ITEP-TH-42/01
hep-th/0110032

POLYNOMILAL POISSON ALGEBRAS WITH REGULAR STRUCTURE OF SYMPLECTIC LEAVES

ALEXANDRE ODESSKII AND VLADIMIR RUBTSOV

Abstract. We study polynomial Poisson algebras with some regularity conditions. Linear (Lie-Berezin-Kirillov) structures on dual spaces of semi-simple Lie algebras, quadratic Sklyanin elliptic algebras of [3],[4] as well as polynomial algebras recently described by Bondal-Dubrovin-Ugaglia ([7],[11]) belong to this class. We establish some simple determinantal relations between the brackets and Casimirs in this algebras. These relations imply in particular that for Sklyanin elliptic algebras the sum of Casimir degrees coincides with the dimension of the algebra. We are discussing some interesting examples of these algebras and in particular we show that some of them arise naturally in Hamiltonian integrable systems. Among these examples is a new class of two-body integrable systems admitting an elliptic dependence both on coordinates and momenta.

1. Introduction

We shall understand under polynomial Poisson structures those ones whose brackets are polynomial in terms of local coordinates on underlying Poisson manifold. The typical example of a such structure is the famous Sklyanin algebra.

Remind that a Poisson structure on a manifold M(for instance it does not play an important role is it smooth or algebraic) is given by a bivector antisymmetric tensor field $\pi \in \Lambda^2(TM)$ defining on the corresponded algebra of functions on M a structure of (infinite dimensional) Lie algebra by means of the Poisson brackets

$$\{f, g\} = \langle \pi, df \wedge dg \rangle.$$

The Jacobi identity for this brackets is equivalent to an analogue of (classical) Yang-Baxter equation namely to the "Poisson Master Equation": $[\pi, \pi] = 0$, where the brackets $[,] : \Lambda^p(TM) \times \Lambda^q(TM) \mapsto \Lambda^{p+q-1}(TM)$ are the only Lie super-algebra structure on $\Lambda^\cdot(TM)$ given by the so-called Schouten brackets. We refer for all this facts for example to the book [19].

There is an analogue of the Darboux theorem [21] which gives a local description of any Poisson manifold. Namely, there are coordinates $(q_1, \dots, q_l, p_1, \dots, p_l, x_1, \dots, x_k)$ near any point $m \in M$ such the bivector field π reads as

$$\pi = \sum_{i=1}^{l} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i>j}^{k} f_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

Date: july 2001.

such that $f(m) = 0$.

The case $k = 0$ is corresponded to a symplectic structure and the opposite case $l = 0$ is called usually totally degenerated Poisson structure. Let us remind the important notion of the Casimir functions of π (or briefly speaking *Casimirs*). A function $F \in Fun(M)$ is a Casimir of the Poisson structure π if $\{F, G\} = 0$ for all functions $G \in Fun(M)$. It is clear that if the rank of the structure is constant in this neighbourhood of m (m is called a regular point) then the Casimirs in the neighbourhood are the functions depending only on $x_1, \ldots, x_k$ and Poisson manifold admits a foliation by symplectic leaves, i.e. is a unification of submanifolds

$$x_1 = c_1, \ldots, x_k = c_k$$

and c_i are constants

such that π is non- degenerate on each of them. In general the dimension of the leaves is constant only the open dense subset of regular points and may vary outside.

We will discuss in this paper such polynomial Poisson structures which have some regularity properties for their symplectic leaves. We will postpone the exact definition of these properties but remark that there are lot of interesting classes of Poisson structures which acquire them. The most familiar is of course the famous Lie- Berezin - Kirillov Poisson structure on the dual space $\mathfrak{g}^*$ to a semi-simple Lie algebra $\mathfrak{g}$. This structure is linear in the coordinate functions $x_1, \ldots, x_n$ on $\mathfrak{g}^*$ corresponding to a base $X_1, \ldots, X_n$ of $\mathfrak{g}$ and is given by the structure constants of $\mathfrak{g}$. Reciprocally, any linear structure

$$\{x_i, x_j\} = C_{ij}^k x_k$$

arises from a Lie algebra. We should restrict to the semi-simple Lie algebras if we want to get a regular Poisson Lie-Berezin - Kirillov bracket.

The next wide class of interesting Poisson structures with regular symplectic leaves is provided by a subclass of *quadratic* Poisson algebras introducing by E. Sklyanin in [17] and in more general context describing in [3],[4] as a quasi-classical limit of associative algebras with quadratic relations which are flat deformations of function algebra on $\mathbb{C}^n$. These algebras are associated with elliptic curves and have the following curious property: they have polynomial Casimirs and for all of them the dimension of the algebra is equal to the sum of the degrees of Casimir polynomial ring generators. One of the aim of the paper is to give an explanation to this property which follows from a simple general determinantal relation between alternated products of the coordinate function Poisson brackets and the minors of the Jacobi matrix of Casimirs with respect to this coordinates (theorem 3.1 below).

Another motivation to study this type of Poisson polynomial algebras comes from the world of Hamiltonian integrable systems. We will show that such Poisson algebras arise naturally as Hamiltonian structures attached to different physically interesting models: from the trivial Arnold-Euler-Nahm top till the recently described ([5],[16]) double-elliptic $SU(2)$-model. The polynomial Poisson algebra which provide a natural Hamiltonian structure to this model is homogeneous and has the degree 3.

The paper is organized as follows. We start with the general remarks concerning the polynomial Poisson algebra. We remind some well (and maybe less) known definitions, discuss the holomorphic prolongation of the affine Poisson structures and remind the generalized Sklyanin elliptic algebras, following [3].

In the next section we define the algebras with a regular structure of symplectic leaves and prove the mentioned theorem (Theorem 4.1) which is just a straightforward calculation. The less evident example of the theorem 4.1 relation is given by the generalized Sklyanin algebra q_5.

The chapters 3 and 4 are devoted to different examples. We discuss in ch.3 the examples of amazing *non- polynomial* changes of variables which preserve the polynomial character of the Poisson structures associated with elliptic curves. This changes are going back to "Mirror transformation" of Calabi-Yau manifolds. The form and sens of this change in "quantum" setting of [3] is an interesting open question.

In the chapter 4 we discuss the examples of polynomial Poisson structures associated with K3 surfaces (both smooth or singular). The interesting feature of this structures is that they are of degree 3 in coordinate functions and it is still unclear how to "quantiz them or what are the analogues of generalized Sklyanin algebras to this case? This analogues (if they are exist) should be called "Mukai algebras". At the moment the only known examples are the "quantum" projective plane of [14] or some "non-commutatives" K3 surfaces ([15], which are the result of deformation of toroidal orbifolds and don't provide an interesting example of Poisson algebra quantization).

Finally, in the last section we give the examples of some natural integrable systems which are Hamiltonian with respect to the regular Poisson structures under discussion. In fact, one of the examples, the double elliptic or "DELL" system which is cubic in the natural Hamilton setting, gave the initial motivation to study this series of structures. The detailed study of this interesting example associated with 6d SUSY gauge theory with matter hypermultiplets will be done in our joint paper with H.Braden and A. Gorsky. Our paper may be considered as an account to the polynomial Poisson structures which will be used for the DELL. This is a particular explanation of the illustrative character of the paper. We often had restricted on the level of the evident examples to do the paper accessible to a wide circle of readers (in the first for physicists).

2. Poisson algebras on polynomial rings

2.1. **General remarks.** We will discuss some examples of polynomial Poisson algebras which are defined on an affine part of some algebraic varieties. Typically, this varieties embedded as (complete) intersections in (weighted) projective spaces. We should remark that the considerations of intersection varieties in weighted projective spaces are equivalent sometimes to intersections in the products of usual $\mathbb{C}P^n$ - the fact which is well -known to string theory physicists who had studied the Calabi -Yau complete intersections as a compactification scenario to superstring vacua and the mirror symmetry as a map between the moduli spaces of weighted Calabi - Yau manifolds ([22],[23]).

