The categorical theory of relations and quantization

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Abstract

In this paper we develops a categorical theory of relations and use this formulation to define the notion of quantization for relations. Categories of relations are defined in the context of symmetric monoidal categories. They are shown to be symmetric monoidal categories in their own right and are found to be isomorphic to certain categories of $A-A$ bicomodules. Properties of relations are defined in terms of the symmetric monoidal structure. Equivalence relations are shown to be commutative monoids in the category of relations. Quantization in our view is a property of functors between monoidal categories. This notion of quantization induce a deformation of all algebraic structures in the category, in particular the ones defining properties of relations like transitivity and symmetry.

Contents

1 Introduction

The concept of quantization is somewhat mysterious and rather ill defined. It first appeared in a rudimentary form in the work of Max Planck [12] .. Its role there was as a purely technical device to solve a problem central to the physics of radiation at the time, the so called ultraviolet catastrophe for the blackbody radiation spectrum. Planck's original idea was shortly thereafter used by Einstein to explain the photoelectric effect [5] and was further developed by N. Bohr into what we today call the Old Quantum Theory. This theory explained with greater precision than ever before the position of the spectral lines for the hydrogen atom. The theory was however rather ad hoc and it was difficult to generalize the theory to more complicated atomic systems. The next step forward was introduced by Louise De Broglie [2], [3],[4], . He generalized the already well known wave-particle duality for light to matter and postulated that electrons confined to an atom would display wavelike properties. The idea of wave-particle duality inspired E. Schrødinger in 1926 to write down a wave equation for matter waves. A different view on the notion of quantization was introduced by Heisenberg [6][14] in 1925 through his matrix mechanics. These two approaches was soon shown to be equivalent. From a modern point of view the difference in the two approaches lies in Schrødingers use of the Hamiltonian formulation of classical mechanics and of Heisenbergs use of a formulation of classical mechanics in terms of Poissont brackets. Schrødinger's approach gave rise to the canonical quantization procedure. This procedure has been applied successfully to many systems but contain ambiguities, like variable ordering, and has invariance problems. The method of Geometric Quantization [7] was introduced in order to resolve these problems. Heisenbergs approach to quantization although equivalent to Schrødingers approach at an elementary level, has a distinctly more algebraic flavor than the wave mechanics of Schrødinger. Here the structure of a physical system is represented in terms of an algebra of observables. Representations of this algebra of observables are possible models of the system in question. Whereas algebras derived from a classical description of the system are commutative, the algebras representing quantized systems are in general noncommutative although still associative. Deformation quantization [1],[13] is a collection of tools and methods that have been developed in order to find quantized version of classical systems by deforming the algebraic description of the system within some class of algebras. What is clear from the existence of all these different approaches is that the notion of quantization is not well defined. The various approaches agree for simple systems, but they have different domains of applicability and even for a single approach several possible quantizations are possible for a given system. What are the properties, or constraints, a system need in order for the notion of quantization to be applicable? Is quantization one thing or several different things? What is the relation between constraints and quantizations? These are just some of the questions that comes to mind. This paper will not give a definite answer to any of these questions but will introduce a mathematical framework that emphasize the idea that quantization is something that depends on constraints and that these constraints may not belong to the domain of mechanics or not even to physics. In fact we believe that quantization has its natural description in terms of a theory of representation for constraints. We also believe that at the present time the only mathematical framework with the right kind of generality for the formulation of a representation theory of constraints is Category Theory [8]. Constraints will in this framework take the form of relations between natural transformations and a representation of the constraints will be a category that supports all given functors and natural transformation with the assumed relations. Quantizations will be related to morphisms in the category of possible representations of a given set of constraints. What we describe here is of course a lot of bones with very little flesh. The goal of this paper is to put a little more flesh on the bones. This we will do by developing a theory for the quantization of relations along the lines described above. This theory illustrate our view of quantization, but is also of independent interest since it gives a framework for the quantization of logic and machines as described in the classical theory of computing. In these days when the whole domain of classical computing is in the process of being quantized a wider point of view on the process of quantization is certainly needed. The categorical approach to quantization has been introduced by one of the authors in [9],[10],[11].

2 Categorical framework

In this first chapter we formulate the basic categorical machinery that we need in order to categorize the notion of relation. In the first subsection we introduce the notion of a semimonoidal and a monoidal category. In line with our general ideas of constraints and representations both notions are defined entirely in terms of functors and natural transformations. This leads to a slightly more general notion of monoidal category than the usual one although we does not pursue this here. Symmetries for monoidal categories is introduced as a further set of constraints on monoidal categories. A certain derived relation for the natural transformations defining a symmetric monoidal category is described and shown to be equivalent to the usual Yang-Baxter equation. This new formulation of the Yang-Baxter equation is essential when we later in this paper introduce a generalization of the usual notion of symmetry that we need in order to formulate commutativity in the context of relations. We lay the groundwork for this generalization by showing how the Yang-Baxter equation is intimately connected to a action by a certain S_2-graded group. In the last subsection in this part of the paper we introduce the notion of M-categories and C-categories. These categories have exactly the constraints needed in order to formulate and develop a theory of relations.

2.1 Symmetric monoidal categories

A semimonoidal category is a category that has a product that is associative up to a natural isomorphism. A semimonoidal category is a monoidal category if there is a object that is a unit for the product up to a natural isomorphism. Properties of categories are most clearly expressed in terms of functors and natural transformations. We now review this formulation. On any category we have defined the identity functor 1_C. Let us assume that there also is a bifunctor $\otimes : C \times C \longrightarrow C$ defined on C.

Definition 1 *A semimonoidal category is a triple $\langle\, C, \otimes, \alpha\rangle$ where C is a category, $\otimes : C\times\ C \longrightarrow C$ is a bifunctors,*

$$\alpha : \otimes \circ (1_C \times \otimes) \longrightarrow \otimes \circ (\otimes \times 1_C)$$

is a natural isomorphism and where the following relation holds

$$\begin{aligned}(\alpha \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (\alpha \circ 1_{1_C \times 1_C \times \otimes}) = (1_\otimes \circ (\alpha \times 1_{1_C}))\\ \cdot (\alpha \circ 1_{1_C \times \otimes \times 1_C}) \cdot (1_\otimes \circ (1_{1_C} \times \alpha))\end{aligned}$$

A semimonoidal category is strict if $\otimes \circ (1_C \times \otimes) = \otimes \circ (\otimes \times 1_C)$ and $\alpha = 1_{\otimes \circ (1_C \times \otimes)}$. The relation on α given in the previous definition is the object-free formulation of the usual MacLane coherence condition for the associativity constraint α.

For any category C we have defined two bifunctors $P : C \times C \longrightarrow C$ and $Q : C \times C \longrightarrow C$. These are the projection on the first and second factor, $P(X, Y) = X$ and $Q(X, Y) = Y$ with obvious extension to arrows. Let e be a fixed object in the category C and define a constant functor $K_e : C \longrightarrow C$ by $K_e(X) = e$ and $K_e(f) = 1_e$. Using these functors we can give a definition of a monoidal category entirely in terms of functors and natural transformations.

Definition 2 *A monoidal category is a 6-tuple $\langle C\ , \otimes, K_e, \alpha, \beta, \gamma\rangle$ such that $\langle C\ , \otimes, \alpha\rangle$ is a semimonoidal category and where*

$$\begin{aligned}\beta : \otimes \circ (K_e \times 1_C) \longrightarrow Q\\ \gamma : \otimes \circ (1_C \times K_e) \longrightarrow P\end{aligned}$$

are natural isomorphisms such that the following relations holds

$$\begin{aligned}(1_\otimes \circ (\gamma \times 1_{1_C})) \cdot (\alpha \circ 1_{1_C \times K_e \times 1_C}) = (1_\otimes \circ (1_{1_C} \times \beta))\\ \beta \circ 1_{1_C \times K_e} = \gamma \circ 1_{K_e \times 1_C}\end{aligned}$$

A monoidal category is strict if $\langle C, \otimes, \alpha \rangle$ is a strict semimonoidal category and if $\otimes \circ (K_e \times 1_C) = Q$, $\otimes \circ (1_C \times K_e) = P$ and $\beta = 1_Q$,$\gamma = 1_P$.

Note that $\langle C, P, 1_{P\circ(1_C, \times P)} \rangle$ and $\langle C, Q, 1_{Q\circ(1_C, \times Q)} \rangle$ both are strict semimonoidal categories. None of them can be made into a monoidal category by selecting a unit e. However if $\otimes$ is part of a monoidal structure on C then we can reduce the product to projections by fixing the first and second argument to be the unit object.

Our definition in fact deviate somewhat from the standard formulation in terms of objects. Recall that a monoidal category in the usual sense is a 6-tuple $\langle C, \otimes, e, \alpha', \beta', \gamma' \rangle$ where $\alpha'_{X,Y,Z} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z$, $\beta'_X : e \otimes X \longrightarrow X$ and $\gamma'_X : X \otimes e \longrightarrow X$ are isomorphisms in C that are natural in X, Y, and Z and where the following MacLane Coherence [8] conditions are satisfied

$$\begin{tikzcd}
X \otimes (Y \otimes (Z \otimes T)) \arrow[r, "\alpha'_{X,Y,Z\otimes T}"] \arrow[d, "1_X \otimes \alpha'_{Y,Z,T}"] & (X \otimes Y) \otimes (Z \otimes T) \arrow[r, "\alpha'_{X\otimes Y,Z,T}"] & ((X \otimes Y) \otimes Z) \otimes T \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[rr, "\alpha'_{X,Y\otimes Z,T}"'] & & (X \otimes (Y \otimes Z)) \otimes T \arrow[u, "\alpha'_{X,Y,Z} \otimes 1_T"']
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (e \otimes Y) \arrow[rr, "\alpha' X, e, Y"] \arrow[dr, "1_X \otimes \beta'_Y"'] & & (X \otimes e) \otimes Y \arrow[dl, "\gamma'_X \otimes 1_Y"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
e \otimes e \arrow[r, shift left, "\beta'_e"] \arrow[r, shift right, "\gamma'_e"'] & e
\end{tikzcd}$$

It is easy to see that if we define

$$\begin{aligned}
\beta_{X,Y} &= \beta'_Y \\
\gamma_{X,Y} &= \gamma'_X \\
\alpha_{X,Y,Z} &= \alpha'_{X,Y,Z}
\end{aligned}$$

for all objects X and Y in C, then $\langle \otimes, K_e, \alpha, \beta, \gamma \rangle$ is a monoidal category as defined in 2. If we assume that C is a category such that for all pairs of objects there exists at least one arrow $f : X \longrightarrow X'$. Then $K_e(f) = 1_e$ and naturality of β implies the commutativity of the following diagram

$$\begin{tikzcd}
e \otimes Y \arrow[r, "\beta_{X,Y}"] \arrow[d, "1e \otimes 1_Y"'] & Y \arrow[d, "1_Y"] \\
e \otimes Y \arrow[r, "\beta_{X',Y}"'] & Y
\end{tikzcd}$$

We thus get $\beta_{X,Y} = \beta_{X',Y}$. In a similar way we find $\gamma_{X,Y} = \gamma_{X,Y'}$. This gives us a monoidal category in the usual sense if we define $\beta'_X = \beta_{Y,X}$ and $\gamma'_X = \gamma_{X,Y}$. Our aim in this paper is not to investigate generalizations of the notion of a monoidal category and we will therefore assume that solutions to the relations in 2 satisfy $\beta_{X,Y} = \beta_{X',Y}$ and $\gamma_{X,Y} = \gamma_{X,Y'}$.

We will need to express categorically the process of changing order in a product with several factors. For any category C we have the transposition functor $\tau : C \times C \longrightarrow C \times C$ defined by $\tau(X,Y) = (Y,X)$ and $\tau(f,g) = (g,f)$. A symmetry for a monoidal category is expressed using the functor τ.

Definition 3 *A symmetric monoidal category is a 7-tuple $\langle C, \otimes, K_e, \alpha, \beta, \gamma, \sigma\rangle$ such that $\langle C, \otimes, K_e, \alpha, \beta, \gamma\rangle$ is a monoidal category and where*

$$\sigma : \otimes \longrightarrow \otimes \circ \tau$$

is a natural isomorphism such that the following relations holds

$$\begin{aligned}
\sigma \circ 1_{\otimes \times 1_C} &= (\alpha^{-1} \circ 1_{\tau \times 1_C} \circ 1_{1_C \times \tau}) \cdot (1_\otimes \circ (\sigma \times 1_{1_C}) \\
&\quad \cdot (\alpha \circ 1_{1_C \times \tau}) \cdot (1_\otimes \circ (1_{1_C} \times \sigma)) \cdot \alpha^{-1} \\
\sigma \circ 1_{1_C \times \otimes} &= (\alpha \circ 1_{1_C \times \tau} \circ 1_{\tau \times 1_C}) \cdot (1_\otimes \circ (1_{1_C} \times \sigma) \circ 1_{\tau \times 1_C}) \\
&\quad \cdot (\alpha^{-1} \circ 1_{\tau \times 1_C}) \cdot (1_\otimes \circ (\sigma \times 1_{1_C})) \cdot \alpha \\
\beta &= (\gamma \circ 1_\tau) \cdot (\sigma \circ 1_{K_e \times 1_C}) \\
\gamma &= (\beta \circ 1_\tau) \cdot (\sigma \circ 1_{1_C \times K_e}) \\
\sigma \circ 1_\tau &= \sigma^{-1}
\end{aligned}$$

A symmetric monoidal category is strict if the underlying monoidal category $\langle C, \otimes, K_e, \alpha, \beta, \gamma\rangle$ is strict.

The conditions in the definition are not independent.

Proposition 4 *Let $\langle C, \otimes, K_e, \alpha, \beta, \gamma\rangle$ be a monoidal category and let $\sigma : \otimes \longrightarrow \otimes \circ \tau$ be a natural isomorphism such that $\sigma \circ 1_\tau = \sigma^{-1}$. Then the following two conditions are equivalent*

$$\begin{aligned}
\sigma \circ 1_{\otimes \times 1_C} &= (\alpha^{-1} \circ 1_{\tau \times 1_C} \circ 1_{1_C \times \tau}) \cdot (1_\otimes \circ (\sigma \times 1_{1_C})) \\
&\quad \cdot (\alpha \circ 1_{1_C \times \tau}) \cdot (1_\otimes \circ (1_{1_C} \times \sigma)) \cdot \alpha^{-1} \\
\sigma \circ 1_{1_C \times \otimes} &= (\alpha \circ 1_{1_C \times \tau} \circ 1_{\tau \times 1_C}) \cdot (1_\otimes \circ (1_{1_C} \times \sigma) \circ 1_{\tau \times 1_C}) \\
&\quad \cdot (\alpha^{-1} \circ 1_{\tau \times 1_C}) \cdot (1_\otimes \circ (\sigma \times 1_{1_C})) \cdot \alpha
\end{aligned}$$

Proof. We have the following relations $\tau \circ \tau = 1_{C \times C}$ and $\tau \circ (1_C \times \otimes) = (\otimes \times 1_C) \circ (1_C \times \tau) \circ (\tau \times 1_C)$. Using these functorial relations we have

$$\begin{aligned}
&\sigma \circ 1_{1_C \times \otimes} \\
&= \sigma \circ 1_\tau \circ 1_\tau \circ 1_{1_C \times \otimes} \\
&= \sigma \circ 1_\tau \circ 1_{\otimes \times 1_C} \circ 1_{1_C \times \tau} \circ 1_{\tau \times 1_C} \\
&= (\sigma \circ 1_{\otimes \times 1_C})^{-1} \circ 1_{1_C \times \tau} \circ 1_{\tau \times 1_C}
\end{aligned}$$

We thus have a relations between $\sigma \circ 1_{1_C \times \otimes}$ and $\sigma \circ 1_{\otimes \times 1_C}$. The equivalence of the two conditions stated in the proposition follows directly from this relation. ■

The third and fourth relations are also equivalent

Proposition 5 *Let $\langle C\ , \otimes, K_e, \alpha, \beta, \gamma\rangle$ be a monoidal category and let $\sigma : \otimes \longrightarrow \otimes \circ \tau$ be a natural isomorphism such that $\sigma \circ 1_\tau = \sigma^{-1}$. Then the following two conditions are equivalent*

$$\begin{aligned}\beta &= (\gamma \circ 1_\tau) \cdot (\sigma \circ 1_{K_e \times 1_C})\\ \gamma &= (\beta \circ 1_\tau) \cdot (\sigma \circ 1_{1_C \times K_e})\end{aligned}$$

Proof. Let the first condition be given. Then we have

$$\begin{aligned}&\beta \circ 1_\tau\\ &= ((\gamma \circ 1_\tau) \cdot (\sigma \circ 1_{K_e \times 1_C})) \circ 1_\tau\\ &= (\gamma \circ 1_\tau \circ 1_\tau) \cdot (\sigma \circ 1_{K_e \times 1_C} \circ 1_\tau)\\ &= \gamma \cdot (\sigma \circ 1_\tau \circ 1_{1_C \times K_e})\\ &= \gamma \cdot (\sigma^{-1} \circ 1_{1_C \times K_e})\end{aligned}$$

and this is equivalent to the last condition. ■

The symmetry conditions have a consequence that will play an important role.

Proposition 6 *Let $\langle C\ , \otimes, K_e, \alpha, \beta, \gamma, \sigma\rangle$ be a symmetric monoidal category. Then the following equation holds*

$$(\alpha \circ 1_{1_C \times \tau} \circ 1_{\tau \times 1_C} \circ 1_{1_C \times \tau}) \cdot (\sigma \circ (\sigma \times 1_{1_C})) \cdot \alpha = \sigma \circ (1_{1_C} \times \sigma)$$

Proof. We have

$$\begin{aligned}&\sigma \circ (1_{1_C} \times \sigma)\\ &= (\sigma \circ (1_{1_C} \times 1_{\otimes \circ \tau})) \cdot (1_\otimes \circ (1_{1_C} \times \sigma))\\ &= (\sigma \circ 1_{1_C \times \otimes} \circ 1_{1_C \times \tau}) \cdot (1_\otimes \circ (1_{1_C} \times \sigma))\\ &= (((\alpha \circ 1_{1_C \times \tau} \circ 1_{\tau \times 1_C}) \cdot (1_\otimes \circ (1_{1_C} \times \sigma) \circ 1_{1_C \times \tau})\\ &\cdot (\alpha^{-1} \circ 1_{\tau \times 1_C}) \cdot (1_\otimes \circ (\sigma \times 1_{1_C})) \cdot \alpha) \circ 1_{1_C \times \tau}) \cdot (1_\otimes \circ (1_{1_C} \times \sigma))\\ &= (\alpha \circ 1_{1_C \times \tau} \circ 1_{\tau \times 1_C} \circ 1_{1_C \times \tau}) \cdot (1_\otimes \circ (1_{1_C} \times \sigma) \circ 1_{\tau \times 1_C} \circ 1_{1_C \times \tau})\\ &\cdot (\alpha^{-1} \circ 1_{\tau \times 1_C} \circ 1_{1_C \times \tau}) \cdot (1_\otimes \circ (\sigma \times 1_{1_C}) \circ 1_{1_C \times \tau}) \cdot (\alpha \circ 1_{1_C \times \tau})\\ &\cdot (1_\otimes \circ (1_{1_C} \times \sigma))\\ &= (\alpha \circ 1_{1_C \times \tau} \circ 1_{\tau \times 1_C} \circ 1_{1_C \times \tau}) \cdot (1_\otimes \circ (1_{1_C} \times \sigma) \circ 1_{\tau \times 1_C} \circ 1_{1_C \times \tau})\\ &\cdot (\sigma \circ 1_{\otimes \times 1_C}) \cdot \alpha\\ &= (\alpha \circ 1_{1_C \times \tau} \circ 1_{\tau \times 1_C} \circ 1_{1_C \times \tau}) \cdot (1_{\otimes \circ \tau} \circ (\sigma \times 1_{1_C})) \cdot (\sigma \circ 1_{\otimes \times 1_C}) \cdot \alpha\\ &= (\alpha \circ 1_{1_C \times \tau} \circ 1_{\tau \times 1_C} \circ 1_{1_C \times \tau}) \cdot (\sigma \circ (\sigma \times 1_{1_C})) \cdot \alpha\end{aligned}$$

■

If we introduce the expressions for $\sigma \circ 1_{\otimes \times 1_C}$ and $\sigma \circ 1_{1_C \times \otimes}$ into the equation from the previous proposition we get a equation that is cubic in σ. This equation is the well known Yang-Baxter equation. In terms of object it takes in the strict case the following form

$$\begin{aligned}&(1_Z \otimes \sigma_{X,Y}) \circ (\sigma_{X,Z} \otimes 1_Y) \circ (1_X \otimes \sigma_{Y,Z})\\&= (\sigma_{Y,Z} \otimes 1_X) \circ (1_Y \otimes \sigma_{X,Z}) \circ (\sigma_{X,Y} \otimes 1_Z)\end{aligned}$$

The equation from the previous proposition is clearly equivalent to the Yang-Baxter equation in a symmetric monoidal category. We will call this equation also for the Yang-Baxter equation. A certain generalization of this equation will play a fundamental role in our theory of relations. This generalization is based on characterization of symmetries in terms of a group action.

