q \longmapsto \Delta(h) \cdot q \cdot h^{-\otimes 2}$. The space$\mathcal{Q}H^2(H)$ of $U(H)$-orbits is called the 2nd nonlinear cohomology of H. The linearization of nonlinear cohomologies (the "tangent space" to $\mathcal{Q}H^2(H)$ at id) coincides with the second Hochschild cohomology group of the coalgebra H (see, **??**).

3.2.4 Symmetries

Let G be a Lie group and let $H = \mathcal{D}(G)$ be the corresponding Hopf algebra of distributions. We shall consider here special type of quantizations. Namely, by *exponential quantizations* we mean quantizations given by quantizers of the form $q(t) = \exp(tK), t \geq 0$, for some element $K \in H \otimes H$.

In other words, $q(t)$ is the solution of the differential equation

$$\begin{aligned} &\frac{\partial K}{\partial t} = K * q,\ t \geq 0, \\ &q(0) = 1 \end{aligned}$$

in the algebra of distributions on $G \times G$.

We shall distinguish two types of quantizations. The first one, we call them *Moyal quantizations,* K is skew-symmetric and $q(t)$ is invertible[cf. [1],[4]]. The second type, we call them *Nelson quantizations,* K is symmetric and $q(t)$ is not invertible. Strictly speaking they are not quantizations in our sense because they produce natural transformations but not natural isomorphisms. One may consider this type transformations to deforms algebras, modules, internal homomorphisms, but not coalgebras, bialgebras and braidings.

The picture produces by the Moyal type quantizations corresponds to the Heisenberg approach to quantum mechanics. On the other hand, the Nelson type quantizations correspond to the stochastic quantizations.

To write down the coherence conditions for the *kernel* K we shall consider K as 2-cochain: $K : C^{\infty}(G) \otimes C^{\infty}(G) \to \mathbb{C}$.

Then the coherence conditions for K mean:

(1) K is a $\mathrm{Ad}\, G$-invariant,
(2) $K(1, f) = K(f, 1) = 0,\ \forall f \in C^{\infty}(G)$,
(3) $K(fg, h) + h(e) K(f, g) = K(f, gh) + f(e) K(g, h),\ \forall f, g, h \in C^{\infty}(G)$.

In other words, $K \in C^2_{\text{Hochschild}}(C^{\infty}(G), \mathbb{C})$ is a normalized $\mathrm{Ad}\, G$-invariant Hochschild 2-cocycle.

Assume now that G acts on a manifold M and let $\widehat{K}$ be the image of a kernel K under the action. We identify $\widehat{K}$ with an operator into $C^{\infty}(M \times M)$. To find a quantized product $f \underset{q}{\cdot} g = F$, $f, g \in C^{\infty}(M)$ one should solve the following differential equation

$$\begin{aligned} &\frac{\partial F}{\partial t} = \widehat{K}(F),\ t \geq 0, \\ &F|_{t=0} = f \otimes g \end{aligned}$$

on $M \times M$.

It is easy to check that any $\mathrm{Ad}\, G$-invariant tensor $\tau : T^*_e(G) \otimes T^*_e(G) \to \mathbb{C}$ determines a distribution K_{τ} , $K_{\tau}(f, g) = \tau(d_e f \otimes d_e g)$ which satisfy the coherence condition.

The classical Moyal quantization we obtain for the abelian group $G = \mathbb{R}^{2n}$

with K_τ corresponding to the standard symplectic structure [[1],[13]]. The more general type of quantizations we shall get for an arbitrary algebra Lie $\mathfrak{G}$ of symmetries by taking bivectors $\tau \in \Lambda^2(\mathfrak{H})^W$ on a Cartan subalgebra $\mathfrak{H}$ which are invariants of the Weyl group W.

The Casimir elements $\tau \in S^2(T_e^*(G))$ give us examples of stochastic type quantizations.

The second approach to use symmetries for quantizations is based on the Grothendieck algebra of G [[9]].