Let us consider $n-2$ polynomials Q_i in $\mathbb{C}^n$ with coordinates $x_i, i=1,...,n$. For any polynomial $\lambda \in \mathbb{C}[x_1,...,x_n]$ we can define a bilinear differential operation

$$\{,\}:\mathbb{C}[x_1,...,x_n]\otimes\mathbb{C}[x_1,...,x_n]\mapsto\mathbb{C}[x_1,...,x_n]$$

by the formula

$$\{f,g\}=\lambda\frac{df\wedge dg\wedge dQ_1\wedge...\wedge dQ_{n-2}}{dx_l\wedge dx_2\wedge...\wedge dx_n},\ f,g\in\mathbb{C}[x_1,...,x_n]. \tag{1}$$

This operation gives a Poisson algebra structure on $\mathbb{C}[x_1,...,x_n]$ as a partial case of more general $n-m$-ary Nambu operation given by an antysymmetric $n-m$-polyvector field η :

$$\langle\eta, df_1\wedge...\wedge df_{n-m}\rangle=\big\{f_1,...,f_{n-m}\big\},$$

depending on m polynomial "Casimirs" $Q_1,\ldots,Q_m$ and λ
such that

$$\{f_1,\ldots,f_{n-m}\}=\lambda\frac{df_1\wedge\ldots\wedge df_{n-m}\wedge dQ_1\wedge...\wedge dQ_m}{dx_l\wedge dx_2\wedge...\wedge dx_n},\ f_i\in\mathbb{C}[x_1,...,x_n]. \tag{2}$$

and

$$\big\{,...,\big\}:\mathbb{C}[x_1,...,x_n]^{\otimes n-m}\mapsto\mathbb{C}[x_1,...,x_n]$$

such that the three properties are valid:
1)antisymmetricity:

$$\big\{f_1,...,f_{n-m}\big\}=(-1)^{\sigma}\big\{f_{\sigma(1)},...,f_{\sigma(n-m)}\big\},\sigma\in Symm_{n-m};$$

2) coordinate-wise "Leibnitz rule" for any $h\in\mathbb{C}[x_1,...,x_n]$:

$$\big\{f_1h,...,f_{n-m}\big\}=f_1\big\{h,...,f_{n-m}\big\}+h\big\{f_1,...,f_{n-m}\big\};$$

3) The "Fundamental Identity" (which replaces the Jacobi):

$$\Big\{\big\{f_1,...,f_{n-m}\big\},f_{n-m+1},\cdots,f_{2(n-m)-1}\Big\}+$$
$$\Big\{f_{n-m},\big\{f_1,...,(f_{n-m})^{\vee}f_{n-m+1}\big\},f_{n-m+2},\cdots,f_{2(n-m)-1}\Big\}+$$
$$+\Big\{f_{n-m},...,f_{2(n-m)-2},\big\{f_1,...,f_{n-m-1},f_{2(n-m)-1}\big\}\Big\}=$$

$$\{f_1, ..., f_{n-m-1}, \{f_{n-m}, ..., f_{2(n-m)-1}\}\}$$

for any $f_1, ..., f_{2(n-m)-1} \in \mathbb{C}[x_1, ..., x_n]$.

This structure is a natural generalization of the Poisson structure (which corresponds to $n - m = 2$) was introduced by Y.Nambu in 1973([25]) and was recently extensively studied by L.Tachtajan [26].

The most natural example of the Nambu - Poisson structure is so-called "canonical" Nambu-Poisson structure on $\mathbb{C}^m$ with coordinates $x_1, ..., x_m$:

$$\{f_1, ..., f_m\} = Jac(f_1, ..., f_m) = \frac{\partial(f_1, ..., f_m)}{\partial(x_1, ..., x_m)}.$$

We should remark also that the formula (1) for the Poisson brackets takes place in more general setting when the polynomials Q_i are replaced, say, by rational functions but the resulting brackets are still polynomials. More generally, this formula is valid for the power series rings.

The polynomials $Q_i, i = 1, ..., n-2$ are Casimir functions for the bracket (1) and any Poisson structure in $\mathbb{C}^n$ with $n-2$ generic Casimirs Q_i are written in this form.

All this facts are not new and very well known probably since Nambu and Sklyanin papers. They had reappared recently ([2],[12]) in the different frameworks because of general interest to classification and quantization problems for Poisson structures.

The case $n = 4$ in (1) corresponds to the classical (generalized) Sklyanin quadratic Poisson algebra. The very Sklyanin algebra is associated with the following two quadrics in $\mathbb{C}^4$:

$$Q_1 = x_1^2 + x_2^2 + x_3^2, \tag{3}$$

$$Q_2 = x_4^2 + J_1x_1^2 + J_2x_2^2 + J_3x_3^2. \tag{4}$$

The Poisson brackets (1) with $\lambda = 1$ between the affine coordinates looks as follows

$$\{x_i, x_j\} = (-1)^{i+j} det\left(\frac{\partial Q_k}{\partial x_l}\right), l \neq i, j, \, i > j. \tag{5}$$

The expression (1) has an advantage before (5) because it is compatible with the more general situations when the intersected varieties are embedded in the weighted projective spaces or in the product of the projective spaces. We will consider an example of such situation below.

The natural question arises: to extend the brackets (1) or (5) from $\mathbb{C}^n$ to the projective space $\mathbb{C}P^n$.

We can state the following

Proposition 2.1. *Let $X_1, ..., X_n$ are coordinates on $\mathbb{C}^n$ considering as an affine part of the corresponding projective space $\mathbb{C}P^n$ with the homogeneous coordinates $(x_0 : x_1 : \dots : x_n)$, $X_i = \frac{x_i}{x_0}$ then if*

$\{X_i, X_j\}$ extends to a holomorphic Poisson structure on $\mathbb{C}P^n$ then the maximal degree of the structure (= the length of monomes in X_i) is 3 and

$$X_k\{X_i, X_j\}_3 + X_i\{X_j, X_k\}_3 + X_j\{X_k, X_i\}_3 = 0, i \neq j \neq k, \tag{6}$$

i.e. $\{X_i, X_j\}_3 = X_iY_j - X_jY_i$, with $degY_i = 2$

Indeed, under the change of the coordinates we have

$$\{1/X_i, X_j/X_i\} = -1/X_i{}^3\{X_i, X_j\}$$

hence $deg\{X_i, X_j\} \leq 3$.

Moreover, we can conclude that

$$\{X_i, X_j\} = \{X_i, X_j\}_0 + \{X_i, X_j\}_1 + \{X_i, X_j\}_2 + \{X_i, X_j\}_3,$$

where $deg\{X_i, X_j\}_k = k, k = 0, 1, 2, 3$.

The 6 is a corollary of the following identities:

$$\{X_j/X_i, X_k/X_i\} = 1/X_i{}^2\{X_j, X_k\} - X_k/X_i{}^3\{X_j, X_i\} - X_j/X_i{}^3\{X_i, X_k\},$$

The statement is proved.

In general, the homogeneous Poisson algebras which are described by (1) or (5) have no projective extensions because they don't satisfy for $n \geq 4$ to the conditions of the proposition.

The Poisson algebras of the type (1), (5) have the following property: they have only symplectic leaves of "small" dimensions ("small" means in fact 0 and 2).

The following description of "small" dimensional symplectic leaf structures is a fairly direct corollary of the leaf definition

Proposition 2.2. *Let $\{x_i, x_j\} = \langle \pi, dx_i \wedge dx_j \rangle = p_{ij}$ be an affine Poisson structure. Then π has only "small dimenisional" symplectic leaves iff p_{ij} is written in "Plucker form" or $p_{ij} = \alpha_i\beta_j - \alpha_j\beta_i$ where α_i, β_j are some functions (not necessarily polynomials).*

2.2. **Poisson algebras associated to elliptic curves.** Another wide class of the polynomial Poisson algebras arises as a quasi-classical limit $q_{n,k}(\mathcal{E})$ of the associative quadratic algebras $Q_{n,k}(\mathcal{E}, \eta)$ which were introduced in a cycle of papers [3, 4]. Here $\mathcal{E}$ is an elliptic curve and n, k are integer numbers without common divisors ,such that $1 \leq k < n$ while η is a complex number and $Q_{n,k}(\mathcal{E}, 0) = \mathbb{C}[x_1, ..., x_n]$.

