GEOMETRIC STRUCTURES ON SOLUTION SPACES OF INTEGRABLE DISTRIBUTIONS

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ABSTRACT. In this paper we will investigate invariant tensors of completely integrable distributions, in particular the Cartan distribution associated with a ordinary differential equation. For second order equations examples of invariant 1-forms, symplectic structure, metric structure and curvature is presented.

1. VECTOR FIELDS, SYMMETRIC 2-FORMS AND CURVATURE.

Let P be a completely integrable distribution on a manifold M, specified by either of the the locally free modules

$$\Delta(P) = \{X \in D(M) \mid X_m \in P_m \ \forall m \in M\}$$

$$\text{or} \qquad \mathrm{Ann}P = \{\theta \in \Omega^1(M) \mid \theta(X) = 0 \ \forall X \in \Delta(P)\}$$

We denote

$$\mathcal{D}(P) = \mathrm{Sym}(P)/\Delta(P)$$

where $\mathrm{Sym}(P)$ is the collection of symmetries of P. We define the space of solutions of P to be

$$\mathcal{S} = M/\sim$$

where points $x, y \in M$ are equivalent if they belong to a connected integral manifold of M.

$\mathcal{F}(P) = \{f \in C^\infty(M) \mid L_X(f) = 0\}$ is an $\mathbb{R}$-algebra, the algebra of *first integrals.* Elements of $\mathcal{D}(P)$ act as *derivations* $\mathcal{F}(P) \to \mathcal{F}(P)$, by acting on functions by any representative in $\mathrm{Sym}(P)$. This action is well defined with respect to choice of representative. $\mathcal{D}(P)$ also inherits the Lie-algebra structure of $\mathrm{Sym}(P)$, with respect to the commutator bracket

$$[\cdot,\cdot] \, : \, \mathcal{D}(P) \longrightarrow \mathcal{D}(P)$$

taken on representatives modulo $\Delta(P)$. The operation is well-defined.

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Definition 1.1. *A symmetric 2-form g is P-*invariant *if*

$$g(X,\,\cdot\,) = 0 \quad \text{and} \quad L_X(g) = 0 \quad \forall X \in \Delta(P)$$

The set of invariant 2-forms is an $\mathcal{F}(P)$-module, denoted $S^2(P)$, and elements of $S^2(P)$ act as symmetric bilinear forms on $\mathcal{D}(P)$ into $\mathcal{F}(P)$, the action well defined on representatives of classes in $\mathcal{D}(P)$. We say that g is positive if $g(X_a, X_a) > 0$ for any $X_a \notin P_a$, $a \in M$. A positive g will induce a *connection*

$$\nabla : \mathcal{D}(P) \times \mathcal{D}(P) \longrightarrow \mathcal{D}(P)$$

with the Levi-Civita properties, that is:

(1) $\nabla_{\overline{X}}(f\overline{Y} + h\overline{Z}) = f\nabla_{\overline{X}}\overline{Y} + h\nabla_{\overline{X}}\overline{Z} + \overline{X}(f)\cdot\overline{Y} + \overline{X}(h)\cdot\overline{Z}$
(2) $\nabla_{f\overline{X}+h\overline{Y}}\overline{Z} = f\nabla_{\overline{X}}\overline{Z} + h\nabla_{\overline{Y}}\overline{Z}$
(3) $\big[\overline{X},\overline{Y}\big] = \nabla_{\overline{X}}\overline{Y} - \nabla_{\overline{Y}}\overline{X}$
(4) $\overline{X}\,\big(g(\overline{Y},\overline{Z})\big) = g(\,\nabla_{\overline{X}}\overline{Y},\overline{Z}\,) + g(\,\overline{Y},\nabla_{\overline{X}}\overline{Z}\,)$

for any $\overline{X},\overline{Y}$ and $\overline{Z} \in \mathcal{D}(P)$, $f,h \in \mathcal{F}(P)$.

