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Double-Elliptic Dynamical Systems from Generalized Mukai-Sklyanin Algebras

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Abstract

We consider the double-elliptic generalisation of dynamical systems of Calogero-Toda-Ruijsenaars type using finite-dimensional Mukai-Sklyanin algebras. The two-body system, which involves an elliptic dependence both on coordinates and momenta, is investigated in detail and the relation with Nambu dynamics is mentioned. We identify the 2D complex manifold associated with the double elliptic system as an elliptically fibered rational ("1/2K3 ") surface. Some generalisations are suggested which provide the ground for a description of the N-body systems. Possible applications to SUSY gauge theories with adjoint matter in $d = 6$ with two compact dimensions are discussed.

1 Introduction

Hamiltonian systems of both Calogero-Toda-Ruijsenaars type and spin chains can naturally be associated with finite or infinite dimensional groups via Hamiltonian reduction. The Hamiltonian reduction procedure produces a unified description of such systems in terms of free motion on the group-like symplectic manifold restricted to a fixed level of the moment map [1]. The line of generalisation from rational to trigonometric to elliptic models has a clear algebraic counterpart in terms of zero loop, one-loop and two-loop structures [2]. One further has to distinguish two different possible generalisations, where either coordinates or momenta take values on the elliptic curve. It is useful to introduce the notion of "dual" system here [3, 4], where the coordinates and action variables effectively get interchanged. The attempt to give a mathematically rigorous definition to the duality of integrable systems has been recently given in [5].

The line of generalisation in terms of increasing the number of loops to produce say the trigonometric Ruijsenaars models from the rational ones is not the only one that has been considered. The same results can be obtained for the rational and trigonometric Ruijsenaars models moving in another direction [3]. Namely, there it was shown that these models admit the realisation as motion on a finite dimensional quantum group. However a similar interpretation for the elliptic models was lacking. Below we shall suggest a realisation of the elliptic models as motion associated with certain finite dimensional algebras.[1] It seems that the proper algebraic object for the general double-elliptic system is the generalised Mukai-Sklyanin algebra. This was introduced for two quadrics in $\mathbb{C}^4$ in [7] and may be easily generalised to n quadrics in $\mathbb{C}^{n+2}$ in such a way that the famous Mukai symplectic Poisson brackets are obtained by considering the corresponding $K3$ surfaces as affine symplectic leaves of the n quadric structure [8, 9].

In this paper we shall use the algebra induced by four quadrics in $\mathbb{C}^6$ which yields the phase space for the two-body double-elliptic problem. This model was given explicitly in [4] in Darboux variables. We shall show that this formulation of the double elliptic system is equivalent to the one suggested in [3]. All other two-body elliptic models such as the elliptic Ruijsenaars and Calogero models, as well as the periodic Toda chain, can be derived from the double-elliptic system via one or other degeneration. Recall that the elliptic Calogero system is an example of a Hitchin system [10]. More general integrable systems corresponding to Hitchin systems for curves with many marked points were discussed in [14, 13, 15].

Of course the real problem to be solved is a derivation of the integrable many-body system. In the relatively simple cases known so far, one could just consider a higher rank group, which can be finite or infinite dimensional. This program works perfectly up to the elliptic Ruijsenaars model which is the most complicated system treated successfully in this way [16]. However the next step to the double-elliptic system appears to be very complicated technically and so one has to look for another technique. One method, which

[1]The preliminary results of this paper were presented in the review [6].

suggests a complementary description, is the separation of variables. This approach yields a parametrization of the phase space of the many-body system as the symmetric power of some complicated two dimensional complex manifold [17]. The most difficult part of analysis in this picture is the proper identification of the two-dimensional complex manifold corresponding to the two-particle problem.

The problem of a geometrical description of the double-elliptic systems, as well as being intrinsically interesting from the point of integrability, has important applications to the description of low-energy effective actions in SUSY gauge theories in different dimensions. Via Seiberg-Witten theory it is known that the classical dynamics of integrable many-body systems addresses quantum issues in SUSY gauge theories (see [18, 19] and the references therein for a review). The most complicated SUSY systems treated in this manner thus far have been the elliptic Ruijsenaars model, corresponding to a D=5 SUSY theory with adjoint matter and with one compact dimension[20, 21], and theories with fundamental matter in D=5, 6 [22]. The double-elliptic system supplies the next step to a D=6 SUSY system with adjoint matter, and where two dimensions are compact. The two elliptic moduli to be found in the gauge theory can be identified with the complexified coupling constant and the ratio of the radii in the fifth and sixth dimensions. Therefore the answers we obtain will be of the immediate use for this theory. The pure D=6 SUSY theory with two compact dimensions corresponds to the degeneration of the double-elliptic system to the "elliptic Toda" one.

Let us also note that a clear representation of the integrable systems corresponding to D=6 SUSY gauge theories is also necessary to formulate the dualities from [3, 4] precisely. The point is that these dualities relate gauge theories in different dimensions. For instance, the systems dual to the elliptic Calogero or periodic Toda system (D=4) correspond to D=6 gauge models. Another example involves the relationship between Ruijsenaars (D=5) models and the trigonometric Calogero (D=4) ones. It seems that this duality will be very helpful in the nonperturbative (De)-construction of dimensions recently discussed in [23].

The paper is organised as follows. We start with general remarks concerning the generalised Mukai-Sklyanin algebra. Then we identify the four-dimensional manifolds responsible for the elliptic models including the double elliptic one. The possible generalisation to the n-body problems will be suggested. Finally we apply the double-elliptic system to the description of the $SU(2)$ SUSY gauge models with the adjoint matter in D=6 where two dimensions are assumed to be compact.

2 Two-body system

2.1 General facts

The generalised Mukai-Sklyanin algebra provides a Poisson structure on the intersection of n polynomials Q_i in $\mathbb{C}^{n+2}$. The case of $n = 2$ quadrics corresponds to the Sklyanin algebra while the case $n = 3$ corresponds to the Poisson structure attributed to a K3 surface. The Poisson bracket in affine coordinates looks as follows:

$$\{x_i, x_j\} = \epsilon_{ijk_1 \ldots k_n} \frac{\partial Q_1}{\partial x_{k_1}} \cdots \frac{\partial Q_n}{\partial x_{k_n}}, \tag{1}$$

where x_i are affine coordinates in $\mathbb{C}^{n+2}$. The corresponding four dimensional manifold is "Poisson noncommutative", a point we shall amplify in due course. The polynomials Q_i themselves yield the Casimirs of the algebras.

We choose the following system of four quadrics in $\mathbb{C}^6$ which provides the phase space for the two-body double-elliptic system

$$\begin{aligned} x_1^2 - x_2^2 &= 1 \\ x_1^2 - x_3^2 &= k^2 \\ -g^2 x_1^2 + x_4^2 + x_5^2 &= 1 \\ -g^2 x_1^2 + x_4^2 + \tilde{k}^{-2} x_6^2 &= \tilde{k}^{-2}. \end{aligned} \tag{2}$$

Here the first pair of equations yields the "affinization" of the projective embedding of the elliptic curve into $\mathbb{C}P^3$, while the second pair provides the elliptic curve which locally is fibered over the first elliptic curve. If the coupling constant g vanishes the system is just two copies of elliptic curves embedded in $\mathbb{C}^3 \times \mathbb{C}^3$. Let us emphasise that the coupling constant amounts to an additional noncommutativity between the coordinates beyond the standard noncommutativity of coordinates and momenta.

We should also warn that we will use the same letters to denote the homogeneous and non-homogeneous coordinates in all non-confusing cases.

2.2 Poisson structure

The relevant Poisson brackets for this particular system of quadrics reads

$$\begin{gathered} \{x_1, x_2\} = \{x_1, x_3\} = \{x_2, x_3\} = 0, \\ \{x_5, x_1\} = -x_2 x_3 x_4 x_6, \qquad \{x_5, x_2\} = -x_1 x_3 x_4 x_6, \qquad \{x_5, x_3\} = -x_1 x_2 x_4 x_6, \\ \{x_5, x_4\} = -g^2 \tilde{k}^{-2} 2 x_1 x_2 x_3 x_6, \qquad \{x_5, x_6\} = 0. \end{gathered}$$

(Various factors of two have been incorporated into the definitions of the quadrics to avoid unnecessary factors in these Poisson brackets.) We should remark that the Poisson structure is singular and can't be extended up to a holomorphic structure on the whole $\mathbb{C}P^6$ because of the arguments of [9]. Namely, let $X_1, ..., X_n$ be coordinates on $\mathbb{C}^n$ considered as an affine part of the corresponding projective space $\mathbb{C}P^n$ with the homogeneous coordinates $(x_0 : x_1 : \cdots : x_n)$, $X_i = \frac{x_i}{x_0}$. Then if $\{X_i, X_j\}$ extends to a holomorphic Poisson structure on $\mathbb{C}P^n$ the maximal degree of the structure (= the length of monomials in X_i) is 3 and

$$X_k \{X_i, X_j\}_3 + X_i \{X_j, X_k\}_3 + X_j \{X_k, X_i\}_3 = 0, i \neq j \neq k. \tag{3}$$

Thus $\{X_i, X_j\}_3 = X_i Y_j - X_j Y_i$, with $deg Y_i = 2$.

The nontrivial commutation relations between coordinates on the distinct tori correspond to the standard phase space Poisson brackets while the nontrivial bracket $\{x_5, x_4\}$ is the additional noncommutativity of the momentum space mentioned earlier.