Let $\mathcal{E} = \mathbb{C}/\Gamma$ be an elliptic curve defined by a lattice $\Gamma = \mathbb{Z} \oplus \tau\mathbb{Z}, \tau \in \mathbb{C}, \Im\tau > 0$. The algebra $Q_{n,k}(\mathcal{E}, \eta)$ has generators $x_i, i \in \mathbb{Z}/n\mathbb{Z}$ subjected to the relations

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)} x_{j-r}x_{i+r} = 0$$

and have the following properties:

1) $Q_{n,k}(\mathcal{E},\eta) = \mathbb{C} \oplus Q_1 \oplus Q_2 \oplus ...$ such that $Q_\alpha * Q_\beta = Q_{\alpha+\beta}$, here $*$ denotes the algebra multiplication. In other words, the algebras $Q_{n,k}(\mathcal{E},\eta)$ are $\mathbb{Z}$ - graded;

2) The Hilbert function of $Q_{n,k}(\mathcal{E},\eta)$ is $\sum_{\alpha\geq 0} \dim Q_\alpha t^\alpha = \frac{1}{(1-t)^n}$.

We see that the algebra $Q_{n,k}(\mathcal{E},\eta)$ for fixed $\mathcal{E}$ is a flat deformation of the polynomial ring $\mathbb{C}[x_1,...,x_n]$.

Let $q_{n,k}(\mathcal{E})$ be the correspondent Poisson algebra. It is shown in [3] that the algebra $q_{n,k}(\mathcal{E})$ has $l = \gcd(n, k+1)$ Casimirs. Let us denote them by $P_\alpha, \alpha \in \mathbb{Z}/l\mathbb{Z}$. Their degrees $\deg P_\alpha$ are equal to n/l.

We should stress, that the algebras $q_{n,k}(\mathcal{E})$ for $n > 4$ have symplectic leaves of dimension > 2 and hence these algebras are not given by the formulas (1), (5) for $n > 4$. The examples of the algebras $q_{3,1}(\mathcal{E}), q_{4,1}(\mathcal{E})$ will be discussed below.

3. Algebras with regular structures of symplectic leaves

Let us remind that the minimal codimension of symplectic leaves of a Poisson algebra is called a rank of this algebra. Now we will describe the class of Poisson algebras on $\mathbb{C}^n$ satisfying to the following properties of regularity:

Definition 3.1. *A Poisson polynomial algebra A on $\mathbb{C}^n$ of rank l is called an algebra with a regular structure of symplectic leaves if:*

1) The center $Z(A)$ is a polynomial ring $\mathbb{C}[Q_1, \cdots, Q_l]$ of Casimirs Q_i;

2) The subvariety $L_{\lambda_1,\cdots,\lambda_l} := \bigcap_i \{Q_i = \lambda_i\}$ is a complete intersection (or the tangent spaces $T_p(\{Q_i = \lambda_i\})$ in generic point $p \in L_{\lambda_1,\cdots,\lambda_l}$ are intersected transversally);

3) Let M^* be a unification $\bigcup_i \{F_i\}$ of symplectic leaves of dimension $\dim F_i < n-l$ then $\dim M^* \leq n-2$;

4) Let L^* be a unification of singularities of $L_{\lambda_1,\cdots,\lambda_l}$ by $\lambda_1, \cdots, \lambda_l$, then $\dim L^* \leq n-2$.

We want to remark that the elliptic algebras $q_{n,k}(\mathcal{E})$ are satisfied to the definition as follows from the description of their symplectic leaves in [4]. For all of them as well as for the others algebras with the property 3.1 the following theorem holds

Theorem 3.1.

$$\underbrace{\pi \wedge \pi \wedge \cdots \wedge \pi}_{(n\text{-}l)/2} = \lambda (dQ_1 \wedge \cdots \wedge dQ_l)^*,\ \lambda \in \mathbb{C}^*,$$

where l is the dimension of the Poisson center (the number of Casimirs) and $Q_1, \cdots, Q_l$ are the Casimirs and $*$ means duality between l-forms and l-polyvectors established by the standard choice of a volume form $dx_1 \wedge \cdots \wedge dx_n$.

In coordinates the expression (3.1) is re-written as

$$\lambda \det \begin{pmatrix} \frac{\partial Q_1}{\partial x_{i_{n-l+1}}} & \cdots & \frac{\partial Q_1}{\partial x_{i_n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial Q_l}{\partial x_{i_{n-l+1}}} & \cdots & \frac{\partial Q_l}{\partial x_{i_n}} \end{pmatrix} = Alt_{(i_1,\cdots,i_n)} \left(\{x_{i_1}, x_{i_2}\} \cdots \{x_{i_{n-l-1}}, x_{i_{n-l}}\} \right),$$

where $(i_1, \cdots i_n)$ is an even permutation of $(1, \cdots, n), \lambda \in \mathbb{C}^*$−non-zero constant.

Proof of 3.1 For any $n-l$ polynomial functions $f_1, \cdots, f_{n-l}$ the formula 3.1 may be re-written in the following way

$$w^{(n-l)/2}(f_1, \cdots, f_{n-l}) = \lambda \frac{df_1 \wedge \cdots \wedge df_{n-l} \wedge dQ_1 \wedge \cdots \wedge dQ_l}{dx_1 \wedge \cdots \wedge dx_n}, \;\; \lambda \in \mathbb{C}^*. \tag{7}$$

Let Ω_1(resp. Ω_2) be left(resp. right) part of (7).

It is clear that $\imath_{Q_\alpha} \Omega_i = 0$ for any $\alpha, i = 1, 2$, and hence writing the tensors Ω_i in coordinates

$$\Omega_i = \sum_{(\alpha_1, \cdots \alpha_{n-l})} w^i_{\alpha_1, \cdots, \alpha_l} \partial_{\alpha_1} \wedge \cdots, \wedge \partial_{\alpha_{n-l}}$$

we obtain a system of linear equations to the coefficients $w^i_{\alpha_1, \cdots, \alpha_l}$:

$$\imath_{Q_\alpha} \Omega_i = \sum_{\alpha_1} w^i_{\alpha_1 \cdots \alpha_{n-l}} Q'_{\alpha x_{\alpha_1}} \partial_{\alpha_2} \wedge \cdots \wedge \partial_{\alpha_{n-l}} = 0.$$

We obtain that both sides of the (7) are proportional one to another (where λ is a non-zero function) from this system. This function λ is rational and hence there are two polynomials without common divisors p_1, p_2 such that $\lambda = \frac{p_1}{p_2}$. But the first polynomial p_1 has the set of zeroes lying in M^* hence $codim M^* = 1$. Taking in to the account 3) of 3.1 we obtain $p_1 = const$.

The same arguments show that the zeroes of p_2 contain in (or coincide with) L^* and $p_2 = const_1$. Hence the result.

Remark 1 The statement of 3.1 is still valid to *any* Poisson algebra of rank l with l independent central elements Q_α if we admit as the factor λ an arbitrary rational function.

Remark 2 Under the conditions of the theorem if A is a quadratic Poisson algebra our formula gives the following relation between the degrees of Casimirs and the dimension of A:

$$\sum_\alpha \deg Q_\alpha = dim A = n$$

Moreover , for all known elliptic algebras which are graded deformations of $\mathbb{Z}^n$-graded polynomial ring with finite number of generators, the sum of generator degrees is equal to the sum of degrees of Casimirs, For example, $n = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \deg x_i = \sum_{j \in \mathbb{Z}/l\mathbb{Z}} \deg Q_j$.