Defining ∇ on representatives of elements in $\mathcal{D}(P)$, and requiring that it is invariant with respect to choice of representatives will, together with the above requirements (1)-(4) determine ∇ completely. Let m=codim$P$, and $\mathcal{G}$ be an m-dimensional transversal subalgebra of $\mathcal{D}(P)$ generated by $\{\overline{X}_i\}_{i=1}^m$ where the commutators are

$$[\overline{X}_i, \overline{X}_j] = \sum_{s=1}^{m} c_{ij}^s \overline{X}_s$$

We can find a (local) basis $\{X_i\}_{i=1}^{dimM}$ of $\mathcal{D}(M)$ so that $\{X_i\}_{i=1}^m$ is a set of representatives of $\{\overline{X}_i\}$. We define ∇ in term of the functions Γ_{ij}^s by the equations

$$\nabla_{X_i} X_j = \sum_{s=1}^{\dim M} \Gamma_{ij}^s X_s$$

We must have $\Gamma_{ij}^s = 0$ for $i,j,s > m$, then

$$\nabla_{X_i+Y} X_j = \nabla_{X_i} X_j = \nabla_{X_i}(X_j + Y)$$

for any $Y \in \Delta(P)$, by requirements (1) and (2). Furthermore, properties (3) and (4) are equivalent to

$$\begin{aligned} &(3)' \quad \Gamma_{ijl} = \Gamma_{jil} + c_{ijl} \\ \text{and} \quad &(4)' \quad X_i(g_{jl}) = \Gamma_{ijl} + \Gamma_{ilj} \end{aligned}$$

where $g_{ij} = g(X_i, X_j)$, $\Gamma_{ijl} = \sum_{l=1}^m \Gamma_{ij}^s g_{sl}$ and $c_{ijl} = \sum_{l=1}^m c_{ij}^s g_{sl}$. Combining the two equations for different permutations of indices makes us arrive at the following:

$$\Gamma_{ijl} = \frac{1}{2}\left[\sum_k [c_{ij}^k g_{kl} - c_{jl}^k g_{ki} + c_{li}^k g_{kj}] \ + \ X_j(g_{il}) - X_l(g_{ij}) + X_i(g_{jl})\right] \quad (1)$$

Thus

$$\Gamma_{ij}^s = \frac{1}{2}\sum_{l=1}^m [c_{ijl} - c_{jli} + c_{lij} + X_j(g_{il}) - X_l(g_{ij}) + X_i(g_{jl})]\, g^{ls} \qquad (2)$$

where g^{ls} are entries of of $(g_{ij})^{-1}$. This gives us the local expression of ∇ by means of g and the local basis $\{X_i\}$. We can now define the *curvature operator*

$$R(X,Y) \stackrel{\text{def}}{=} [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} : \mathcal{D}(P) \longrightarrow \mathcal{D}(P)$$

and the *curvature tensor* R of the distribution P

$$R(X,Y,Z,W) \stackrel{\text{def}}{=} g(R(X,Y)(Z),W)$$

2. The algebra of invariant differential forms.

Definition 2.1. *We say that a k-form $\theta \in \Omega^1(M)$ is P-invariant if*

$$i_X(\theta) = 0 \ \text{ and } \ i_X(d\theta) \ = \ 0$$

for all $X \in \Delta(P)$.

The set of invariant k-forms form a $\mathcal{F}(P)$-module, denoted $\Omega^k(P)$.

Proposition 2.1. *Locally any element $\theta \in \Omega^l(P)$ is on the form*

$$\theta = \sum \alpha_{(i_1,\dots,i_l)}\theta_{i_1} \wedge \ldots \wedge \theta_{i_l}$$

where the θ_{i_j}-s form a local basis of Ann(P).