We shall illustrate this method for the simplest non-trivial group S_3. The representation algebra $K(S_3)$ is generated by the non-trivial (sign) representation λ_1 , the two-dimensional irreducible representation λ_2 and the following relations $\lambda_1^2 = 1,\ \lambda_1\lambda_2 = \lambda_2,\ \lambda_2^2 = 1 + \lambda_1 + \lambda_2$. To define quantizations we should fix isomorphisms $\omega_{ij} : \lambda_i \otimes \lambda_j \to \lambda_i \otimes \lambda_j$ in such a way that the coherence conditions hold. Let us write $\omega_{ij} = \sum_k \omega_{ij}^k \cdot \lambda_k$ where ω_{ij}^k is the restriction of ω_{ij} on λ_k-component of $\lambda_i \otimes \lambda_j$. In our case

$$\omega_{11} = a^2 \cdot 1, \omega_{12} = \omega_{21} = a \cdot \lambda_2, \omega_{22} = ab \cdot 1 + b \cdot \lambda_1 + c \cdot \lambda_2$$

for some $a, b, c \in \mathbb{C}^*$.

Natural isomorphisms of the category can be represented by elements $h = h_1 \cdot \lambda_1 + h_2 \cdot \lambda_2,\ \ h_1, h_2 \in \mathbb{C},$ with the following action on the quantizations

$$h(\omega_{11}) = h_1^{-2}\omega_{11},\ h(\omega_{12}) = h_1^{-1}\omega_{12},\ h(\omega_{22}) = h_2^{-1}c \cdot \lambda_2 + h_1 h_2^{-2} b \cdot \lambda_1 + h_2^{-2} ab \cdot 1.$$

Therefore, one has 1-parameter family of nontrivial quantizations of the category.

4 Differential Operators in ABC-Categories

In this section we discuss differential operators in monoidal categories equipped with braiding. The main reason for such type of generalization is based on the fact that the quantizations of "classical" differential operators lead us to this type of differential operators .

4.1 **ABC**-*categories*

To build up suitable calculus we need the following extra ingredients for basic category $\mathcal{C}$. Namely, we shall assume that category under consideration satisfies the following three conditions:

A $\mathcal{C}$ is an abelian monoidal category (in the sense that $\mathcal{C}$ is abelian monoidal and the tensor product bifunctor is biadditive and right exact).
B $\mathcal{C}$ is a braided category. That is, $\mathcal{C}$ is equipped with a braiding σ.
C $\mathcal{C}$ is a closed category. That is, the internal homomorphism bifunctor $X, Y \longmapsto \mathrm{Hom}\,(X, Y)$ is determined for all objects of the category.

We use abbreviation **ABC**-category for a category satisfying the above properties.

4.2 Symmetrizations and Differential Approximations

Let (A, μ) be a σ-commutative algebra in an **ABC**-category and let (X, μ_X^l, μ_X^r) be a A-A-bimodule. We say that X is *σ-symmetric if* $\mu_X^l = \mu_X^r \circ \sigma_{X,A}$ and $\mu_X^r = \mu_X^l \circ \sigma_{A,X}$. For any A-A-bimodule X we define a *σ-symmetric part* X_σ of X as maximal σ-symmetric submodule. That is,

$$X_\sigma = \left\{ x \in X \middle| \left(\mu_X^r - \mu_X^l \circ \sigma_{A,X}\right)(x \otimes A) = 0, \left(\mu_X^l - \mu_X^r \circ \sigma_{X,A}\right)(x \otimes A) = 0 \right\}.$$

Let us consider a quotient bimodule X / X_σ and define $X_\sigma^{(1)}$ as preimage of $(X / X_\sigma)_\sigma$ with respect to the canonical projection $X \to X / X_\sigma$. Proceeding in this way we get a filtration of the bimodule X by bimodules $X_\sigma^{(i)}$, $i = 0, 1, \ldots$;

$$0 = X_\sigma^{(-1)} \subset X_\sigma^{(0)} = X_\sigma \subset X_\sigma^{(1)} \subset \cdots \subset X_\sigma^{(i)} \subset X_\sigma^{(i+1)} \subset \cdots \subset X_\sigma^{(*)} \subset X.$$

We call a bimodule $X_\sigma^{(*)} = \bigcup_{i \geq 0} X_\sigma^{(i)}$ a *differential approximation of* X. We say that A-A-bimodule X is *differential* if $X = X_\sigma^{(*)}$.

The graded object $\mathrm{Gr}_* X = \sum X_\sigma^{(i)} / X_\sigma^{(i-1)}$ is a σ-symmetric A-A-bimodule.