2.2 Symmetries and group action

Let S_2 be the group of permutation of two elements with the single generator given by t. Let $\tau : C \times C \longrightarrow C \times C$ be the transposition bifunctor. The functors $T_1 = 1_C$, $T_2 = \tau$ and $T_3 = (1_C \times \tau) \circ (\tau \times 1_C) \circ (1_C \times \tau)$ defines action of the group S_2 on the categories $C, C^2 = C \times C$ and $C^3 = C \times C \times C$. Let $[C^2, C]$ and $[C^3, C]$ be the category of bifunctors and trifunctors on C with natural transformations as arrows. We can induce an action of S_2 on the functor categories $[C^2, C]$ and $[C^3, C]$ in the usual way by defining for objects F and arrows α in $[C^i, C], i = 2, 3$

$$\begin{aligned}tF &= F \circ T_i\\ta &= \alpha \circ 1_{T_i}\end{aligned}$$

It is easy to see that this really defines an action of S_2. Let us first consider the case when C is a semimonoidal category with product $\otimes$ and associativity constraint α. Note that

$$\begin{aligned}&t(\otimes \circ (1_C \times \otimes))\\&= \otimes \circ (1_C \times \otimes) \circ (\tau \times 1_C) \circ (1_C \times \tau) \circ (\tau \times 1_C)\\&= t \otimes \circ(\otimes \times 1_C) \circ (\tau \times 1_C)\\&= t \otimes \circ(t \otimes \times 1_C)\end{aligned}$$

In a similar way we find that $t(\otimes \circ (\otimes \times 1_C)) = t \otimes \circ(1_C \times t\otimes)$. We have here used the fact that $(1_C \times \tau) \circ (\tau \times 1_C) \circ (1_C \times \tau) = (\tau \times 1_C) \circ (1_C \times \tau) \circ (\tau \times 1_C)$. We therefore have a natural isomorphism

$$t\alpha^{-1} : t \otimes \circ(1_C \times t\otimes) \longrightarrow t \otimes \circ(t \otimes \times 1_C)$$

This is in fact an associativity constraint as the next proposition show

Proposition 7 $\langle C, t\otimes, t\alpha^{-1} \rangle$ *is a semimonoidal category*

Proof. Let $g = (1_C \times \tau \times 1_C) \circ (\tau \times \tau) \circ (1_C \times \tau \times 1_C) \circ (\tau \times \tau)$. Then we have

$$
\begin{aligned}
&(t\alpha^{-1} \circ 1_{t\otimes \times 1_C \times 1_C}) \cdot (t\alpha^{-1} \circ 1_{1_C \times 1_C \times t\otimes}) \\
&= (\alpha^{-1} \circ 1_{T_3} \circ 1_{t\otimes \times 1_C \times 1_C}) \cdot (\alpha^{-1} \circ 1_{T_3} \circ 1_{1_C \times 1_C \times t\otimes}) \\
&= (\alpha^{-1} \circ 1_{t\otimes \times 1_C \times 1_C} \circ 1_g) \cdot (\alpha^{-1} \circ 1_{1_C \times 1_C \times t\otimes} \circ 1_g) \\
&= [(\alpha \circ 1_{1_C \times 1_C \times t\otimes} \circ 1_g) \cdot (\alpha \circ 1_{t\otimes \times 1_C \times 1_C} \circ 1_g)]^{-1} \\
&= [((\alpha \circ 1_{1_C \times 1_C \times t\otimes}) \cdot (\alpha \circ 1_{t\otimes \times 1_C \times 1_C})) \circ 1_g]^{-1} \\
&= [((1_\otimes \circ (\alpha \times 1_{1_C})) \cdot (\alpha \circ 1_{1_C \times \otimes \times 1_C}) \cdot (1_\otimes \circ (1_{1_C} \times \alpha))) \circ 1_g]^{-1} \\
&= ((1_\otimes \circ (1_{1_C} \times \alpha^{-1})) \cdot (\alpha^{-1} \circ 1_{1_C \times \otimes \times 1_C}) \cdot (1_\otimes \circ (\alpha^{-1} \times 1_{1_C}))) \circ 1_g \\
&= (1_\otimes \circ (1_{1_C} \times \alpha^{-1}) \circ 1_g) \cdot (\alpha^{-1} \circ 1_{1_C \times \otimes \times 1_C} \circ 1_g) \cdot (1_\otimes \circ (\alpha^{-1} \times 1_{1_C}) \circ 1_g) \\
&= (1_{t\otimes} \circ (t\alpha^{-1} \times 1_{1_C})) \cdot (t\alpha^{-1} \circ 1_{1_C \times t\otimes \times 1_C}) \cdot (1_{t\otimes} \circ (1_{1_C} \times t\alpha^{-1}))
\end{aligned}
$$

■

Let us assume that there exists a natural isomorphism $\sigma : \otimes \longrightarrow t\otimes$ and let α be a associativity constraint for a semimonoidal category $\langle C, \otimes, \alpha \rangle$. Then $t\alpha^{-1} : t\otimes \circ (1_C \times t\otimes) \longrightarrow t\otimes \circ (t\otimes \times 1_C)$ is a associativity constraint for a semimonoidal category $\langle C, \otimes, t\alpha^{-1} \rangle$. On the other hand we have natural isomorphisms

$$
\begin{aligned}
\sigma \circ (1_{1_C} \times \sigma) &: t\otimes \circ (1_C \times t\otimes) \longrightarrow \otimes \circ (1_C \times \otimes) \\
\sigma \circ (\sigma \times 1_{1_C}) &: t\otimes \circ (t\otimes \times 1_C) \longrightarrow \otimes \circ (\otimes \times 1_C)
\end{aligned}
$$

We therefore have a natural isomorphism $\widehat{\alpha} : \otimes \circ (1_C \times \otimes) \longrightarrow \otimes \circ (\otimes \times 1_C)$ where we have defined

$$
\widehat{\alpha} = (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \cdot t\alpha^{-1} \cdot (\sigma \circ (1_{1_C} \times \sigma))
$$

This new isomorphism also an associativity constraint.

Proposition 8 *$\langle C, \otimes, \widehat{\alpha} \rangle$ is a semimonoidal category.*

Proof. We only need to show that the MacLane coherence condition hold for $\widehat{\alpha}$. Let us first observe that

$$
\begin{aligned}
&(\sigma \circ (1_{1_C} \times \sigma) \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ 1_{1_C \times 1_C \times \otimes}) \\
&= (\sigma \circ (1_{1_C} \times \sigma) \circ (1_{\otimes} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ (1_{1_C} \times 1_{1_C} \times 1_{\otimes})) \\
&= (\sigma \circ ((1_{1_C} \circ 1_{\otimes}) \times (\sigma \circ (1_{1_C} \times 1_{1_C})))) \\
&\cdot (\sigma^{-1} \circ ((\sigma^{-1} \circ (1_{1_C} \times 1_{1_C})) \times (1_{1_C} \circ 1_{\otimes}))) \\
&= (\sigma \circ (1_{\otimes} \times \sigma)) \cdot (\sigma^{-1} \circ (\sigma^{-1} \times 1_{\otimes})) \\
&= (1_{t\otimes} \circ (\sigma^{-1} \times \sigma)) \\
&= (1_{t\otimes} \circ (\sigma^{-1} \times 1_{t\otimes})) \cdot (1_{t\otimes} \circ (1_{t\otimes} \times \sigma)) \\
&= (1_{t\otimes} \circ ((1_{1_C} \circ \sigma^{-1}) \times (1_{t\otimes} \circ (1_{1_C} \times 1_{1_C})))) \\
&\cdot (1_{t\otimes} \circ ((1_{t\otimes} \circ (1_{1_C} \times 1_{1_C})) \times (1_{1_C} \circ \sigma))) \\
&= (1_{t\otimes} \circ (1_{1_C} \times 1_{t\otimes}) \circ (\sigma^{-1} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot (1_{t\otimes} \circ (1_{t\otimes} \times 1_{1_C}) \circ (1_{1_C} \times 1_{1_C} \times \sigma))
\end{aligned}
$$

Let $g = (1_C \times \tau \times 1_C) \circ (\tau \times \tau) \circ (1_C \times \tau \times 1_C) \circ (\tau \times \tau)$.

Using the previous identity we have for the left hand side of the coherence condition

$$
\begin{aligned}
&(\widehat{\alpha} \circ 1_{\otimes \times 1_C \times 1_C}) \circ (\widehat{\alpha} \circ 1_{1_C \times 1_C \times \otimes}) \\
&= (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (t\alpha^{-1} \circ 1_{\otimes \times 1_C \times 1_C}) \\
&\cdot (\sigma \circ (1_{1_C} \times \sigma) \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ 1_{1_C \times 1_C \times \otimes}) \\
&(t\alpha^{-1} \circ 1_{1_C \times 1_C \times \otimes}) \cdot (\sigma \circ (1_{1_C} \times \sigma) \circ 1_{1_C \times 1_C \times \otimes})
\end{aligned}
$$

$$
\begin{aligned}
&= (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (t\alpha^{-1} \circ (1_\otimes \times 1_{1_C} \times 1_{1_C})) \\
&\cdot (1_{t\otimes} \circ (1_{1_C} \times 1_{t\otimes}) \circ (\sigma^{-1} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot (1_{t\otimes} \circ (1_{t\otimes} \times 1_{1_C}) \circ (1_{1_C} \times 1_{1_C} \times \sigma)) \\
&\cdot (t\alpha^{-1} \circ (1_{1_C} \times 1_{1_C} \times 1_\otimes)) \cdot (\sigma \circ (1_{1_C} \times \sigma) \circ 1_{1_C \times 1_C \times \otimes}) \\
&= (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ (1_\otimes \times 1_{1_C} \times 1_{1_C})) \cdot (t\alpha^{-1} \circ (\sigma^{-1} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot (t\alpha^{-1} \circ (1_{1_C} \times 1_{1_C} \times \sigma)) \cdot (\sigma \circ (1_{1_C} \times \sigma) \circ (1_{1_C} \times 1_{1_C} \times 1_\otimes)) \\
&= (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ (\sigma^{-1} \times 1_{1_C} \times 1_{1_C})) \cdot (\alpha^{-1} \circ 1_{T_3} \circ (1_{t\otimes} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot (\alpha^{-1} \circ 1_{T_3} \circ (1_{1_C} \times 1_{1_C} \times 1_{t\otimes})) \cdot (\sigma \circ (1_{1_C} \times \sigma) \circ (1_{1_C} \times 1_{1_C} \times \sigma)) \\
&= (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ (\sigma^{-1} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot (\alpha^{-1} \circ 1_{1_C \times 1_C \times \otimes} \circ 1_g) \cdot (\alpha^{-1} \circ 1_{\otimes \times 1_C \times 1_C} \circ 1_g) \\
&\cdot (\sigma \circ (1_{1_C} \times \sigma) \circ (1_{1_C} \times 1_{1_C} \times \sigma)) \\
&= (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ (\sigma^{-1} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot ([(\alpha \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (\alpha \circ 1_{1_C \times 1_C \times \otimes})]^{-1} \circ 1_g) \\
&\cdot (\sigma \circ (1_{1_C} \times \sigma) \circ (1_{1_C} \times 1_{1_C} \times \sigma)) \\
&= (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ (\sigma^{-1} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot ([(1t \circ (\alpha \times 1_{1_C})) \cdot (\alpha \circ 1_{1_C \times \otimes \times 1_C}) \cdot (1_\otimes \circ (1_{1_C} \times \alpha))]^{-1} \circ 1_g) \\
&\cdot (\sigma \circ (1_{1_C} \times \sigma) \circ (1_{1_C} \times 1_{1_C} \times \sigma)) \\
&= (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ (\sigma^{-1} \times 1_{1_C} \times 1_{1_C})) \cdot (1_\otimes \circ (1_{1_C} \times \alpha^{-1}) \circ 1_g) \\
&\cdot (\alpha^{-1} \circ 1_{1_C \times \otimes \times 1_C} \circ 1_g) \cdot (1_\otimes \circ (\alpha^{-1} \times 1_{1_C}) \circ 1_g) \\
&\cdot (\sigma \circ (1_{1_C} \times \sigma) \circ (1_{1_C} \times 1_{1_C} \times \sigma))
\end{aligned}
$$

For evaluating the righthand side of the MacLane condition we need the following two identities

$$
\begin{aligned}
&(1_\otimes \circ ((\sigma \circ (1_{1_C} \times \sigma)) \times 1_{1_C})) \cdot (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ 1_{1_C \times \otimes \times 1_C}) \\
&= (1_\otimes \circ (\sigma \times 1_{1_C}) \circ (1_{1_C} \times 1_\otimes \times 1_{1_C})) \\
&\cdot (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ (1_{1_C} \times 1_\otimes \times 1_{1_C})) \\
&= (\sigma^{-1} \circ (1_{t\otimes} \times 1_{1_C}) \circ (1_{1_C} \times \sigma \times 1_{1_C})) \\
&= (\sigma^{-1} \circ (1_{t\otimes} \times 1_{1_C}) \circ (1_{1_C} \times 1_{t\otimes} \times 1_{1_C})) \\
&\cdot (1_{t\otimes} \circ (1_{t\otimes} \times 1_{1_C}) \circ (1_{1_C} \times 1_\otimes \times 1_{1_C}))
\end{aligned}
$$

and

$$
\begin{aligned}
&(\sigma \circ (1_{1_C} \times \sigma) \circ 1_{1_C \times \otimes \times 1_C}) \cdot (1_\otimes \circ (1_{1_C} \times (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}))))\\
&= (\sigma \circ (1_{1_C} \times \sigma) \circ (1_{1_C} \times 1_\otimes \times 1_{1_C}))\\
&\cdot (1_\otimes \circ (1_{1_C} \times \sigma^{-1}) \circ (1_{1_C} \times \sigma^{-1} \times 1_{1_C}))\\
&= (\sigma \circ (1_{1_C} \times 1_{t\otimes}) \circ (1_{1_C} \times \sigma^{-1} \times 1_{1_C}))\\
&= (1_{t\otimes} \circ (1_{1_C} \times 1_{t\otimes}) \circ (1_{1_C} \times \sigma^{-1} \times 1_{1_C}))\\
&\cdot (\sigma \circ (1_{1_C} \times 1_{t\otimes}) \circ (1_{1_C} \times 1_{t\otimes} \times 1_{1_C}))
\end{aligned}
$$

Using these identities we have for the righthand side of the MacLane condition

$$
\begin{aligned}
&(1_{\otimes} \circ (\widehat{\alpha} \times 1_{1_C})) \cdot (\widehat{\alpha} \circ 1_{1_C \times \otimes \times 1_C}) \cdot (1_{\otimes} \times (1_{1_C} \times \widehat{\alpha}))\\
&= (1_{\otimes} \circ ([(\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \cdot t\alpha^{-1} \cdot (\sigma \circ (1_{1_C} \times \sigma))] \times 1_{1_C}))\\
&\cdot ([(\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \cdot t\alpha^{-1} \cdot (\sigma \circ (1_{1_C} \times \sigma))] \circ 1_{1_C \times \otimes \times 1_C})\\
&\cdot (1_{\otimes} \circ (1_{1_C} \times [(\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \cdot t\alpha^{-1} \cdot (\sigma \circ (1_{1_C} \times \sigma))]))\\
&= (1_{\otimes} \circ ((\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \times 1_{1_C}) \cdot (1_{\otimes} \circ (t\alpha^{-1} \times 1_{1_C}))\\
&\cdot (1_{\otimes} \circ ((\sigma \circ (1_{1_C} \times \sigma)) \times 1_{1_C})) \cdot (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ 1_{1_C \times \otimes \times 1_C})\\
&\cdot (t\alpha^{-1} \circ 1_{1_C \times \otimes \times 1_C}) \cdot (\sigma \circ (1_{1_C} \times \sigma) \circ 1_{1_C \times \otimes \times 1_C})\\
&\cdot (1_{\otimes} \circ (1_{1_C} \times (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})))) \cdot (1_{\otimes} \circ (1_{1_C} \times t\alpha^{-1}))\\
&\cdot (1_{\otimes} \circ (1_{1_C} \times (\sigma \circ (1_{1_C} \times \sigma))))\\
&= (1_{\otimes} \circ (\sigma^{-1} \times 1_{1_C}) \circ (\sigma^{-1} \times 1_{1_C} \times 1_{1_C})) \cdot (1_{\otimes} \circ (t\alpha^{-1} \times 1_{1_C}))\\
&\cdot (\sigma^{-1} \circ (1_{t\otimes} \times 1_{1_C}) \circ (1_{1_C} \times 1_{t\otimes} \times 1_{1_C}))\\
&\cdot (1_{t\otimes} \circ (1_{t\otimes} \times 1_{1_C}) \circ (1_{1_C} \times \sigma \times 1_{1_C})) \cdot (t\alpha^{-1} \circ (1_{1_C} \times 1_{\otimes} \times 1_{1_C}))\\
&\cdot (1_{t\otimes} \circ (1_{1_C} \times 1_{t\otimes}) \circ (1_{1_C} \times \sigma^{-1} \times 1_{1_C}))\\
&\cdot (\sigma \circ (1_{1_C} \times 1_{t\otimes}) \circ (1_{1_C} \times 1_{t\otimes} \times 1_{1_C})) \cdot (1_{\otimes} \circ (1_{1_C} \times t\alpha^{-1}))\\
&\cdot (1_{\otimes} \circ (1_{1_C} \times \sigma) \circ (1_{1_C} \times 1_{1_C} \times \sigma))\\
&= (1_{\otimes} \circ (\sigma^{-1} \times 1_{1_C}) \circ (\sigma^{-1} \times 1_{1_C} \times 1_{1_C})) \cdot (1_{\otimes} \circ (t\alpha^{-1} \times 1_{1_C}))\\
&\cdot (\sigma^{-1} \circ ((1_{t\otimes} \circ (1_{1_C} \times 1_{t\otimes})) \times 1_{1_C})) \cdot (1_{t\otimes} \circ (1_{t\otimes} \times 1_{1_C}) \circ (1_{1_C} \times \sigma \times 1_{1_C}))\\
&\cdot (t\alpha^{-1} \circ (1_{1_C} \times 1_{\otimes} \times 1_{1_C})) \cdot (1_{t\otimes} \circ (1_{1_C} \times 1_{t\otimes}) \circ (1_{1_C} \times \sigma^{-1} \times 1_{1_C}))\\
&\cdot (\sigma \circ (1_{1_C} \times (1_{t\otimes} \circ (1_{t\otimes} \times 1_{1_C})))) \cdot (1_{\otimes} \circ (1_{1_C} \times t\alpha^{-1}))\\
&\cdot (1_{\otimes} \circ (1_{1_C} \times \sigma) \circ (1_{1_C} \times 1_{1_C} \times \sigma))\\
&= (1_{\otimes} \circ ((\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \times 1_{1_C})) \cdot (\sigma^{-1} \circ (t\alpha^{-1} \times 1_{1_C}))\\
&\cdot (t\alpha^{-1} \circ (1_{1_C} \times 1_{t\otimes} \times 1_{1_C})) \cdot (\sigma \circ (1_{1_C} \times t\alpha^{-1}))\\
&\cdot (1_{\otimes} \circ (1_{1_C} \times (\sigma \circ (1_{1_C} \times \sigma))))\\
&= (\sigma^{-1} \circ ((\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \times 1_{1_C})) \cdot (1_{t\otimes} \circ (t\alpha^{-1} \times 1_{1_C}))\\
&\cdot (t\alpha^{-1} \circ (1_{1_C} \times 1_{t\otimes} \times 1_{1_C})) \cdot (1_{t\otimes} \circ (1_{1_C} \times t\alpha^{-1}))\\
&\cdot (\sigma \circ (1_{1_C} \times (\sigma \circ (1_{1_C} \times \sigma))))\\
&= (\sigma^{-1} \circ ((\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \times 1_{1_C})) \cdot (1_{\otimes} \circ (1_{1_C} \times \alpha^{-1}) \circ 1_g)\\
&\cdot (\alpha^{-1} \circ 1_{1_C \times \otimes \times 1_C} \circ 1_g) \cdot (1_{\otimes} \circ (\alpha^{-1} \times 1_{1_C}) \circ 1_g)\\
&\cdot (\sigma \circ (1_{1_C} \times (\sigma \circ (1_{1_C} \times \sigma))))
\end{aligned}
$$

The lefthand side and the righthand side are thus equal and this proves the proposition. ■

Let us define $S_{C,\otimes} = \{\alpha \mid \langle C, \otimes, \alpha\rangle$ is a semimonoidal category $\}$. Then the previous proposition show that for each natural isomorphism $\sigma : \otimes \longrightarrow t\otimes$ we have a mapping of $S_{C,\otimes}$ to itself.

Let us next consider the case of a monoidal category $\langle C, \otimes, K_e, \alpha, \beta, \gamma \rangle$. Using the natural isomorphism σ we can define new natural isomorphisms

$$\widehat{\beta} = (t\gamma) \cdot (\sigma \circ 1_{K_e \times 1_C}) : \otimes \circ (K_e \times 1_C) \longrightarrow Q$$
$$\widehat{\gamma} = (t\beta) \cdot (\sigma \circ 1_{1_C \times K_e}) : \otimes \circ (1_C \times K_e) \longrightarrow P$$

For $\widehat{\alpha}$ and the two natural isomorphisms $\widehat{\beta}$ and $\widehat{\gamma}$ we have

Proposition 9 *$\langle C, \otimes, K_e, \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma} \rangle$ is a monoidal category*

Proof. The First MacLane coherence condition has already been verified. For the second MacLane condition we need the identities

$$\begin{aligned}
&(1_\otimes \circ ((\sigma \circ 1_{1_C \times K_e}) \times 1_{1_C})) \cdot ((\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \circ 1_{1_C \times K_e \times 1_C})\\
&= (1_\otimes \circ (\sigma \times 1_{1_C}) \circ (1_{1_C} \times 1_{K_e} \times 1_{1_C}))\\
&\cdot (\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C}) \circ (1_{1_C} \times 1_{K_e} \times 1_{1_C}))\\
&= (\sigma^{-1} \circ (1_{t\otimes} \times 1_{1_C}) \circ (1_{1_C} \times 1_{K_e} \times 1_{1_C}))\\
&= (\sigma^{-1} \circ (1_{t\otimes} \circ (1_{1_C} \times 1_{K_e}) \times 1_{1_C}))
\end{aligned}$$

and

$$\begin{aligned}
&(1_{t\otimes} \circ (t\beta \times 1_{1_C})) \cdot (t\alpha^{-1} \circ 1_{1_C \times K_e \times 1_C})\\
&= (1_\otimes \circ (1_{1_C} \times \beta) \circ 1_{T_3}) \cdot (\alpha^{-1} \circ 1_{1_C \times K_e \times 1_C} \circ 1_{T_3})\\
&= (((1_\otimes \circ (1_{1_C} \times \beta)) \cdot (\alpha^{-1} \circ 1_{1_C \times K_e \times 1_C})) \circ 1_{T_3})\\
&= (1_\otimes \circ (\gamma \times 1_{1_C}) \circ 1_{T_3})
\end{aligned}$$