Remark 3 In the case when the condition 3) in 3.1 is failed the following definition is useful:

Definition 3.2. *An element $Q \in A$ is called quasi-central or quasi-Casimir if the bracket $\{Q, f\}$ is divided by Q for any $f \in A$.*

It is clear that if Q_1 and Q_2 are quasi-central elements of A then their product Q_1Q_2 is also quasi-central. Let $codim M^* = 1$ and $\{M_i\}$ are irreducible components of M^* of dimension $n-1$. If M_i is given by the equations $Q_i = 0$, then it is clear that $\{Q_i\}$ are quasi-Casimirs. Conversely, let undecomposable Q be a quasi-central but non-central element, then the hyper-surface $Q = 0$ contains in M^*.

The last curious and possibly useful observation is that for any quasi-central Q the bracket $\{log Q, .\}$ is an (exterior) derivation of A.

To illustrate the theorem we show that it is true for the algebra $q_5(\mathcal{E})$ from [3].

Example We have the polynomial ring with 5 generators $x_i, i \in \mathbb{Z}/5\mathbb{Z}$ enabled with the following Poisson bracket:

$$\{x_{i+1}, x_{i+4}\} = -1/5(k^2 + 3/k^3)x_{i+1}x_{i+4} + 2x_{i+2}x_{i+3} + kx_i^2;$$

$$\{x_{i+2}, x_{i+3}\} = 1/5(3k^2 - 1/k^3)x_{i+2}x_{i+3} + 2/kx_{i+1}x_{i+4} - 1/k^2x_i^2.$$

Here $i \in \mathbb{Z}/5\mathbb{Z}$ and $k \in \mathbb{C}$ is a parameter of the curve $\mathcal{E}_\tau = \mathbb{C}/\Gamma$, i.e. some function of τ.

The center $Z(q_5(\mathcal{E}))$ is generated by the polynomial

$$\begin{gathered}
P = -1/k\left(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5\right) + \\
(1/k^5 - 3)\left(x_0^3x_1x_4 + x_1^3x_0x_2 + x_2^3x_1x_3 + x_3^3x_2x_4 + x_2^3x_0x_3\right) + \\
+\left(k^3 + 3/k^3\right)(x_0^3x_2x_3 + x_1^3x_3x_4 + x_2^3x_0x_4 + x_3^3x_1x_0 + x_4^3x_1x_2\big) - \\
-(2k + 1/k^4)\left(x_0x_1^2x_4^2 + x_1x_2^2x_0^2 + x_2x_0^2x_4^2 + x_3x_1^2x_0^2 + x_4x_1^2x_2^2\right) + \\
+(k^2 - 2/k^3)\left(x_0x_2^2x_3^2 + x_1x_3^2x_4^2 + x_2x_0^2x_4^2 + x_3x_1^2x_0^2 + x_4x_1^2x_2^2\right) + \\
+(k^4 + 16/k - 1/k^6)x_0x_1x_2x_3x_4.
\end{gathered}$$

It is easy to check that for any $i \in \mathbb{Z}/5\mathbb{Z}$

$$\{x_{i+1}, x_{i+2}\}\{x_{i+3}, x_{i+4}\}+\{x_{i+3}, x_{i+1}\}\{x_{i+2}, x_{i+4}\}+\{x_{i+2}, x_{i+3}\}\{x_{i+1}, x_{i+4}\} = 1/5\frac{\partial P}{\partial x_i}.$$

4. Examples of regular leave algebras, $n = 3$

4.1. **Elliptic algebras.** Let

$$P(x_1, x_2, x_3) = 1/3\left(x_1^3 + x_2^3 + x_3^3\right) + kx_1x_2x_3, \tag{8}$$

then

$$\begin{aligned}
\{x_1, x_2\} &= kx_1x_2 + x_3^2 \\
\{x_2, x_3\} &= kx_2x_3 + x_1^2 \\
\{x_3, x_1\} &= kx_3x_1 + x_2^2.
\end{aligned}$$

The quantum counterpart of this Poisson structure is the algebra $\mathcal{Q}_3(\mathcal{E}, \eta)$, where $\mathcal{E} \subset \mathbb{C}P^2$ is an elliptic curve given by $P(x_1, x_2, x_3) = 0$.

4.2. **"Mirror transformation".** The interesting feature of this algebra is that their polynomial character is preserved even after the following changes of variables:

a)let

$$y_1 = x_1, y_2 = x_2 x_3^{-1/2}, y_3 = x_3^{3/2}. \tag{9}$$

The polynomial P in the coordinates (y_1, y_2, y_3) has the form

$$P^{\vee}(y_1, y_2, y_3) = 1/3 \left(y_1^3 + y_2^3 y_3 + y_3^3\right) + k y_1 y_2 y_3 \tag{10}$$

and the Poisson bracket is also polynomial (which is not evident at all!) and has the same form:

$\{y_i, y_j\} = \frac{\partial P^{\vee}}{\partial y_k}$, where $(i, j, k) = (1, 2, 3)$.

If we put $\deg y_1 = 2, \deg y_2 = 1, \deg y_3 = 3$ then the polynomial $P^{\vee}$ is also homogeneous in (y_1, y_2, y_3) and defines an elliptic curve $P^{\vee} = 0$ in the weighted projective space $\mathbb{W}P_{2,1,3}$.

b)now let $z_1 = x_1^{-3/4} x_2^{3/2}, z_2 = x_1^{1/4} x_2^{-1/2} x_3, z_3 = x_1^{3/2}$.

The polynomial P in the coordinates (z_1, z_2, z_3) has the form $P(z_1, z_2, z_3) = 1/3\left(z_3^2 + z_1^2 z_3 + z_1 z_2^3\right) + k z_1 z_2 z_3$ and the Poisson bracket is also polynomial (which is not evident at all!) and has the same form: $\{z_i, z_j\} = \frac{\partial P}{\partial z_k}$, where $(i, j, k) = (1, 2, 3)$.

If we put $\deg z_1 = 1, \deg z_2 = 1, \deg z_3 = 2$ then the polynomial P is also homogeneous in (z_1, z_2, z_3) and defines an elliptic curve $P = 0$ in the weighted projective space $\mathbb{W}P_{1,1,2}$.

The origins of the strange non-polynomial change of variables (9) lie in the construction of "mirror" dual Calabi - Yau manifolds ([22]) and the torus (8) has (10) as a "mirror dual". Of course, the mirror map is trivial for 1-dimensional Calabi - Yau manifolds. Curiously, mapping (9) being a Poisson map if we complete the polynomial ring in a proper way and allow the non - polynomial functions gives rise to a new "relation" on quantum level: the quantum elliptic algebra $Q_3(\mathcal{E}^{\vee})$ corresponded to (10) has complex structure $(\tau + 1)/3$ when (8) has τ. Hence, these two algebras are different. The "quantum" analogue of the mapping (9) is still obscure and needs further studies.

4.3. **Markov polynomials: "rational degeneration".** We consider as a Casimir now an example of non-homogeneous polynomial ("Markov polynomial of degree 3") $P(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 3x_1 x_2 x_3$. The corresponding Poisson algebra $\{x_i, x_j\} = \frac{\partial P}{\partial x_k}, (i, j, k) = (1, 2, 3)$ appeared recently as a Hamiltonian structure on the space of Stokes matrices in the theory of isomonodromic deformations (see [10],[11],[8]) as well as a quasi-classical limit in quantization of Teichmuller space by [6]. We should remark that the $n > 3$ generalization of this example associated with "higher" degree invariants studied in full generality by A. Bondal ([7]) have a regular structure of symplectic leaves.

For example, the next Bondal-Dubrovin-Ugaglia ([7], [11]) affine Poisson algebra in $\mathbb{C}^6$ with two Casimirs of degree 4 and 2 which has a 4-dimensional symplectic leaf given by their intersection satisfies to 3.1:

Let $\mathbb{C}^6$ is identified with the six-dimensional space of Stokes matrices of the form

$$\begin{pmatrix} 1 & p & q & r \\ 0 & 1 & x & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with Poisson algebra linear-quadratic relations (see [7]).