Proof. Let $\{\theta_i\}_{i=1}^n$ be a local basis of $\Omega^1(M)$, $n = \dim M$, such that $\{\theta_i\}_{i=1}^m$ generates Ann(P). Take $\{E_j\}_{j=1}^n$ to be the dual local basis of $\mathcal{D}(M)$ such that $\{E_j\}_{j=m+1}^n$ generates $\Delta(P)$ and $\theta_i(E_j) = \delta_{ij}$. Any element θ of $\Omega^l(P)$ is on the form

$$\theta = \sum_{1\leq i_1<\ldots<i_l\leq n} \alpha_{(i_1,\dots,i_l)}\theta_{i_1} \wedge \ldots \wedge \theta_{i_l}$$

for some $\alpha_{(i_1,\dots,i_l)} \in C^\infty(M)$. By requiring that $i_{E_s}\theta = 0$, starting with $s = n$, and down to $s = m+1$ we see that $\alpha_{(i_1,..,i_l)} = 0$ whenever any $i_j > m$. □

Corollary 2.1.

$$\Omega^l(P) = 0 \quad \mathit{for} \quad l > m = \mathit{codim} P$$

Futhermore, we define $\Omega^0(P) = \mathcal{F}(P)$ and $\Omega^s(P) = 0$ for $s < 0$, and get the following.

Theorem 2.1.
$$\Omega^{\cdot}(P) = \bigoplus_{s \in Z} \Omega^s(P)$$
is a Z-graded σ-commutative algebra with the usual wedge product
$$\wedge : \Omega^s(P) \times \Omega^t(P) \longrightarrow \Omega^{s+t}(P)$$
σ-commutativity means that $\omega \wedge \theta = \sigma(s,t)(\theta \wedge \omega)$ where $\sigma(s,t) = (-1)^{st}$. Also, the differential d is a derivation of degree $+1$ of $\Omega^{\cdot}(P)$.

Proof. Given $\theta \in \Omega^s(P)$, $\omega \in \Omega^t(P)$ and $X \in \Delta(P)$ we have that
$$i_X(\theta \wedge \omega) = (i_X\theta) \wedge \omega + (-1)^s \theta \wedge (i_X\omega) = 0$$
Moreover, $d(\theta \wedge \omega) = d\theta \wedge \omega + (-1)^s \theta \wedge d\omega$, and by a calculation similar to the one above we get $i_X\, d(\theta \wedge \omega) = 0$ which implies that $\theta \wedge \omega \in \Omega^{s+t}(P)$. Direct calculation shows that $d\theta \in \Omega^{s+1}(P)$ whenever $\theta \in \Omega^s(P)$. Each $\Omega^s(P)$ is a $\mathcal{F}(P)$-module; we have that
$$i_X(f\omega) = f(i_X\omega) = 0$$
for every $X \in \Delta(P), \omega \in \Omega^s(P), f \in \mathcal{F}(P)$, and
$$i_X d(f\omega) = i_X(df \wedge \omega) = 0$$
since $df \wedge \omega \in \Omega^{s+1}(P)$. Thus $f\omega \in \Omega^s(P)$. It is obvious that $\Omega^s(P)$ is closed under addition of forms. ☐

Each invariant l-form $\theta \in \Omega^l(P)$ defines a $\mathcal{F}(P)$-linear map
$$\theta \;:\; \mathcal{D}(P) \times \ldots \times \mathcal{D}(P) \;\longrightarrow\; \mathcal{F}(P)$$
by $\theta(\overline{X_1}, \ldots, \overline{X_l}) \stackrel{\text{def}}{=} \theta(X_1, \ldots, X_l)$ for any choice of representatives X_i of $\overline{X_i} \in \mathcal{D}(P)$.

The derivation $d = d_s \,:\, \Omega^s(P) \longrightarrow \Omega^{s+1}(P)$ provides the notion of cohomology: with the l-th cohomology group of P defined by
$$H^s(P) \stackrel{\text{def}}{=} \text{Ker } d_s / \text{ Im}\, d_{s-1}$$

Proposition 2.2. *With respect to the multiplication induced by the wedge product,*
$$H^{\cdot}(P) \;=\; \bigoplus_{l \in \mathbb{Z}} H^l(P)$$
is a $\mathbb{Z}$-graded σ-commutative algebra with
$$[\,\theta\,] \wedge [\,\omega\,] \stackrel{def}{=} [\,\theta \wedge \omega\,]$$
for any choice of representatives $\theta \in \ker d_s$, $\omega \in \ker d_t$.