Using these two identities we have

$$\begin{aligned}
&(1_\otimes \circ (\widehat{\gamma} \times 1_{1_C})) \cdot (\widehat{\alpha} \circ 1_{1_C \times K_e \times 1_C})\\
&= (1_\otimes \circ ((t\beta \cdot (\sigma \circ 1_{1_C \times K_e})) \times 1_{1_C}))\\
&\cdot (((\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \cdot t\alpha^{-1} \cdot (\sigma \circ (1_{1_C} \times \sigma))) \circ 1_{1_C \times K_e \times 1_C})\\
&= (1_\otimes \circ (t\beta \times 1_{1_C})) \cdot (1_\otimes \circ ((\sigma \circ 1_{1_C \times K_e}) \times 1_{1_C}))\\
&\cdot ((\sigma^{-1} \circ (\sigma^{-1} \times 1_{1_C})) \circ 1_{1_C \times K_e \times 1_C}) \cdot (t\alpha^{-1} \circ 1_{1_C \times K_e \times 1_C})\\
&\cdot ((\sigma \circ (1_{1_C} \times \sigma)) \circ 1_{1_C \times K_e \times 1_C})\\
&= (1_\otimes \circ (t\beta \times 1_{1_C})) \cdot (\sigma^{-1} \circ (1_{t\otimes} \circ (1_{1_C} \times 1_{K_e}) \times 1_{1_C})) \cdot (t\alpha^{-1} \circ 1_{1_C \times K_e \times 1_C})\\
&\cdot ((\sigma \circ (1_{1_C} \times \sigma)) \circ 1_{1_C \times K_e \times 1_C})\\
&= (\sigma^{-1} \circ (1_P \times 1_{1_C})) \cdot (1_{t\otimes} \circ (t\beta \times 1_{1_C})) \cdot (t\alpha^{-1} \circ 1_{1_C \times K_e \times 1_C})\\
&\cdot ((\sigma \circ (1_{1_C} \times \sigma)) \circ 1_{1_C \times K_e \times 1_C})\\
&= (\sigma^{-1} \circ (1_P \times 1_{1_C})) \cdot (1_\otimes \circ (\gamma \times 1_{1_C}) \circ 1_{T_3}) \cdot ((\sigma \circ (1_{1_C} \times \sigma)) \circ 1_{1_C \times K_e \times 1_C})\\
&= (\sigma^{-1} \circ 1_{P \times 1_C}) \cdot (1_{t\otimes} \circ (1_{1_C} \times t\gamma)) \cdot (\sigma \circ (1_{1_C} \times \sigma) \circ 1_{1_C \times K_e \times 1_C})\\
&= (\sigma^{-1} \circ 1_{1_C \times Q}) \cdot (1_{t\otimes} \circ (1_{1_C} \times t\gamma)) \cdot (\sigma \circ (1_{1_C} \times (\sigma \circ (1_{K_e} \times 1_{1_C}))))\\
&= (1_\otimes \circ (1_{1_C} \times [t\gamma \cdot (\sigma \circ 1_{K_e \times 1_C})]))\\
&= (1_\otimes \circ (1_{1_C} \times \widehat{\beta}))
\end{aligned}$$

For the last MacLane condition we have

$$\begin{aligned}
&\widehat{\beta} \circ 1_{1_C \times K_e} \\
&= (t\gamma \circ 1_{1_C \times K_e}) \cdot (\sigma \circ 1_{K_e \times 1_C} \circ 1_{1_C \times K_e}) \\
&= (\gamma \circ 1_\tau \circ 1_{1_C \times K_e}) \cdot (\sigma \circ 1_{K_e \times K_e}) \\
&= (\gamma \circ 1_{K_e \times 1_C} \circ 1_\tau) \cdot (\sigma \circ 1_{K_e \times K_e}) \\
&= (\beta \circ 1_{1_C \times K_e} \circ 1_\tau) \cdot (\sigma \circ 1_{K_e \times K_e}) \\
&= (t\beta \circ 1_{K_e \times 1_C}) \cdot (\sigma \circ 1_{1_C \times K_e} \circ 1_{K_e \times 1_C}) \\
&= (t\beta \cdot (\sigma \circ 1_{1_C \times K_e})) \circ 1_{K_e \times 1_C} \\
&= \widehat{\gamma} \circ 1_{K_e \times 1_C}
\end{aligned}$$

■

Let $M_{C,\otimes,e} = \{(\alpha, \beta, \gamma) \mid \langle C, \otimes, K_e, \alpha, \beta, \gamma\rangle$ is a monoidal category$\}$. Then the previous proposition show that for each natural isomorphism $\sigma : \otimes \longrightarrow t\otimes$ we have a map

$$T_t(\sigma) : M_{C,\otimes,e} \longrightarrow M_{C,\otimes,e}$$

defined by $T_t(\sigma)(\alpha, \beta, \gamma) = (\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$. Let us next for each $\rho : \otimes \longrightarrow \otimes$ define a map on elements in $M_{C,\otimes,e}$

$$T_1(\rho)(\alpha, \beta, \gamma) = (\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma})$$

where we have

$$\begin{aligned}
\widetilde{\alpha} &= (\rho^{-1} \circ (\rho^{-1} \times 1_{1_C})) \cdot \alpha \cdot (\rho \circ (1_{1_C} \times \rho)) \\
\widetilde{\beta} &= \beta \cdot (\rho \circ 1_{K_e \times 1_C}) \\
\widetilde{\gamma} &= \gamma \cdot (\rho \circ 1_{1_C \times K_e})
\end{aligned}$$

For this map we have

Proposition 10 $T_1(\rho) : M_{C,\otimes,e} \longrightarrow M_{C,\otimes,e}$

The proof of this proposition is similar to the one for the map $T_1(\sigma)$ and is not reproduced here.

Let

$$G_{C,\otimes,e} = \{T_t(\sigma), T_1(\rho) \mid \sigma : \otimes \longrightarrow t\otimes, \rho : \otimes \longrightarrow \otimes \text{ , } \sigma, \rho \text{ natural isomorphisms}\}$$

¿From the construction it is evident that all maps in $G_{C,\otimes,e}$ are bijections. The next proposition show that $G_{C,\otimes,e}$ is closed under composition of maps.

Proposition 11 *Let $\sigma_1, \sigma_2 : \otimes \longrightarrow t\otimes$ and $\rho_1, \rho_2 : \otimes \longrightarrow \otimes$ be natural isomorphisms. Then we have*

$$\begin{aligned}
T_t(\sigma_2) \circ T_t(\sigma_1) &= T_1(t\sigma_1 \cdot \sigma_2) \\
T_t(\sigma_1) \circ T_1(\rho_1) &= T_t(\rho_1 \cdot t\sigma_1) \\
T_1(\rho_1) \circ T_t(\sigma_1) &= T_t(\sigma_1 \cdot \rho_1) \\
T_1(\rho_2) \circ T_1(\rho_1) &= T_1(\rho_1 \cdot \rho_2)
\end{aligned}$$

The proof of this proposition is routine and is left out. The set $G_{C,\otimes,e}$ is thus closed under composition and contains the identity map $T_1(1_\otimes) = 1_{M_{C,\otimes,e}}$.All maps in the set $G_{C,\otimes,e}$ are invertible by construction and $G_{C,\otimes,e}$ is closed under the operation of taking the inverse of a map. We have

$$\begin{aligned} T_1(\rho) \circ T_1(\rho^{-1}) &= 1_{M_{C,\otimes,e}} \\ T_t(\sigma) \circ T_t((t\sigma)^{-1}) &= 1_{M_{C,\otimes,e}} \end{aligned}$$

The previous propositions can now be restated in the following way.

Corollary 12 *The set $M_{C,\otimes,e}$ of monoidal structures on C corresponding to a fixed product $\otimes$ and unit e is invariant under the action of the S_2-graded group $G_{C,\otimes,e}$.*

We can use the S_2-graded group $G_{C,\otimes,e}$ to give an interpretation of the notion of a symmetric monoidal category.

Proposition 13 *Let $\langle C, \otimes, K_e, \alpha, \beta, \gamma, \sigma\rangle$ be a symmetric monoidal category. Then $H = \{T_t(\sigma), 1_\otimes\}$ is a S_2 graded subgroup of $G_{C,\otimes,e}$ and $(\alpha, \beta, \gamma) \in M_{C,\otimes,e}$ is a fixpoint for the action of H.*

This gives an interpretation of the Yang-Baxter equation and the two unit conditions in terms of invariance with respect to the action by the group H. No such interpretation appears to be possible for the first two conditions from the definition 3, of a symmetry. These two conditions appear to be of a technical nature.

2.3 σ-commutative comonoids in symmetric monoidal categories

Recall that a comonoid in a monoidal category is a triple $\langle A, \delta_A, \epsilon_A\rangle$ where A is a object in the category and $\delta_A : A \longrightarrow A \otimes A$ and $\epsilon_A : A \longrightarrow e$ are morphisms in the category such that the following diagrams commute

$$\begin{tikzcd}
A \otimes (A \otimes A) \arrow[d, "\alpha_{A,A,A}"'] & A \otimes A \arrow[l, "1_A \otimes \delta_A"'] \arrow[dl, "\delta_A \otimes 1_A"] & A \arrow[l, "\delta_A"'] \\
(A \otimes A) \otimes A & &
\end{tikzcd}$$

$$\begin{tikzcd}
e \otimes A \arrow[dr, "\beta_A"'] & A \otimes A \arrow[l, "\epsilon_A \otimes 1_A"'] & A \otimes e \arrow[l, "1_A \otimes \epsilon_A"'] \arrow[dl, "\gamma_A"] \\
& A \arrow[u, "\delta_A"] &
\end{tikzcd}$$

The simpler structure $\langle A, \delta_A \rangle$ is called a cosemigroup. The morphism ϵ_A is the counit for the comonoid and δ_A is called the coproduct.

Before we proceed with formal developments we will first consider some examples of these constructions. Let us first consider the case of sets. The category $Sets$ is a monoidal category with cartesian product, $\times$ as bifunctor. The neutral object is the one point set $e = \{*\}$. The associativity constraints $\alpha_{A,B,C} : A \times (B \times C) \longrightarrow (A \times B) \times C$ and unit constraints $\beta_A : e \otimes A \longrightarrow A$ and $\gamma_A : A \otimes e \longrightarrow A$ given by

$$\begin{aligned} \alpha_{A,B,C}(x, (y, z)) &= ((x, y), z) \\ \beta_A(*, x) &= x \\ \gamma_A(x, *) &= x \end{aligned}$$

Finite sets offer many examples of cosemigroups. Let $A = \{a, b, c\}$ and define a map $\delta_A : A \longrightarrow A \times A$ by

$$\begin{aligned} \delta_A(a) &= (a, a) \\ \delta_A(b) &= (b, a) \\ \delta_A(c) &= (a, c) \end{aligned}$$

A direct calculation show that $\langle A, \delta_A \rangle$ is a cosemigroup. There is only one possible map $\epsilon_A : A \longrightarrow e$ since $e = \{*\}$ is terminal is $Sets$ and this is the map $\epsilon_A(x) = *$ for all $x \in A$. But for this map we find

$$[\beta_A \circ (\epsilon_A \otimes 1_A) \circ \delta_A](b) = [\beta_A \circ (\epsilon_A \otimes 1_A)](b, a) = \beta_A(*, a) = a$$

so $\langle A, \delta_A, \epsilon_A \rangle$ is not a comonoid.

Let A be any set. Define the map $\delta_A : A \longrightarrow A \times A$ by

$$\delta_A(x) = (x, x)$$

This is the diagonal map in $Sets$. We then have

$$\begin{aligned} [\alpha_{A,A,A} \circ (1_A \times \delta_A) \circ \delta_A](x) &= [\alpha_{A,A,A} \circ (1_A \times \delta_A)](x, x) = ((x, x), x) \\ [(\delta_A \times 1_A) \circ \delta_A](x) &= (\delta_A \times 1_A)(x, x) = ((x, x), x) \end{aligned}$$

so $\langle A, \delta_A \rangle$ is a cosemigroup. The only possible counit satisfy

$$\begin{aligned} [\beta_A \circ (\epsilon_A \otimes 1_A) \circ \delta_A](x) &= [\beta_A \circ (\epsilon_A \otimes 1_A)](x, x) = \beta_A(*, x) = x \\ [\gamma_A \circ (1_A \otimes \epsilon_A) \circ \delta_A](x) &= [\gamma_A \circ (1_A \otimes \epsilon_A)](x, x) = \gamma_A(x, *) = x \end{aligned}$$

so $\langle A, \delta_A, \epsilon_A \rangle$ is a comonoid. Let $\delta_A : A \longrightarrow A \times A$, $\epsilon_A : A \longrightarrow \{*\}$ be any comonoid structure on A. We have $\delta_A(a) = (f(a), g(a))$ and $\epsilon_A(a) = *$. The first counit condition $\beta_A \circ (\epsilon_A \times 1_A) \circ \delta_A = 1_A$ gives $g(a) = a$ for all a. Similarly the second counit condition gives $f(a) = a$ for all a. So the previous example in fact gives the only possible comonoid structure in this category. We will always assume that the objects in $Sets$ are comonoid with this structure.

As our next example let us consider a pointed set. This is a set A with a chosen point $x_0 \in A$. Define a map $\delta_A : A \longrightarrow A \times A$ by $\delta_A(x) = (x_0, x)$. Then we have

$$[(1_A \times \delta_A) \circ \delta_A](x) = (1_A \times \delta_A)(x_0, x) = (x_0, x_0, x)$$
$$[(\delta_A \times 1_A) \circ \delta_A](x) = (\delta_A \times 1_A)(x_0, x) = (x_0, x_0, x)$$

so $\langle A, \delta_A \rangle$ is a cosemigroup. It is not a comonoid because the only possible map $\epsilon_A : A \longrightarrow e$ gives

$$[\gamma_A \circ (1_A \otimes \epsilon_A) \circ \delta_A](x) = [\gamma_A \circ (1_A \otimes \epsilon_A)](x_0, x) = \gamma_A(x_0, *) = x_0$$

so if there are any elements in A different from x_0 then A is not a comonoid. This construction only gives a comonoid when $A = e$. This fact is true for any monoidal category.

Let us next consider the category $Vect_k$. This is the category of vectorspaces over a field k with morphisms given by linear maps. This category is monoidal with product bifunctor given by the tensorproduct of vectorspaces $\otimes = \otimes_k$. The neutral object is k. The associativity constraint α and unit constraints β and γ for this case are the linear maps $\alpha_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$, $\beta_A : k \otimes A \longrightarrow A$ and $\gamma_A : A \otimes k \longrightarrow A$ given on generators by

$$\alpha_{A,B,C}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z$$
$$\beta_A(r \otimes x) = rx$$
$$\gamma_A(x \otimes r) = rx$$

Let A be any finite dimensional vectorspace in $Vect_k$. Let Ω be a finite index set and let $\{a_i\}_{i \in \Omega}$ be a basis for A indexed by Ω. Then $\{a_i \otimes a_{i'}\}_{i,i' \in \Omega}$ is a basis for $A \otimes A$. Define a linear map $\delta_A : A \longrightarrow A \otimes A$ by

$$\delta_A(a_i) = a_i \otimes a_i$$

Then evidently $\langle A, \delta_A \rangle$ is a cosemigroup. Define a linear map $\epsilon_A : A \longrightarrow k$ on generators by $\epsilon_A(a_i) = 1 \in k$. Then we have

$$[\beta_A \circ (\epsilon_A \otimes 1_A) \circ \delta_A](a_i) = [\beta_A \circ (\epsilon_A \otimes 1_A)](a_i, a_i) = \beta_A(1 \otimes a_i) = a_i$$
$$[\gamma_A \circ (1_A \otimes \epsilon_A) \circ \delta_A](a_i) = [\gamma_A \circ (1_A \otimes \epsilon_A)](a_i, a_i) = \gamma_A(a_i \otimes 1) = a_i$$

so $\langle A, \delta_A, \epsilon_A \rangle$ is a comonoid. In contrast to the case of $Sets$ we can have many nonisomorphic comonoid structures on a given object in $Vect_k$. Let δ_A : $A \longrightarrow A \otimes A$ and $\epsilon_A : A \longrightarrow k$ be linear maps. We have thus

$$\delta_A(a_i) = \sum_{j,k} r^i_{j,k} a_j \otimes a_k$$
$$\epsilon_A(a_i) = q_i$$

where all indices run from 1 to m, the dimension of A.

Then $\langle A, \delta_A, \epsilon_A \rangle$ is a comonoid if $\{r^i_{j,k}\}$ and $\{q_i\}$ are solutions of the following system of quadratic equations.

$$\sum_j (r^i_{j,k} r^j_{l,n} - r^i_{l,j} r^j_{n,k}) = 0 \quad \text{for all } i, k, l, n.$$

$$\sum_j r^i_{j,k} q_j = \delta_{i,k} \text{ for all } i, k.$$

$$\sum_j r^i_{k,j} q_j = \delta_{i,k} \text{ for all } i, k.$$

For $m = 2$ this system have four different families of solutions. One of these families is the following

$$\begin{aligned}
&\delta_A(a_1) = a_1 \otimes a_1 \\
&\delta_A(a_2) = -x a_1 \otimes a_1 + a_1 \otimes a_2 + a_2 \otimes a_1 \\
&q_A(a_1) = 1 \\
&q_A(a_2) = x
\end{aligned}$$

where x is a arbitrary element of k.

Let now G be a finite group and let $A = \mathcal{F}(G)$ be the vectorspace of k valued functions on G.

Note that since G is finite we have $\mathcal{F}(G \times G) \approx \mathcal{F}(G) \otimes_k \mathcal{F}(G)$. Define a linear map $\delta_{\mathcal{F}(G)} : \mathcal{F}(G) \longrightarrow \mathcal{F}(G) \otimes_k \mathcal{F}(G)$ by

$$\delta_{\mathcal{F}(G)}(f)(x, y) = f(xy)$$

This clearly makes $\mathcal{F}(G)$ into a cosemigroup. The linear map $\epsilon_{\mathcal{F}(G)}$: $\mathcal{F}(G) \longrightarrow k$

$$\epsilon_{\mathcal{F}(G)}(f) = f(e)$$

where $e \in G$ is the unit of the group G, makes $\langle\ \mathcal{F}(G), \delta_{\mathcal{F}(G)}, \epsilon_{\mathcal{F}(G)} \rangle$ into a comonoid. Note that this conclusion depends strongly on the identification $\mathcal{F}(G \times G) \approx \mathcal{F}(G) \otimes_k \mathcal{F}(G)$. For infinite groups this relation does not hold in general but for some infinite groups it does. For these cases we also get comonoids.

The tensorproduct is not the only monoidal structure on $Vect_k$. Let $\oplus$ be the direct sum of vectorspaces. This is a monoidal structure with the neutral object given by the zero dimensional vectorspace $e = \{0\}$. The maps α, β and γ are the standard identifications used for the direct sum. The symmetry is the linear map $\sigma(u, v) = (v, u)$. These structures defines the structure of a symmetric monoidal category on $Vect_k$. A cosemigroup is a pair $\langle A, \delta_A \rangle$ with $\delta_A : A \rightarrow A \oplus A$ a coassociative linear map. Any such map is determined by a pair of linear maps $f, g : A \to A$ through $\delta_A(a) = (f(a), g(a))$. The coassociativity gives the following conditions on the maps f and g.

$$\begin{aligned} f \circ f &= f \\ g \circ g &= g \\ f \circ g &= g \circ f \end{aligned}$$

So any pair of commuting projectors on A define the structure of a cosemigroup on A. There are thus in general many nontrivial cosemigroup structures on a linear space. The comonoidstructure is however much more restrictive. This is because the neutral object for $\oplus$ is also the terminal object for the category. This means that there is only one possible counit for any comonoid. It is straight forward to see that the counit property for the only possible counit gives $f = g = 1_A$. So there is only one comonoid structure on A and this is the diagonal map

$$\delta_A(a) = (a, a)$$

In all the examples we have seen that coproduct for the comonoids have been monomorphisms. This is true in general

Proposition 14 *Let $\langle B, \delta_B, \epsilon_B \rangle$ be a comonoid. Then the coproduct is a monomorphism.*

Proof. Let D be any object in $\mathcal{C}$ and let $\varphi, \psi : D \longrightarrow B$ be two morphisms in $\mathcal{C}$ such that $\delta_B \circ \varphi = \delta_B \circ \psi$. Then we have

$$\begin{aligned} \psi &= 1_B \circ \psi \\ &= \beta_B \circ (\epsilon_B \otimes 1_B) \circ \delta_B \circ \psi \\ &= \beta_B \circ (\epsilon_B \otimes 1_B) \circ \delta_B \circ \varphi \\ &= 1_B \circ \varphi \\ &= \varphi \end{aligned}$$

so δ_B is by definition mono. ∎

We will in general only be interested in comonoids where the coproduct has the additional property of being commutative. Only such comonoids carry enough structure to support a full theory of relations. We express this property by using the symmetry σ.

Definition 15 *A comonoid $\langle A, \delta_A, \epsilon_A \rangle$ in a symmetric monoidal category is σ-commutative if $\sigma_{A,A} \circ \delta_A = \delta_A$.*

2.4 C-categories and M-categories

In *Sets* each object is a σ-commutative comonoid in one and only one way . For the case of a general symmetric monoidal category we have seen that objects may have several σ-commutative comonoid structures defined on them. We need to preserve the unique relation between objects and structures when we generalize from *Sets*. This relation is expressed in terms of functors and natural transformations. To any category C we have associated a set of functors. These are the projection functors $P : C \times C \longrightarrow C$ and $Q : C \times C \longrightarrow C$,the diagonal functor $\Delta : C \longrightarrow C \times C$ defined by $\Delta(X) = (X, X)$ and the transposition functor $\tau : C \times C \longrightarrow C \times C$. Let e be a fixed object in the category C. To this object we associate the constant functor $K_e : C \longrightarrow C$. Finally let us assume that $\otimes : C \times C \longrightarrow C$ is a bifunctor and let $H = (1_C \times \tau \times 1_C) \circ (\Delta \times \Delta)$. We are now ready to define the notion of a C-category.