2.3 Equations of motion

Let us note that the triple x_1, x_2, x_3 can be considered in the elliptic parametrization

$$x_1 = \tfrac{1}{sn(q|k)} \tag{4}$$

$$x_2 = \tfrac{cn(q|k)}{sn(q|k)} \tag{5}$$

$$x_3 = \tfrac{dn(q|k)}{sn(q|k)} \tag{6}$$

with the other three coordinates of (2) expressed by the natural Jacobi functions uniformizing the second ("momenta") elliptic curve:

$$x_4 = \alpha sn(\beta p|\tfrac{\tilde{k}\alpha}{\beta}) \tag{7}$$

$$x_5 = \alpha cn(\beta p|\tfrac{\tilde{k}\alpha}{\beta}) \tag{8}$$

$$x_6 = \beta dn(\beta p|\tfrac{\tilde{k}\alpha}{\beta}). \tag{9}$$

Here

$$\alpha(q|k) = \sqrt{1 + \frac{g^2}{sn^2(q|k)}}, \tag{10}$$

$$\beta(q|k, \tilde{k}) = \sqrt{1 + \frac{g^2 \tilde{k}}{sn^2(q|k)}}, \tag{11}$$

The Hamiltonian of the double-elliptic system in the form of [4]

$$H(p, q) = \alpha(q|k)\, cn(p\beta(q|k, \tilde{k})|\frac{\tilde{k}\alpha(q|k)}{\beta(q|k)}) \tag{12}$$

coincides[2] with x_5.

We can check directly (see the Appendix) that the canonical equations of motion can be re-written as the following polynomial system

$$\begin{array}{r}
\dot{x}_1 = \{x_1, x_5\} = x_2 x_3 x_4 x_6 \\
\dot{x}_2 = \{x_2, x_5\} = x_1 x_3 x_4 x_6 \\
\dot{x}_3 = \{x_3, x_5\} = x_1 x_2 x_4 x_6 \\
\dot{x}_4 = \{x_4, x_5\} = g^2 x_1 x_2 x_3 x_6 \\
\dot{x}_5 = \{x_5, x_5\} \equiv 0 \\
\dot{x}_6 = \{x_6, x_5\} = 0.
\end{array} \tag{13}$$

[2]We have chosen a value of the coupling constant g to be pure imaginary and rescaled by $\sqrt{2}$ with respect to the choice of [4].

Due to the commutation relations x_6 is a constant, $x_6 = c$ say, which has to be related to the energy $x_5 = E$ to provide the consistency of the system which comes from the last two equations of (2):

$$\tilde{k}^2(1 - E^2) = 1 - c^2. \tag{14}$$

This consistency condition reduces the number of quadrics in the system because of the coincidence of the last two equations.

To get an explicit form of the spectral curve which corresponds to the solution to the equations of motion let us put $y = x_1x_2x_3x_4$ and obtain the following equation

$$y^2 = x_1^2(x_1^2 - 1)(x_1^2 - k^2)(g^2x_1^2 - (E^2 - 1)). \tag{15}$$

Taking $s = x_1^2$ and assuming the energy level to be "generic" $(E^2 \neq 1, g^2 + 1, g^2k^2 + 1)$ we obtain that the curve under consideration is a two-sheeted covering of the following hyperelliptic form of an elliptic curve

$$y^2 = s(s - 1)(s - k^2)(g^2s - (E^2 - 1))$$

with the branching points $0, 1, k^2, (E^2-1)/g^2$. We can immediately integrate the system by the hyperelliptic integral:

$$dt = \frac{2x_1dx_1}{x_1x_2x_3x_4} = \frac{2x_1dx_1}{y} = \frac{ds}{y},$$

$$t = \int \frac{ds}{y} = 2\int \frac{dx_1}{\sqrt{(x_1^2 - 1)(x_1^2 - k^2)(g^2x_1^2 - (E^2 - 1))}}.$$

Remark 1. Let us suppose that the coupling constant g in the hamiltonian (12) is equal to 0 (the case of "free" motion). Then the expression of (12) reduces to

$$H(p, q) = cn(p|\tilde{k}) \tag{16}$$

and the parametrization of the second curve is simplified as

$$x_4 = sn(p|\tilde{k}) \tag{17}$$
$$x_5 = cn(p|\tilde{k}) \tag{18}$$
$$x_6 = dn(p|\tilde{k}). \tag{19}$$

We have the following "decoupled" family of the quadrics in $\mathbb{C}^3 \times \mathbb{C}^3$:

$$\begin{aligned} x_1^2 - x_2^2 &= 1 \\ x_1^2 - x_3^2 &= k^2 \\ x_4^2 + x_5^2 &= 1 \\ \tilde{k}^2x_4^2 + x_6^2 &= 1. \end{aligned}$$

The geometrical picture of this family is of course an affine part of the direct product of the elliptic curves $\mathcal{E}_q(k)$ parametrized by x_1, x_2, x_3 and $\mathcal{E}_p(\tilde{k})$ parametrized by x_4, x_5, x_6.

The hamiltonian again is identified with x_5 and we obtain a new involutive coordinate x_4. The canonical equations of the "free" motion may be written down immediately:

$$\begin{aligned}\dot{x}_1 &= \{x_1, x_5\} = x_2x_3x_4x_6\\ \dot{x}_2 &= \{x_2, x_5\} = x_1x_3x_4x_6\\ \dot{x}_3 &= \{x_3, x_5\} = x_1x_2x_4x_6\\ \dot{x}_4 &= \{x_4, x_5\} = 0\\ \dot{x}_5 &= \{x_5, x_5\} \equiv 0\\ \dot{x}_6 &= \{x_6, x_5\} = 0.\end{aligned}$$

and after restriction on a level $x_5 = E$ we can easily recognise in the above system the classical Arnold-Euler-Nahm top for $SU(2)$ in the variables x_1, x_2, x_3.

Remark 2. Let us consider the "singular energy level" $E = {}^{\pm}1$. Then we have the following nodal rational curve

$$y^2 = g^2s^2(s-1)(s-k^2)$$

which may be reduced to the following canonical form

$$Y^2 = g^2(1-S)(1-k^2S)$$

by the transformation

$$S = 1/s,\ Y = y/s^2.$$

Remark 3. We note that the spectral curve of the double elliptic system (15) has a clear physical meaning from the point of view of D=6 gauge theory. It is this curve that provides the information concerning the low energy effective action as well as the spectrum of BPS states. The energy level E corresponds to the coordinate on the Coulumb branch of the moduli space in the corresponding gauge theory. The masses of the BPS particles can be derived calculating the integrals of the meromorphic differentials over the cycles on the spectral curve. The singular values of the energy, at which points the curve degenerates, correspond to the singularities on the Coulumb branch of the moduli space when the mass of some BPS particle vanishes.

2.4 Another choice of the double-elliptic ansatz

The choice of another solution for the ansatz of [4] should give an another reparametrisation for the double-elliptic phase space. It is clear that all such choices are related to the initial by an appropriate modular transformation of the Jacobi moduli $k, \tilde{k}, k_{eff}, \tilde{k}_{eff}$ (see [4]). In fact, we are able to check the following statement: The second choice of the double-elliptic ansatz in [4] provides us with a polynomial Hamilton system which is equivalent to the first.

Indeed, the Hamiltonian of the double-elliptic system in the form of the second solution of [4] is

$$H(p,q) = \alpha(q|k)\, cn(p|\tilde{k}\alpha(q|k)) \tag{20}$$

where now $\alpha(q|k) = \sqrt{1 + \frac{g^2}{cn^2(q|k)}}$. Taking the the following parametrizations of the coordinate and momentum tori

$$x_1 = \tfrac{1}{cn(q|k)}, \qquad x_2 = \tfrac{sn(q|k)}{cn(q|k)}, \qquad x_3 = \tfrac{dn(q|k)}{cn(q|k)};$$
$$y_1 = sn(p|\tilde{k}\alpha(q,k)), \qquad y_2 = cn(p|\tilde{k}\alpha(q,k)), \qquad y_3 = dn(p|\tilde{k}\alpha(q,k))$$

and then denoting by

$$x_4 = y_3, \qquad x_5 = \sqrt{1+g^2x_1^2}\,y_1, \qquad x_6 = \sqrt{1+g^2x_1^2}\,y_2$$

we obtain the four quadrics in $\mathbb{C}^6$:

$$\begin{aligned} x_1^2 - x_2^2 &= 1 \\ x_1^2 - x_3^2 &= k^2x_2^2 \\ -g^2x_1^2 + x_5^2 + x_6^2 &= 1 \\ \tilde{k}^2x_5^2 + x_4^2 &= 1. \end{aligned}$$

Again the double-elliptic hamiltonian coincides with x_5 and the system associated with the quadrics has a form similar to (13):

$$\begin{aligned} \dot{x}_1 = \{x_1, x_5\} &= -x_2x_3x_4x_6 \\ \dot{x}_2 = \{x_2, x_5\} &= -x_1x_3x_4x_6 \\ \dot{x}_3 = \{x_3, x_5\} &= -k'^2x_1x_2x_4x_6 \\ \dot{x}_6 = \{x_6, x_5\} &= -g^2x_1x_2x_3x_4 \\ \dot{x}_5 = \{x_5, x_5\} &\equiv 0 \\ \dot{x}_4 = \{x_4, x_5\} &= 0. \end{aligned}$$

Here $k'^2 + k^2 = 1$ is the complementary modulus and the variable x_4 plays the same role as x_6 in the case (13) and also can be eliminated to obtain an almost identical polynomial form. As was argued in [4], this choice of solution better the study of degenerations of the model (see below chapter 4).