Let $\phi : M \to N$ be a smooth map of manifolds, equipped with integable distributions P and Q respectively. If $\phi^*(\text{Ann}(Q)) \subset \text{Ann}(P)$, we say that ϕ is a *morphism of distributions.*

Theorem 2.2. *A morphism of distributions $\phi : (M,P) \longrightarrow (N,Q)$ induces a graded algebra-homomorphism*

$$\phi^* : H^{\cdot}(Q) \longrightarrow H^{\cdot}(P)$$

by $\phi^*([\,\theta\,]) \stackrel{\text{def}}{=} [\,\phi^*(\theta)\,]$.

3. Equations of symmetry and cosymmetry

It is well known that associated with the ODE

$$y^{(k)} = F(x, y, y', .., y^{(k-1)}) \tag{3}$$

is the Cartan distribution $\mathcal{C}$ generated by the Cartan forms

$$\omega_0 = dp_0 - p_1 dx, \;\; \omega_1 = dp_1 - p_2 dx, \;\ldots\;, \;\; \omega_{k-1} = dp_{k-1} - F dx$$

$F = F(x, p_0, ..., p_{k-1})$, or, alternatively, by the characteristic line field

$$D = \frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial p_0} + ... + p_{k-1} \frac{\partial}{\partial p_{k-2}} + F \frac{\partial}{\partial p_{k-1}}$$

We know that symmetries modulo characteristic symmetries are all on the form :

$$X_\phi = \phi \partial_{p_0} + D(\phi)\partial_{p_1} + ... + D^{k-1}(\phi)\partial_{p_{k-1}}$$

where $\phi = \phi(x, p_0, p_1, ..., p_{k-1})$ solves the *Lie Equation*

$$L(\phi) = [D^k - \sum_{l=0}^{k-1} \frac{\partial F}{\partial p_l} D^l](\phi) = 0 \tag{4}$$

Theorem 3.1. *Any invariant* 1*-form $\theta \in \Omega^1(\mathcal{C})$ is on the form*

$$\theta = \theta_\psi = \psi \omega_{k-1} + \sum_{l=2}^{k} H_l(\psi)\omega_{k-l}$$

where $\psi = \psi(x, p_0, p_1, ..., p_{k-1})$ solves the adjoint equation

$$L^*(\psi) = [(D^k) - \sum_{r=0}^{k-1} (-1)^{r+k} D^r \cdot \frac{\partial F}{\partial_{p_r}}](\psi) = 0 \tag{5}$$

and where

$$H_l = (-1)^{l-1} D^{l-1} - \sum_{s=0}^{l-2} (-1)^s D^s \frac{\partial F}{\partial p_{k-l+1+s}}$$

(with $p_{-1} = x$ in this formula.)

Proof. The requirement $i_D(\theta) = 0$ implies that $\theta = \sum_{=0}^{k-1} \alpha_i \omega_i$ for some functions α_i since the Cartan forms generate $\mathrm{Ann}(\mathcal{C})$. The second requirement then becomes $i_X d\theta = L_D\theta = 0$, and direct calculation of $L_D\theta$ gives exactly the equations on the α_i-s of the theorem. □

We will denote invariant 1-forms by *cosymmetries*, as they are dual objects to symmetries.

4. Second order ODEs.

For the case $k = 2$ the characteristic field is

$$D = \partial_x + p\partial_u + F(x,u,p)\partial_p$$

and the Lie Equation becomes

$$L(\phi) \;=\; (\; D^2 - F_p D - F_u\;)(\;\phi\;) \;=\; 0 \tag{6}$$

A solution ϕ of (6) generates a symmetry

$$X = X_\phi = \phi\partial_u + D(\phi)\partial_p$$

The adjoint equation becomes :

$$L^*(\psi) \;=\; (\,D^2 + D\cdot F_p - F_u\,)(\psi) \;=\; 0 \tag{7}$$

A solution ψ of (7) generates a cosymmetry

$$\theta_\psi \;=\; (-D - F_p)(\psi)\,\omega_0 \;+\; \psi\,\omega_1$$

For metric structure we get the following requirements. Denote by F_u, F_p $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial p}$ respectively.