Definition 16 *A C-category is a collection* $\langle\ C, \otimes, K_e, \alpha, \beta, \gamma, \sigma, \delta, \epsilon\rangle$ *where* $\langle\ C, \otimes, K_e, \alpha, \beta, \gamma, \sigma\rangle$ *is a symmetric monoidal category and where* δ, ϵ *are natural transformations*

$$
\begin{aligned}
\delta &: 1_C \longrightarrow \otimes \circ \Delta \\
\epsilon &: 1_C \longrightarrow K_e
\end{aligned}
$$

such that the following relations holds

$$
\begin{aligned}
(1_{\otimes} \circ (\delta \times 1_{1_C}) \circ 1_{\Delta}) \cdot \delta &= (\alpha \circ 1_{(1_C \times \Delta) \circ \Delta}) \cdot (1_{\otimes} \circ (1_{1_C} \times \sigma) \circ 1_{\Delta}) \cdot \delta \\
1_{1_C} &= (\beta \circ 1_{\Delta}) \cdot (1_{\otimes} \circ (\epsilon \times 1_{1_C}) \circ 1_{\Delta}) \cdot \delta \\
1_{1_C} &= (\gamma \circ 1_{\Delta}) \cdot (1_{\otimes} \circ (1_{1_C} \times \epsilon) \circ 1_{\Delta}) \cdot \delta \\
\delta &= (\sigma^{-1} \circ 1_{\Delta}) \cdot \delta \\
\delta \circ 1_{\otimes} &= (\alpha^{-1} \circ 1_{\otimes \times 1_C \times 1_C} \circ 1_H) \cdot (1_{\otimes} \circ (\alpha \times 1_{1_C}) \circ 1_H) \\
&\quad \cdot (1_{\otimes \circ (\otimes \times 1_C)} \circ (1_{1_C} \times \sigma \times 1_{1_C}) \circ 1_{\Delta \times \Delta}) \\
&\quad \cdot (1_{\otimes} \circ (\alpha^{-1} \times 1_{1_C}) \circ 1_{\Delta \times \Delta}) \cdot (\alpha \circ 1_{\otimes \times 1_C \times 1_C} \circ 1_{\Delta \times \Delta}) \\
&\quad \cdot (1_{\otimes} \circ (\delta \times \delta)) \\
\epsilon \circ 1_{\otimes} &= (\beta \circ 1_{1_C \times K_e}) \cdot (1_{\otimes} \circ (\epsilon \times \epsilon))
\end{aligned}
$$

The four first relations ensure that for each object in C there is fixed a unique commutative comonoid structure. The last two relations say that if a object X can be decomposed as $X = A \otimes B$, then we can express the unique comonoid structure on X in terms of the comonoid structures on A and B. For the strict case they take the following form in terms of objects

$$
\begin{aligned}
\delta_{A \otimes B} &= (1_A \otimes \sigma_{A,B} \otimes 1_B) \circ (\delta_A \otimes \delta_B) \\
\epsilon_{A \otimes B} &= \epsilon_A \otimes \epsilon_B
\end{aligned}
$$

A M-category is the dual of a C-category. We get its defining equations by reversing all arrows. It is a category where for each object there is fixed a unique monoid structure and where the monoid structure on a object of the form $X = A \otimes B$ can be expressed in terms of the structures on A and B.

3 Categorical theory of relations

In this part of the paper we use the categorical framework described in the previous section to define a category of relations and develop its properties. We first define the notion of a relation and a corelation in a C-category. In a similar way relations and corelations can be developed in a M-category. The notions of C-categories and M-categories are dual concepts so that any definitions made or propositions proved in one of them hold in a dualized version in the other. Since the notion of relation and corelation also are dual of each other it is clear that it is enough to develop the theory of relations in C-categories. The other cases follow by duality. We start this section by defining relations on a object A in a C-category C in terms of arrows and collect such arrows into a category of relations $\mathcal{R}^A(C)$. This category of relations is then shown to be isomorphic to the category $\mathcal{S}^A(C)$ of $A-A$ bicomodules in C. A semimonoidal structure $\boxtimes^A$is introduced in this category and by isomorphism into the category of relations. This semimonoidal structure is then used to introduce a bifunctor $\otimes^A$on $\mathcal{S}^A(C)$ and by isomorphism on $\mathcal{R}^A(C)$. This bifunctor is used to introduce a monoidal structure on the category of relations. Certain properties of relations like transitivity and reflexivity are formulated in algebraic terms using the monoidal structure. In the final sections a generalized notion of symmetry is introduced, this notion of symmetry use in a essential way the formulation of the Yang-Baxter equation in terms of action of a S_2 graded group. The new notion of symmetry is then used to further categorize properties of relations. Equivalence relations appears as commutative and associative algebras with units.

3.1 Relations

Let $\langle C, \otimes, k_e, \alpha, \beta, \gamma, \sigma, \delta, \epsilon \rangle$ be a C-category and let A be a object in C.

Definition 17 *A relation on A is a arrow in C with codomain $A \otimes A$.*

Note that we will use the same symbol for an arrow in C and the corresponding morphism of relations. Also note that a given arrow $f : B \longrightarrow B'$ in C can give rise to more than one morphism of relations. This can happen because we might have $r_1 = r_1' \circ f$ and $r_2 = r_2' \circ f$ where $r_1, r_2 : B \longrightarrow A \otimes A$ and $r_1', r_2' : B' \longrightarrow A \otimes A$ are two pairs of relations on A. In this sense we can write $1_r = 1_B$ where B is the domain of the arrow r. Let us now consider a few examples of this construction.

Let us first consider the case of *Sets*. This is a C-category with $\delta_X(x) = (x, x)$ and $\epsilon_X(x) = *$ for all objects $X \in C$. Let A and B be sets and let $r : B \longrightarrow A \times A$ be a map of sets. We can write $r(b) = (f(b), g(b))$. We have

$$\begin{aligned}
&[(r \times r) \circ \delta_B](b) \\
&= (f(b), f(b), g(b), g(b)) \\
&= (\delta_A \times \delta_A)(f(b), g(b)) \\
&= [\delta_{A \times A} \circ r](b)
\end{aligned}$$

so r is a arrow in the C-category $Sets$ and is therefore a relation in $Sets$ in our sense. A relation on A in the usual sense is a subset of $A \times A$. This is equivalent to assuming that the map r is a monomorphism. In general the map r assign to each element in B a source and a target. Several elements in B can be assigned the same source and target. In fact we observe that in $Sets$ a relation in our sense is the same as a directed labelled graph.

Let us next consider the C-category $Vect_k$ with direct sum as monoidal structure and δ and ϵ defined as for $Sets$. A relation on a linear space A is any linear map $r : B \longrightarrow A \oplus A$. Let $L : A \longrightarrow A$ be a endomorphism on A. Let $B = A$ and define $r : B \longrightarrow A \oplus A$ by

$$r(a) = (a, L(a))$$

Then r is a linear map and therefore defines a relation on A in our sense. Note that the image of A under r is by definition the graph of the linear map L. More generally, let L be a linear subspace of $A \oplus A$. Let $B = L$ and $r : B \longrightarrow A \oplus A$ the inclusion map. Then r is evidently a relation on A. In general a relation on A is like a graph, where the set of vertices and the set of labels have a vectorspace structure and the source and target maps respect these structures.

As with any categorical concept the notion of a relation has a dual.

Definition 18 *A corelation on a A is a arrow in C with domain $A \otimes A$.*

Let $r : S \longrightarrow \Omega \times \Omega$ be a relation on Ω in $Sets$. We assume now that the sets S and Ω are finite. The algebraic description of the sets S and Ω are given by the space of k valued functions $B = \mathcal{F}(S)$ on S and $A = \mathcal{F}(\Omega)$ on Ω. Let c defined by $c(f \otimes g)(x) = (f \otimes g)(r(x))$. Then $c : A \otimes A \longrightarrow B$ is a linear map and by duality a morphism of the induced algebra structures on B and $A \otimes A$.

Therefore the algebraic image of the relation r in $Sets$ is a corelation c in $Vect_k$. This example show that corelations arise naturally by algebraization of relations in $Sets$. Note that in general a corelation $c : A \otimes A \longrightarrow B$ in $Vect_k$ with the tensor product as monoidal structure is in algebra usually called a $A \otimes A$ algebra.

3.2 Categories of relations

Let $r : B \longrightarrow A \otimes A$ and $r' : B' \longrightarrow A \otimes A$ be two relations on A. A morphism $f : r \longrightarrow r'$ is a arrow $f : B \longrightarrow B'$ in C such that the following diagram commute

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "r"'] & & B' \arrow[dl, "r'"] \\ & A \otimes A & \end{tikzcd}$$

Let $\mathcal{R}^A(C)$ be the category of relations on A. This is a category whose objects are relations and morphisms are morphisms of relations as just defined.

It is evident that to each diagram in $\mathcal{R}^A(C)$ there is a corresponding diagram of arrows in C and commutativity of diagrams in $\mathcal{R}^A(C)$ follows from commutativity of the corresponding diagrams in C. For now there is no restriction on the object A or the arrows that are relations on A. We will introduce further restrictions as we develop the properties of the category of relations.

Morphisms of corelations are defined by dualizing the corresponding diagrams for morphisms of relations. Corelations and morphisms of corelations form the category of corelations on A, $\mathcal{R}_A(C)$.

We will now proceed to develop some formal properties of the category $\mathcal{R}^A(C)$. The corresponding dualized properties holds for the category $\mathcal{R}_A(C)$.

Let r be a object in $\mathcal{R}^A(C)$ with domain B. Define two arrows $\delta^l : B \longrightarrow A \otimes B$ and $\delta^r : B \longrightarrow B \otimes A$ in C by

$$\delta^l = (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B$$
$$\delta^r = (1_B \otimes \beta_A) \circ (1_B \otimes (\epsilon_A \otimes 1_A)) \circ (1_B \otimes r) \circ \delta_B$$

Define $\theta^l : B \longrightarrow (A \otimes A) \otimes B$ and $\theta^r : B \longrightarrow B \otimes (A \otimes A)$ by

$$\theta^l = (r \otimes 1_B) \circ \delta_B$$
$$\theta^r = (1_B \otimes r) \circ \delta_B$$

We first prove the identities

Lemma 19

$$(\delta_{A\otimes A} \otimes 1_B) \circ \theta^l = \alpha_{A\otimes A, A\otimes A, B} \circ (1_{A\otimes A} \otimes \theta^l) \circ \theta^l$$
$$(1_B \otimes \delta_{A\otimes A}) \circ \theta^r = \alpha_{B, A\otimes A, A\otimes A} \circ (\theta^r \otimes 1_{A\otimes A}) \circ \theta^r$$
$$(\theta^l \otimes 1_{A\otimes A}) \circ \theta^r = \alpha_{A\otimes A, B, A\otimes A} \circ (1_{A\otimes A} \otimes \theta^r) \circ \theta^l$$

Proof. Since r is a morphism in C we have the diagram

$$\begin{tikzcd}
B \times B \arrow[r, "r \otimes r"] & A \otimes A \otimes A \otimes A \\
B \arrow[u, "\delta_B"] \arrow[r, "r"'] & A \otimes A \arrow[u, "\delta_{A\otimes A}"']
\end{tikzcd}$$

But then we have

$$\begin{aligned}
&\alpha_{A\otimes A, A\otimes A, B} \circ (1_{A\otimes A} \otimes \theta^l) \circ \theta^l \\
&= \alpha_{A\otimes A, A\otimes A, B} \circ (1_{A\otimes A} \otimes (r \otimes 1_B)) \circ (1_{A\otimes A} \otimes \delta_B) \circ (r \otimes 1_B) \circ \delta_B \\
&= \alpha_{A\otimes A, A\otimes A, B} \circ (r \otimes (r \otimes 1_B)) \circ (1_B \otimes \delta_B) \circ \delta_B \\
&= \alpha_{A\otimes A, A\otimes A, B} \circ (r \otimes (r \otimes 1_B)) \circ \alpha^{-1}_{B,B,B} \circ (\delta_B \otimes 1_B) \circ \delta_B \\
&= ((r \otimes r) \circ \delta_B \otimes 1_B) \circ \delta_B \\
&= (\delta_{A\otimes A} \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B
\end{aligned}$$

The proof of the second relation proceeds in a similar way. For the third we have

$$
\begin{aligned}
&(\theta^l \otimes 1_{A\otimes A}) \circ \theta^r \\
&= ((r \otimes 1_B) \otimes 1_{A\otimes A}) \circ (\delta_B \otimes 1_{A\otimes A}) \circ (1_B \otimes r) \circ \delta_B \\
&= ((r \otimes 1_B) \otimes r) \circ (\delta_B \otimes 1_B) \circ \delta_B \\
&= ((r \otimes 1_B) \otimes r) \circ \alpha_{B,B,B} \circ (1_B \otimes \delta_B) \circ \delta_B \\
&= \alpha_{A\otimes A,B,A\otimes A} \circ (r \otimes (1_B \otimes r)) \circ (1_B \otimes \delta_B) \circ \delta_B \\
&= \alpha_{A\otimes A,B,A\otimes A} \circ (r \otimes \theta^r) \circ \delta_B
\end{aligned}
$$

■

we can now prove the following

Proposition 20 *Let r be a relation. Then the following diagrams commute*

$$
\begin{tikzcd}
& A \otimes B \arrow[r, "\delta_A \otimes 1_B"] & (A \otimes A) \otimes B \\
B \arrow[ur, "\delta^l"] \arrow[dr, "\delta^l"'] & & \\
& A \otimes B \arrow[r, "1_A \otimes \delta^l"'] & A \otimes (A \otimes B) \arrow[uu, "\alpha_{A,A,B}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
e \otimes B \arrow[dr, "\beta_B"'] & A \otimes B \arrow[l, "\epsilon_A \otimes 1_B"'] \\
& B \arrow[u, "\delta^l"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
& B \otimes A \arrow[r, "1_B \otimes \delta_A"] & B \otimes (A \otimes A) \arrow[dd, "\alpha_{B,A,A}"] \\
B \arrow[ur, "\delta^r"] \arrow[dr, "\delta^r"'] & & \\
& B \otimes A \arrow[r, "\delta^r \otimes 1_A"'] & (B \otimes A) \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes A \arrow[r, "1_B \otimes \epsilon_A"] & B \otimes e \arrow[dl, "\gamma_B"] \\
B \arrow[u, "\delta^r"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& A \otimes B \arrow[r, "1_A \otimes \delta^r"] & A \otimes (B \otimes A) \arrow[dd, "\alpha_{A,B,A}"] \\
B \arrow[ur, "\delta^l"] \arrow[dr, "\delta^r"'] & & \\
& B \otimes A \arrow[r, "\delta^l \otimes 1_A"'] & (A \otimes B) \otimes A
\end{tikzcd}
$$

Proof. Using the lemma and the naturality of α and δ we have

$$
\begin{aligned}
&\alpha_{A,A,B} \circ (1_A \otimes \delta^l) \circ \delta^l \\
&= \alpha_{A,A,B} \circ (1_A \otimes (\gamma_A \otimes 1_B)) \circ (1_A \otimes ((1_A \otimes \epsilon_A) \otimes 1_B)) \\
&\circ (1_A \otimes \theta^l) \circ (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ \theta^l \\
&= (((\gamma_A \circ (1_A \otimes \epsilon_A)) \otimes (\gamma_A \circ (1_A \otimes \epsilon_A))) \otimes 1_B) \\
&\circ \alpha_{A\otimes A, A\otimes A, B} \circ (1_{A\otimes A} \otimes \theta^l) \circ \theta^l \\
&= (((\gamma_A \circ (1_A \otimes \epsilon_A)) \otimes (\gamma_A \circ (1_A \otimes \epsilon_A))) \otimes 1_B) \\
&\circ (\delta_{A\otimes A} \otimes 1_B) \circ \theta^l \\
&= (\delta_A \otimes 1_B) \circ ((\gamma_A \circ (1_A \otimes \epsilon_A)) \otimes 1_B) \circ \theta^l \\
&= (\delta_A \otimes 1_B) \circ \delta^l \\
&= (\delta_A \otimes 1_B) \circ (1_A \otimes \epsilon_A \otimes 1_B) \circ \theta^l \\
&= (\delta_A \otimes 1_B) \circ \delta^l
\end{aligned}
$$

Since ϵ is natural and r a morphism in C we have the identities

$$
\begin{aligned}
\epsilon_{A\otimes e} &= \epsilon_A \circ \gamma_A \\
\epsilon_{A\otimes A} &= \epsilon_{A\otimes e} \circ (1_A \otimes \epsilon_A) \\
\epsilon_B &= \epsilon_{A\otimes A} \circ r
\end{aligned}
$$

But then we have

$$
\begin{aligned}
&\beta_B \circ (\epsilon_A \otimes 1_B) \circ \delta^l \\
&= \beta_B \circ (\epsilon_A \otimes 1_B) \circ (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B \\
&= \beta_B \circ (\epsilon_{A\otimes e} \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B \\
&= \beta_B \circ (\epsilon_{A\otimes A} \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B \\
&= \beta_B \circ (\epsilon_B \otimes 1_B) \circ \delta_B \\
&= 1_B
\end{aligned}
$$

so the first pair of diagrams are commutative. The proof of the commutativity of the second pair of diagrams is similar. For the last diagram we have

$$\begin{aligned}
&\alpha_{A,B,A} \circ (1_A \otimes \delta^r) \circ \delta^l \\
&\alpha_{A,B,A} \circ (1_A \otimes (1_B \otimes \beta_A)) \circ (1_A \otimes (1_B \otimes (\epsilon_A \otimes 1_A))) \\
&\circ (1_A \otimes \theta^r) \circ (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ \theta^l \\
&= ((1_A \otimes 1_B) \otimes \beta_A) \circ ((1_A \otimes 1_B) \otimes (\epsilon_A \otimes 1_A)) \\
&\circ ((\gamma_A \otimes 1_B) \otimes 1_{A\otimes A}) \circ (((1_A \otimes \epsilon_A) \otimes 1_B) \otimes 1_{A\otimes A}) \\
&\circ \alpha_{A\otimes A,B,A\otimes A} \circ (1_{A\otimes A} \otimes \theta^r) \circ \theta^l \\
&= (((\gamma_A \circ (1_A \otimes \epsilon_A)) \otimes 1_B) \otimes (\beta_A \circ (\epsilon_A \otimes 1_A))) \\
&\circ (\theta^l \otimes 1_{A\otimes A}) \circ \theta^r \\
&= (\delta^l \otimes 1_A) \circ (1_B \otimes (\beta_A \circ (\epsilon_A \otimes 1_A))) \circ \theta^r \\
&= (\delta^l \otimes 1_A) \circ \delta^r
\end{aligned}$$

so this diagram is also commutative. ■

The previous proposition show that the pair $\{\delta^l, \delta^r\}$ define the structure of a $A - A$ bicomodule on B.

Definition 21 *Let $\delta^l : B \longrightarrow A \otimes B, \delta^r : B \longrightarrow B \otimes A$ be arrows in C. Then $\{\delta^l, \delta^r\}$ define the structure of a $A - A$ bicomodule on B if the diagrams in proposition 20 commute*

We call the object B in the previous definition the underlying object for the $A - A$ bicomodule $\delta = \{\delta^l, \delta^r\}$.

Let now δ and γ be $A - A$ bicomodules with underlying objects B and E. A morphism $f : \delta \longrightarrow \gamma$ of $A - A$ bicomodules is a arrow $f : B \longrightarrow E$ in C such that the following diagrams commute

$$\begin{tikzcd}
A \otimes B \arrow[r, "1_A \otimes f"] & A \otimes E \\
B \arrow[u, "\delta^l"] \arrow[r, "f"'] & E \arrow[u, "\gamma^l"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes A \arrow[r, "f \otimes 1_A"] & E \otimes A \\
B \arrow[u, "\delta^r"] \arrow[r, "f"'] & E \arrow[u, "\gamma^r"']
\end{tikzcd}$$

We now form a new category where objects are $A-A$ bicomodules and where morphisms are morphisms of $A - A$ bicomodules. Let this category be named $\mathcal{S}^A(C)$.

3.3 Relations in terms of $A - A$ bicomodules.

To each object r in $\mathcal{R}^A(C)$ there corresponds a object δ in $\mathcal{S}^A(C)$. For morphisms of relations we have the following.

Proposition 22 *Let r and s be two relations with domains B and E and let $f : r \longrightarrow s$ be a morphism of relations. Let δ and γ be the objects in $\mathcal{S}^A(C)$ corresponding to r and s. Then the corresponding arrow $f : B \longrightarrow E$ in C defines a morphism $f : \delta \longrightarrow \gamma$ in $\mathcal{S}^A(C)$.*

Proof. Since f is a morphism in C and also a morphism from r to s we have the following commutative diagrams

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "r"'] & & E \arrow[dl, "s"] \\ & A \otimes A & \end{tikzcd} \qquad \begin{tikzcd} B \otimes B \arrow[r, "f \otimes f"] & E \otimes E \\ B \arrow[u, "\delta_B"] \arrow[r, "f"'] & E \arrow[u, "\delta_E"'] \end{tikzcd}$$

Using these identities we have e then have

$$\begin{aligned}
&(1_A \otimes f) \circ \delta^l \\
&= (1_A \otimes f) \circ (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B \\
&= (\gamma_A \otimes 1_E) \circ ((1_A \otimes 1_e) \otimes f) \circ (s \otimes 1_B) \circ (f \otimes 1_B) \circ \delta_B \\
&= (\gamma_A \otimes 1_E) \circ ((1_A \otimes \epsilon_A) \otimes 1_E) \circ (s \otimes 1_B) \circ (f \otimes f) \circ \delta_B \\
&= (\gamma_A \otimes 1_E) \circ ((1_A \otimes \epsilon_A) \otimes 1_E) \circ (s \otimes 1_B) \circ \delta_E \circ f \\
&= \gamma^l \circ f
\end{aligned}$$

In a similar way we prove the identity $(f \otimes 1_A) \circ \delta^r = \gamma^r \circ f$. ∎

The previous definition show that we have a well defined functor $\Phi : \mathcal{R}^A(C) \longrightarrow \mathcal{S}^A(C)$, where $\Phi(r)$ is the $A - A$ bicomodule corresponding to r and where $\Phi(f) = f$. We will next construct a functor from $\mathcal{S}^A(C)$ to $\mathcal{R}^A(C)$.

Let δ be a object in $\mathcal{S}^A(C)$. define a morphism $r : B \longrightarrow A \otimes A$ by

$$r = (1_A \otimes \beta_A) \circ (1_A \otimes (\epsilon_B \otimes 1_A)) \circ (1_A \otimes \delta^r) \circ \delta^l$$

We have proved that β_A and ϵ_B are arrows in C and have therefore the following result.