3 Duality

Let us now describe the "duality" property for the dynamical system under consideration. We shall formulate the duality as an involution of the four dimensional manifold. First perform the following involution of homogeneous coordinates

$$x_1 \to x_0, \qquad x_0 \to x_1, \qquad x_2 \to x_5 \tag{21}$$

which yields the following intersection of four quadrics

$$\begin{aligned} x_0^2 - x_5^2 &= x_1^2 \\ x_0^2 - x_3^2 &= k^2x_1^2 \\ -g^2x_0^2 + x_4^2 + x_2^2 - x_1^2 &= 0 \\ -g^2\tilde{k}^2x_0^2 + \tilde{k}^2x_4^2 + x_6^2 &= x_1^2. \end{aligned}$$

The dual Hamiltonian of [4] is

$$H_{dual} = cn(q|k) = \frac{x_2}{x_1} \tag{22}$$

In terms if the affine coordinates $z_i, 1 \leq \ i \leq \ 6$:

$$z_1 = \frac{x_0}{x_1}, z_2 = \frac{x_5}{x_1}, z_3 = \frac{x_3}{x_1}, z_4 = \frac{x_4}{x_1}, z_5 = \frac{x_2}{x_1}, z_6 = \frac{x_6}{x_1}.$$

our “dual” system of quadrics has “affine” form

$$\begin{aligned} z_1^2 - z_5^2 &= 1 \\ z_1^2 - z_3^2 &= \tilde{k}^2 \\ -g^2 z_1^2 + z_4^2 + z_2^2 &= 1 \\ -g^2 k^2 z_1^2 + k^2 z_4^2 + z_6^2 &= 1. \end{aligned}$$

The dual Hamiltonian is encoded in the variable z_2 and corresponding Mukai-Sklyanin Poisson algebra gives the “dual” Hamilton system

$$\begin{aligned} \dot{z}_1 = \{z_1, z_2\} &= -z_3 z_4 z_5 z_6 \\ \dot{z}_2 = \{z_2, z_2\} &\equiv 0 \\ \dot{z}_3 = \{z_3, z_2\} &= -z_1 z_4 z_5 z_6 \\ \dot{z}_4 = \{z_4, z_2\} &= -g^2 z_1 z_3 z_5 z_6 \\ \dot{z}_5 = \{z_5, z_2\} &= -z_1 z_3 z_4 z_6 \\ \dot{z}_6 = \{z_6, z_2\} &= 0 \end{aligned}$$

which is clearly equivalent (“self-dual”) to our initial double-elliptic system (13).

Remark It is worth noting that the dual Hamiltonian (22) in the initial (“canonical”) coordinates after the Fourier rotation $p \to -q, \ q \to p$ goes to the Hamiltonian (16) of the “free” ($g = 0$) model corresponding to an Arnold-Euler-Nahm $SU(2)$ top. Below we will show that there is a transformation of the “non-free” hamiltonian system with general double-elliptic Hamiltonian (12) into a system describing a pair of identical $SU(2)$-Arnold-Euler-Nahm tops.

3.1 Algebro-geometric remarks

The system of four quadrics (2) can be considered as an affine part, S_s, of the intersection surface $\bar{S}_s \subset \mathbb{C}P^6$:

$$\begin{aligned} x_1^2 - x_2^2 &= x_0^2 \\ x_1^2 - x_3^2 &= k^2 x_0^2 \\ -g^2 x_1^2 + x_4^2 + x_5^2 &= x_0^2 \\ -g^2 x_1^2 + x_4^2 + \tilde{k}^{-2} x_6^2 &= \tilde{k}^{-2} x_0^2. \end{aligned}$$

This is a singular complex surface (i.e. a two-dimensional algebraic variety) which depends on the vector $\bar{s} = (1, k^2, g^2, \tilde{k}^{-2})$. A straight forward computation shows that (up to a non-zero factor) the Jacobian for the above system is given by the following matrix:

$$\begin{pmatrix} x_1 & -x_2 & 0 & 0 & 0 & 0 & -x_0 \\ x_1 & 0 & -x_3 & 0 & 0 & 0 & -k^2 x_0 \\ -g^2 x_1 & 0 & 0 & x_4 & x_5 & 0 & -x_0 \\ 0 & 0 & 0 & 0 & -x_5 & \tilde{k}^{-2} x_6 & (\tilde{k}^{-2} - 1) x_0 \end{pmatrix}.$$

There are therefore the two following sets of singular points of $\bar{S}_s$:

$$1) \qquad x_0 = x_5 = x_6 = 0;\ x_4^2 = g^2 x_1^2;\ x_1^2 = x_2^2 = x_3^2,$$

which contains 8 points and

$$2) \qquad x_0 = x_1 = x_2 = x_3 = 0;\ x_4^2 + x_5^2 = 0;\ x_4^2 + \tilde{k}^{-2} x_6^2 = 0,$$

containing another 4 points.

We will argue below that our intersection surface belongs to the class of so-called Adler-van Moerbeke surfaces in $\mathbb{C}P^6$ which project to a (singular) Kummer surface, which in its turn gives a Del Pezzo (elliptically fibered rational) surface with 12 singular elliptic fibers ("1/2K3" surface). Our philosophy here is to "embed" the 1-dimensional integrable system, associated with a complex curve (in our case this is the "base" elliptic curve of the coordinates x_1, x_2, x_3) into a system associated with a 2-dimensional complex variety along with the deformation of Hitchin systems proposed in [34].

Our model has evidently only one degree of freedom so we are interested in a section of the surface given by the "energy level" $H = z$, which is a hyperplane section in our case. The canonical divisor $K_S \equiv (4\times2-4-3)H = H$ is given by a hyperplane section H of the surface $\bar{S}_s$.

We will be interested in the hyperplane section $\tilde{\mathcal{C}}_z$ of the surface by $x_5 = zx_0$, given by a fixed ("generic") complex number z : $\tilde{\mathcal{C}}_z = \bar{S}_s \bigcap \{x_5 = zx_0\} \subset \mathbb{C}P^5$.

Let us compute the genus of this (singular) curve taking into account that it passes through the 8 of the twelve singularities enlisted above. The canonical divisor is calculated as $K_{\tilde{\mathcal{C}}_z} = (8-6)H = \mathcal{O}_{\tilde{\mathcal{C}}_z}(2)$ and $\deg \tilde{\mathcal{C}}_z = ((\tilde{\mathcal{C}}_z^2) + (\tilde{\mathcal{C}}_z, K_S) = 2^5$. We have by the adjunction formula

$$g(\tilde{\mathcal{C}}_z) = 1/2((\tilde{\mathcal{C}}_z^2) + (\tilde{\mathcal{C}}_z, K_S)) + 1 - \delta = 2^4 + 1 - 8 = 9$$

and we have that the genus of the (normalised) curve is 9.

Let consider the corresponding ("projected")family of curves in $\mathbb{C}P^4$ because the last equation of the quadrics is a superficial one due to the consistency condition between x_5 and x_6 (see 2.3).

The family $\tilde{C}_z$ represents a singular family of plane curves

$$\tilde{C}_z \ : \ y^2 = x^2(x^2-1)(x^2-k^2)(g^2x^2-(z^2-1)),$$

with $x = x_1$, $y = x_1x_2x_3x_4$. This family is incomplete and has a double point at the origin for each "generic" value of z. A completion of the "generic" curves in the family $\tilde{C}_z$ may be achieved in two different ways which we will now briefly discuss.

The first is by adding a point at infinity which is a completion of the affine curve in $\mathbb{C}^2$ to a singular curve $C_z \subset \mathbb{C}P^2$ of genus $g(C_z) = 2$. We have

$$C_z \ : \ \tilde{Y}^2\tilde{Z}^6 = (\tilde{X}^2-\tilde{Z}^2)(\tilde{X}^2-k^2\tilde{Z}^2)(g^2\tilde{X}^2-(z^2-1)\tilde{Z}^2),$$

with $(\tilde{X} : \tilde{Y} : \tilde{Z})$ homogeneous coordinates on $\mathbb{C}P^2$ such that $x = \frac{\tilde{X}}{\tilde{Z}}, y = \frac{\tilde{Y}}{\tilde{Z}}$.

The curve C_z is a double covering of $\mathbb{C}P^1$, $\pi_C : C_z \to \mathbb{C}P^1$, branching at 6 points. Its normalization E_z has genus 1 because the family C_z is a double cover of a family of elliptic curves (an "elliptic pencil"):

$$E_z \ : Y^2 = (1-X)(1-k^2X)(g^2-(z^2-1)X),$$

which is in turn a double cover of $\mathbb{C}P^1$, $\pi_E : E_z \to \mathbb{C}P^1$ with the invariants

$$g_2(z) = 1/12\{(k^4-k^2+1)(z^2-1)^2 - k^2g^2(k^2+1)(z^2-1) + k^4g^4\} \in \mathcal{O}_z(4);$$

$$g_3(z) = 1/48k^2(z^2-1)[(z^2-1)+(k^2+1)g^2][(k^2+1)(z^2-1)+k^2g^2]-$$
$$-1/216[(k^2+1)(z^2-1)+k^2g^2]^3 - g^2/16k^4(z^2-1)^2 \in \mathcal{O}_z(6).$$

For the second we can consider the complete curve $\bar{C}_z$ of genus 3 which is a normalization of the fiber product

$$C_z \times_{\mathbb{C}P^1} E_z := \{(p,q) \in C_z \times E_z | \pi_C(p) = \pi_E(q)\}.$$

The curve $\bar{C}_z$ is an "étale double covering" of C_z.

Let us discuss the requirement for the intersection surface $S(\bar{S}_s)$ to be an abelian surface. In what follows we will suppose that it is. The curve C_z is irreducible of genus 2 curve with one double point admitting an elliptic involution. It lies on the surface S. We describe the beautiful geometric relations between the triple of the families of curves $\bar{C}_z, C_z$ and E_z and the surface S which is based on [24].