Theorem 4.1. *Any invariant symmetric 2 -form is on the form*

$$g = L_{00}(\eta)\,\omega_0^2 + 2L_{01}(\eta)\,\omega_0\cdot\omega_1 + \eta\,\omega_1^2$$

where η, a generating function of g, is the solution of the equation

$$\begin{aligned} L_{11}(\eta) \;=\; \big[\,D^3 \;+\; [\,3F_p\,]\,D^2 \;+\; [\,5D(F_p) + 2F_p^2 - 4F_u\,]\,D + \\ +[\,2D^2(F_p) + 4F_pD(F_p) - 2D(F_u) - 4F_pF_u\,]\big]\;(\eta) \;=\; 0 \end{aligned} \tag{8}$$

and the operators L_{00} and L_{01} are

$$L_{00} = \frac{1}{2}\,[\;D^2 + 3F_pD + 2[\,D(F_p) + F_p^2 - F_u\,]\,] \tag{9}$$

$$L_{01} = -\frac{1}{2}(\,D + 2F_p\,) \tag{10}$$

Proof. This follows from the requirements $g(D,\cdot) = 0$ and $L_D(g) = 0$. The first implies that $g = \sum \alpha_{ij}\omega_i\cdot\omega_j$, where the ω_i are the Cartan forms, who generate $\mathrm{Ann}(\mathcal{C})$. The second gives the requirements of the Theorem on the coefficient functions α_{ij} by direct calculation of $L_D(g)$. □

We call L_{11} the *symmetric power* of L^*. Given two solutions ψ_1 , ψ_2 of the L^*-equation, they generate cosymmetries θ_1 and θ_2, which in turn provides us with an invariant symmetric 2-form $g = \theta_1^2 + \theta_2^2$. The functions ψ_1^2 , $\psi_1\psi_2$ and ψ_2^2 are solutions of the L_{11}-equation.

There is a large class of equations that possess a *symplectic structure* in the sense of an invariant 2-form, non-degenerated except on $\Delta(P)$.

Theorem 4.2. *Equations on the form*

$$y'' = \gamma(x)y' + \delta(x, y) \tag{11}$$

where $\gamma(x)$ *and* $\delta(x, y)$ *are arbitrary, have an invariant* 2*-form*

$$\Lambda = e^{\alpha(x)}\omega_1 \wedge \omega_0$$

where α *is any function such that* $\alpha'(x) = -\gamma(x)$*, and* ω_0 *and* ω_1 *are the Cartan forms.*

Proof. We see that $i_D\Lambda = 0$ since $i_D\omega_0 = i_D\omega_1 = 0$. Direct calculation gives $i_D d\Lambda = L_D\Lambda = 0$, hence Λ is $\mathcal{C}$-invariant. □

Note that by Corollary 2.1 we immediately get that $d\Lambda = 0$ since $\Omega^3(\mathcal{C}) = 0$. Λ produces a Poisson structure on the algebra of first integrals, in coordinates

$$\{f, g\} = e^{-\alpha}\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial u} - \frac{\partial f}{\partial u}\frac{\partial g}{\partial p}\right)$$

Also, we have the notion of Hamiltonian vector field; for any first integral f we get a corresponding symmetry $X_f = e^{-\alpha}\left(\frac{\partial f}{\partial p}\partial_u - \frac{\partial f}{\partial u}\partial_p\right)$ that satisfies the condition $i_{X_f}\Lambda + df = 0$.

For equations as in Theorem 4.2 we have the following relation between the associated L and L^*-equations:

$$L(\phi) = 0 \quad \Leftrightarrow \quad L^*(e^{\alpha}\phi) = 0 \tag{12}$$

Thus, knowing a full set of symmetries of the equation gives us a corresponding set of cosymmetries, and vice versa.

5. (Co-)symmetries, connections and curvature: examples.