Proposition 23 *r is a object in $\mathcal{R}^A(C)$.*

Using this result we can define a map of objects $\Psi : \mathcal{S}^A(C) \longrightarrow \mathcal{R}^A(C)$ by $\Psi(\delta) = r$. For morphisms in $\mathcal{S}^A(C)$ we have

Proposition 24 *Let δ, and γ be two objects in $\mathcal{S}^A(C)$ and let $f : \delta \longrightarrow \gamma$ be a morphism. Then the corresponding arrow in C defines a morphism of the objects $r = \Psi(\delta)$ and $s = \Psi(\gamma)$ in $\mathcal{R}^A(C)$.*

Proof. Let the domains of r and s be B and E. We have the following commutative diagrams

$$\begin{tikzcd} A \otimes B \arrow[r, "1_A \otimes f"] & A \otimes E \\ B \arrow[u, "\delta^l"] \arrow[r, "f"'] & E \arrow[u, "\gamma^l"'] \end{tikzcd} \qquad \begin{tikzcd} B \otimes A \arrow[r, "f \otimes 1_A"] & E \otimes A \\ B \arrow[u, "\delta^r"] \arrow[r, "f"'] & E \arrow[u, "\gamma^r"'] \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\epsilon_B"'] & E \arrow[dl, "\epsilon_E"] \\ e & \end{tikzcd}$$

But then we have

$$\begin{aligned}
& s \circ f \\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (\epsilon_E \otimes 1_A)) \circ (1_A \otimes \gamma^r) \circ \gamma^l \circ f \\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (\epsilon_E \otimes 1_A)) \circ (1_A \otimes \gamma^r) \circ (1_A \otimes f) \circ \delta^l \\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (\epsilon_E \otimes 1_A)) \circ (1_A \otimes (f \otimes 1_A)) \circ (1_A \otimes \delta^r) \circ \delta^l \\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (\epsilon_E \circ f \otimes 1_A)) \circ (1_A \otimes \delta^r) \circ \delta^l \\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (\epsilon_B \otimes 1_A)) \circ (1_A \otimes \delta^r) \circ \delta^l \\
&= r
\end{aligned}$$

so f is a morphism of relations. ■

We use this result to extend Ψ to a functor from $\mathcal{S}^A(C)$ to $\mathcal{R}^A(C)$ by defining $\Psi(f) = f$. We will now show that $\mathcal{R}^A(C)$ and $\mathcal{S}^A(C)$ are isomorphic categories. We need the following lemma

Lemma 25

$$\begin{aligned}
(\gamma_A \otimes \beta_A) \circ ((1_A \otimes \epsilon_A) \otimes (\epsilon_A \otimes 1_A)) \circ \delta_{A \otimes A} &= 1_{A \otimes A} \\
(\delta^l \otimes 1_B) \circ \delta_B &= \alpha_{A,B,B} \circ (1_A \otimes \delta_B) \circ \delta^l \\
(1_B \otimes \delta^r) \circ \delta_B &= \alpha^{-1}_{B,B,A} \circ (\delta_B \otimes 1_A) \circ \delta^r
\end{aligned}$$

Proof. for the first part of the lemma we have

$$
\begin{aligned}
&(\gamma_A \otimes \beta_A) \circ ((1_A \otimes \epsilon_A) \otimes (\epsilon_A \otimes 1_A)) \circ \delta_{A\otimes A} \\
&= (\gamma_A \otimes \beta_A) \circ ((1_A \otimes \epsilon_A) \otimes (\epsilon_A \otimes 1_A)) \circ \alpha^{-1}_{A\otimes A,A,A} \\
&\circ (\alpha_{A,A,A} \otimes 1_A) \circ ((1_A \otimes \sigma_{A,A}) \otimes 1_A) \circ (\alpha^{-1}_{A,A,A} \otimes 1_A) \\
&\circ \alpha_{A\otimes A,A,A} \circ (\delta_A \otimes \delta_A) \\
&= (1_A \otimes \beta_A) \circ (\gamma_A \otimes (1_e \otimes 1_A)) \circ \alpha^{-1}_{A\otimes e,e,A} \circ (\alpha_{A,e,e} \otimes 1_A) \\
&\circ ((1_A \otimes \sigma_{e,e}) \otimes 1_A) \circ (\alpha^{-1}_{A,e,e} \otimes 1_A) \circ \alpha_{A\otimes e,A,A} \\
&\circ ((1_A \otimes \epsilon_A) \otimes (\epsilon_A \otimes 1_A)) \circ (\delta_A \otimes \delta_A) \\
&= (1_A \otimes \beta_A) \circ (\gamma_A \otimes (1_e \otimes 1_A)) \circ \alpha^{-1}_{A\otimes e,e,A} \circ (\alpha_{A,e,e} \otimes 1_A) \\
&\circ ((1_A \otimes \sigma_{e,e}) \otimes 1_A) \circ (\alpha^{-1}_{A,e,e} \otimes 1_A) \circ \alpha_{A\otimes e,A,A} \\
&\circ (\gamma_A^{-1} \otimes \beta_A^{-1}) \\
&= (1_A \otimes \beta_A) \circ \alpha^{-1}_{A,e,A} \circ ((1_A \otimes \beta_e) \otimes 1_A) \circ ((1_A \otimes \sigma_{e,e}) \otimes 1_A) \\
&\circ (\alpha^{-1}_{A,e,e} \otimes 1_A) \circ \alpha_{A\otimes e,e,A} \circ (\gamma_A^{-1} \otimes \beta_A^{-1}) \\
&= (1_A \otimes \beta_A) \circ \alpha^{-1}_{A,e,A} \circ ((1_A \otimes \gamma_e) \otimes 1_A) \circ \\
&\circ (\alpha^{-1}_{A,e,e} \otimes 1_A) \circ \alpha_{A\otimes e,e,A} \circ (\gamma_A^{-1} \otimes \beta_A^{-1}) \\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (\gamma_e \otimes 1_A)) \circ \\
&\circ \alpha^{-1}_{A,e\otimes e,A} \circ (\alpha^{-1}_{A,e,e} \otimes 1_A) \circ \alpha_{A\otimes e,e,A} \circ (\gamma_A^{-1} \otimes \beta_A^{-1}) \\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (\gamma_e \otimes 1_A)) \circ (1_A \otimes \alpha_{e,e,A}) \\
&\circ \alpha^{-1}_{A,e,e\otimes A} \circ ((1_A \otimes 1_e) \otimes \beta_A^{-1}) \circ (\gamma_A^{-1} \otimes 1_A) \\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (1_e \otimes \beta_A)) \circ (1_A \otimes (1_e \otimes \beta_A^{-1})) \\
&\circ \alpha^{-1}_{A,e,A} \circ (\gamma_A^{-1} \otimes 1_A) \\
&= (1_A \otimes \beta_A) \circ \alpha^{-1}_{A,e,A} \circ (\gamma_A^{-1} \otimes 1_A) \\
&= (1_A \otimes \beta_A) \circ (1_A \otimes \beta_A^{-1}) \\
&= 1_{A\otimes A}
\end{aligned}
$$

for the second part of the lemma we have

$$
\begin{aligned}
&(\delta^l \otimes 1_B) \circ \delta_B \\
&= ((\gamma_A \otimes 1_B) \otimes 1_B) \circ (((1_A \otimes \epsilon_A) \otimes 1_B) \otimes 1_B) \circ ((r \otimes 1_B) \otimes 1_B) \\
&\circ (\delta_B \otimes 1_B) \circ \delta_B \\
&= ((\gamma_A \otimes 1_B) \otimes 1_B) \circ (((1_A \otimes \epsilon_A) \otimes 1_B) \otimes 1_B) \circ ((r \otimes 1_B) \otimes 1_B) \\
&\circ \alpha_{B,B,B} \circ (1_B \otimes \delta_B) \circ \delta_B \\
&= \alpha_{A,B,B} \circ (\gamma_A \otimes 1_{B\otimes B}) \circ ((1_A \otimes \epsilon_A) \otimes 1_{B\otimes B}) \circ (r \otimes 1_{B\otimes B}) \circ (1_B \otimes \delta_B) \circ \delta_B \\
&= (1_B \otimes \delta_B) \circ (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B \\
&= (1_B \otimes \delta_B) \circ \delta^l
\end{aligned}
$$

The proof of the third part of the lemma is similar to the second part. ■

We now use the lemma to prove the following theorem

Theorem 26 *The functor* $\Phi : \mathcal{R}^A(C) \longrightarrow \mathcal{S}^A(C)$ *is invertible with inverse* $\Psi : \mathcal{S}^A(C) \longrightarrow \mathcal{R}^A(C)$.

Proof. We only need to prove that Φ is bijective with inverse Ψ on objects since Ψ is obviously the inverse of Φ on morphisms.

Let r be a object in $\mathcal{R}^A(C)$ with domain B. Using lemma 25 we have

$$
\begin{aligned}
&(\Psi \circ \Phi)(r)\\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (\epsilon_B \otimes 1_A)) \circ (1_A \otimes (\Phi(r))^r) \circ (\Phi(r))^l\\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (\epsilon_B \otimes 1_A)) \circ (1_A \otimes (1_B \otimes \beta_A))\\
&\circ (1_A \otimes (1_B \otimes (\epsilon_A \otimes 1_A))) \circ (1_A \otimes (1_B \otimes r)) \circ (1_A \otimes \delta_B)\\
&\circ (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B\\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (1_e \otimes \beta_A)) \circ (1_A \otimes (1_e \otimes (\epsilon_A \otimes 1_A)))\\
&\circ (1_A \otimes (1_e \otimes r)) \circ (1_A \otimes (\epsilon_B \otimes 1_B) \circ \delta_B) \circ (\gamma_A \otimes 1_B)\\
&\circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B\\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (1_e \otimes \beta_A)) \circ (1_A \otimes (1_e \otimes (\epsilon_A \otimes 1_A)))\\
&\circ (1_A \otimes (1_e \otimes r)) \circ (1_A \otimes \beta_B^{-1}) \circ (\gamma_A \otimes 1_B)\\
&\circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B\\
&= (1_A \otimes \beta_A) \circ (1_A \otimes (1_e \otimes \beta_A)) \circ (1_A \otimes (1_e \otimes (\epsilon_A \otimes 1_A)))\\
&\circ (1_A \otimes \beta_{A\otimes A}^{-1}) \circ (\gamma_A \otimes 1_{A\otimes A}) \circ ((1_A \otimes \epsilon_A) \otimes 1_{A\otimes A}) \circ (r \otimes r) \circ \delta_B\\
&= (1_A \otimes \beta_A \circ (1_e \otimes \beta_A) \circ \beta_{e\otimes A}^{-1}) \circ (1_A \otimes \beta_{e\otimes A}^{-1}) \circ (\gamma_A \otimes (1_e \otimes 1_A))\\
&((1_A \otimes \epsilon_A) \otimes (\epsilon_A \otimes 1_A)) \circ \delta_{A\otimes A} \circ r\\
&= (\gamma_A \otimes \beta_A) \circ ((1_A \otimes \epsilon_A) \otimes (\epsilon_A \otimes 1_A)) \circ \delta_{A\otimes A} \circ r\\
&= r
\end{aligned}
$$

so $\Psi \circ \Phi = 1_{\mathcal{R}^A(C)}$. Next let δ be any object in $\mathcal{S}^A(C)$ with underlying object B. We have

$$
\begin{aligned}
&(\Phi(\Psi(\delta)))^l \\
&= (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (\Psi(\delta) \otimes 1_B) \circ \delta_B \\
&= (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ ((1_A \otimes \beta_A) \otimes 1_B) \\
&\circ ((1_A \otimes (\epsilon_B \otimes 1_A)) \otimes 1_B) \circ ((1_A \otimes \delta^r) \otimes 1_B) \circ (\delta^l \otimes 1_B) \circ \delta_B \\
&= (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ ((1_A \otimes \beta_A) \otimes 1_B) \\
&\circ ((1_A \otimes (\epsilon_B \otimes 1_A)) \otimes 1_B) \circ ((1_A \otimes \delta^r) \otimes 1_B) \\
&\circ \alpha_{A,B,B} \circ (1_A \otimes \delta_B) \circ \delta^l \\
&= (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_{e \otimes A}) \otimes 1_B) \circ ((1_A \otimes (\epsilon_B \otimes 1_A)) \otimes 1_B) \\
&\circ ((1_A \otimes \delta\rangle) \otimes 1_B) \circ \alpha_{A,B,B} \circ (1_A \otimes \delta_B) \circ \delta^l \\
&= (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_{B \otimes A}) \otimes 1_B) \circ ((1_A \otimes \delta^r) \otimes 1_B) \\
&\circ \alpha_{A,B,B} \circ (1_A \otimes \delta_B) \circ \delta^l \\
&= (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_{B \otimes e}) \otimes 1_B) \circ ((1_A \otimes (1_B \otimes \epsilon_A)) \otimes 1_B) \\
&\circ ((1_A \otimes \delta^r) \otimes 1_B) \circ \alpha_{A,B,B} \circ (1_A \otimes \delta_B) \circ \delta^l \\
&= (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_{B \otimes e}) \otimes 1_B) \circ ((1_A \otimes \gamma_B^{-1}) \otimes 1_B) \\
&\circ \alpha_{A,B,B} \circ (1_A \otimes \delta_B) \circ \delta^l \\
&= (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_B) \otimes 1_B) \circ \alpha_{A,B,B} \circ (1_A \otimes \delta_B) \circ \delta^l \\
&= (\gamma_A \otimes 1_B) \circ \alpha_{A,e,B} \circ (1_A \otimes (\epsilon_B \otimes 1_B)) \circ (1_A \otimes \delta_B) \circ \delta^l \\
&= (1_A \otimes \beta_B) \circ (1_A \otimes (\epsilon_B \otimes 1_B)) \circ (1_A \otimes \delta_B) \circ \delta^l \\
&= \delta^l
\end{aligned}
$$

and similarly we find that $(\Phi(\Psi(\delta)))^r = \delta^r$. This proves that $\Phi \circ \Psi = 1_{\mathcal{S}^A(C)}$.
■

By definition $\delta_A : A \longrightarrow A \otimes A$ is a object in $\mathcal{R}^A(C)$. Its image by Φ is therefore an object in $\mathcal{S}^A(C)$.

Proposition 27 $\Phi(\delta_A) = \{\delta_A, \delta_A\}$

Proof. We have

$$
\begin{aligned}
&\Phi(\delta_A)^l \\
&= (\gamma_A \otimes 1_A) \circ ((1_A \otimes \epsilon_A) \otimes 1_A) \circ (\delta_A \otimes 1_A) \circ \delta_A \\
&= (\gamma_A \circ (1_A \otimes \epsilon_A) \circ \delta_A \otimes 1_A) \circ \delta_A \\
&= (1_A \otimes 1_A) \circ \delta_A \\
&= \delta_A
\end{aligned}
$$

In a similar way we prove that $\Phi(\delta_A)^r = \delta_A$. ■

The object $\{\delta_A, \delta_A\}$ in $\mathcal{S}^A(C)$ will play an important role for a product we will define later and is given a special name.

$$a = \{\delta_A, \delta_A\}$$

In the following we will mostly work in the category $\mathcal{S}^A(C)$ and use the isomorphism to induce the corresponding structures on the category of relations $\mathcal{R}^A(C)$. Note that the category of corelations on A,$\mathcal{R}_A(C)$, is by duality isomorphic to the category of $A-A$ bimodules. Denote this category by $\mathcal{S}_A(C)$.

3.4 The $\boxtimes^A$ product of relations

Let δ and γ be two objects in $\mathcal{S}^A(C)$ with underlying objects B and E. Define two arrows in C $(\delta \boxtimes^A \gamma)^l : B \otimes E \longrightarrow A \otimes (B \otimes E)$ and $(\delta \boxtimes^A \gamma)^r : B \otimes E \longrightarrow (B \otimes E) \otimes A$ by

$$\begin{aligned}
(\delta \boxtimes^A \gamma)^l &= \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \\
(\delta \boxtimes^A \gamma)^r &= \alpha_{B,E,A} \circ (1_B \otimes \gamma^r)
\end{aligned}$$

Then we have

Proposition 28 *$\delta \boxtimes^A \gamma = \{(\delta \boxtimes^A \gamma)^l, (\delta \boxtimes^A \gamma)^r\}$ is a object in $\mathcal{S}^A(C)$.*

Proof. Using the naturality and the MacLane coherence condition for α we have

$$\begin{aligned}
&\alpha_{A,A,B\otimes E} \circ (1_A \otimes (\delta \boxtimes^A \gamma)^l) \circ (\delta \boxtimes^A \gamma)^l \\
&= \alpha_{A,A,B\otimes E} \circ (1_A \otimes \alpha^{-1}_{A,B,E}) \circ (1_A(\delta^l \otimes 1_E)) \circ \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \\
&= \alpha_{A,A,B\otimes E} \circ (1_A \otimes \alpha^{-1}_{A,B,E}) \circ \alpha^{-1}_{A,A\otimes B,E} \circ ((1_A \otimes \delta^l) \circ \delta^l \otimes 1_E) \\
&= \alpha_{A,A,B\otimes E} \circ (1_A \otimes \alpha^{-1}_{A,B,E}) \circ \alpha^{-1}_{A,A\otimes B,E} \circ (\alpha^{-1}_{A,A,B} \otimes 1_E) \\
&\circ ((\delta_A \otimes 1_B) \otimes 1_E) \circ (\delta^l \otimes 1_E) \\
&= \alpha^{-1}_{A\otimes A,B,E} \circ ((\delta_A \otimes 1_B) \otimes 1_E) \circ (\delta^l \otimes 1_E) \\
&= (\delta_A \otimes 1_{B\otimes E}) \circ (\alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \\
&= (\delta_A \otimes 1_{B\otimes E}) \circ (\delta \boxtimes^A \gamma)^l
\end{aligned}$$

and

$$\begin{aligned}
&\beta_{B\otimes E} \circ (\epsilon_A \otimes 1_{B\otimes E}) \circ (\delta \boxtimes^A \gamma)^l \\
&= \beta_{B\otimes E} \circ (\epsilon_A \otimes (1_B \otimes 1_E)) \circ \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \\
&= \beta_{B\otimes E} \circ \alpha^{-1}_{e,B,E} \circ ((\epsilon_A \otimes 1_B) \circ \delta^l \otimes 1_E) \\
&= \beta_{B\otimes E} \circ \alpha^{-1}_{e,B,E} \circ (\beta^{-1}_B \otimes 1_E) \\
&= \beta_{B\otimes E} \circ \beta^{-1}_{B\otimes E} \\
&= 1_{B\otimes E}
\end{aligned}$$

so $B \otimes E$ is a left A comodule. In a similar way we show that $B \otimes E$ is a right A comodule. For the compatibility between the two structures we have

$$
\begin{aligned}
&\alpha_{A,B\otimes E,A} \circ (1_A \otimes (\delta \boxtimes^A \gamma)^r) \circ (\delta \boxtimes^A \gamma)^l \\
&= \alpha_{A,B\otimes E,A} \circ (1_A \otimes \alpha_{B,E,A}) \circ (1_A \otimes (1_B \otimes \gamma^r)) \\
&\circ \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \\
&= \alpha_{A,B\otimes E,A} \circ (1_A \otimes \alpha_{B,E,A}) \circ \alpha^{-1}_{A,B,E\otimes A} \\
&\circ (\delta^l \otimes (1_E \otimes 1_A)) \circ (1_B \otimes \gamma^r) \\
&= (\alpha^{-1}_{A,B,E} \otimes 1_A) \circ \alpha_{A\otimes B,E,A} \\
&\circ (\delta^l \otimes (1_E \otimes 1_A)) \circ (1_B \otimes \gamma^r) \\
&= (\alpha^{-1}_{A,B,E} \otimes 1_A) \circ ((\delta^l \otimes 1_E) \otimes 1_A) \\
&\circ \alpha_{B,E,A} \circ (1_B \otimes \gamma^r) \\
&= ((\delta \boxtimes^A \gamma)^l \otimes 1_A) \circ (\delta \boxtimes^A \gamma)^r
\end{aligned}
$$

■

Using the previous proposition we can define a object map $\boxtimes^A : \mathcal{S}^A(C) \times \mathcal{S}^A(C) \longrightarrow \mathcal{S}^A(C)$ by

$$\boxtimes^A(\delta, \gamma) = \delta \boxtimes^A \gamma$$

Let δ, γ, ρ and θ be objects in $\mathcal{S}^A(C)$ with underlying objects B, E, D and T in C and let $f : \delta \longrightarrow \rho$ and $g : \gamma \longrightarrow \theta$ be two morphisms in $\mathcal{S}^A(C)$. Let $f : B \longrightarrow E$ and $g : D \longrightarrow T$ be the corresponding arrows in C and let $f \otimes g : B \otimes E \longrightarrow D \otimes T$ be their product in C. Define $f \boxtimes^A g = f \otimes g$.

Proposition 29 *$f \boxtimes^A g :^A \delta \boxtimes^A \gamma \longrightarrow \rho \boxtimes^A \theta$ is a morphism in $\mathcal{S}^A(C)$*

Proof. We have previously proved that $f \otimes g$ is a arrow in C. It is also a morphism in $\mathcal{S}^A(C)$

$$
\begin{aligned}
&(\rho \boxtimes^A \theta)^l \circ (f \boxtimes^A g) \\
&= \alpha^{-1}_{A,D,T} \circ (\rho^l \otimes 1_T) \circ (f \otimes g) \\
&= \alpha^{-1}_{B,D,T} \circ (\rho^l \circ f \otimes g) \\
&= \alpha^{-1}_{B,D,T} \circ ((1_A \otimes f) \otimes g) \circ (\delta^l \otimes 1_E) \\
&= (1_A \otimes (f \otimes g)) \circ \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \\
&= (1_A \otimes (f \boxtimes^A g)) \circ (\delta \boxtimes^A \gamma)^l
\end{aligned}
$$

The identity $(\rho \boxtimes^A \theta)^r \circ (f \otimes g) = ((f \boxtimes^A g) \otimes 1_A) \circ (\delta \boxtimes^A \gamma)^r$ is proved in a similar way. ■

Using this proposition we can extend the object map $\boxtimes^A$ to a bifunctor by defining

$$\boxtimes^A(f, g) = f \boxtimes^A g$$

In terms of this bifunctor we have the following immediate consequence of lemma 25

Corollary 30 *Let δ be a object in $\mathcal{S}^A(C)$ with underlying object B. Then $\delta_B : \delta \longrightarrow \delta \boxtimes^A \delta$ is a morphism in $\mathcal{S}^A(C)$.*

In general there exists no neutral object for $\boxtimes^A$. This is clearly seen in the case of $Sets$. Let B be a set and let $f : B \longrightarrow A$ be a injective map of sets. Define a $A - A$ bicomodule structure on B by $\delta^l(x) = (f(x), x)$ and $\delta^r(x) = (x, f(x))$. Assume that ω is a neutral object for $\boxtimes^A$and let the underlying object for ω be S. Then there must exist a isomorphism $h : \delta \boxtimes^A \omega \longrightarrow \delta$ and therefore bijective map $h : B \times S \longrightarrow B$ that is a morphism of $A - A$ bicomodules. But this implies that $f(h(x, s)) = f(x)$ for all x and s. But since f is injective we must have $h(x, s) = x$ and this is not possible if there is more than one element in S. A neutral element ω for $\boxtimes^A$ therefore must have $e = \{*\}$ as underlying object. Any $A - A$ bicomodule structure ω on e must be of the form $\omega^l(*) = (x_0, *)$ and $\omega^r(*) = (*, y_0)$ for some elements $x_0, y_0 \in A$. Let B be a set with more than one point and let $f : B \longrightarrow A$ be a map of sets that is not constant. Define a $A - A$ bicomodule structure on B by $\delta^l(x) = (f(x), x)$ and $\delta^r(x) = (x, f(x))$. If ω is a neutral object for $\boxtimes^A$ there must exist a isomorphism $h : e \times B \longrightarrow B$ that is a morphism of $A - A$ bicomodules. But this implies that for all $b \in B$ we have $f(h(*, b)) = x_0$ and this implies that f is constant since h is bijective. This is a contradiction and this proves that $\boxtimes^A$ does not have a neutral object in the case of $Sets$.