Fix the double covering $\pi : \bar{C}_z \to E_z$ and let a_0 be a branch point. Then we have the natural mapping $\bar{C}_z \to J(\bar{C}_z) := Pic^0(\bar{C}_z)$ by $p \in \bar{C}_z \mapsto \mathcal{O}_{\bar{C}_z}(p-a_0)$. If $\pi(a_0) = q_0$ then the elliptic curve E_z is identified with $Pic^0(E_z)$ by the mapping $q \in E_z \mapsto \mathcal{O}_{E_z}(q-q_0)$ so we have the induced mapping $\pi^* : E_z \to J(\bar{C}_z)$ such that $\ker \pi^* = 0$. The compatibility of the involutions shows that the images of $Pic^0(E_z)$ and $Pic^0(C_z)$ in $Pic^0(\bar{C}_z)$ are complementary. The quotient of $Pic^0(\bar{C}_z) \cong J(\bar{C}_z)$ by the subgroup $\ker(Pic^0(E_z) \to Pic^0(\bar{C}_z))$ is an abelian surface which is called the Prym variety of the involution on $\bar{C}_z$ and denoted by $Prym(\bar{C}_z/E_z)$. Then the surface S is identified with the quotient $J(\bar{C}_z)/\pi^*(E_z)$ and is dual to $Prym(\bar{C}_z/E_z)$ by the "duality theorem" (th.1.12 in [24]). Moreover, both surfaces S and $Prym(\bar{C}_z/E_z)$ are (1,2)- polarized and the following diagram takes place:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & Prym(\bar{C}_z/E_z) \arrow[d] \arrow[dr, "\Longleftrightarrow"] & & \\
0 \arrow[r] & E_z \arrow[r, "\bar{\pi}^*"] & J(\bar{C}_z) \arrow[r] \arrow[d] & S \arrow[r] & 0 \\
 & \bar{C}_z \arrow[r, "\bar{\pi}"] \arrow[ur] & E_z \arrow[d] \arrow[dr, "\pi_E"'] & C_z \arrow[l, "\pi"] \arrow[r] \arrow[d, "\pi_C"'] & J(C_z) \arrow[ul, "\simeq"] \\
 & & 0 & \mathbb{C}P^1 &
\end{tikzcd}$$

Here the symbol $\simeq$ means that the surfaces $Prym(\bar{C}_z/E_z)$ and $J(C_z)$ are isogenous. The fact that $Prym(\bar{C}_z/E_z) \simeq J(C_z)$ is rather standard (in the framework of integrable systems we learnt it from the beautiful book [37]). The symbol $\Longleftrightarrow$ above means the duality between the intersection surface S and the Prym variety $Prym(\bar{C}_z/E_z)$, namely $S^\vee \Longleftrightarrow Prym(\bar{C}_z/E_z)$ and $(Prym(\bar{C}_z/E_z))^\vee = S$ with the duality of (1,2)-polarizations.

On other hand, it is well-known (see [25]) that the Jacobian of a genus 2 curve of "type I" (which is here the case for C_z which has affine equation $y^2 = (x^2-1)(x^2-k^2)(g^2x^2-(z^2-1)))$ is isogenous to a product of two elliptic curves. Hence the dual of the intersection surface S is isogenous to a product of two elliptic curves. Naively this isogeny can be interpreted as the combination of the elementary duality transformation of [4] and the Fourier-Legendre rotation applied to the surface S given by the quadric intersection.

Remark The "non-generic" values of z ($z^2 = 1, g^2+1, g^2k^2+1$) reduce the family $\tilde{C}_z$ to different examples of plane nodal rational curves.

3.2 Comparison with Beauville-Mukai systems

When we fix the consistency condition

$$\tilde{k}^2(1-E^2) = 1 - c^2 \tag{23}$$

a system of the following form emerges:

$$\begin{aligned} x_1^2 - x_2^2 &= 1 \\ x_1^2 - x_3^2 &= k^2 \\ -g^2x_1^2 + x_4^2 + x_5^2 &= 1. \end{aligned}$$

We would like to associate the intersection surface $S(\bar{S}_s)$ with a double covering $\pi : S \mapsto \mathbb{C}P^2$ *a la* Beauville-Takasaki ([26, 27]).

Let us suppose (according to the discussion in [3]) that the value of the Hamiltonian $x_5 = z$ is a coordinate function on the open set $U_0 = \mathbb{C}P^1 \setminus \infty$ and take homogeneous coordinates $(X_0 : X_1 : X_2)$ on $\mathbb{C}P^2$. Consider the surface $S_0 \subset \mathbb{C}P^2 \times \mathbb{A}^1$ ($\mathbb{A}^1$ is an affine part of $\mathbb{C}P^1$) given by the equation:

$$X_0X_2^2 = X_1^3 + f(z)X_0^2X_1 + g(z)X_0^3,$$

where $f(z)$ and $g(z)$ are polynomials. Again let us put $y = x_1x_2x_3x_4$ and obtain the equation

$$y^2 = x_1^2(x_1^2-1)(x_1^2-k^2)(g^2x_1^2-(z^2-1)). \tag{24}$$

from our system of quadrics. With help of the evident linear change

$$X = X - 1/3(\frac{g^2}{z^2-1} - \frac{1+k^2}{k^2})$$

we can replace the curves of the cubic family

$$Y^2 = (1-X)(1-k^2X)(g^2-(z^2-1)X).$$

by an equation of Weierstrass form. If we consider the coordinates Y, X as non-homogeneous for $(X_0 : X_1 : X_2)$ (ie $Y = X_2/X_0,\ X = X_1/X_0$) the equation acquires the Weierstrass form:

$$Y^2 = X^3 + f(z)X + g(z), \tag{25}$$

where $f(z) \in \mathcal{O}(4), g(z) \in \mathcal{O}(6)$.

Let us produce the transition to the chart $U_1 = \mathbb{C}P^1 \setminus 0$ with coordinate $1/z$:

$$\tilde{z} = 1/z, \tilde{X} = X/z^2, \tilde{Y} = Y/z^3.$$

We have

$$\tilde{Y}^2 = \tilde{X}^3 + \tilde{f}(\tilde{z})\tilde{X} + \tilde{g}(\tilde{z}), \tag{26}$$

where $\tilde{f}(\tilde{z}) = f(z)z^{-4}, \tilde{g}(\tilde{z}) = g(z)z^{-6}$. We can consider this as a surface $S_1 \subset \mathbb{C}P^2 \times \mathbb{A}^1$ (where $\mathbb{A}^1 = U_1$) and hence the union $S_0 \bigcup S_1$ correctly defines a surface $f : S \in \mathbb{C}P^1$ which is elliptically fibered. In general it is a singular surface (at the points where the $J-$invariant $J(z) = f(z)^3/\Delta(z)$, with $\Delta = 4f(z)^3 + 27g(z)^2 \in \mathcal{O}(12)$ vanishes).

Therefore, we can suppose that the family of curves $\bar{C}_z$ covers the elliptic pencil lying in (an open part of) a rational elliptic ("1/2 K3") surface S. The rational elliptic fibrations (del Pezzo surfaces) can (roughly speaking) be considered as a Kummer surfaces $\mathcal{K}$ modulo $\mathbb{Z}_2$. The Kummer surface in our case is exactly the 2:1 projection of the intersection (3.1):

$$\begin{array}{cccc}
\tilde{\mathcal{C}}_z & \subset \bar{S}_s & \subset & \mathbb{C}P^6 \\
\downarrow & \downarrow & & \downarrow \\
C_z & \subset \mathcal{K} & \subset & \mathbb{C}P^5. \\
\downarrow & \downarrow & \nearrow & \\
E_z & \subset \mathcal{K}/\mathbb{Z}_2 & &
\end{array}$$

Remark This description fits well to the results of [11] where the full description of Kummer surfaces in $\mathbb{C}P^5$ arising as projections of the Adler-van Moerbeke [12] family of abelian surfaces in $\mathbb{C}P^6$. Our description is corresponded to the first type of the surfaces : a Kummer surface with 12 nodes. The explicit quadratic equations of all 9 types of singular Kummer surfaces with 1:1 correspondence to the Adler-van Moerbeke surfaces can be found in [11].

Let us give another geometric picture of the intersection surface. In "physical" language we are dealing with a family of "supersymmetric cycles" (C_z, L) in S, where $C_z^2 := [C_z].[C_z] = 2g(C_z) - 2 = 2$ (because the genus $g(C_z) = 2$), $[C_z]$ its homology class in $H_2(S, \mathbb{Z})$ and L is a flat $U(1)$-bundle (:= a choice of a point on $J(C_z)$). We consider the linear system $|C_z| := \mathbb{P}(H^0(S, \mathcal{O}(C_z)) = \mathbb{C}P^2$ of all such curves.

The family C_z is singular and we consider a compactification $\bar{J}(C_z)$ of the Jacobian variety $J(C_z)$ of C_z. This compactified Jacobian is a fibration

$$\bar{J}(C_z) \to J(E_z)$$

with singular rational fiber. Here $J(E_z)$ is the Jacobian of the normalisation such that the generalised Jacobian is the extension

$$0 \to \mathbb{C}^* \to J(C_z) \to J(E_z) \to 0.$$

The union $\bigcup \bar{J}(C_z)$ has dimension $2g(C_z) = 4$ and is a fibration over $|C_z| = \mathbb{C}P^2$.

In our case $\mathcal{O}(C_z)$ generates $Pic(S)$ so the family of the compactified Jacobians carries a hyperkahler structure, moreover it is birationally symplectomorphic to $Hilb^{[2]}(S)$ - the Hilbert scheme of 2 points on the surface S (see [26, 28, 17]). This is a framework of the so-called Beauville-Mukai integrable systems. Namely we have a holomorphic Lagrangian fibration $\bigcup \bar{J}(C_z) \to |C_z|$ and an open embedding of $Symm^2(S) \to \bigcup \bar{J}(C_z)$ (choosing a base point p_∞ on C_z we can use the Abel-Jacobi map $AbJ : Symm^2(C_z) \to J(C_z)$, $AbJ(p_1, p_2) = \mathcal{O}_{C_z}(p_1 + p_2 - 2p_\infty)$), such that the following commutative diagram takes place

$$\begin{tikzcd} Symm^2(S) \arrow[r] \arrow[dr] & \bigcup \bar{J}(C_z) \arrow[d] \\ & |C_z| \end{tikzcd} .$$

Hamiltonian in our case implies that we have a degree 6 polynomial defining a hyperelliptic curve of genus 2:

$$Y^2 = (cz^2 + u_1 z + u_2)^3 + f(z)(cz^2 + u_1 z + u_2) + g(z).$$

Here we should consider the constant c as a Casimir following Takasaki's prescription in [27]. Various expressions of the canonical form are considered in Appendix B.