In this section we first investigate equations of the type

$$y'' = y' + f(y)$$

where the function $f(y)$ is non-linear. In the article [DL91] the problem of finding p-linear generating functions of symmetries is treated in full. All equations equipped with a two dimensional Lie-algebra of such symmetries are classified. $F_p = 1$, so Theorem 4.2 implies that

$$L(\phi) = 0 \quad \Leftrightarrow \quad L^*(e^{-x}\phi) = 0$$

Theorem 5.1. [DL91] *Non-linear equations on the form* $y'' = y' + f(y)$ *that possess a two dimensional Lie-algebra of point-symmetries can be divided into classes* (1) *and* (2)*.*

(1) $y'' = y' + ae^{by} - \frac{2}{b}$ with $a, b \in \mathbb{R},\ a, b \neq 0$

(2) $y'' = y' + a(y+b)^c - \frac{(2c+2)}{(c+3)^2}(y+b)$ with $a, b, c \in \mathbb{R}$ and $a \neq 0,\ c \neq 0, 1, -3$

These equations are equipped with the following structure, as listed below.

Type (1)

(i) Solutions of $L(\phi)=0$:

$$\phi_1 = p \quad \text{and} \quad \phi_2 = e^{-x}(p-\frac{2}{b})$$

(ii) Corresponding symmetries:

$$X_1 = p\partial_u + (p + ae^{bu} - \frac{2}{b})\partial_p\ , \qquad X_2 = e^{-x}(p-\frac{2}{b})\partial_u + e^{-x}ae^{bu}\partial_p$$

(iii) Solutions of $L^*(\psi)=0$:

$$\psi_1 = e^{-x}p \quad \text{and} \quad \psi_2 = e^{-2x}(p-\frac{2}{b})$$

(iv) Corresponding cosymmetries:

$$\theta_1 = e^{-x}\,[\,(-p-ae^{bu}+\frac{2}{b})\omega_0 + p\omega_1\,]\,, \qquad \theta_2 = e^{-2x}\,[\,-ae^{bu}\omega_0 + (p-\frac{2}{b})\omega_1\,]$$

(v) First integral:

$$f = \theta_1(X_2) = e^{-2x}[\frac{2a}{b}e^{bu} - p^2 + \frac{4}{b}p - \frac{4}{b^2}]$$

(vi) Symplectic form:

$$\Lambda = e^{-x}\omega_1 \wedge \omega_0$$

(vii) Metric structure:

$$g \;=\; \theta_1^2 + \theta_2^2 \;= e^{-2x}\left[a^2(1+e^{-2x})e^{2bu} + (p-\frac{2}{b})^2 + 2(p-\frac{2}{b})ae^{bu}\right]\omega_0^2 +$$
$$2e^{-2x}\left[ae^{bu}(-p-e^{-2x}(p-\frac{2}{b})) - p^2 + \frac{2}{b}p\right]\omega_0{\cdot}\omega_1 + e^{-2x}\left[p^2 + e^{-2x}(p-\frac{2}{b})^2\right]\omega_1^2$$

(viii) Connection symbols:

$$\Gamma^1_{11} = \Gamma^2_{12} = 2\,\Gamma^2_{21} = -2\,\Gamma^1_{22} = 2$$

(ix) Curvature:

$$R = \frac{-1}{f^2}(\theta_1 \wedge \theta_2)^2$$

Type (2)

(i) Solutions of $L(\phi)=0$:

$$\phi_1 = p \quad \text{and} \quad \phi_2 = e^{\frac{(1-c)}{(c+3)}x}(p - \frac{2}{(c+3)}(u+b)\,)$$

(ii) Corresponding symmetries:

$$X_1 \;=\; p\,\partial_u + (p + a(u+b)^c - \frac{(2c+2)}{(c+3)^2}(u+b))\partial_p$$

$$X_2 = e^{\frac{(1-c)x}{(c+3)}}\,[\,(p - \frac{2}{c+3}(u+b))\partial_u + (\frac{2}{c+3}p + a(u+b)^c - \frac{4}{(c+3)^2}(u+b))\partial_p]$$

(iii) Solutions of $L^*(\psi) = 0$:

$$\psi_1 = e^{-x}p \quad \text{and} \quad \psi_2 = e^{\frac{(-2-2c)}{(c+3)}x}(p - \frac{2}{(c+3)}(u+b))$$