3.5 Semimonoidal structures on the category of relations

Recall that $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ is a semimonoidal category if $M^A : \boxtimes^A \circ (1 \times \boxtimes^A) \longrightarrow \boxtimes^A \circ (\boxtimes^A \times 1)$ is a natural isomorphism such that the first of the MacLane Coherence conditions is satisfied. We will in general assume that the category $\mathcal{S}^A(C)$ is a semimonoidal category with respect to some choice of natural isomorphism M^A.

At least one semimonoidal structure for $\boxtimes^A$ will always exists.

Proposition 31 *Let δ and γ be objects in $\mathcal{S}^A(C)$ with underlying objects B and E in C. Define*

$$M^A_{\delta,\gamma,\rho} = \alpha_{B,E,D}$$

where α is the associativity constraint for the category C. Then $\langle \mathcal{S}^A(C), \boxtimes^A, M^A, S^A \rangle$ is a symmetric semimonoidal category.

Proof. We have proved previously that $\alpha_{B,E,D}$ is a Carrow in C. Next we need to show that the associativity constraint for $\otimes$ on C is in fact a morphism in $\mathcal{S}^A(C)$. If we use the naturality and the MacLane coherence condition for α

we have

$$\begin{aligned}
&((\delta \boxtimes^A \gamma) \boxtimes^A \rho) \circ \alpha_{B,E,D} \\
&= \alpha^{-1}_{A,B\otimes E,D} \circ (\alpha^{-1}_{A,B,E} \otimes 1_D) \circ ((\delta^l \otimes 1_B) \otimes 1_D) \circ \alpha_{B,E,D} \\
&= \alpha^{-1}_{A,B\otimes E,D} \circ (\alpha^{-1}_{A,B,E} \otimes 1_D) \circ \alpha_{A\otimes B,E,D} \circ (\delta^l \otimes (1_E \otimes 1_D)) \\
&= (1_A \otimes \alpha_{B,E,D}) \circ \alpha^{-1}_{A,B,E\otimes D} \circ (\delta_B \otimes 1_{E\otimes D}) \\
&= (1_A \otimes \alpha_{B,E,D}) \circ (\delta \boxtimes^A (\gamma \boxtimes^A \rho))
\end{aligned}$$

M^A is clearly an isomorphism and is a natural transformation if the following identity holds $((f \boxtimes^A g) \boxtimes^A h) \circ M^A_{\delta,\gamma,\rho} = M^A_{\delta',\gamma',\rho'} \circ (f \boxtimes^A (g \boxtimes^A h))$ for all morphisms $f : \delta \longrightarrow \delta', g : \gamma \longrightarrow \gamma'$ and $h : \rho \longrightarrow \rho'$in $\mathcal{S}^A(C)$. But the corresponding identity in C is $((f \otimes g) \otimes h) \circ \alpha_{B,E,D} = \alpha_{B',E',D'} \circ (f \otimes (g \otimes h))$ and this identity holds because α is a natural transformation. ■

The previous proposition leads us to the following definition

Definition 32 *The semimonoidal $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ category is external if for all objects δ, γ and ρ we have*

$$M^A_{\delta,\gamma,\rho} = \alpha_{B,E,D}$$

where α is the associativity constraint for the product $\otimes$ on the category C and where B,E and D are the underlying objects for δ, γ and ρ.

Since $\mathcal{S}^A(C)$ is isomorphic to the category of relations, a semimonoidal structure on $\mathcal{S}^A(C)$ will induce one on the category of relations. Let the product in $\mathcal{R}^A(C)$ corresponding to $\boxtimes^A$ be $\boxdot^A : \mathcal{R}^A(C) \times \mathcal{R}^A(C) \longrightarrow \mathcal{R}^A(C)$. We thus have

$$\boxdot^A = \Psi \circ \boxtimes^A \circ (\Phi \times \Phi)$$

We have the following explicit expression for the product

Proposition 33 *Let r and s be two objects in $\mathcal{R}^A(C)$. Then we have*

$$r \boxdot^A s = (\gamma_A \otimes \beta_A) \circ ((1_A \otimes \epsilon_A) \otimes (\epsilon_A \otimes 1_A)) \circ (r \otimes s)$$

Proof. Since φ and Ψ are isomorphisms with $\Phi = \Psi^{-1}$we only need to verify that

$$\Phi(r \boxdot^A s) = \Phi(r) \boxtimes^A \Phi(s)$$

for all objects r and s in $\mathcal{R}^A(C)$. Note that the naturality of ϵ gives the following relations

$$\begin{aligned}
&\epsilon_A \circ \beta_A \circ (\epsilon_A \otimes 1_A) \\
&= \epsilon_{e\otimes A} \circ (\epsilon_A \otimes 1_A) \\
&= \epsilon_{A\otimes A} \\
&= \epsilon_{A\otimes e} \circ (1_A \otimes \epsilon_A) \\
&= \epsilon_A \circ \gamma_A \circ (\epsilon_A \otimes 1_A)
\end{aligned}$$

W then have

$$
\begin{aligned}
&(\Phi(r \boxdot^A s))^l \\
&= (\gamma_A \otimes 1_{B\otimes E}) \circ ((1_A \otimes \epsilon_A) \otimes 1_{B\otimes E}) \circ ((t \boxdot^A s) \otimes 1_{B\otimes E}) \circ \delta_{B\otimes E} \\
&= (\gamma_A \otimes 1_{B\otimes E}) \circ ((1_A \otimes \epsilon_A) \otimes 1_{B\otimes E}) \circ ((\gamma_A \otimes \beta_A) \otimes 1_{B\otimes E}) \\
&\circ (((1_A \otimes \epsilon_A) \otimes (\epsilon_A \otimes 1_A)) \otimes 1_{B\otimes E}) \circ ((r \otimes s) \otimes 1_{B\otimes E}) \circ \delta_{B\otimes E} \\
&= (\gamma_A \otimes 1_{B\otimes E}) \circ ((1_A \otimes \epsilon_A) \otimes 1_{B\otimes E}) \circ ((\gamma_A \otimes \beta_A) \otimes 1_{B\otimes E}) \\
&\circ (((1_A \otimes \epsilon_A) \otimes (\epsilon_A \otimes 1_A)) \otimes 1_{B\otimes E}) \circ ((r \otimes s) \otimes 1_{B\otimes E}) \\
&\circ \alpha^{-1}_{B\otimes E,B,E} \circ (\alpha_{B,E,B} \otimes 1_E) \circ ((1_B \otimes \sigma_{B,E}) \otimes 1_E) \\
&\circ (\alpha^{-1}_{B,B,E} \otimes 1_E) \circ \alpha_{B\otimes B,E,E} \circ (\delta_B \otimes \delta_E) \\
&= \alpha^{-1}_{A,B,E} \circ ((\gamma_A \otimes 1_B) \otimes 1_E) \circ (\alpha_{A,e,B} \otimes 1_E) \circ ((1_A \otimes \sigma_{B,e}) \otimes 1_E) \\
&\circ ((1_A \otimes (1_B \otimes \epsilon_A)) \otimes 1_E) \circ (\alpha^{-1}_{A,B,A} \otimes 1_E) \circ \alpha_{A\otimes B,A,E} \\
&\circ ((\gamma_A \otimes 1_B) \otimes (\beta_A \otimes 1_E)) \circ (((1_A \otimes \epsilon_A) \otimes 1_B) \otimes ((\epsilon_A \otimes 1_A) \otimes 1_E)) \\
&\circ ((r \otimes 1_B) \otimes (s \otimes 1_E)) \circ (\delta_B \otimes \delta_E) \\
&= \alpha^{-1}_{A,B,E} \circ ((1_A \otimes \gamma_B) \otimes 1_E) \circ (\alpha^{-1}_{A,B,e} \otimes 1_E) \circ \alpha_{A\otimes B,e,E} \circ (1_{A\otimes B} \otimes (\epsilon_A \otimes 1_E)) \\
&\circ ((\gamma_A \otimes 1_B) \otimes (\beta_A \otimes 1_E)) \circ (((1_A \otimes \epsilon_A) \otimes 1_B) \otimes ((\epsilon_A \otimes 1_A) \otimes 1_E)) \\
&\circ ((r \otimes 1_B) \otimes (s \otimes 1_E)) \circ (\delta_B \otimes \delta_E) \\
&= (1_A \otimes (\gamma_B \otimes 1_E)) \circ \alpha^{-1}_{A,B\otimes e,E} \circ (\alpha^{-1}_{A,B,e} \otimes 1_E) \circ \alpha_{A\otimes B,e,E} \circ (1_{A\otimes B} \otimes (\epsilon_A \otimes 1_E)) \\
&\circ ((\gamma_A \otimes 1_B) \otimes (\beta_A \otimes 1_E)) \circ (((1_A \otimes \epsilon_A) \otimes 1_B) \otimes ((\epsilon_A \otimes 1_A) \otimes 1_E)) \\
&\circ ((r \otimes 1_B) \otimes (s \otimes 1_E)) \circ (\delta_B \otimes \delta_E) \\
&= (1_A \otimes (\gamma_B \otimes 1_E)) \circ (1_A \otimes \alpha_{B,e,E}) \circ \alpha^{-1}_{A,B,e\otimes E} \circ (1_{A\otimes B} \otimes (\epsilon_A \otimes 1_E)) \\
&\circ ((\gamma_A \otimes 1_B) \otimes (\beta_A \otimes 1_E)) \circ (((1_A \otimes \epsilon_A) \otimes 1_B) \otimes ((\epsilon_A \otimes 1_A) \otimes 1_E)) \\
&\circ ((r \otimes 1_B) \otimes (s \otimes 1_E)) \circ (\delta_B \otimes \delta_E) \\
&= \alpha^{-1}_{A,B,E} \circ (\Phi(r)^l \otimes \beta_E \circ ((\epsilon_A \circ \beta_A \circ (\epsilon_A \otimes 1_A)) \circ s) \otimes 1_E) \circ \delta_E) \\
&= \alpha^{-1}_{A,B,E} \circ (\Phi(r)^l \otimes \beta_E \circ ((\epsilon_A \circ \gamma_A \circ (1_A \otimes \epsilon_A)) \circ s) \otimes 1_E) \circ \delta_E) \\
&= \alpha^{-1}_{A,B,E} \circ (\Phi(r)^l \otimes \beta_E \circ (\epsilon_A \otimes 1_E) \circ \Phi(s)^l) \\
&= \alpha^{-1}_{A,B,E} \circ (\Phi(r)^l \otimes 1_E) \\
&= \Phi(r)^l \boxtimes^A \Phi(s)^l
\end{aligned}
$$

The proof of the identity $(\Phi(r \boxdot^A s))^r = (\Phi(r) \boxtimes^A \Phi(s))^r$ is similar. ■

By duality the category of C-corelations $\mathcal{R}_A(C)$ is isomorphic to the category of $A-A$ bimodules $\mathcal{S}_A(C)$.

3.6 The tensor product of relations

For categories of bimodules over rings we have a standard construction of a tensor product. This construction is categorical in nature and can in a natural way be generalized to the category of $A-A$ bimodules $\mathcal{S}_A(C)$. By dualizing this construction we arrive at our definition of a tensor product of $A-A$ bicomodules. The isomorphism $\Psi : \mathcal{S}^A(C) \longrightarrow \mathcal{R}^A(C)$ is used to define the tensor product of relations.

The following lemma is fundamental for the construction of the tensor product.

Lemma 34 *Let δ be a object in $\mathcal{S}^A(C)$. Then $\delta^l : \delta \longrightarrow a \boxtimes^A \delta$ and $\delta^r : \delta \longrightarrow \delta \boxtimes^A a$ are morphisms in $\mathcal{S}^A(C)$ and if $f : \delta \longrightarrow \gamma$ is a morphism in $\mathcal{S}^A(C)$ the following diagrams in $\mathcal{S}^A(C)$ are commutative.*

$$\begin{tikzcd}
a \boxtimes \delta \arrow[r, "1_a \boxtimes^A f"] & a \boxtimes^A \gamma & \delta \boxtimes^A a \arrow[r, "f \boxtimes^A 1_a"] & \gamma \boxtimes^A a \\
\delta \arrow[u, "\delta^l"] \arrow[r, "f"'] & \gamma \arrow[u, "\gamma^l"'] & \delta \arrow[u, "\delta^r"] \arrow[r, "f"'] & \gamma \arrow[u, "\gamma^r"']
\end{tikzcd}$$

Proof. There are four diagrams that need to be commutative for the first part of the lemma to be true. It is seen by inspection that this set of diagrams is included in the set of diagrams defining δ to be a $A-A$ bicomodule. The second part of the lemma is clearly true since the diagrams in C corresponding to the given diagrams are the conditions for f to be a morphism of the $A-A$ bicomodules δ and γ. ■

Let now δ and γ be any pair of objects in $\mathcal{S}^A(C)$. ¿From the previous lemma we can conclude that $\mathcal{P}^A_{\delta,\gamma}$ given by

$$\begin{tikzcd}
\delta \boxtimes^A (a \boxtimes^A \gamma) \arrow[rr, "M^A_{\delta,a,\gamma}"] & & (\delta \boxtimes^A a) \boxtimes \gamma \\
& \delta \boxtimes \gamma \arrow[ul, "1_\delta \boxtimes^A \gamma^l"] \arrow[ur, "\delta^r \boxtimes^A 1_\gamma"'] &
\end{tikzcd}$$

is a diagram in $\mathcal{S}^A(C)$. The limit of this diagram, when it exists, is determined by a object in $\mathcal{S}^A(C)$ denoted by $\delta \otimes^A \gamma$ and a morphism $\pi^A_{\delta,\gamma} : \delta \otimes^A \gamma \longrightarrow \delta \boxtimes^A \gamma$.

Definition 35 *Let δ and γ be two objects in $\mathcal{S}^A(C)$. The tensor product of δ and γ is given by*

$$\otimes^A(\delta, \gamma) = \delta \otimes^A \gamma$$

The following property of $\pi^A_{\delta,\gamma}$ is important.

Proposition 36 *$\pi^A_{\delta,\gamma}$ is a monomorphism.*

Proof. Let ρ be a object in $\mathcal{S}^A(C)$ and let $f, g : \rho \longrightarrow \delta \otimes^A \gamma$ be a pair of morphisms such that $\pi^A_{\delta,\gamma} \circ f = \pi^A_{\delta,\gamma} \circ g$. Define $h = \pi^A_{\delta,\gamma} \circ g$. Then $\langle \rho, h \rangle$ is a cone on $\mathcal{P}^A_{\delta,\gamma}$ and therefore the equation

$$\pi^A_{\delta,\gamma} \circ f = h$$

has a unique solution. But both f and g are solutions and therefore by uniqueness we can conclude that $f = g$ and this proves that $\pi^A_{\delta,\gamma}$ is a monomorphism. $\blacksquare$

We now want to extend the tensorproduct to morphisms. Let δ,γ,θ and ρ be objects in $\mathcal{S}^A(C)$ and let $f : \delta \longrightarrow \theta$ and $g : \gamma \longrightarrow \rho$ be morphisms. Then we have

Lemma 37 *$(f \boxtimes^A g) \circ \pi^A_{\delta,\gamma}$ is a cone on the diagram $\mathcal{P}^A_{\theta,\rho}$.*

Proof. We have

$$
\begin{aligned}
& M^A_{\theta,a,\rho} \circ (1_\theta \boxtimes^A \rho^l) \circ (f \boxtimes^A g) \circ \pi^A_{\delta,\gamma} \\
&= M^A_{\theta,a,\rho} \circ (f \boxtimes^A (\rho^l \circ g)) \circ \pi^A_{\delta,\gamma} \\
&= M^A_{\theta,a,\rho} \circ (f \boxtimes^A (1_a \boxtimes^A g) \circ \gamma^l) \circ \pi^A_{\delta,\gamma} \\
&= M^A_{\theta,a,\rho} \circ (f \boxtimes^A (1_a \boxtimes^A g)) \circ (1_\delta \boxtimes^A \gamma^l) \circ \pi^A_{\delta,\gamma} \\
&= ((f \boxtimes^A 1_a) \boxtimes^A g) \circ M^A_{\delta,a,\gamma} \circ (1_\delta \boxtimes^A \gamma^l) \circ \pi^A_{\delta,\gamma} \\
&= ((f \boxtimes^A 1_a) \boxtimes^A g) \circ (\delta^r \boxtimes^A 1_\gamma) \circ \pi^A_{\delta,\gamma} \\
&= ((f \boxtimes^A 1_a) \circ \delta^r \boxtimes^A g) \circ \pi^A_{\delta,\gamma} \\
&= ((\theta^r \circ f) \boxtimes^A g) \circ \pi^A_{\delta,\gamma} \\
&= (\theta^r \boxtimes^A 1_\rho) \circ (f \boxtimes^A g) \circ \pi^A_{\delta,\gamma}
\end{aligned}
$$

$\blacksquare$

Let $f \otimes^A g : \delta \otimes^A \gamma \longrightarrow \theta \otimes^A \rho$ be the unique morphism that exists by the universality of the cone $\theta \otimes^A \rho$. For this morphism we have the commutative diagram

$$
\begin{tikzcd}
\delta \boxtimes^A \gamma \arrow[r, "f \boxtimes^A g"] & \theta \boxtimes^A \rho \\
\delta \otimes^A \gamma \arrow[u, "\pi^A_{\delta,\gamma}"] \arrow[r, "f \otimes^A g"'] & \theta \otimes^A \rho \arrow[u, "\pi^A_{\theta,\rho}"']
\end{tikzcd}
$$

In general $\delta \otimes^A \gamma$ will not exist for all pairs of objects in $\mathcal{S}^A(C)$. In order for it to exists and have reasonable properties we need to restrict the notion relation as we have defined it. Our first restriction is to assume that $\otimes^A$ is defined for all pairs of objects in $\mathcal{S}^A(C)$. Our second restriction involves the arrow $\pi^A_{\delta,\gamma}$. Let δ, γ and ρ be relations. We require that the morphism $\pi^A_{\delta,\gamma} \boxtimes^A 1_\rho$ is mono. We have proved that $\pi^A_{\delta,\gamma}$ is always mono, but requiring that $\pi^A_{\delta,\gamma} \boxtimes^A 1_\rho$ is mono is a nontrivial restriction in general. It can be thought of a some kind of "flatness" condition on A.

Given the above restrictions we can define a map $\otimes^A : \mathcal{S}^A(C) \times \mathcal{S}^A(C) \longrightarrow \mathcal{S}^A(C)$ by

$$
\begin{aligned}
\otimes^A(\delta, \gamma) &= \delta \otimes^A \gamma \\
\otimes^A(f, g) &= f \otimes^A g
\end{aligned}
$$

Proposition 38 *The map $\otimes^A$ is a bifunctor*

Proof. Let $\delta, \delta', \delta'', \gamma, \gamma'$ and γ'' be four objects in $\mathcal{S}^A(C)$ and let $f : \delta \longrightarrow \delta', f' : \delta' \longrightarrow \delta'', g : \gamma \longrightarrow \gamma'$ and $g' : \gamma' \longrightarrow \gamma''$ be morphisms. By universality we know that the equation

$$\pi_{\delta'',\gamma''} \circ \varphi = ((f' \circ f) \boxtimes^A (g' \circ g)) \circ \pi^A_{\delta,\gamma}$$

has a unique solution. One solution is by definition $(f' \circ f) \otimes^A (g' \circ g)$. But we also have

$$\begin{aligned}
&\pi^A_{\delta'',\gamma''} \circ ((f' \otimes^A g') \circ (f \otimes^A g)) \\
&= (f' \boxtimes^A g') \circ \pi^A_{\delta',\gamma'} \circ (f \otimes^A g) \\
&= (f' \boxtimes^A g') \circ (f \boxtimes^A g) \circ \pi^A_{\delta,\gamma} \\
&= ((f' \circ f) \boxtimes^A (g' \circ g)) \circ \pi^A_{\delta,\gamma}
\end{aligned}$$

By uniqueness we must have

$$(f' \circ f) \otimes^A (g' \circ g) = (f' \otimes^A g') \circ (f \otimes^A g)$$

Also by universality the following equation has a unique solution.

$$\pi^A_{\delta,\gamma} \circ \varphi = 1_{\delta \boxtimes^A \gamma} \circ \pi^A_{\delta,\gamma}$$

One solution is clearly $1_{\delta \otimes^A \gamma}$. But we have

$$\begin{aligned}
&\pi^A_{\delta,\gamma} \circ (1_\delta \otimes^A 1_\gamma) \\
&= (1_\delta \boxtimes^A 1_\gamma) \circ \pi^A_{\delta,\gamma} \\
&= 1_{\delta \boxtimes^A \gamma} \circ \pi^A_{\delta,\gamma}
\end{aligned}$$

so by uniqueness we have $1_{\delta \otimes^A \gamma} = 1_\delta \otimes^A 1_\gamma$. ■

We will call $\delta \otimes^A \gamma$ for the tensor product of the $A - A$ bicomodules δ and γ.

We defined the map $\otimes^A$ using universal cones in the category $\mathcal{S}^A(C)$ but we will now prove that it can be constructed from universal cones in the category C.