If we choose two Casimirs

$$P_1 = p^2 + q^2 + r^2 + x^2 + y^2 + z^2 - pqx - pry - qrz - xyz + prxz \qquad (11)$$
$$P_2 = pz + xr - qy \qquad (12)$$

then it is straightforward to check for example the following relation

$$\{x, y\}\{p, z\} + \{y, z\}\{p, x\} + \{z, x\}\{p, y\} = \det \begin{pmatrix} \frac{\partial P_1}{\partial q} & \frac{\partial P_1}{\partial r} \\ \frac{\partial P_2}{\partial q} & \frac{\partial P_2}{\partial r} \end{pmatrix} =$$

$$\det \begin{pmatrix} -y & x \\ 2q - px - rz & 2r - py - qz + pxz \end{pmatrix}.$$

4.4. **Polynomial extension of Askey - Wilson algebra.** A classical (Poisson) Askey-Wilson ([31]) algebra is described in our terms by the following Casimir polynomial

$$P(x, y, z) = z^2 - F(x, y), F(x, y) = ax^2y^2 + g(x, y), \qquad (13)$$

where $g(x, y) = a_1x^2y + a_2xy^2 + a_3x^2 + a_4y^2 + a_5xy + a_6x + a_7y$.

The terminology is explained by the fact that the "quantum" analogue of the algebra admits a natural representations by Askey-Wilson polynomials.

It is easy to obtain a polynomial extension of the Askey-Wilson algebra with the same rule (13) but with $F(x, y)$ of arbitrary degree with the only constraint that $P(x, y, z)$ is a Casimir.

For example, if we admit in $F(x, y)$ the maximal degree terms x^4, y^4 we have the extension of Askey-Wilson algebra which is equivalent to the standard Sklyanin algebra (3) (4) (see [31]).

We assume the locus $P(x, y, z) = 0$ for (13) to be a curve in the weighted projective space $\mathbb{W}P_{1,1,2}$ with the variables of $deg\ x = deg\ y = 1, deg\ z = 2$. Then the relation of the **Remark 2** in the theorem 3.1 is still holds.

5. K3 - surfaces

5.1. **Cone of a canonical curve.** Let $\mathcal{C}$ be a curve of genus $g > 1$ which is not a hyper-elliptic. Let $K_{\mathcal{C}}$ be the canonical bundle of $\mathcal{C}$. We consider the canonical embedding $\xi : \mathcal{C} \to \mathbb{C}P^{g-1}$ and the cone $\mathbb{K}_{\mathcal{C}} \subset \mathbb{C}^g$ of $\xi(\mathcal{C})$. Let $\tilde{\mathbb{K}}_{\mathcal{C}}$ be a covering of $\mathbb{K}_{\mathcal{C}}$ which correspond to the covering $\mathcal{H} \mapsto \mathcal{E}$ where $\mathcal{H}$ is the upper half-plane with the coordinate τ and

$$\tilde{\mathbb{K}}_{\mathcal{C}} = \mathcal{H} \times \mathbb{C}^*.$$

We will denote by $d\tau$ the corresponding coordinate on $\mathbb{C}^*$. It is well-known that $\mathbb{K}_{\mathcal{C}} = \tilde{\mathbb{K}}_{\mathcal{C}}/\Gamma$ where $\Gamma \subset \mathfrak{sl}_2(\mathbb{R})$ is a discrete subgroup which acts naturally on the coordinates $(\tau, d\tau)$. Let us describe a Poisson structure on $\tilde{\mathbb{K}}_{\mathcal{C}}$ by the relation $\{\tau, d\tau\} = (d\tau)^2$. It is easy to check that this structure is $\mathfrak{sl}_2(\mathbb{R})$-invariant and symplectic. Hence the cone $\mathbb{K}_{\mathcal{C}}$ is a homogeneous symplectic. For $g = 3$ the cone $\mathbb{K}_{\mathcal{C}} \subset \mathbb{C}^3$ is given by the equation $p(x_1, x_2, x_3) = 0$, where $p(x_1, x_2, x_3)$ is a homogeneous polynomial of the degree 4. We supply $\mathbb{C}^3$ with the polynomial Poisson structure as above: $\{x_i, x_j\} = \frac{\partial p}{\partial x_k}, (i, j, k) = (1, 2, 3)$.

Proposition 5.1. $\mathbb{K}_{\mathcal{C}} \subset \mathbb{C}^3$ *is a symplectic leaf of this polynomial structure and the restriction of the structure to* $\mathbb{K}_{\mathcal{C}}$ *has the form*

$$\{\tau, d\tau\} = \lambda(d\tau)^2, \ \lambda \in \mathbb{C}^*.$$

We have analogues of this description for the cases $g = 4, 5$.

5.2. **General discussion.** Let us compactify the precedent situation in considering the one -point compactification of the cone or the completion $\mathbb{P}(K_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}})$. We obtain a compact two-dimensional complex manifold with trivial anti-canonical bundle or a holomorphic embedding of the curve C in a $K3$-surface. We will describe the restriction of the Poisson structures on the $K3$-surface as a symplectic leave. It is an interesting example of non-quadratic affine brackets.

All $K3$ have symplectic holomorphic form due to S. Mukai ([1]) and all examples of the projective models of $K3$ has realization of the Mukai symplectic brackets as the restrictions of polynomial Poisson structures on $\mathbb{C}^n, n = 3, 4, 5$ arising on the complete intersections of Casimirs which realize these models. These brackets have the degree 3. It is worst to remark that in the accordance with the proposition 2.1 the brackets can not be extended to holomorphic Poisson structures on $\mathbb{C}P^n, n = 3, 4, 5$. We will give a direct verification that a non-trivial obstruction exists in this case.

It is well-known that the K3-surfaces are the only two-dimensional compact complex manifolds with zero 1-st Betti number and a holomorphic symplectic structure. We will give a simple arguments related the polynomial Poisson brackets with Mukai description of the holomorphic symplectic structure on algebraic K3 surfaces.

The construction of Mukai ([1]) is based on the following simple facts: all algebraic K3 have a projective embedding, for any $\mathbb{C}P^n$ with a system of homogeneous coordinates $(X_0 : X_1 : ... : X_n)$ and the standard n-form

$$\Omega = \sum_{i=0}^{n} (-1)^i X_i dX_0 \wedge ... \wedge d\hat{X}_i \wedge ... \wedge dX_n,$$

which is used to define the Mukai holomorphic two-form by the residue form $Res_{\mathcal{S}}\left(\frac{\Omega}{P}\right)$ along the K3 -surface $\mathcal{S}$ where the projective embedding of $\mathcal{S}$ given by zero loci P, which is either quartic $(n = 3)$ or the transversal intersection of a quadric and a cubic $(n = 4)$ or the transversal intersection of three quadrics for $n = 5$.

We can check straightforwardly that the polynomial Poisson structure on $\mathbb{C}^n$ for $n = 3, 4, 5$ given above being restricted to the surface $\mathcal{S}$ coincides with the Mukai structure being written in the affine coordinates.

5.3. **Example 1: Fermat quartic.** Let $P_4(X_0, X_1, X_2, X_3) = 0, degP_4 = 4$ be a quartic K3 surface in $\mathbb{C}P^3$. Taking an open domain $U_0 = \{X_0 \neq 0\}$ we have that the form $\alpha = \frac{\Omega}{X_0^4}$ which is written in the affine part as $\alpha_0 = dx_1 \wedge dx_2 \wedge dx_3, x_i = \frac{X_i}{X_0}$ is a holomorphic 3-form on $\mathcal{S} \setminus U_0$ and has simple poles along the intersection $\mathcal{S} \bigcap U_0$. Hence in the affine part $\frac{\Omega}{P} = \frac{\alpha_0}{P(1,x_1,x_2,x_3)}$ and the residue $\omega = Res_{\mathcal{S}} \frac{\Omega}{P_4}$ is given (for example) by

$$\omega = \frac{dx_1 \wedge dx_2}{\frac{\partial P_4}{\partial x_3}} \tag{14}$$

and as we have seen in the affine chart U_0 the bracket is given by $\{x_1, x_2\} = -\frac{\partial P_4}{\partial x_3}$.