(iv) Corresponding cosymmetries:

$$\theta_1 = e^{-x}\,[\,(-p - a(u+b)^c + \frac{(2c+2)}{(c+3)^2}(u+b))\,\omega_0 \;+\; p\,\omega_1]$$

$$\theta_2 = e^{\frac{(-2-2c)x}{c+3}}\;[\,(-\frac{2}{c+3}p - a(u+b)^c + \frac{4}{(c+3)^2}(u+b))\,\omega_0 \;+\; (p - \frac{2(u+b)}{c+3})\,\omega_1\,]$$

(v) First integral:

$$f = \theta_1(X_2) =$$

$$e^{\frac{(-2-2c)x}{c+3}}\frac{(1+c)}{(c+3)}[-p^2 + \frac{4}{(c+3)}p(u+b) - \frac{4}{(c+3)^2}(u+b)^2 + \frac{2a}{(1+c)}(u+b)^{c+1}]$$

(vi) Symplectic form:

$$\Lambda = e^{-x}\omega_1 \wedge \omega_0$$

(vii) Metric structure:

$$g = \theta_1^2 + \theta_2^2 = \Bigg[\,[\,e^{\frac{(-2-2c)x}{c+3}}(\,a(u+b)^c + \frac{2}{c+3}p - \frac{4}{(c+3)^2}(u+b))\,]^2 \;+$$

$$+\,[e^{-x}(\,a(u+b)^c - \frac{2+2c}{(c+3)^2}(u+b) + p)\,]^2\,\Bigg]\;\omega_0^2 \;-$$

$$-2\Bigg[\,[\,e^{\frac{(-4-4c)x}{c+3}}(\,a(u+b)^c + \frac{2}{c+3}p - \frac{4}{(c+3)^2}(u+b))\,](\,p - \frac{2}{c+3}(u+b)\,) +$$

$$+e^{-2x}[\,a(u+b)^c - \frac{2+2c}{(c+3)^2}(u+b) + p\,]\Bigg]\;\omega_0\cdot\omega_1 \;+$$

$$+\;e^{-2x}\left[p^2 + e^{\frac{(2-2c)x}{c+3}}\,(p - \frac{2}{c+3}(u+b))^2\right]\;\omega_1^2$$

(viii) Connection symbols:

$$\Gamma^1_{11} = \Gamma^2_{12} = 2\frac{(1+c)}{(c+3)}\;,\;\;\Gamma^2_{21} = -\Gamma^1_{22} = 1$$

(ix) Curvature:

$$R = \frac{(1-c)}{(c+3)}\frac{1}{f^2}(\theta_1 \wedge \theta_2)^2$$

The last example is the *harmonic oscillator equation*

$$y'' + cy = 0 \tag{13}$$

where $c \in \mathbb{R}$. We have the following structures:

(i) Symmetries:

$$X_1 = p\partial_u - cu\partial_p \ , \quad X_2 = u\partial_u + p\partial_p$$

(ii) Cosymmetries:

$$\theta_1 = cu\omega_0 + p\omega_1 \ , \quad \theta_2 = -p\omega_0 + u\omega_1$$

(iii) First integral:

$$f = \theta_1(X_2) = cu^2 + p^2$$

(iv) Symplectic structure:

$$\Lambda = \omega_1 \wedge \omega_0$$

(v) Metric structures:

$$g_1 \ = \ \theta_1^2 + \theta_2^2 \ = (c^2u^2 + p^2)\omega_0^2 + +2up(c-1)\omega_0 \cdot \omega_1 + (u^2 + p^2)\omega_1^2$$

$$\text{and in addition} \quad g_2 \ = c\omega_0^2 + \omega_1^2 \quad \text{by Theorem 4.1}$$

(vi) Connection symbols and curvature:

$$\Gamma_{ij}^k = 0 \ \text{ for both } g_1 \text{ and } g_2$$

$$R_1 = R_2 = 0$$

References

[DL91] S. V. Duzhin and V. V. Lychagin. Symmetries of distributions and quadrature of ordinary differential equations. *Acta Applicandae Mathematicae*, 24:29–57, 1991.