Let δ and γ be two objects in $\mathcal{S}^A(C)$ with underlying objects B and E in C. Let the diagram $\mathcal{P}^A_{B,E}$ in C be given by

$$\begin{tikzcd}
B \otimes (A \otimes E) \arrow[rr, "\alpha_{B,A,E}"] & & (B \otimes A) \otimes E \\
& B \otimes E \arrow[ul, "1_B \otimes \gamma^l"] \arrow[ur, "\delta^r \otimes 1_E"'] &
\end{tikzcd}$$

Assume that there exists a universal cone $\langle X, h\rangle$ on the diagram $\mathcal{P}^A_{B,E}$ in C. Define

$$\begin{aligned}\Theta^l &= (\gamma_A \otimes 1_X) \circ ((1_A \otimes \epsilon_{B\otimes E}) \otimes 1_X) \circ (\alpha^{-1}_{A,B,E} \otimes 1_X) \\ &\circ ((\delta^l \otimes 1_E) \otimes 1_X) \circ (h \otimes 1_X) \circ \delta_X \\ \Theta^r &= (1_X \otimes \beta_A) \circ (1_X \otimes (\epsilon_{B\otimes E} \otimes 1_A)) \circ (1_X \otimes \alpha_{B,E,A}) \\ &\circ (1_X \otimes (1_B \otimes \gamma^r)) \circ (1_X \otimes h) \circ \delta_X\end{aligned}$$

Proposition 39 *$\Theta = \{\Theta^l, \Theta^r\}$ is a $A - A$ bicomodule with underlying object X.*

Proof. Let us define morphisms $L, M : X \longrightarrow A$ by

$$\begin{aligned}L &= \gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \circ \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \circ h \\ M &= \beta_A \circ (\epsilon_{B\otimes E} \otimes 1_A) \circ \alpha_{B,E,A} \circ (1_B \otimes \gamma^r) \circ h\end{aligned}$$

Then $\Theta^l = (L \otimes 1_X) \circ \delta_X$ and $\Theta^r = (1_X \otimes M) \circ \delta_X$ and we have for the left structure

$$\begin{aligned}&\alpha_{A,A,X} \circ (1_A \otimes \Theta^l) \circ \Theta^l \\ &= \alpha_{A,A,X} \circ (1_A \otimes (L \otimes 1_X) \circ \delta_X) \circ (L \otimes 1_X) \circ \delta_X \\ &= \alpha_{A,A,X} \circ (1_A \otimes (L \otimes 1_X)) \circ (1_A \otimes \delta_X) \circ (L \otimes 1_X) \circ \delta_X \\ &= \alpha_{A,A,X} \circ (1_A \otimes (L \otimes 1_X)) \circ (L \otimes (1_X \otimes 1_X)) \circ (1_X \otimes \delta_X) \circ \delta_X \\ &= \alpha_{A,A,X} \circ (1_A \otimes (L \otimes 1_X)) \circ (L \otimes (1_X \otimes 1_X)) \circ \alpha^{-1}_{X;X;X} \circ (\delta_X \otimes 1_X) \circ \delta_X \\ &= ((1_A \otimes L) \otimes 1_X) \circ ((L \otimes 1_X) \otimes 1_X) \circ (\delta_X \otimes 1_X) \circ \delta_X \\ &= ((L \otimes L) \circ \delta_X \otimes 1_X) \circ \delta_X \\ &= (\delta_A \circ L \otimes 1_X) \circ \delta_X \\ &= (\delta_A \otimes 1_X) \circ \Theta^l\end{aligned}$$

The proof for the right structure is similar. For the compatibility of the left and right structure we have

$$\begin{aligned}&(\Theta^l \otimes 1_A) \circ \Theta^r \\ &= ((L \otimes 1_X) \circ \delta_X \otimes 1_A) \circ (1_X \otimes M) \circ \delta_X \\ &= ((L \otimes 1_X) \otimes 1_A) \circ (\delta_X \otimes 1_A) \circ (1_X \otimes M) \circ \delta_X \\ &= ((L \otimes 1_X) \otimes M) \circ (\delta_X \otimes 1_X) \circ \delta_X \\ &= ((L \otimes 1_X) \otimes M) \circ \alpha^{-1}_{X;X;X} \circ (1_X \otimes \delta_X) \circ \delta_X \\ &= \alpha^{-1}_{A,X,A} \circ (L \otimes (1_X \otimes M) \circ \delta_X) \circ \delta_X \\ &= \alpha^{-1}_{A,X,A} \circ (1_A \otimes \Theta^r) \circ \Theta^l\end{aligned}$$

■

We will next show that h is a morphism in $\mathcal{S}^A(C)$. For this we need the following lemma

Lemma 40

$$(\gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \otimes (\beta_B \circ (\epsilon_A \otimes 1_B) \otimes 1_E) \circ \alpha_{A,B,E}) \circ \delta_{A\otimes(B\otimes E)} = 1_{A\otimes(B\otimes E)}$$

Proof. Let T be defined as

$$\begin{aligned}T = {} & ((1_A \otimes \sigma_{e,e}) \otimes 1_{B\otimes E}) \circ (\alpha^{-1}_{A,e,e} \otimes 1_{B\otimes E}) \circ \alpha_{A\otimes e,e,B\otimes E} \\ & \circ ((1_A \otimes \epsilon_A) \otimes (\epsilon_{B\otimes E} \otimes 1_{B\otimes E})) \circ (\delta_A \otimes \delta_{B\otimes E})\end{aligned}$$

Then we have

$$\begin{aligned}
& (\gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \otimes (\beta_B \circ (\epsilon_A \otimes 1_B) \otimes 1_E) \circ \alpha_{A,B,E}) \\
& = (\gamma_A \otimes (\beta_B \otimes 1_E)) \circ ((1_A \otimes \epsilon_{B\otimes E}) \otimes ((\epsilon_A \otimes 1_B) \otimes 1_E)) \\
& \circ (1_{A\otimes(B\otimes E)} \otimes \alpha_{A,B,E}) \circ \delta_{A\otimes(B\otimes E)} \\
& = (\gamma_A \otimes (\beta_B \otimes 1_E)) \circ (1_{A\otimes e} \otimes \alpha_{e,B,E}) \circ ((1_A \otimes \epsilon_{B\otimes E}) \otimes (\epsilon_A \otimes 1_{B\otimes E})) \\
& \circ \alpha^{-1}_{A\otimes(B\otimes E),A,B\otimes E} \circ (\alpha_{A,B\otimes E,A} \otimes 1_{B\otimes E}) \circ ((1_A \otimes \sigma_{A,B\otimes E}) \otimes 1_{B\otimes E}) \\
& \circ (\alpha^{-1}_{A,A,B\otimes E} \otimes 1_{B\otimes E}) \alpha_{A\otimes A,B\otimes E,B\otimes E} \circ (\delta_A \otimes \delta_{B\otimes E}) \\
& = (\gamma_A \otimes (\beta_B \otimes 1_E)) \circ (1_{A\otimes e} \otimes \alpha_{e,B,E}) \circ \alpha_{A\otimes e,e,B\otimes E} \circ (\alpha_{A,e,e} \otimes 1_{B\otimes E}) \circ T \\
& = (\gamma_A \otimes (\beta_B \otimes 1_E)) \circ \alpha^{-1}_{A\otimes e,\eta p B,E} \circ (\alpha_{A\otimes e,e,B} \otimes 1_E) \circ \alpha_{(A\otimes e)\otimes e,B,E} \\
& \circ (\alpha_{A,e,e} \otimes 1_{B\otimes E}) \circ T \\
& = \alpha^{-1}_{A,B,E} \circ ((\gamma_A \otimes 1_B) \otimes 1_E) \circ ((1_{A\otimes e} \otimes \beta_B) \otimes 1_E) \circ (\alpha_{A\otimes e,e,B} \otimes 1_E) \\
& \circ \alpha_{(A\otimes e)\otimes e,B,E} \circ (\alpha_{A,e,e} \otimes 1_{B\otimes E}) \circ T \\
& = \alpha^{-1}_{A,B,E} \circ ((\gamma_A \otimes 1_B) \otimes 1_E) \circ ((\gamma_{A\otimes e} \otimes 1_B) \otimes 1_E) \circ \alpha_{(A\otimes e)\otimes e,B,E} \\
& \circ (\alpha_{A,e,e} \otimes 1_{B\otimes E}) \circ T \\
& = (\gamma_A \otimes 1_{B\otimes E}) \circ (\gamma_{A\otimes e} \otimes 1_{B\otimes E}) \circ (\alpha_{A,e,e} \otimes 1_{B\otimes E}) \circ T \\
& = (\gamma_A \otimes 1_{B\otimes E}) \circ ((\gamma_A \otimes 1_e) \otimes 1_{B\otimes E}) \circ (\alpha_{A,e,e} \otimes 1_{B\otimes E}) \circ T \\
& = (\gamma_A \otimes 1_{B\otimes E}) \circ ((1_A \otimes \beta_e) \otimes 1_{B\otimes E}) \circ ((1_A \otimes \sigma_{e,e}) \otimes 1_{B\otimes E}) \\
& \circ (\alpha^{-1}_{A,e,e} \otimes 1_{B\otimes E}) \circ \alpha_{A\otimes e,e,B\otimes E} \circ ((1_A \otimes \epsilon_A) \circ \delta_A \otimes (\epsilon_{B\otimes E} \otimes 1_{B\otimes E}) \circ \delta_{B\otimes E}) \\
& = (\gamma_A \otimes 1_{B\otimes E}) \circ ((1_A \otimes \gamma_e) \otimes 1_{B\otimes E}) \circ (\alpha^{-1}_{A,e,e} \otimes 1_{B\otimes E}) \circ \alpha_{A\otimes e,e,B\otimes E} \\
& \circ (\gamma_A^{-1} \otimes \beta^{-1}_{B\otimes E}) \\
& = (\gamma_A \otimes 1_{B\otimes E}) \circ ((\gamma_A \otimes 1_e) \otimes 1_{B\otimes E}) \circ \alpha_{A\otimes e,e,B\otimes E} \circ (\gamma_A^{-1} \otimes \beta^{-1}_{B\otimes E}) \\
& = (\gamma_A \otimes 1_{B\otimes E}) \circ \alpha_{A,e,B\otimes E} \circ (\gamma_A \otimes (1_e \otimes 1_{B\otimes E})) \circ (\gamma_A^{-1} \otimes \beta^{-1}_{B\otimes E}) \\
& = (\gamma_A \otimes \beta_{B\otimes E}) \circ (\gamma_A^{-1} \otimes \beta^{-1}_{B\otimes E}) \\
& = 1_{A\otimes(B\otimes E)}
\end{aligned}$$

■

We can now prove that h is a morphism in $\mathcal{S}^A(C)$.

Proposition 41 *h defines a monomorphism in $\mathcal{S}^A(C)$ with domain Θ and codomain $\delta \boxtimes^A \gamma$*

Proof. The fact that h is a monomorphism in C follows from the universality as i did for $\pi^A_{\delta,\gamma}$ in proposition 36.

For the left structure we have

$$
\begin{aligned}
&(1_A \otimes h) \circ \Theta^l \\
&(1_A \otimes h) \circ (L \otimes 1_X) \circ \delta_X \\
&= (L \otimes 1_{B\otimes E}) \circ (1_X \otimes h) \circ \delta_X \\
&= (\gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \otimes 1_{B\otimes E}) \circ (\alpha^{-1}_{A,B,E} \otimes 1_{B\otimes E}) \circ ((\delta^l \otimes 1_E) \otimes 1_{B\otimes E}) \\
&\circ (h \otimes h) \circ \delta_X \\
&= (\gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \otimes 1_{B\otimes E}) \circ (\alpha^{-1}_{A,B,E} \otimes 1_{B\otimes E}) \circ ((\delta^l \otimes 1_E) \otimes 1_{B\otimes E}) \\
&\circ \delta_{B\otimes E} \circ h \\
&= (\gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \otimes 1_{B\otimes E}) \circ (\alpha^{-1}_{A,B,E} \otimes 1_{B\otimes E}) \\
&\circ ((1_{A\otimes B} \otimes 1_E) \otimes (\beta_B \otimes 1_E) \circ ((\epsilon_A \otimes 1_B) \otimes 1_E)) \circ ((\delta^l \otimes 1_E) \otimes (\delta^l \otimes 1_E)) \\
&\circ \delta_{B\otimes E} \circ h \\
&= (\gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \otimes (\beta_B \circ (\epsilon_A \otimes 1_B) \otimes 1_E) \circ \alpha_{A,B,E}) \circ (\alpha^{-1}_{A,B,E} \otimes \alpha^{-1}_{A,B,E}) \\
&\circ \delta_{(A\otimes B)\otimes E} \circ (\delta^l \otimes 1_E) \circ h \\
&= (\gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \otimes (\beta_B \circ (\epsilon_A \otimes 1_B) \otimes 1_E) \circ \alpha_{A,B,E}) \circ \delta_{A\otimes(B\otimes E)} \\
&\circ \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \circ h \\
&= \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \circ h \\
&= (\delta \boxtimes^A \gamma)^l \circ h
\end{aligned}
$$

In a similar way we show the identity $(h \otimes 1_A) \circ \Theta^r = (\delta \boxtimes^A \gamma)^r \circ h$. ∎

Proposition 42 *Let the semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, \alpha\rangle$ be external. Then $\langle \Theta, h\rangle$ is a universal cone on $\mathcal{P}^A_{\delta,\gamma}$.*

Proof. It is evident that $\langle \Theta, h\rangle$ is a cone on the diagram $\mathcal{P}^A_{\delta,\gamma}$. Let $\langle \theta, u\rangle$ be any cone on $\mathcal{P}^A_{\delta,\gamma}$. Let $\varphi, \psi : \theta \longrightarrow \Theta$ be two morphisms in $\mathcal{S}^A(C)$ such that

$$
\begin{aligned}
h \circ \varphi = \theta \\
h \circ \psi = \theta
\end{aligned}
$$

Then $h \circ \varphi = h \circ \psi$ and since h is mono we have $\varphi = \psi$. Therefore the equation $h \circ \varphi = \theta$ has at most one solution.

The fact that $\langle \theta, u\rangle$ is a cone gives us the relation

$$
M^A_{\delta,a,\gamma} \circ (1_\delta \boxtimes^A \gamma^l) \circ u = (\delta^l \boxtimes^A 1_\gamma) \circ u
$$

If the underlying objects for δ, γ and θ are B, E and D, then $u : D \longrightarrow B \otimes E$ and the previous identity corresponds to the following
identity in C

$$
(1_B \otimes \gamma^l) \circ u = (\delta^l \otimes 1_E) \circ u
$$

and therefore $\langle D, u\rangle$ is a cone on the diagram $\mathcal{P}^A_{B,E}$ in C. By universality there exists a unique morphism $\varphi : D \longrightarrow X$ in Csuch that

$$h \circ \varphi = u$$

The fact that φ is a morphism in C and u and δ_D are morphisms in $\mathcal{S}^A(C)$ gives us the following four commutative diagrams

$$\begin{tikzcd}
D \arrow[r, "u"] \arrow[d, "\theta^l"'] & B \otimes E \arrow[d, "\delta^l \otimes 1_E"] \\
A \otimes D \arrow[r, "1_A \otimes u"'] & A \otimes B \otimes E
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, "\delta_D"] \arrow[d, "\theta^l"'] & D \otimes D \arrow[d, "\delta^l \otimes 1_D"] \\
A \otimes D \arrow[r, "1_A \otimes \delta_D"'] & A \otimes D \otimes D
\end{tikzcd}$$

$$\begin{tikzcd}
D \arrow[r, "\phi"] \arrow[d, "\delta_D"'] & X \arrow[d, "\delta_X"] \\
D \otimes D \arrow[r, "\phi \otimes \phi"'] & X \otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[rr, "u"] \arrow[dr, "\epsilon_D"'] & & B \otimes E \arrow[dl, "\epsilon_{B\otimes E}"] \\
& e &
\end{tikzcd}$$

If we define L as in proposition 39 we have the following identities

$$\begin{aligned}
&L \circ \varphi \\
&= \gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \circ \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \circ h \circ \varphi \\
&= \gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \circ \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \circ u \\
&= \gamma_A \circ (1_A \otimes \epsilon_{B\otimes E}) \circ (1_A \otimes u) \circ \theta^l \\
&= \gamma_A \circ (1_A \otimes \epsilon_D) \circ \theta^l
\end{aligned}$$

But then we have

$$\begin{aligned}
&\Theta^l \circ \varphi \\
&= (L \otimes 1_X) \circ \delta_X \circ \varphi \\
&= (L \otimes 1_X) \circ (\varphi \otimes \varphi) \circ \delta_D \\
&= (L \circ \varphi \otimes \varphi) \circ \delta_D \\
&= (1_A \otimes \varphi) \circ (\gamma_A \otimes 1_D) \circ ((1_A \otimes \epsilon_D) \otimes 1_D) \circ (\theta^l \otimes 1_D) \circ \delta_D \\
&= (1_A \otimes \varphi) \circ (\gamma_A \otimes 1_D) \circ \alpha_{A,e,D} \circ (1_A \otimes (\epsilon_D \otimes 1_D)) \circ (1_A \otimes \delta_D) \circ \theta^l \\
&= (1_A \otimes \varphi) \circ (1_A \otimes \beta_D \circ (\epsilon_D \otimes 1_D) \circ \delta_D) \circ \theta^l \\
&= (1_A \otimes \varphi) \circ \theta^l
\end{aligned}$$

In a similar way we prove the identity $\Theta^r \circ \varphi = (\varphi \otimes 1_A) \circ \theta^r$. This proves that φ is a morphism in $\mathcal{S}^A(C)$ and therefore that the equation

$$h \circ \varphi = u$$

has a unique solution in $\mathcal{S}^A(C)$. ■

This proposition show that $\delta \otimes^A \gamma \approx \Theta$ since universal cones are determined up to isomorphism. This is the way the tensor product is usually computed.

We now have two bifunctors $\boxtimes^A$ and $\otimes^A$ defined on $\mathcal{S}^A(C)$. These two structures are related at the functorial level as the next proposition show.

Proposition 43 *$\pi^A_{\delta,\gamma}$ are the components of a natural monomorphism*

$$\pi^A : \otimes^A \longrightarrow \boxtimes^A$$

Proof. The proposition follows directly from the commutative diagram 3.6. ■

3.7 Monoidal structures on the category of relations

We have seen that $\boxtimes^A$ defines a semimonoidal structure on $\mathcal{S}^A(C)$ with associativity constraint M^A. We will in the following only consider the case when the product $\otimes^A$ defines a monoidal structure on $\mathcal{S}^A(C)$ with neutral object a. This is a further restriction on the category $\mathcal{S}^A(C)$ and thus on the category of relations. Recall that the pair $\otimes^A, a$ defines a monoidal structure on $\mathcal{S}^A(C)$ if for all objects δ, γ and ρ there exists isomorphisms

$$\begin{aligned} m^A_{\delta,\gamma,\rho} &: \delta \otimes^A (\gamma \otimes^A \rho) \longrightarrow (\delta \otimes^A \gamma) \otimes^A \rho \\ l^A_\delta &: a \otimes^A \delta \longrightarrow \delta \\ r^A_\delta &: \delta \otimes^A a \longrightarrow \delta \end{aligned}$$

that are natural in δ, γ and ρ and such that the MacLane coherence conditions are satisfied. The coherence conditions are a set of equations for the morphisms m^A, l^A and r^A and these equations may have no solutions, a unique solution or many solutions depending on the category C and the coalgebra A.

Definition 44 *A monoidal structure $\langle \mathcal{S}^A(C), \otimes^A, a, m^A, l^A, r^A \rangle$ on the category $\mathcal{S}^A(C)$ is induced if for all objects δ, γ and ρ in $\mathcal{S}^A(C)$ the following diagrams commute*

$$\begin{tikzcd} a \otimes^A \delta \arrow[rr, "l^A_\delta"] \arrow[dr, "\pi^A_{a,\delta}"'] & & \delta \arrow[dl, "\delta^l"] \\ & a \boxtimes^A \delta & \end{tikzcd} \qquad \begin{tikzcd} \delta \otimes^A a \arrow[rr, "r^A_\delta"] \arrow[dr, "\pi^A_{\delta,a}"'] & & \delta \arrow[dl, "\delta^r"] \\ & \delta \boxtimes^A a & \end{tikzcd}$$

$$\begin{tikzcd} \delta \boxtimes^A (\gamma \boxtimes^A \rho) \arrow[r, "M^A_{\delta,\gamma,\rho}"] & (\delta \boxtimes^A \gamma) \boxtimes \rho \\ \delta \boxtimes^A (\gamma \otimes^A \rho) \arrow[u, "1_\delta \boxtimes^A \pi^A_{\gamma,\rho}"] & (\delta \otimes^A \gamma) \boxtimes^A \rho \arrow[u, "\pi^A_{\delta,\gamma} \otimes 1_\rho"'] \\ \delta \otimes^A (\gamma \otimes^A \rho) \arrow[u, "\pi^A_{\delta,\gamma\otimes^A\rho}"] \arrow[r, "m^A_{\delta,\gamma,\rho}"'] & (\delta \otimes^A \gamma) \otimes^A \rho \arrow[u, "\pi^A_{\delta\otimes^A\gamma,\rho}"'] \end{tikzcd}$$

We will in the following derive a necessary and sufficient condition for induced constraints to exist in the external case. Let us assume that the semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ is external. Let $\mathcal{P}^A_{a,\delta}$ be the diagram

$$\begin{tikzcd}
a \boxtimes^A (a \boxtimes^A \delta) \arrow[rr, "M^A_{a,a,\delta}"] & & (\delta \boxtimes^A a) \boxtimes \gamma \\
& a \boxtimes \delta \arrow[ul, "1_a \boxtimes^A \delta^l"] \arrow[ur, "\delta_A \boxtimes^A 1_\delta"'] &
\end{tikzcd}$$

Then $\langle \delta, \delta^l \rangle$ is clearly a cone on this diagram since this is equivalent to the condition that δ^l is a left comodule structure on the underlying object of δ. But we have also a stronger condition.

Proposition 45 *Let the semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ be external. Then $\langle \delta, \delta^l \rangle$ is a universal cone on $\mathcal{P}^A_{a,\delta}$.*

Proof. In order to prove that $\langle \delta, \delta^l \rangle$ is a universal cone we must show that the equation

$$\delta^l \circ \varphi = f$$

has a unique solution $\varphi : \gamma \longrightarrow \delta$ for any $f : \gamma \longrightarrow a \boxtimes^A \delta$ such that

$$(\delta_A \otimes 1_\delta) \circ f = M^A_{a,a,\delta} \circ (1_a \otimes \delta^l) \circ f$$

Since $\beta_B \circ (\epsilon_A \otimes 1_B) \circ \delta^l = 1_B$ the equation can have only one solution and this solution must be

$$\varphi = \beta_B \circ (\epsilon_A \otimes 1_B) \circ f$$

The universality is proved if we can show that this is in fact a solution and also a morphism in $\mathcal{S}^A(C)$.

$$\begin{aligned}
&\delta^l \circ \varphi \\
&= \delta^l \circ \beta_B \circ (\epsilon_A \otimes 1_B) \circ f \\
&= \beta_{A \otimes B} \circ (1_e \otimes \delta^l) \circ (\epsilon_A \otimes 1_B) \circ f \\
&= \beta_{A \otimes B} \circ (\epsilon_A \otimes (1_A \otimes 1_B)) \circ (1_A \otimes \delta^l) \circ f \\
&= \beta_{A \otimes B} \circ (\epsilon_A \otimes (1_A \otimes 1_B)) \circ \alpha^{-1}_{A,A,B} \circ (\delta_A \otimes 1_B) \circ f \\
&= \beta_{A \otimes B} \circ \alpha^{-1}_{e,A,B} \circ (\beta^{-1}_A \otimes 1_B) \circ f \\
&= f
\end{aligned}$$

so φ is a solution. Here we have used the identity $\beta_{A \otimes B} = (\beta_A \otimes 1_B) \circ \alpha_{e,A,B}$.