Proposition 5.2. *The polynomial structures on* $\mathbb{C}^n, n = 3, 4, 5$ *given by the polynomials* P_4 *for* $n = 3$*, and by complete intersections of a quadric and cubic* P_2, P_3 *for* $n = 4$ *or by an intersection of three quadrics* P_2, Q_2, R_2 *for* $n = 5$ *have no holomorphic extensions to the projective spaces* $\mathbb{C}P^n, n = 3, 4, 5$

To prove it, for $n = 3$ by the proposition 2.1 we can verify that

$$X_3\{X_1, X_2\}_3 + X_1\{X_2, X_3\}_3 + X_2\{X_3, X_1\}_3 = -(X_3\frac{\partial P_4}{\partial x_3} + X_1\frac{\partial P_4}{\partial x_1} + X_2\frac{\partial P_4}{\partial x_2}) = C \neq 0,$$

Strictly speaking we had checked only that the structure on $K3$ is not extended by the *given* polynomial formulas. In fact it is easy to show that the brackets has no *any* extension from the surface to the whole $\mathbb{C}P^3$. Indeed, if such extension exists it should be polynomial and coincide on $K3$ with our brackets and hence their difference equals to 0 on the surface and so it should have in affine coordinates the degree at least 4, because it should have the polynomial P_4 as a divisor.

5.4. **Example2 : "singular" $K3$ in a product of projective spaces.** Another interesting example of the formulas to a Poisson structure on $K3$ is based on the so called "splitting principle " which goes back to the construction of Calabi-Yau varieties and their mirror dual in [22]. Roughly speaking, the "splitting principle " permits us to consider the $K3$ surface (possibly singular) embedded as a hyper-surface in a weighted projective space like a part of two-dimensional variety in a product of usual (or more generally also weighted) projective spaces. We are able to apply the coordinate formulas in this situation and we are sure that more rigorous and conceptual approach is associated with the proper generalization of the residue theory to toric varieties (see [24]).

Let us consider, following to [9] a hyper-surface

$$P = y_1(y_1^3 + y_3^6 + y_5^3) - y_2(y_2^3 + y_4^6 - y_5^3) = 0$$

in $\mathbb{W}P_{1,1,2,2,2}[8]$ which becomes a singular $K3$ in $\mathbb{C}P^3$

$$P = x_1(x_1^3 + x_3^3 + x_4^3) - x_2(x_2^3 + x_3^3 - x_4^3) = 0 \tag{15}$$

after the re-definitions $y_4 = y_3$ and $x_1 = y_1, x_2 = y_2, x_3 = y_3^2, x_4 = y_5$. This surface in the patch $x_1 \neq 0$ is given by the affine equation

$$P = 1 + X_3^3 + X_4^3 - X_2^4 - X_2X_3^3 + X_2X_4^3 = 0 \tag{16}$$

and in the accordance with the prescription

$$\{X_2, X_3\} = -\frac{\partial P}{\partial X_4} = -3X_4^2(X_2 + 1).$$

We can re-write this surface as an intersection

$$P_1 = z_1x_1 + z_2x_2 \tag{17}$$

$$P_2 = z_1\left(x_2^3 + x_3^3 - x_4^3\right) + z_2\left(x_1^3 + x_3^3 + x_4^3\right) \tag{18}$$

in the product

$$\begin{array}{c} \mathbb{C}P_1 \\ \mathbb{C}P_3 \end{array} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \tag{19}$$

and in the domain $U_{11} = (\{z_1 \neq 0\}) \bigcap (\{x_1 \neq 0\})$ with the affine coordinates $Z = \frac{z_2}{z_1}, X_i = \frac{x_i}{x_1}, i = 2, 3, 4$ we have , for example for the intersection

$$P_1 = 1 + ZX_2 = 0 \tag{20}$$

$$P_2 = X_2^3 + X_3^3 - X_4^3 + Z\left(1 + X_3^3 + X_4^3\right) = 0 \tag{21}$$

$$\{X_2, X_3\} = \frac{dX_2 \wedge dX_3 \wedge dP_1 \wedge dP_2}{dX_2 \wedge dX_3 \wedge dX_4 \wedge dZ} = -\frac{dX_2 \wedge dX_3}{\mathrm{Jac}\left(\frac{P_1, P_2}{X_4, Z}\right)} = -3X_4^2(X_2 + 1).$$

We can see that this brackets are not homogeneous (and it is not amazing - we takes deal with the *weighted homogeneous* coordinates - but they have the degree 3 like it should be for the "usual" $K3$ surfaces.

Remark Like it was observed in subsection 4.2 the general definition of the "splitting principle " leads also to *rational* non -polynomial mapping between products of curves,surfaces etc.in products of (weighted) projective spaces. So it is not clear at the moment which relation it has with the algebras $Q_{n,k}(\mathcal{E})$ and their tensor products. The understanding of a proper sense of the "splitting principle on "quantum" level may shed some light on the question of a "quantization" of $K3$ surfaces and fibrations.

6. Integrable systems.

In this section we collect some examples when the discussed polynomial Poisson structures naturally appears in Hamiltonian systems. The notion of Liouville integrability is ambiguous for algebraic varieties and lies out of the scope of our interests in the paper (see for the discussion [13]).We would like only to show that the naive construction of the affine Poisson brackets sometimes may be very useful and can clarify the properties of the initial Hamiltonian systems. As a general reference to the subject we propose the reviews of [27].

6.1. **Quadratic algebras,** $n = 4$**.** We will consider a couple of generic quadratic forms in $\mathbb{C}^4$. Let

$$\begin{aligned} p_1 &= 1/2\,(x_1^2 + x_3^2) + kx_2x_4 \\ p_2 &= 1/2\,(x_2^2 + x_4^2) + kx_1x_3, \end{aligned}$$

then the brackets are read

$$\begin{aligned} \{x_i, x_{i+1}\} &= k^2 x_i x_{i+1} - x_{i+2}x_{i+3} \\ \{x_i, x_{i+2}\} &= k\left(x_{i+3}^2 - x_{i+1}^2\right), \end{aligned}$$

where $i \equiv 0, 1, 2, 3(mod\ 4)$.

The quantization of this relations is the elliptic algebra $\mathcal{Q}_4(\mathcal{E}, \eta)$, where $\mathcal{E} \subset \mathbb{C}P^3$ given by the relations $p_1 = 0; p_2 = 0$.

To relate a natural integrable system with this Poisson structure we need to re-write the couple in the "standard" Sklyanin form (3),(4) and to take the coordinate x_4 as a Hamiltonian. Then we obtain from the Sklyanin algebra relations the following Hamiltonian system:

$$\begin{aligned} \dot{x}_1 &= \{x_1, H\} = (J_2 - J_3)x_2x_3 \\ \dot{x}_2 &= \{x_2, H\} = (J_1 - J_3)x_1x_3 \\ \dot{x}_3 &= \{x_3, H\} = (J_1 - J_2)x_1x_2. \end{aligned}$$

This is so-called "top-like "(or elliptic rotator) representation of the classical two-particle elliptic S. Ruijsenaars model. This observation (on the quantum level is due to I. Krichever and A. Zabrodin ([18])) will be discussed in our subsequent paper with H.Braden and A. Gorsky ([20]).