4.3 σ-Differential Operators

Now we apply the differential approximation procedure to bimodules of internal homomorphisms and obtain bimodules of σ-differential operators. Namely, the differential approximations $(\mathrm{Hom}\,(X, Y))_\sigma^{(i)}$ we denote by $\mathrm{Diff}_i^\sigma\,(X, Y)$ and call modules of σ-differential operators of the order $\leq i$. The composition of internal homomorphisms induces a composition of σ-differential operators $\mathrm{Diff}_i^\sigma\,(Y, Z) \otimes \mathrm{Diff}_j^\sigma\,(X, Y) \to \mathrm{Diff}_{i+j}^\sigma\,(X, Z)$.

The corresponding graded object $\mathrm{Gr}_*(\mathrm{Hom}(X,Y))$ we call a *symbol module* and denote by $\mathrm{Smbl}_*^\sigma(X,Y)$. All of them are symmetric A-A-bimodules and the symbol algebra $\mathrm{Smbl}_*^\sigma(A,A)$ is, an addition, σ-commutative algebra.

Follow the classical scheme we can define σ-Poisson structure into the symbol algebra. Namely, let us denote by $\mathrm{smbl}_i^\sigma(\Delta)$ the symbol of $\Delta \in \mathrm{Diff}_i^\sigma(A,A)$, $\mathrm{smbl}_i^\sigma(\Delta) = \Delta \bmod \mathrm{Diff}_{i-1}^\sigma(A,A)$. Then the σ-commutativity of symbol algebra $\mathrm{Smbl}_*^\sigma(A,A)$ means that the σ-commutator

$$[\Delta,\nabla]^\sigma = \mu^c(\Delta \otimes \nabla - \sigma_{\mathrm{Diff}_j^\sigma(A,A),\mathrm{Diff}_i^\sigma(A,A)}(\nabla \otimes \Delta))$$

of operators $\Delta \in \mathrm{Diff}_i^\sigma(A,A)$, $\nabla \in \mathrm{Diff}_j^\sigma(A,A)$ belongs to $\mathrm{Diff}_{i+j-1}^\sigma(A,A)$.

The symbol $\mathrm{smbl}_{i+j-1}^\sigma([\Delta,\nabla]^\sigma)$ depends on symbols of Δ and ∇ and determines *σ-Poisson bracket*

$$\left\{\mathrm{smbl}_i^\sigma(\Delta), \mathrm{smbl}_j^\sigma(\nabla)\right\} = \mathrm{smbl}_{i+j-1}^\sigma([\Delta,\nabla]^\sigma)$$

and *σ-Poisson structure* into the symbol algebra $\mathrm{Smbl}_*^\sigma(A,A)$.

4.4 Quantizations of Differential Operators

The quantization procedure for differential operators is based on the following observation. Let $\left(X,\mu^l,\mu^r\right)$ be A-A-bimodule over σ-commutative algebra A and let $\mathcal{Q}$ be a quantization. Then the differential approximations for $\left(X,\mu^l,\mu^r\right)$ and $\left(X,\mu_q^l,\mu_q^r\right)$ considered as A_q-A_q-bimodule coincide as objects but differ in the bimodule structures.

Applying this remark to the bimodules of internal homomorphisms we get that the quantization of internal homomorphisms induced a quantization of σ-differential operators to σ^q-differential operators. This quantization preserves order and composition of differential operators and therefore isomorphisms between corresponding σ-Poisson algebras of symbols,[[7]].

Linear differential equations in a monoidal category can be viewed as modules over the algebra of σ-differential operators and therefore can be quantized due to the general scheme. In the same manner non-linear differential equations correspond to algebras over the differential operators and the same scheme can be applied. Note that the results of these quantizations are differential equations but with the quantized σ.

As a toy example, let us consider ODEs which possess the solvable two-dimensional Lie algebra $\mathfrak{G} = (X,Y)$. of symmetries. Then applying the above quantization scheme to the ODEs as objects of category $\mathfrak{G}$-modules and choose $\tau = X \wedge Y \in \Lambda^2(\mathfrak{G})$ one gets a quantization of the ODEs.

Take for example the more common algebra of point transformations generated by the translation $\frac{\partial}{\partial q}$ and the scale transformation $u\frac{\partial}{\partial u}$. Then $\tau = u\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial q} \in \Lambda^2(\mathfrak{G})$ and, say, the action of the quantized operator $\mathcal{Q}_h(\frac{\partial}{\partial q})$ on functions $f \in C^\infty(\mathbb{R})$ shall be the following $\mathcal{Q}_h(\frac{\partial}{\partial q})f = \frac{\partial f}{\partial q} + t\frac{\partial^2 f}{\partial q^2} + o(t)$.

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