3.3 Comparison of two descriptions of the double-elliptic system

Let us now compare the above description of the double-elliptic system with that suggested in [3]. The two dimensional manifold which provides the phase space in that paper was identified with the double covering of an elliptically fibered two-dimensional manifold (K3 surface). The suggested Hamiltonian was the coordinate on the base of the fibration while the Hamiltonian of the dual system was identified with the coordinate of the fiber. It was supposed in [3] that generically the K3 surface contained an incomplete hyperelliptic genus five curve, and that is why Takasaki [27] in order to obtain an embedding of the one degree of freedom double-elliptic system in to the scheme of Beauville had to set four of his five parameters $u_i, i = 1, \ldots, 5$ to be equal zero. But, as was shown above, the manifold of [4] has the structure of a double covering of an elliptically fibered rational surface. Moreover in these terms the Hamiltonian of the system is the coordinate on the fiber, while the Hamiltonian for the dual system is the coordinate on the base. Therefore we could consider the same argument as for the double-elliptic system from [3] in the case of the rational elliptic fibrations, (incomplete) hyperelliptic curves of genus 2 (a typical fiber of the dual system) and the double cover looks like a hyperelliptic Jacobian of the genus 2 curve. This gives us a direct embedding of our picture in the Beauville-Mukai systems *a la* Takasaki [27], taking the Hilbert scheme $S^{[2]}$ (or rather $Symm^2(S)$) with the symplectic form $\frac{dx^{(1)}\wedge dz^{(1)}}{y^{(1)}} + \frac{dx^{(2)}\wedge dz^{(2)}}{y^{(2)}}$ and the Abel -Jacobi map

$$AbJ(p_1, p_2) = \left(\int_{p_\infty}^{p_1} \frac{zdz}{y} + \int_{p_\infty}^{p_2} \frac{zdz}{y}; \int_{p_\infty}^{p_1} \frac{dz}{y} + \int_{p_\infty}^{p_2} \frac{dz}{y} \right) = (\phi_1, \phi_2)$$

and

$$AbJ^* \left(\frac{dx^{(1)} \wedge dz^{(1)}}{y^{(1)}} + \frac{dx^{(2)} \wedge dz^{(2)}}{y^{(2)}} \right) = du_1 \wedge d\phi_1 + du_2 \wedge d\phi_2.$$

Here we suppose that $p_i = (z^{(i)}, y^{(i)}) \in S, i = 1, 2$ We can illustrate the situation by the following diagram of double-elliptic system with the interchange of the direct and dual systems.

$$\begin{array}{ccccccc} S & \simeq & J(C_x) & \Longleftrightarrow & J(C_z) & \simeq & S^\vee \\ \swarrow^{q\in E_x} & & \searrow^{\varphi^D \in C_x} & & \swarrow_{\varphi\in C_z} & & \searrow^{Q\in E_z = J(E_x)} \\ x \in \mathbb{C}P^1 & & x \in \mathbb{C}P^1 & & z \in \mathbb{C}P^1 & & z \in \mathbb{C}P^1 \end{array}$$

Moreover, now we are able to explain the "naive" definition of duality [4] arising from the anticanonical condition:

$$dP \wedge dQ = -dp \wedge dq.$$

Consider two "dual" elliptic fibrations associated with the double-elliptic system and with its dual in terms of the families of projective Weierstrass cubics in $\mathbb{C}P^2$:

$$S \subset \mathbb{C}P^2 : y^2 = z^3 + f(x)z + g(x), x \in \mathbb{C}P^1,$$

and

$$S^\vee \subset \mathbb{C}P^2 : y^2 = x^3 + \tilde{f}(z)x + \tilde{g}(z), z \in \mathbb{C}P^1.$$

Here z, y are nonhomogeneous coordinates in $\mathbb{C}P^2$, with canonical symplectic form $\Omega = \frac{dz\wedge dx}{y}$ and similarly x, y are nonhomogeneous coordinates in $\mathbb{C}P^2$, with the canonical symplectic form $\Omega = \frac{dx\wedge dz}{y}$.

Now the local action coordinate $I(x)$ is computed [3] as

$$dI(x) = T(x)dx = \left(1/2\pi \oint_{A_x} dz/y \right) dx,$$

where $[A_x] \in H_1(\mathcal{E}_x, \mathbb{Z})$ is a chosen A-cycle. Analogously, the dual action coordinate $I^D(z)$ satisfies

$$dI^D(z) = T^D(z)dz = \left(1/2\pi \oint_{L_z} dx/y \right) dz,$$

where $[L_z] \in H_1(\mathcal{C}_z, \mathbb{Z})$.

Now we have the following chain of transformations:

$$\Omega = \frac{dz \wedge dx}{y} = -\frac{dx \wedge dz}{y},$$

$$\frac{dz}{y} \wedge \frac{dI}{T(x)} = -\frac{dx}{y} \wedge \frac{dI^D}{T^D(z)},$$

or

$$\frac{dz}{yT(x)} \wedge dI = -\frac{dx}{yT^D(z)} \wedge dI^D.$$

Using the expressions for the angle variables $d\varphi = \frac{dz}{yT(x)}$ and $d\varphi^D = \frac{dx}{yT^D(z)}$ we obtain

$$d\varphi \wedge dI = -d\varphi^D \wedge dI^D.$$

Keeping in mind the interpretation of [4], the equality above amounts to

$$K(k)dp^{Jac} \wedge dI = -K(\tilde{k})dp^{Jac^\vee} \wedge dI^D,$$

or

$$dP \wedge dQ = -dp \wedge dq,$$

the starting point of [4].

Remark The duality transformation $x \to z$, $z \to x$ between two elliptic pencils can naively be interpreted as a local "mirror transformation" between two "1/2K3" surfaces [29]: thinking of (z, x) of the first cubic as the Kahler and complex parameters we, obtain that (x, z) on the second becomes the corresponding pair of parameters. Hence the duality at hand changes the "Kahler" and "complex" structures of the cubic surfaces.

3.4 Two coupling constants

There exist natural generalizations of the double-elliptic system so far described involving additional parameters expressing noncommutativity. In terms of the intersection of quadrics the number of the independent coupling constants amounts from the number of their independent coefficients. One simple case involves such counting for the noncommutative T^4. In this case all couplings follow from noncommutativity, hence the maximal number of parameters can be counted from the antisymmetric matrix θ_{ij} defining the Poisson bracket

$$\{x_i, x_j\} = \theta_{ij} \tag{27}$$

and equals to six [42]. In the stringy framework the couplings correspond to the fluxes of rank-two field switched on along different directions in four-dimensional manifold.

One interesting example of two coupling constants comes from the following system of quadrics

$$\begin{aligned} \tilde{g}^2 x_4^2 + x_1^2 - x_2^2 &= 1 \\ \tilde{g}^2 x_4^2 + x_1^2 - x_3^2 &= k^2 \\ -g^2 x_1^2 + x_4^2 + x_5^2 &= 1 \\ -g^2 x_1^2 + x_4^2 + x_6^2 &= \tilde{k}^{-2} \end{aligned}$$

which is written in the most symmetric form. This system yields the following set of the Poisson brackets

$$
\begin{array}{ccc}
\{x_1, x_2\} = \tilde{g}^2 x_4 x_5 x_3 x_6, & \{x_1, x_3\} = \tilde{g}^2 x_4 x_5 x_2 x_6, & \{x_2, x_3\} = 0 \\
\{x_5, x_1\} = -x_2 x_3 x_4 x_6, & \{x_5, x_2\} = -x_1 x_3 x_4 x_6, & \{x_5, x_3\} = -x_1 x_2 x_4 x_6 \\
\{x_5, x_4\} = -g^2 x_1 x_2 x_3 x_6, & \{x_5, x_6\} = 0, & \{x_6, x_4\} = -g^2 x_1 x_2 x_3 x_5.
\end{array}
$$

Now if x_5 is taken as the Hamiltonian then the coupling g corresponds to the "interaction" of the coordinates, while $\tilde{g}$ corresponds to the "interaction" of momenta. The duality now acquires a more symmetric form and correspond just to the simultaneous interchange of x_2 and x_5 and the couplings. One could also consider examples of self-dual systems by choosing, say, $x_2 x_5$ or $x_2 + x_5$ as Hamiltonians.

4 Degenerations

Let us briefly consider possible degenerations and limits of the model. If one sends $\tilde{k}$ to zero the dynamical system degenerates to the elliptic Ruijsenaars model and the corresponding four dimensional manifold reduces to $[T^2 \times \mathbb{C}^*]_g$ which will be explicitly described below. At the next step one can reduce the model to the elliptic Calogero model with the manifold $[T^2 \times \mathbb{C}]_g$. Alternately, if k is sent to zero the system reduces to the one which is dual to the elliptic Ruijsenaars and Calogero models respectively.

The constructions involving quadrics provide a very explicit description of the noncommutative manifolds above. Indeed let us consider limit $\tilde{k} \to 0$ which amounts to the system of three quadrics in $\mathbb{C}^5$

$$
\begin{aligned}
x_1^2 - x_2^2 &= 1 \\
x_1^2 - x_3^2 &= k^2 \\
-g^2 x_1^2 + x_4^2 + x_5^2 &= 1.
\end{aligned}
$$

Geometrically this looks like a cylinder which fibered over the elliptic curve. The corresponding deformed four-manifold for the Calogero system $[T^2 \times \mathbb{C}]_g$ is a "pinched" $\mathbb{C}P^1$ fibering over the elliptic curve.