Remark The "usual" Euler - Nahm top is obtained in this scheme as a system on a co-adjoint orbit of $SO(3, \mathbb{C}) = SL_2(\mathbb{C})$ given by the Casimir $Q_1 = 1/2(x_1^2 +$

$x_2^2+x_3^2)=c_1$ with the second quadric $Q_2=1/2(J_1x_1^2+J_2x_2^2+J_3x_3^2)$ taking as a Hamiltonian. Namely,

$$\begin{aligned}\dot{x}_1=\{x_1,H\}=J_2x_2\{x_1,x_2\}+J_3x_3\{x_1,x_3\}=(J_3-J_2)x_2x_3\\ \dot{x}_2=\{x_2,H\}=J_1x_1\{x_2,x_1\}+J_3x_3\{x_2,x_3\}=(J_1-J_3)x_1x_3\\ \dot{x}_3=\{x_3,H\}=J_1x_1\{x_3,x_1\}+J_2x_2\{x_3,x_2\}=(J_2-J_1)x_1x_2,\end{aligned}$$

where we have used the linear regular Poisson structure associated with the curve $Q_1=1/2(x_1^2+x_2^2+x_3^2)=c_1$ in $\mathbb{C}^3:\{x_i,x_j\}=\epsilon_{ijk}\frac{\partial Q_1}{\partial x_k}=\epsilon_{ijk}x_k$.

6.2. **Double elliptic ("DELL") system.** There is a deep relation between integrable systems and $N=2$ SUSY gauge theories which goes back to Witten and Seiberg ([29]). The classically known multi-particle integrable Hamiltonian models (both - continues - Calogero-Mother and differences - Rujisenaars systems)get some new insights coming from the physical background. The description of $6d$ gauge theory with the adjoint matter fields had motivated an appearence of double elliptic (DELL) system unifying the Calogero -Ruijsenaars family and their *duals* and admitting an elliptic dependence both in coordinates and in momenta in it Hamiltonian ([5]),([16]). We will show that there is an example of $K3-$like regular Poisson structure which provides a natural Hamiltonian description of the $SU(2)$ DELL system.

The Hamiltonian of the two-particle DELL in center masses frame ($SU(2)$-case) in the form of ([16]) is given by the function

$$H(p,q)=\alpha(q|k)cn(p\beta(q|k,\tilde{k})|\frac{\tilde{k}\alpha(q|k)}{\beta(q|k)}) \tag{22}$$

where $\alpha(q|k)=\sqrt{1+\frac{g^2}{sn^2(q|k)}}$ and $\beta(q|k,\tilde{k})=\sqrt{1+\frac{g^2\tilde{k}}{sn^2(q|k)}}$, coincides with x_5.

We choose the following system of four quadrics in $\mathbb{C}^6$ which provides the phase space for two-body double elliptic system

$$\begin{aligned}x_1^2-x_2^2=1\\ x_1^2-x_3^2=k^2\\ -g^2x_1^2+x_4^2-x_5^2=1\\ -g^2x_1^2+x_4^2+\tilde{k}^{-2}x_6^2=\tilde{k}^{-2}\end{aligned}$$

The first pair of the equations yields the "affinization" of projective embedding of the elliptic curve into $\mathbb{C}P^3$ and the second pair provides the elliptic curve which locally is fibered over the first elliptic curve. If the coupling constant g vanishes the system is just two copy of elliptic curves embedded in $\mathbb{C}P^3\times\mathbb{C}P^3$. Let us emphasize that the coupling constant amounts to the additional non - commutativity between the coordinates compared to the standard non-commutativity of coordinates and momenta.

The relevant Poisson brackets for this particular system of quadrics reads

$$\{x_1, x_2\} = \{x_1, x_3\} = \{x_2, x_3\} = 0 \quad (23)$$
$$\{x_5, x_1\} = -x_2x_3x_4x_6 \quad (24)$$
$$\{x_5, x_2\} = -x_1x_3x_4x_6 \quad (25)$$
$$\{x_5, x_3\} = -x_1x_2x_4x_6 \quad (26)$$
$$\{x_5, x_4\} = -g^2x_1x_2x_3x_6 \quad (27)$$
$$\{x_5, x_6\} = 0 \quad (28)$$

We should remark that the Poisson structure is singular and can't be extended up to a holomorphic structure on the whole $\mathbb{C}P^6$ because of (6).

Due to the commutation relations x_6 is constant which has to be related to the constant energy to provide the consistency of the system.

The nontrivial commutation relations between coordinates on the distinct tori correspond to the standard phase space Poisson brackets while the nontrivial bracket $\{x_5, x_4\}$ means the additional non-commutativity of the momentum space . Let us note that the triple x_1, x_2, x_3 can be considered in the elliptic parametrization

$$x_1 = \frac{1}{sn(q|k)}$$
$$x_2 = \frac{cn(q|k)}{sn(q|k)}$$
$$x_3 = \frac{dn(q|k)}{sn(q|k)}$$

by Jacobi sine, cosine and dn functions on the "coordinate" elliptic curve (the torus with local coordinate q).

Theorem 6.1. *The Hamilton system with the DELL Hamiltonian (22) is equivalent to the following Hamiltonian system with respect to (23)and with the Hamiltonian x_5 :*

$$\dot{x}_1 = x_2x_3x_4x_6$$
$$\dot{x}_2 = x_1x_3x_4x_6$$
$$\dot{x}_3 = x_1x_2x_4x_6$$
$$\dot{x}_4 = g^2x_1x_2x_3x_6$$
$$\dot{x}_6 = 0.$$

For the proof see ([20]).

This form of DELL system manifests its algebraic nature and immediately provides its explicit integration by hyper-elliptic integral.

We are able to give another polynomial description of the system observing that it has a form of D. Fairlie "elegant" integrable system ([28]) for $n = 4$ (we can skip the unimportant coordinate x_6):

$$\dot{x}_1 = x_2x_3x_4 \tag{29}$$
$$\dot{x}_2 = x_1x_3x_4 \tag{30}$$
$$\dot{x}_3 = x_1x_2x_4 \tag{31}$$
$$\dot{x}_4 = g^2x_1x_2x_3. \tag{32}$$

This system admits a beautiful description as a *decoupled* pair of Euler -Nahm tops after the following change of variables:

$$\begin{aligned} u_+ &= x_3x_4 + g^2x_1x_2 \\ v_+ &= x_2x_4 + g^2x_1x_3 \\ w_+ &= x_1x_4 + g^2x_3x_2 \\ u_- &= x_3x_4 - g^2x_1x_2 \\ v_- &= x_2x_4 - g^2x_1x_3 \\ w_- &= x_1x_4 - g^2x_3x_2. \end{aligned}$$

In the terms of the new variables the DELL system (29) is equivalent to

$$\dot{u}_+ = v_+w_+ \tag{33}$$
$$\dot{v}_+ = w_+u_+ \tag{34}$$
$$\dot{w}_+ = u_+v_+. \tag{35}$$

and to

$$\dot{u}_- = v_-w_- \tag{36}$$
$$\dot{v}_- = w_-u_- \tag{37}$$
$$\dot{w}_- = u_-v_-. \tag{38}$$

Geometrically this change of variables means a passage to the direct (”decoupled”) product of two elliptic curves $\mathcal{E}_+ \times \mathcal{E}_-$ given by the Casimirs of the models 33 and 36

$$\mathcal{E}_+ : \qquad u_+^2 - v_+^2 = k^2(E^2-1) \tag{39}$$
$$u_+^2 - w_+^2 = (k^2-2)(E^2-1) \tag{40}$$

and

$$\mathcal{E}_- : \qquad u_-^2 - v_-^2 = k^2(E^2-1) \tag{41}$$
$$u_-^2 - w_-^2 = (k^2-2)(E^2-1). \tag{42}$$

This result reminds the theorem of R.Ward ([30]) that the second order differential operator with Lam potential $n(n+1)/2sn^2(q|k)$ is decoupled to a couple of the first order matrix operators $\partial + A, \partial - A$ if the matrix $A = (A_1, A_2, A_3)$ satisfies to the Nahm system

$$\dot{A}_i = \epsilon_{ijk}[A_j, A_k].$$

6.3. **DELL as an example of Nambu - Hamilton system.** The polynomial Poisson structure describing the DELL system provides also an example of the Nambu -Poisson structure (**Subsection 1.1**).