Let us discuss the relation with the Sklyanin algebra [7]. To this end let us interpret the Poisson bracket relations in the Sklyanin algebra

$$
\{S_\alpha, S_0\} = 2(J_\beta - J_\gamma)\, S_\beta S_\gamma \tag{28}
$$

$$
\{S_\alpha, S_\beta\} = 2 S_0 S_\gamma \tag{29}
$$

as an example of the Mukai-Sklyanin algebra generated by two quadrics in $\mathbb{C}P^4$. The quadrics

$$
2Q_1 = \textstyle\sum_{n=1}^3 S_n^2 \tag{30}
$$

$$
2Q_2 = S_0^2 + \textstyle\sum_{n=1}^3 J_n S_n^2 \tag{31}
$$

coincide with the center of the Poisson bracket algebra. Hence the Sklyanin algebra fits into the general scheme.

Now recall the observation made in [30] that the Hamiltonian of the elliptic Ruijsenaars system coincides with the generator of the Sklyanin algebra S_0. On the other hand, following our approach, the double-elliptic Hamiltonian coincides with the coordinate x_5. This means that in the limit of vanishing $\tilde{k}$ the system of four quadrics in $\mathbb{C}P^6$ reduces to a system of two quadrics in $\mathbb{C}P^3$ corresponding to the Sklyanin algebra.

The degeneration to the trigonometric Ruijsenaars model can be performed in a similar way and the corresponding Hamiltonian can be expressed in terms of the coordinates which follow from the realisation of $U_q(SL(2))$ in terms of the intersection of two quadrics in $\mathbb{C}^4$. The explicit degeneration in this direction goes as follows. Let us first consider the algebraic structure intermediate between the Sklyanin algebra and $U_q(SL(2))$. Following [31], we define generators A, B, C, D of the degenerate Sklyanin algebra at "small" $h = e^{i\pi\tau}(\Im\tau \to 0)$:

$$
\begin{aligned}
A &= -\frac{h^{-1/2}}{2\sin\, 2\pi\eta}(\cos\, \pi\eta\, S_0 + i\sin\, \pi\eta\, S_3),\\
D &= -\frac{h^{-1/2}}{2\sin\, 2\pi\eta}(\cos\, \pi\eta\, S_0 - i\sin\, \pi\eta\, S_3),\\
C &= -\frac{h^{1/2}}{2\sin\, 2\pi\eta}(S_1 - iS_2),\\
B &= -\frac{h^{-3/2}}{8\sin\, 2\pi\eta}(S_1 + iS_2).
\end{aligned}
\tag{32}
$$

The generators A, B, C, D satisfy the quadratic algebra [31]:

$$
\begin{aligned}
&DC = e^{2\pi i\eta}CD, \qquad CA = e^{2\pi i\eta}AC,\\
&AD - DA = -2i\, \sin^3 2\pi\eta\, C^2,\\
&BC - CB = \frac{A^2 - D^2}{2i\sin\, 2\pi\eta},\\
&AB - e^{2\pi i\eta}BA = e^{2\pi i\eta}DB - BD = \frac{i}{2}\sin\, 4\pi\eta\,(CA - DC).
\end{aligned}
\tag{33}
$$

The Casimir elements are the following quadrics

$$
\begin{aligned}
Q_1 &= e^{2\pi i\eta}AD - \sin^2 2\pi\eta\, C^2,\\
Q_2 &= \frac{e^{-2\pi i\eta}A^2 + e^{2\pi i\eta}D^2}{4\sin^2 2\pi\eta} - BC - \cos\, 2\pi\eta\, C^2.
\end{aligned}
\tag{34}
$$

and this algebra fits within the general formula for a generalised Mukai-Sklyanin algebra. Some generalisations of this picture for the trigonometric many-body system can be found in [32]. It would be extremely interesting to present the elliptic Ruijsenaars-Schneider many-body systems explicitly in a similar manner in terms of the quadratic elliptic algebras of [33] or their tensor products.

To get the $U_q(SL(2))$ itself from the algebra above it is necessary to consider the contraction $\epsilon \to 0$:

$$
B \to B,\ A \to \epsilon A,\ D \to \epsilon D,\ C \to \epsilon^2 C.
$$

and take the following quadrics given by the Casimirs

$$
\begin{aligned}
Q_1 &= AD \\
Q_2 &= \frac{q^{-2}A^2 + q^2 D^2}{4(q-q^{-1})} - BC,
\end{aligned}
\tag{35}
$$

where $q = \exp i\pi\eta$. The Poisson brackets following from the algebra of the quadrics reproduce the standard Poisson structure of $U_q(SL(2))$.

One further remark concerns the degeneration to the Calogero model. In principle we could obtain this by two paths. First, we can represent the system in terms of the quadratic Hamiltonian with linear Poisson bracket. To do this it is useful to consider the limit of large value of the Casimir Q_2 and express S_0 in terms of other generators of the Sklyanin algebra. Expanding the root to first order we immediately get the representation of the elliptic Calogero system as an elliptic rotator

$$
H = \sum_{n=1}^{3} J_i S_i^2 \tag{36}
$$

which reduces to the standard form after a redefinition of the coordinate. However we could also consider the quadratic algebra using the degeneration of the action of x_0 on the function $f(u)$

$$
x_0 f(u) = \frac{\theta_{11}(\eta)(\theta_{11}(2u - l\eta) f(u+\eta) - \theta_{11}(-2u-l) f(u-\eta))}{\theta_{11}(2u)} \tag{37}
$$

where l is the spin of representation corresponding to the coupling constant in the dynamical system. To get the elliptic Calogero system we have to send $\eta \to 0$ hence the difference operator becomes the differential one.

For completeness let us note that the periodic Toda two-body system can also be described in terms of a top on the quadratic algebra. To this end consider the following Poisson algebra

$$
\{A_1, A_2\} = 2A_3(4 - A_2), \qquad \{A_3, A_2\} = A_2, \qquad \{A_1, A_3\} = A_1 + A_3^2.
$$

Then the Hamiltonian of the Toda system is the linear function

$$
H_{Toda} = 1/2(A_1 + A_2) \tag{38}
$$

This picture can also be considered as the further degeneration from the elliptic Calogero model via the Inozemtzev limit.

5 Other incarnations of the double-elliptic system

5.1 Double-elliptic systems and decoupled Nahm tops

We are now able to give another polynomial description of the double-elliptic system observing that it has the form of Fairlie's "elegant" integrable system

[35] for $n = 4$. Ignoring the unimportant coordinate x_6 we have,

$$\dot{x}_1 = x_2 x_3 x_4 \tag{39}$$
$$\dot{x}_2 = x_1 x_3 x_4 \tag{40}$$
$$\dot{x}_3 = x_1 x_2 x_4 \tag{41}$$
$$\dot{x}_4 = g^2 x_1 x_2 x_3. \tag{42}$$

This system admits a beautiful description as a *decoupled* system of Euler-Nahm tops after the following change of variables:

$$\begin{array}{lll} u_+ = x_3x_4 + gx_1x_2, & v_+ = x_2x_4 + gx_1x_3, & w_+ = x_1x_4 + gx_3x_2 \\ u_- = x_3x_4 - gx_1x_2, & v_- = x_2x_4 - gx_1x_3, & w_- = x_1x_4 - gx_3x_2. \end{array}$$

In the terms of the new variables the double-elliptic system (39) is equivalent to

$$\dot{u}_+ = v_+ w_+ \tag{43}$$
$$\dot{v}_+ = w_+ u_+ \tag{44}$$
$$\dot{w}_+ = u_+ v_+. \tag{45}$$

and to

$$\dot{u}_- = v_- w_- \tag{46}$$
$$\dot{v}_- = w_- u_- \tag{47}$$
$$\dot{w}_- = u_- v_-. \tag{48}$$

Geometrically this change of variables means a passage from the intersection of the quadrics to the direct ("decoupled") product of two elliptic curves $\mathcal{E}_+ \times \mathcal{E}_-$ given by the Casimirs of the models (43) and (46)

$$\begin{array}{c} \mathcal{E}_+ : u_+^2 - v_+^2 = k^2(E^2 - 1) \\ u_+^2 - w_+^2 = (k^2 - 2)(E^2 - 1) \end{array}$$

and

$$\begin{array}{c} \mathcal{E}_- : u_-^2 - v_-^2 = k^2(E^2 - 1) \\ u_-^2 - w_-^2 = (k^2 - 2)(E^2 - 1). \end{array}$$

This result is reminiscent of the result of of Ward [36] that the second order differential operator with Lamé potential $n(n+1)/2sn^2(q|k)$ can be factorized into a product of the first order matrix operators $(\partial + A)(\partial - A)$ with the matrix $A = (A_1, A_2, A_3)$ satisfying the Nahm equations

$$\dot{A}_i = \epsilon_{ijk}[A_j, A_k].$$

This property manifests a hidden supersymmetry underlying the quantum mechanics with Lamé potential. The factorization of the double-elliptic system into two decoupled tops suggests the existence of a hidden SUSY in this case too.