The constructions involving quadrics gives a non-trivial example of the Nambu - Hamilton dynamical system. Namely, the system of three quadrics in $\mathbb{C}^5$

$$\begin{aligned} x_1^2 - x_2^2 &= 1 \\ x_1^2 - x_3^2 &= k^2 \\ -g^2x_1^2 + x_4^2 + x_5^2 &= 1. \end{aligned}$$

admits a section by the choice of the level $x_5 = E$ and the 1-parametric intersection of three quadrics in $\mathbb{C}^4$

$$\begin{aligned} Q_1 = x_1^2 - x_2^2 &= 1 \\ Q_2 = x_1^2 - x_3^2 &= k^2 \\ Q_3 = -g^2x_1^2 + x_4^2 &= 1 - E^2. \end{aligned}$$

defines the Nambu -Hamilton system (which is nothing but our old DELL system!):

$$\frac{dx_i}{dt} = \big\{Q_1, Q_2, Q_3, x_i\big\}.$$

7. Acknowledgements

We are thankful to H. Braden and A.Gorsky for the collaboration and stimulating discussions. Our special thanks to A. Morozov whose interest was very inspiring to us. The numerous discussions of V.R. with A. Marshakov, A. Mironov, N.Nekrasov and A.Turbiner were very helpful. Both authors are indebted to B. Enriquez for his friendly participation and useful discussions.

A.O. is grateful to University of Angers where the big part of this paper was written for invitation and very stimulating working atmosphere. He had acknowledged the partial support of the grants RFBR 99-01-01669, RFBR 00-15-96579,CRDF RP1-2254 and INTAS 00-00055. V.R. is grateful to Boris Khesin for helpful discussions and warm hospitality during his stay at Toronto. Work of V.R. is supported partially by INTAS 99-1705, RFBR 01-01-00549 and by the grant for support of scientific schools 00-15-96557.

References

[1] S. Mukai, Moduli of vector bundles on $K3$ surfaces and symplectic manifolds. (Japanese) Sugaku Expositions 1 (1988), no. 2, 139–174. Sugaku 39 (1987), no. 3, 216–235;

[2] de Azcrraga, J. A.; Perelomov, A. M.; Prez Bueno, J. C. The Schouten-Nijenhuis bracket, cohomology and generalized Poisson structures. J. Phys. A 29 (1996), no. 24, 7993–8009;

[3] B. Feigin and A. Odesskii, Sklyanin's elliptic algebras,Functional Anal. Appl. 23 (1989), no. 3, 207–214 (1990);

[4] B. Feigin and A. Odesskii, Vector bundles on an elliptic curve and Sklyanin algebras. Topics in quantum groups and finite-type invariants, 65–84, Amer. Math. Soc. Transl. Ser. 2, 185, Amer. Math. Soc., Providence, RI, 1998;

[5] Fock, V; Gorsky, A; Nekrasov, N; Rubtsov, V Duality in integrable systems and gauge theories, J. High Energy Phys. 2000, no. 7, Paper 28, 40 pp;

[6] Chekhov, L. O.; Fock, V. V. Observables in 3D gravity and geodesic algebras. Quantum groups and integrable systems (Prague, 2000). Czechoslovak J. Phys. 50 (2000), no. 11, 1201–1208;

[7] Bondal, A. A symplectic groupoid of triangular bilinear forms and the braid group; IHES prepr. M2000/02;

[8] Boalch, Ph; Preprint math.DG/0011062; Stokes Matrices and Poisson Lie Groups;

[9] Lynker, M.; Schimmrigk, R. Conifold transitions and aspects of duality; Nuclear Phys. B 484 (1997), no. 3, 562–582;

[10] Dubrovin, B. Differential geometry of the space of orbits of a Coxeter group. Surveys in differential geometry: integrable systems, 181–211, Surv. Differ. Geom., IV, Int. Press, Boston, MA, 1998;

[11] Ugaglia, M. On a Poisson structure on the space of Stokes matrices. Internat. Math. Res. Notices 1999, no. 9, 473–493;

[12] Khimshiashvili, G. On one class of exact Poisson structures. Proc. A. Razmadze Math. Inst. 119 (1999), 111–120;

[13] Vanhaecke, P. Integrable systems in the realm of algebraic geometry. Lecture Notes in Mathematics, 1638. Springer-Verlag, Berlin, 1996, 218 pp;

[14] Stafford, J. T., Van den Bergh, M. Noncommutative curves and noncommutative surfaces. Bull. Amer. Math. Soc. (N.S.) 38 (2001), no. 2, 171–216;

[15] Kim Hoil, Lee Chang-Yeong
Noncommutative $K3$ Surfaces, hepth/0105265;

[16] Braden, H. W.; Marshakov, A.; Mironov, A.; Morozov, A. On double-elliptic integrable systems. I. A duality argument for the case of SU(2), hepth/9906240; Nuclear Phys. B 573 (2000), no. 1-2, 553–572;

[17] Sklyanin, E. K. Some algebraic structures connected with the Yang-Baxter equation, Funktsional. Anal. i Prilozhen. 16 (1982), no. 4, 27–34;

[18] Krichever, I; Zabrodin, A. Spin generalization of the Ruijsenaars-Schneider model, the nonabelian two-dimensionalized Toda lattice, and representations of the Sklyanin algebra, Russian Math. Surveys 50 (1995), no. 6, 1101–1150;

[19] Vaisman, I Lectures on the geometry of Poisson manifolds. Progress in Mathematics, 118. Birkhuser Verlag, Basel, 1994. 205pp;

[20] H.Braden, A. Gorsky, A.Odesskii, V. Rubtsov in preparation;

[21] Weinstein, A The local structure of Poisson manifolds. J. Differential Geom. 18 (1983), no. 3, 523–557;

[22] Candelas, P.; Dale, A. M.; Lutken, C. A.; Schimmrigk, R. Complete intersection Calabi-Yau manifolds. Nuclear Phys. B 298 (1988), no. 3, 493–525.

[23] Candelas, Philip; de la Ossa, Xenia C.; Green, Paul S.; Parkes, Linda A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. Nuclear Phys. B 359 (1991), no. 1, 21–74.

[24] Cattani, E; Cox, D; Dickenstein, A Residues in toric varieties, Compositio Math. 108 (1997), no. 1, 35–76;

[25] Nambu, Y. Generalized Hamiltonian dynamics. Phys. Rev. D (3) 7 (1973), 2405–2412;

[26] Takhtajan, L. On foundation of the generalized Nambu mechanics. (English. English summary) Comm. Math. Phys. 160 (1994), no. 2, 295–315;

[27] Adler, M.; van Moerbeke, P. Kowalewski's asymptotic method, Kac-Moody Lie algebras and regularization. Comm. Math. Phys. 83 (1982), no. 1, 83–106;

[28] Fairlie, D. B. An elegant integrable system. Phys. Lett. A 119 (1987), no. 9, 438–440;

[29] Donagi, R. Witten, E. Supersymmetric Yang-Mills theory and integrable systems. Nuclear Phys. B 460 (1996), no. 2, 299–334;

[30] Ward, R. S. The Nahm equations, finite-gap potentials and Lam functions. J. Phys. A 20 (1987), no. 10, 2679–2683;

[31] Korovnichenko, A. , Zhedanov, A. Dual algebras with non-linear Poisson brackets, in S.Pakuliak and G. von Gehlen (eds.) Integrable Structures of Exactly Solvable Two-Dimensional Modes of Quantum Field Theory, NATO Science series, v.35, 265-272, Kluwer Academic Publishers, 2001

A.O.:Landau Institute of Theoretical Physics, 2, Kosygina str.,117334 Moscow , Russia

Dépt. de Mathématiques, Univ. d'Angers, 2, Bd. Lavoisier, 49045 Angers, France

V.R.: Dépt. de Mathématiques, Univ. d'Angers, 2, Bd. Lavoisier, 49045 Angers, France

ITEP, 25 Bol. Cheremushkinskaia, 117259 Moscou, Russie