5.2 The Double-elliptic system as an example of a Nambu-Hamilton system

Our polynomial parametrization of the double-elliptic system also provides and example of a Nambu-Poisson structure. Such a structure was introduced by Nambu in 1973 [38] as a natural generalisation of the Poisson structure (which corresponds below to $n = 2$). Such structures were recently extensively studied in [39]. Recall that a Nambu-Poisson manifold M^n is is a manifold endowed with an antisymmetric n-vector $\eta \in \Lambda^n(TM)$. The field η defines an n-ary operation:

$$\{,...,\} : C^\infty(M)^{\otimes n} \mapsto C^\infty(M)$$

such that the three properties are valid:

1. antisymmetry:
$$\{f_1, ..., f_n\} = (-1)^\sigma \{f_{\sigma(1)}, ..., f_{\sigma(n)}\}, \qquad \sigma \in S_n;$$
2. coordinate-wise "Leibnitz rule" for any $h \in C^\infty(M)$:
$$\{f_1 h, ..., f_n\} = f_1\{h, ..., f_n\} + h\{f_1, ..., f_n\};$$
3. The "Fundamental Identity" (which replaces the Jacobi Identity):
$$\{\{f_1, ..., f_n\}, f_{n+1}, ..., f_{2n-1}\} + \{f_n, \{f_1, ..., (f_n)^\vee f_{n+1}\}, f_{n+2}, ..., f_{2n-1}\}+$$
$$+\{f_n, ..., f_{2n-2}, \{f_1, ..., f_{n-1}, f_{2n-1}\}\} = \{f_1, ..., f_{n-1}, \{f_n, ..., f_{2n-1}\}\}$$
for any $f_1, ..., f_{2n-1} \in C^\infty(M)$.

Dynamics on a Nambu-Poisson manifold is defined by $n-1$ Hamiltonians $H_1, ..., H_{n-1}$ giving the Nambu-Hamiltonian system

$$\frac{dx}{dt} = \{H_1, H_2, ..., H_{n-1}, x\}.$$

The most common example of a Nambu-Poisson structure is the so-called "canonical" Nambu-Poisson structure on $\mathbb{C}^n$ with coordinates $x_1, ..., x_n$:

$$\{f_1, ..., f_n\} = Jac(f_1, ..., f_n) = \frac{\partial(f_1, ..., f_n)}{\partial(x_1, ..., x_n)}$$

Our constructions involving quadrics give non-trivial examples of a Nambu-Hamilton dynamical system. Namely, the system of three quadrics in $\mathbb{C}^5$

$$x_1^2 - x_2^2 = 1, \qquad x_1^2 - x_3^2 = k^2, \qquad -g^2 x_1^2 + x_4^2 + x_5^2 = 1.$$

admits a section by the choice of the level $x_5 = E$ and the 1-parameter intersection of three quadrics in $\mathbb{C}^4$

$$\begin{aligned} Q_1 = x_1^2 - x_2^2 &= 1 \\ Q_2 = x_1^2 - x_3^2 &= k^2 \\ Q_3 = -g^2 x_1^2 + x_4^2 &= 1 - E^2. \end{aligned}$$

define a Nambu-Hamilton system, which is our double-elliptic system:

$$\frac{dx_i}{dt} = \{Q_1, Q_2, Q_3, x_i\}.$$

6 N-body systems

In this section we discuss the challenging problem of constructing the many-body generalisations of double-elliptic systems. We will not present an explicit realisation of such systems, but some problems and proposals concerning the matter will be given. Let us emphasise that any finite dimensional two-body system can be generalised in at least in two directions.

The first one involves spin chain type systems. The starting point to be generalised is the Sklyanin Lax operator expressed in terms of the generators of the Sklyanin algebra obeying the $RLL = LLR$ relations with Belavin's elliptic R-matrix. The corresponding N-body system is the XYZ spin chain. Now we would like to construct the local Lax operator expressed in terms of the generators of a generalised Mukai-Sklyanin algebra of n quadrics for $n > 2$. It can be shown that it is impossible to introduce a local Lax operator whose matrix elements are such generators assuming that Belavin's R matrix intertwines it. Therefore to pursue this line of generalisation one has to develop some generalisation of the elliptic R-matrix.

Another line of generalisation would be to seek the putative many-body double-elliptic system as a generalised Hitchin system. Hitherto the most complicated system treated in this way has been the elliptic Ruijsenaars model which can be obtained via the Hamiltonian reduction procedure from the following moment map equation [16]

$$g(z)h(z)g^{-1}(z)h^{-1}(qz) = \mu \tag{49}$$

where g and h are elements of affine $SL(N)$ and μ is the fixed level of the moment map.

To obtain the double-elliptic system one presumably has to solve the following equation

$$g(z)h(z)g^{-1}(\tilde{q}z)h^{-1}(qz) = \mu \tag{50}$$

which can be considered as the moment map equation for the Heisenberg double of the double affine algebra. The symplectic manifold to be constructed is the generalisation of the cotangent bundle to the double affine group which provides the phase space for the elliptic Ruijsenaars model. If the solution to this equation could be found (in some particular gauge) the Hamiltonian of the many-body double-elliptic system would be given by $\int Trg(z)$. The results obtained in [32] are of help for this approach.

To link the Heisenberg double approach with the geometrical considerations of this paper it is instructive to adopt the fundamental group interpretation of the moment map equations. The simplest equation of such type corresponding to the trigonometric Ruijsenaars model looks like

$$ghg^{-1}h^{-1} = \mu. \tag{51}$$

This has no coordinate dependence and just represents the relation following from the fundamental group on the torus with one marked point. To discuss the double-elliptic case in a similar manner we have to consider connections on the whole four dimensional manifold. Actually we are interested in a twisted bundle which in the simplest example of the torus results in t'Hooft's consistency condition for the twists [40]

$$\Omega_i(z+a_i)\Omega_j(z+a_j)\Omega_i^{-1}(z+a_i)\Omega_j^{-1}(z+a_j) = Z_{ij}. \tag{52}$$

Here Z_{ij} belongs to the center of the group and represents the matrix of fluxes. The torus case is too simple to generate the double-elliptic system. However if we solve (52) then the hamiltonian $\int Tr\Omega_i$ would provide a very degenerate example of such a "double-elliptic " system.

One more powerful approach is based on the separation of variables procedure. Since we have identified the phase space of the two-body double-elliptic system one could imagine that the Hilbert scheme of the n-points on this manifold would solve the problem. However, a serious technical difficulty arises since there is no effective description of Hilbert schemes of points on generic four dimensional manifolds. A related issue involves the identification of integrable many-body systems in terms of noncommutative instantons [41]. To obtain the description of the double-elliptic system in terms of noncommutative instantons the ADHM description of instantons on commutative and noncommutative K3 manifolds should be developed. (See [42] for steps in this direction.)

7 The relation to D=6 SUSY gauge theories with massive adjoint matter

Let us briefly discuss the possible application of double-elliptic systems. It is known that there is the map between classical integrable many-body systems of Hitchin type and their generalisations ([19, 18] and the references therein) with supersymmetric gauge theories in $d = 4$, 5, 6 dimensions with the different matter content. This relation can be briefly formulated as follows. The solution to the classical equation of motion of the integrable system provides the spectral curve embedded into the higher dimensional compactification manifold. The gauge theory is defined on the worldvolume of the M5 brane wrapped around the spectral curve. The particular integrable system is in one to one correspondence with known supersymmetric gauge theories. Namely, the space of integrals of motion is mapped into the Coulomb branch of the moduli space in the gauge theory, while the modular parameters of the spectral curve of the integrable system amount to the running coupling constants on the field theory side.

Previously the elliptic Ruijsenaars integrable system, of Hitchin type, was identified with an $N = 1$ $d = 5$ SUSY gauge theory with one compact dimension and adjoint hypermultiplet [20, 22, 21]. The double-elliptic systems we are looking for presumably correspond to the next step, namely an $N = 1$, $d = 6$ gauge theory with adjoint hypermultiplet when two dimensions are compactified. There are two natural elliptic moduli in the gauge theory: one is the complexified coupling constant while the second is the modulus of the compact torus in the fifth and sixth dimensions. These are to be identified with the moduli k, and $\tilde{k}$ in the quadrics above. The coupling constant g as before can be related to the mass of the adjoint field. We suggest that the genus 2 curve (15) we have found for the two-body double-elliptic system plays the role of the Seiberg-Witten curve providing the renormalised coupling constant in $d = 6$ gauge theory with two compact dimensions.

The Coulomb branch of the moduli space of this theory can be also described, via an additional compactification of one further dimension, and consideration of the resulting $2+1$ theory with $N = 4$ SUSY as the theory of NS5 branes compactified on $\mathbb{T}^3$ [42]. The moduli space of this theory can be mapped into the moduli space of noncommutative instantons on $\mathbb{T}^4$ and K3 . The noncommutativity corresponds to the fluxes of B field along some directions. Generically fluxes can be switched on in all directions, corresponding to the complete set of twists in the language of [42]. The system considered in this paper corresponds to a flux only along the fiber of the elliptic fibration which plays the role of the coupling, constant in the dynamical system and the mass of adjoint in the field theory. To link this with the integrable system one could adopt the description of the instanton moduli space via the spectral curve endowed with the spectral bundle [43]. This description emerges after a T duality transformation of the initial four dimensional manifold along two dimensions.

It has been argued that the instanton spectral curve coincides exactly with the Seiberg-Witten curve describing the Coulomb branch of the moduli space of some $N = 2$ theory. The reason for this relation is that moduli space of the instantons on noncommutative $\mathbb{T}^4$ is equivalent to the moduli space of the little string theory on $\mathbb{T}^3$. The twists correspond to the structure of the $\mathbb{T}^2$ fibration over the the base $\mathbb{T}^2$. In the generic situation there are twists over the fiber and base tori. Geometrically the situation matches with the picture of the intersection of the quadrics above. The coupling constants correspond to the twists and it is therefore natural to conjecture that the double-elliptic n body system (or possibly a degenerated version) is related to n, $U(1)$ instantons on noncommutative $\mathbb{T}^4$.

8 Conclusion

In this paper we have elaborated in some detail the simplest double-elliptic systems and their relation to the generalised Mukai-Sklyanin finite dimensional algebras. The previously discovered Hamiltonians for these systems were placed in an algebro-geometric setting allowing various connections with Nahm tops to be made.

The generalisation to the many-body case still remains a challenging problem. We have however identified several mathematical problems to be solved to make progress in this direction. Namely, it is highly desirable to find the generalisation of the R-matrix related to the generic algebra of quadrics, to develop the ADHM description of Hilbert scheme of points on intersection of the quadrics and to study further the ADHM description of instantons on noncommutative K3 manifolds. It is also necessary to find the "$SU(N)$" generalisation of the algebra of quadrics which would involve a larger number of generators but keeping the power of the generators involved (cubic, quartic e.t.c) the same as for “SU(2)” case. Such a generalisation is known [33] only for the Sklyanin algebras of 2 quadrics. The corresponding quadratic Feigin-Odesskii algebras involve an arbitrary number of generators and can be built from the vector bundles on the elliptic curve. The last problem to be mentioned is that of finding a precise formulation of the notion of the Heisenberg double to a double affine algebra. It would be also very interesting to combine the ideas from this paper with the recent two-dimensional generalisation of the Calogero and Hitchin type systems suggested in [44],[45].

Besides the problems above, which can be considered as primarily mathematical ones, there are a number of issues relevant for the physics of D=6 gauge theories. First, let us emphasise that the genus two spectral curve (15) provides the low energy effective action and the spectrum of BPS particles in D=6 theory. It would be interesting to pursue further the duality arguments relating the theories in different dimensions. One more interesting possibility along this way involves the idea of the deconstruction of dimensions, starting from D=4 theory. The deconstruction of D=6 theories with two compact dimensions has been considered recently [46]. We expect that the double elliptic system considered here could be derived from the “double lattice” of the integrable models corresponding to D=4 if the nonperturbative effects would be taken into account. This correspondence has been proven for the D=5 models with one compact dimension which can be derived from the “lattice” provided by the quiver type D=4 theory [23].

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10 Appendix

We give for completeness the computations showing that the double-elliptic Hamiltonian system in canonical variables is rewritten in the form (2) under the change of variables (3-8).

10.1 A Hamiltonian equations

We begin with the following relations

$$\frac{\partial\,\mathrm{sn}(u|k)}{\partial k} = \frac{\mathrm{cn}(u|k)\,\mathrm{dn}(u|k)}{kk'^2}\left[-E(u) + k'^2 u + k^2\,\mathrm{sn}(u|k)\,\mathrm{cd}(u|k)\right]$$

$$\frac{\partial\,\mathrm{cn}(u|k)}{\partial k} = \frac{\mathrm{sn}(u|k)\,\mathrm{dn}(u|k)}{kk'^2}\left[E(u) - k'^2 u - k^2\,\mathrm{sn}(u|k)\,\mathrm{cd}(u|k)\right]$$

$$\frac{\partial\,\mathrm{dn}(u|k)}{\partial k} = \frac{k\,\mathrm{sn}(u|k)\,\mathrm{cn}(u|k)}{k'^2}\left[E(u) - k'^2 u - \mathrm{sn}(u|k)\,\mathrm{dc}(u|k)\right]$$

where $k'^2 = 1 - k^2$, with $\mathrm{cd}(u|k) = \frac{\mathrm{cn}(u|k)}{\mathrm{dn}(u|k)}$ and so forth. Here $E(u) = E(u,k)$ is the elliptic integral of the second kind

$$E(u) = \int_0^{\mathrm{sn}(u|k)} \sqrt{\frac{1-k^2t^2}{1-t^2}}dt = \int_0^u \mathrm{dn}^2(v|k)dv.$$

Only the first of these relations needs to be established, with the latter two following from partial differentiation of the identities

$$\mathrm{sn}^2(u|k) + \mathrm{cn}^2(u|k) = 1$$

and

$$\mathrm{dn}^2(u|k) + k^2\,\mathrm{sn}^2(u|k) = 1$$

respectively.

To establish the first we take the partial derivative with respect to k of the definition

$$u = \int_0^{\mathrm{sn}(u|k)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

giving

$$0 = \frac{1}{\mathrm{cn}(u|k)\,\mathrm{dn}(u|k)}\frac{\partial\,\mathrm{sn}(u|k)}{\partial k} + k\int_0^{\mathrm{sn}(u|k)} \frac{t^2dt}{(1-k^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Upon change of variable we have

$$-\frac{1}{k\,\mathrm{cn}(u|k)\,\mathrm{dn}(u|k)}\frac{\partial\,\mathrm{sn}(u|k)}{\partial k} = \int_0^u \frac{\mathrm{sn}^2(u|k)}{\mathrm{dn}^2(u|k)}du$$

which is straightforwardly integrated to yield the first identity.

Let us set

$$\alpha = \sqrt{1 + \frac{g^2}{\mathrm{sn}^2(q|k)}}, \qquad \beta = \sqrt{1 + \frac{g^2\tilde{k}^2}{\mathrm{sn}^2(q|k)}}, \qquad \Lambda = \frac{\tilde{k}\alpha}{\beta}.$$

The equations of motion for the Hamiltonian $H = \alpha(q)\,\mathrm{cn}(\beta(q)p|\Lambda)$ yields

$$\dot{q} = \frac{\partial H}{\partial p} = -\alpha\beta\,\mathrm{sn}(\beta p|\Lambda)\,\mathrm{dn}(\beta p|\Lambda)$$

and

$$\dot{p} = -\frac{\partial H}{\partial q} = -\partial_q \alpha \,\mathrm{cn}(\beta p|\Lambda) + p\alpha \,\mathrm{sn}(\beta p|\Lambda)\,\mathrm{dn}(\beta p|\Lambda)\partial_q \beta - \alpha \partial_\Lambda \,\mathrm{cn}(\beta p|\Lambda)\partial_q \Lambda.$$

Here

$$\partial_q \Lambda = -g^2 \tilde{k}(1-\tilde{k}^2)\frac{\mathrm{cn}(q|k)\,\mathrm{dn}(q|k)}{\mathrm{sn}^3(q|k)\alpha\beta^3}.$$

We note that $\dot{p}$ is invariant under $p \to p + \frac{2nK}{\beta}$ even though the explicit momentum terms and the elliptic integral of the second kind in the last term individually are not.

Now introduce the quadrics

$$\begin{aligned} x_1^2 - x_2^2 &= 1 \\ x_1^2 - x_3^2 &= k^2 \\ \alpha^2 &= 1 + g^2 x_1^2 \\ \beta^2 &= 1 + g^2 \tilde{k}^2 x_1^2 \end{aligned}$$

which may be realised by

$$x_1 = \frac{1}{\mathrm{sn}(q|k)}, \quad x_2 = \frac{\mathrm{cn}(q|k)}{\mathrm{sn}(q|k)}, \quad x_3 = \frac{\mathrm{dn}(q|k)}{\mathrm{sn}(q|k)}.$$

Consider further the quadrics

$$\begin{aligned} y_1^2 + y_2^2 &= 1 \\ \Lambda^2 y_1^2 + y_3^2 &= 1 \end{aligned}$$

realised by

$$y_1 = \mathrm{sn}(\beta p|\Lambda), \quad y_2 = \mathrm{cn}(\beta p|\Lambda), \quad y_3 = \mathrm{dn}(\beta p|\Lambda).$$

Upon setting

$$x_4 = \alpha y_1, \quad x_5 = \alpha y_2, \quad x_6 = \beta y_3$$

we recover (2) and find that $H = x_5$. Further

$$\dot{x}_4 = \mathrm{sn}(\beta p|\Lambda)\partial_q \alpha \dot{q} + \alpha \,\mathrm{cn}(\beta p|\Lambda)\,\mathrm{dn}(\beta p|\Lambda)\left(p\partial_q \beta \dot{q} + \beta \dot{p}\right) + \alpha \partial_\Lambda \,\mathrm{sn}(\beta p|\Lambda)\partial_q \Lambda \dot{q}.$$

We see

$$\dot{x}_4 = g^2 x_1 x_2 x_3 x_6$$

For this we don't actually need the explicit expressions for $\partial_\Lambda \,\mathrm{sn}(\beta p|\Lambda)$ and $\partial_\Lambda \,\mathrm{cn}(\beta p|\Lambda)$ for these appear in the combination

$$-\alpha^2 \beta \partial_q \Lambda \,\mathrm{dn}(\beta p|\Lambda)\left(\mathrm{sn}(\beta p|\Lambda)\partial_\Lambda \,\mathrm{sn}(\beta p|\Lambda) + \mathrm{cn}(\beta p|\Lambda)\partial_\Lambda \,\mathrm{cn}(\beta p|\Lambda)\right)$$

which vanishes as a result of the first identity given earlier.

10.2 B Canonical 2-form in different parametrizations

Let us compute the expressions for the canonical 2-form $dp \wedge dq$ given in the local coordinates of the initial elliptic curves as a 2-form on the phase surface in the different parametrizations.

We begin with the following identity in the region $x_5 \neq 0$

$$\Omega = dp \wedge dq = \frac{dx_1 \wedge dx_4}{x_2 x_3 x_5 x_6}.$$

Indeed, we have from the obvious relations

$$dp \wedge dq = \frac{dH \wedge dq}{\frac{\partial H}{\partial p}} = \frac{dx_5 \wedge dx_1}{x_2 x_3 x_4 x_6}.$$

and (4),(10) and (11) amount to $dx_5 \wedge dx_1 = \frac{x_4}{x_5} dx_1 \wedge dx_4$.

In the notation of Section 2

$$\Omega = \frac{dX \wedge dz}{x_6 Y}.$$

This follows from the identities

$$\Omega = \frac{dz \wedge dx_1}{x_2 x_3 x_4 x_6} = \frac{dz \wedge dx}{x_1 x_2 x_3 x_4 x_6} = \frac{dz \wedge dx}{y x_6} = \frac{dz \wedge d(1/X)}{Y/X^2 x_6} = \frac{dX \wedge dz}{Y x_6}.$$

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