By construction φ is a arrow in C, but we also have

$$\begin{aligned}
&(1_A \otimes \varphi) \circ \gamma^l \\
&= (1_A \otimes \beta_B) \circ (1_A \otimes (\epsilon_A \otimes 1_B)) \circ (1_A \otimes f) \circ \gamma^l \\
&= (1_A \otimes \beta_B) \circ (1_A \otimes (\epsilon_A \otimes 1_B)) \circ (1_A \otimes \delta^l) \circ f \\
&= (1_A \otimes \beta_B) \circ (1_A \otimes (\epsilon_A \otimes 1_B)) \circ \alpha^{-1}_{A,A,B} \circ (\delta_A \otimes 1_B) \circ f \\
&= (1_A \otimes \beta_B) \circ \alpha^{-1}_{A,e,B} \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (\delta_A \otimes 1_B) \circ f \\
&= (\gamma_A \otimes 1_B) \circ (\gamma_A^{-1} \otimes 1_B) \circ f \\
&= f \\
&= \delta^l \circ \varphi
\end{aligned}$$

so φ is a morphism in $\mathcal{S}^A(C)$. ■

We have the following two corollaries to the previous proposition

Corollary 46 *Let the semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ be external and let the underlying object for δ be B. If induced unit constraints exists they must be of the form*

$$\begin{aligned}
l^A_\delta &= \beta_B \circ (\epsilon_A \otimes 1_B) \circ \pi^A_{a,\delta} \\
r^A_\delta &= \gamma_B \circ (1_B \otimes \epsilon_A) \circ \pi^A_{\delta,a}
\end{aligned}$$

Corollary 47 *Let the semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ be external. Then the morphism $\delta^l : \delta \longrightarrow a \otimes^A \delta$ is a monomorphism.*

Proof. Let γ be any object in $\mathcal{S}^A(C)$ and let $f, g : \gamma \longrightarrow \delta$ be any pair of morphisms. Assume that $\delta^l \circ f = \delta^l \circ g$ and define $h = \delta^l \circ g$. Then both f and h satisfy the equation

$$\delta^l \circ \varphi = h$$

By universality we can conclude that $f = g$. ■

Proposition 48 *Let the semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ be external. Then induced unit and associativity constraints are unique if they exist.*

Proof. By definition a external unit constraint l^A_δ is a solution of the equation

$$\delta^l \circ l^A_\delta = \pi^A_{a,\delta}$$

But by universality this equation has a unique solution. The uniqueness of r^A_δ is proved in a similar way. For $m^A_{r,s,t}$ we note that the morphism $t = (\pi^A_{\delta,\gamma} \otimes 1_\rho) \circ \pi^A_{\delta \otimes^A \gamma, \rho}$ is mono. Let f and g be two morphisms such that the third diagram in definition 44 commutes. Then we have

$$\begin{aligned}
t \circ f = (\pi^A_{\delta,\gamma} \otimes 1_\rho) \circ \pi^A_{\delta \otimes^A \gamma, \rho} \circ f = \alpha_{B,E,D} \circ (1_\delta \otimes \pi^A_{\gamma,\rho}) \circ \pi^A_{\delta, \gamma \otimes^A \rho} \\
t \circ g = (\pi^A_{\gamma,\delta} \otimes 1_\rho) \circ \pi^A_{\delta \otimes^A \gamma, \rho} \circ g = \alpha_{B,E,D} \circ (1_\delta \otimes \pi^A_{\gamma,\rho}) \circ \pi^A_{\delta, \gamma \otimes^A \rho}
\end{aligned}$$

so $t \circ f = t \circ g$. But t is mono and therefore $f = g$ ■

We can now give sufficient conditions for the existence of induced unit and associativity constraints.

Theorem 49 *Let the semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ be external and assume that for all δ, γ and ρ there exists a isomorphism $m^A_{\delta,\gamma,\rho} : \delta \otimes^A (\gamma \otimes^A \rho) \longrightarrow (\delta \otimes^A \gamma) \otimes^A \rho$ such that the following diagram commute.*

$$\begin{tikzcd}
\delta \boxtimes^A (\gamma \boxtimes^A \rho) \arrow[r, "M^A_{\delta,\gamma,\rho}"] & (\delta \boxtimes^A \gamma) \boxtimes \rho \\
\delta \boxtimes^A (\gamma \otimes^A \rho) \arrow[u, "1_\delta \boxtimes^A \pi^A_{\gamma,\rho}"] & (\delta \otimes^A \gamma) \boxtimes^A \rho \arrow[u, "\pi^A_{\delta,\gamma} \otimes 1_\rho"'] \\
\delta \otimes^A (\gamma \otimes^A \rho) \arrow[u, "\pi^A_{\delta,\gamma \otimes^A \rho}"] \arrow[r, "m^A_{\delta,\gamma,\rho}"'] & (\delta \otimes^A \gamma) \otimes^A \rho \arrow[u, "\pi^A_{\delta \otimes^A \gamma,\rho}"']
\end{tikzcd}$$

Then a induced monoidal structure $\langle \mathcal{S}^A(C), \otimes^A, a, m^A, l^A, r^A \rangle$ on the category $\mathcal{S}^A(C)$ exists.

Proof. Let us first prove that $m^A_{\delta,\gamma,\rho}$ are the components of a natural isomorphism. Let δ', γ' and ρ' be three other relations and let $f : \delta \longrightarrow \delta^{'}, g : \gamma \longrightarrow \gamma'$ and $h : \rho \longrightarrow \rho'$ be three morphisms. For any three relations δ, γ and ρ define $\varphi_{\delta,\gamma,\rho} = (\pi^A_{\delta,\gamma} \otimes 1_\rho) \circ \pi^A_{\delta \otimes^A \gamma,\rho}$ and $\psi_{\delta,\gamma,\rho} = (1_\delta \otimes^A \pi^A_{\gamma,\rho}) \circ \pi^A_{\delta,\gamma \otimes^A \rho}$. Then we have

$$\begin{aligned}
&\varphi_{\delta',\gamma',\rho'} \circ ((f \otimes^A g) \otimes^A h) \circ m^A_{\delta,\gamma,\rho} \\
&= ((f \boxtimes^A g) \boxtimes^A h) \circ \varphi_{\delta,\gamma,\rho} \circ m^A_{\delta,\gamma,\rho} \\
&= ((f \boxtimes^A g) \boxtimes^A h) \circ M^A_{\delta,\gamma,\rho} \circ \psi_{\delta,\gamma,\rho} \\
&= M^A_{\delta',\gamma',\rho'} \circ (f \boxtimes^A (g \boxtimes^A h)) \circ \psi_{\delta,\gamma,\rho} \\
&= M^A_{\delta',\gamma',\rho'} \circ \psi_{\delta',\gamma',\rho'} \circ (f \otimes^A (g \otimes^A h)) \\
&= \varphi_{\delta',\gamma',\rho'} \circ m^A_{\delta',\gamma',\rho'} \circ (f \otimes^A (g \otimes^A h))
\end{aligned}$$

¿From this the naturality of $m^A_{\delta,\gamma,\rho}$ follows because $\varphi_{\delta',\gamma',\rho'}$ is mono. We have thus far proved that we have a natural isomorphism

$$m^A_{\delta,\gamma,\rho} : \delta \otimes^A (\gamma \otimes^A \rho) \longrightarrow (\delta \otimes^A \gamma) \otimes^A \rho$$

A induced left unit constraint is a natural isomorphism in $\mathcal{S}^A(C)$ that satisfy the equation

$$\delta^l \circ \varphi = \pi^A_{a,\delta}$$

By definition $a \otimes^A \delta$ is a universal cone on the diagram $\mathcal{P}^A_{a,\delta}$. But δ^l is also a universal cone on this diagram so there exists a isomorphism $\varphi : a \otimes^A \delta \longrightarrow \delta$

such that $\delta^l \circ \varphi = \pi^A_{a,\delta}$. We have seen that the only solution of this equation in $\mathcal{S}^A(C)$ is given by $l^A_\delta = \beta_B \circ (\epsilon_A \otimes 1_B) \circ \pi^A_{a,\delta}$ where the underlying object for δ is B. We can therefore conclude that $l^A_\delta : a \otimes^A \delta \longrightarrow \delta$ is a isomorphism. This isomorphism is natural because if δ and δ' are objects in $\mathcal{S}^A(C)$ with underlying objects B and B' and $f : \delta \longrightarrow \delta'$ is any morphism the naturality of β and π^A give

$$\begin{aligned}
&f \circ l^A_\delta \\
&= f \circ \beta_B \circ (\epsilon_A \otimes 1_B) \circ \pi^A_{a,\delta} \\
&= \beta_{B'} \circ (1_e \otimes f) \circ (\epsilon_A \otimes 1_B) \circ \pi^A_{a,\delta} \\
&= \beta_{B'} \circ (\epsilon_A \otimes f) \circ \pi^A_{a,\delta} \\
&= \beta_{B'} \circ (\epsilon_A \otimes 1_{B'}) \circ (1_A \otimes f) \circ \pi^A_{a,\delta} \\
&= \beta_{B'} \circ (\epsilon_A \otimes 1_{B'}) \circ \pi^A_{a,\delta'} \circ f \\
&= l^A_{\delta'} \circ f
\end{aligned}$$

In a similar way we find a natural isomorphism $r^A_\delta = \gamma_B \circ (1_B \otimes \epsilon_A) \circ \pi^A_{\delta,a}$. The proposition is proved if we can show that these three natural isomorphisms satisfy the MacLane coherence conditions for a monoidal category. We clearly have

$$a^l \circ l^A_a = \pi^A_{a,a} = a^r \circ r^A_a$$

But $a^l = \delta_A = a^r$ and δ_A is a monomorphism so we have $l^A_a = r^A_a$. This is the third MacLane condition. Let us now consider the second MacLane condition. Let δ and γ have underlying objects B and E. Using the formulas for l^A_δ, r^A_δ and the definition of $m^A_{\delta,\gamma,\rho}$ we find

$$\begin{aligned}
&\pi^A_{\delta,\gamma} \circ (r^A_\delta \otimes^A 1_\gamma) \circ m^A_{\delta,a,\gamma} \\
&= (r^A_\delta \otimes 1_E) \circ \pi^A_{\delta \otimes^A a,\gamma} \circ m^A_{\delta,a,\gamma} \\
&= (\gamma_B \circ (1_B \otimes \epsilon_A) \circ \pi^A_{\delta,a} \otimes 1_E) \circ \pi^A_{\delta \otimes^A a,\gamma} \circ m^A_{\delta,a,\gamma} \\
&= (\gamma_B \otimes 1_E) \circ ((1_B \otimes \epsilon_A) \otimes 1_E) \circ (\pi^A_{\delta,a} \boxtimes^A 1_\gamma) \circ \pi^A_{\delta \otimes^A a,\gamma} \circ m^A_{\delta,a,\gamma} \\
&= (\gamma_B \otimes 1_E) \circ ((1_B \otimes \epsilon_A) \otimes 1_E) \circ \alpha_{B,A,E} \circ (1_\delta \boxtimes^A \pi^A_{a,\gamma}) \circ \pi^A_{\delta,a \otimes^A \gamma} \\
&= (\gamma_B \otimes 1_E) \circ \alpha_{B,e,E} \circ (1_B \otimes (\epsilon_A \otimes 1_E)) \circ (1_\delta \boxtimes^A \pi^A_{a,\gamma}) \circ \pi^A_{\delta,a \otimes^A \gamma} \\
&= (1_B \otimes \beta_E) \circ (1_B \otimes (\epsilon_A \otimes 1_E)) \circ (1_B \otimes \pi^A_{a,\gamma}) \circ \pi^A_{\delta,a \otimes^A \gamma} \\
&= (1_B \otimes \beta_E \circ (\epsilon_A \otimes 1_E) \circ \pi^A_{a,\gamma}) \circ \pi^A_{\delta,a \otimes^A \gamma} \\
&= (1_\delta \boxtimes^A l^A_\gamma) \circ \pi^A_{\delta,a \otimes^A \gamma} \\
&= \pi^A_{\delta,\gamma} \circ (1_\delta \otimes^A l^A_\gamma)
\end{aligned}$$

and $\pi^A_{\delta,\gamma}$ is mono so we have

$$(r^A_\delta \otimes^A 1_\gamma) \circ m^A_{\delta,a,\gamma} = (1_\delta \otimes^A l^A_\gamma)$$

and this is the second MacLane condition. The first MacLane condition follow from the assumptions in the Theorem and the fact that $\boxtimes^A$ is a semi-monoidal structure on $\mathcal{S}^A(C)$ with associativity constraint M^A. ∎

Since $\mathcal{S}^A(C)$ is isomorphic to the category of relations a monoidal structure on $\mathcal{S}^A(C)$ will induce one on the category of relations. Let the product in $\mathcal{R}^A(C)$ corresponding to $\otimes^A$ be $\odot^A : \mathcal{R}^A(C) \times \mathcal{R}^A(C) \longrightarrow \mathcal{R}^A(C)$. We thus have

$$\odot^A = \Psi \circ \otimes^A \circ (\Phi \times \Phi)$$

We have the following explicit expression for the product

Proposition 50 *For any pair of objects r and s in$\mathcal{R}^A(C)$ we have*

$$r \odot^A s = (\gamma_A \otimes \beta_A) \circ (1_A \otimes \epsilon_A \otimes \epsilon_A \otimes 1_A) \circ (r \otimes s) \circ \pi^A_{\Phi(r),\Phi(s)}$$

Proof. We have a natural monomorphism

$$\pi^A : \otimes^A \longrightarrow \boxtimes^A$$

Since by definition $\boxdot^A = \Psi \circ \boxtimes^A \circ (\Phi \times \Phi)$ and $\odot^A = \Psi \circ \otimes^A \circ (\Phi \times \Phi)$, horizontal composition of natural transformations give us a natural transformation

$$(1_\Psi \circ \pi^A \circ (1_\Phi \times 1_\Phi)) : \odot^A \longrightarrow \boxdot^A$$

If we evaluate this natural transformation at a pair of objects r and s in $\mathcal{R}^A(C)$ we get the following morphism in $\mathcal{R}^A(C)$

$$\pi^A_{\Phi(r),\Phi(s)} : r \odot^A s \longrightarrow r \boxdot^A s$$

But this means that

$$r \odot^A s = (r \boxdot^A s) \circ \pi^A_{\Phi(r),\Phi(s)}$$

and this is the formula in the proposition if we take into account the formula for $r \boxdot^A s$ that we have derived earlier. ∎

We will now consider a few examples of the tensor product. Let us first assume that the underlying category C is $Sets$ with its unique choice of natural C-category C. We have seen that all possible $A - A$ bicomodule structures $\delta = \{\delta^l, \delta^r\}$on a set B are of the form $\delta^l(x) = (f(x), x)$ and $\delta^r(x) = (x, g(x))$ for some functions $f, g : B \longrightarrow A$. The relation on A corresponding to δ is clearly given by $r(x) = (f(x), g(x))$. Let now $r(x) = (f(x), g(x))$ and $s(y) = (h(y), k(y))$ be two relations with domains B and E and let the corresponding $A - A$ bicomodules be δ and γ. The two maps $(\delta^r \times 1_E)$ and $(1_B \times \gamma^l)$ are given by

$$(\delta^r \times 1_E)(x, y) = (x, g(x), y)$$
$$(1_B \times \gamma^l)(x, y) = (x, h(y), y)$$

In *Sets* the underlying object X for the $A-A$ bicomodule $\delta \otimes^A \gamma$ is the equalizer of the two given maps. We therefore find that

$$X = \{(x, y) |\ g(x) = h(y)\}$$

and

$$(\delta \otimes^A \gamma)^l (x, y) = (f(x), x, y)$$
$$(\delta \otimes^A \gamma)^r (x, y) = (x, y, x, k(y))$$

The map $\pi^A_{\delta,\gamma} : X \longrightarrow B \times E$ is the inclusion map. The relation $r \odot^A s$ corresponding to $\delta \otimes^A \gamma$ is then given by $(r \odot^A s)(x, y) = (f(x), k(y))$.

We have seen that each relation r and s is in fact a directed labelled graph. Each element in B can be thought of as a arrow that has a source and a target in the vertex set A for the graph and similarly for elements in E. Let us define two arrows to be composable if the target of the first is the same as the source of the second. The set X then consists of all composable pairs of arrows from B and E. Two relations on A, $B \subset A \times A$ and $E \subset A \times A$, in the usual sense corresponds to relations r and s in our sense if we let $r(x, x') = (x, x')$ and $s(y, y') = (y, y')$ be the inclusion maps. If we use the same notation as above we find $f(x, x') = x, g(x, x') = x', h(y, y') = y$ and $k(y, y') = y'$.

For this special case we find

$$X = \{((x, y), (y, y'))\}$$
$$(r \odot^A s)((x, y), (y, y')) = (x, y')$$

We then observe that

$$(r \odot^A s)(X) = B \circ E$$

where $B \circ E$ is the usual composition of relations.

Let $t : D \longrightarrow A \times A$ be a third relation with $t(z) = (p(z), q(z))$ and let ρ be the $A - A$ bicomodule corresponding to t. Let X be the underlying object for $(\delta \otimes^A \gamma) \otimes^A \rho$ and Y the underlying object for $\delta \otimes^A (\gamma \otimes^A \rho)$. Direct calculation show that

$$X = \{((x, y), z) |\ g(x) = h(y), k(y) = p(z)\}$$
$$Y = \{(x, (y, z)) |\ g(x) = h(y), k(y) = p(z)\}$$

Define

$$m^A_{\delta,\gamma,\rho}(x, (y, z)) = ((x, y), z)$$
$$l^A_\delta (a, x) = x$$
$$r^A_\delta (x, a) = x$$

It is easy to see that $m^A_{\delta,\gamma,\rho}$ is a morphism of relations. The underlying object for $a \otimes^A \delta$ is easily seen to given by

$$Z = \{(f(x), x) |\ x \in B\}$$

Therefore l_δ^A is clearly a isomorphism and

$$\begin{aligned}
&(\delta^l \circ l_\delta^A)(f(x), x)\\
&= \delta^l(x)\\
&= (f(x), x)\\
&= \pi_{a,\delta}^A(f(x), x)
\end{aligned}$$

In a similar way we show that r_δ^A is a isomorphism that satisfy $\delta^r \circ r_\delta^A = \pi_{\delta,a}^A$. It is easy to see that $\omega_{r,s,t}^A, \beta_r^A$ and γ_r^A satisfy the MacLane coherence conditions. They are therefore the associativity and unit constraints for a monoidal structure $\otimes^A$on $\mathcal{S}^A(C)$. A simple calculation show that they are induced.

In a similar way we can define products of any number of relations. It is evident that the product of n relations consists of strings of composable arrows of length n, one arrow from each relation. Note that δ_A is a relation on A. Let us assume that there exists a morphism of relations $f : \delta_A \longrightarrow r$. This means that for each element $a \in A$ there exists a element $b_a = f(a)$ in B such that a is both the source and target of b_a. If we now take all possible products of the relation r we observe that the result is in fact the (internal) category generated by the graph defined by the relation.

Let us next consider $Vect_k$ with $\oplus$ as monoidal structure. Let A be a linear space and let $r : B \longrightarrow A \oplus A$ and $s : E \longrightarrow A \oplus A$ be relations on A. The domain for the product $r \odot^A s$ is a linear subspace of $B \oplus E$

$$V = \{(u, v)|\ g(u) = h(v)\}$$

$$(r \odot^A s)(u, v) = (f(u), k(v))$$

where $r(u) = (f(u), g(u))$ and $s(v) = (h(v), k(v))$. Let $L : A \longrightarrow A$ and $S : A \longrightarrow A$ be two endomorphism of A and let $B = E = A$ and $r : B \longrightarrow A \oplus A$ $s : E \longrightarrow A \oplus A$ be relations where $r(a) = (a, L(a))$ and $s(a) = (a, S(a))$. We then have $f(a) = a, g(a) = L(a), h(a) = a$ and $k(a) = S(a)$. Therefore the underlying object for $r \odot^A s$ is $X = \{(a, L(a))\}$ and

$$(r \odot^A s)(a, L(a)) = (a, (S \circ L)(a))$$

so the image of X in $A \oplus A$ is the graph of the composition of L and S. More generally let $L \subset A \oplus A$ and $S \subset A \oplus A$ be two linear subspaces and let $r : L \longrightarrow A \oplus A$ and $s : S \longrightarrow A \oplus A$ be the corresponding relations with r and s the inclusion maps. Then the image of the product relation of L and S in $A \oplus A$ is formed by selecting vectors in $u \in L$, decomposing them as $u = a + b$ with $a, b \in A$, selecting vectors $v \in S$ with decomposition $v = b + c$ and finally forming the vectors $w = a + c$.

A monoid in the category of relations is a relation r and two morphisms of relations

$$\begin{aligned}
&\mu_r : r \odot^A r \longrightarrow r\\
&u_r : \delta_A \longrightarrow r
\end{aligned}$$

such that the associativity and unit diagrams commute. Let us consider the case when the basic category is $Sets$. Then we have seen that a relation is a graph with vertex set A and arrow set B. The source and target for any arrow x is given by $f(x), g(x) \in A$ where $r(x) = (f(x), g(x))$. The domain X for the relation $r \odot^A r$ consists of all composable pairs of arrows from the graph B.

$$X = \{(x, x') |\, g(x) = f(x')\ \}$$

The map μ_r will define a associative rule of composition for composable pair of arrows in B. Furthermore the unitmap will provide for each vertex $a \in A$ an arrow $u(a)$ that has a as both source and target and that acts as left and right unit for composition. The structure we have described is clearly a internal category in $Sets$ with objects A and arrows B. Let $B \subset A \times A$ be a transitive and reflexive relation in the usual sense. If we define $r((x, y)) = (x, y)$ then r is clearly a relation in our sense. We have seen that the domain of the relation $r \otimes^A r$ is of the form

$$X = \{((x, y), (y, z)) \mid (x, y) \in B, (y, z) \in B\}$$

and $(r \otimes^A r)(((x, y), (y, z))) = (x, z)$. Define a map of sets $\mu_r : X \longrightarrow A \times A$ by $\mu_r(((x, y), (y, z))) = (x, z)$. But both $(x, y) \in B$ and $(y, z) \in B$ and since B is transitive we have $(x, z) \in B$ and so we have in fact $\mu_r : X \longrightarrow B$. But we also have

$$\begin{aligned}
&(r \circ \mu_r)(((x, y), (y, z)) \\
&= r((x, z)) \\
&= (x, z) \\
&= (r \otimes^A r)(((x, y), (y, z)))
\end{aligned}$$

so $\mu_r : r \otimes^A r \longrightarrow r$. The map μ_r is clearly associative. Define a map of sets $u_r : A \longrightarrow A \times A$ by $u_r(x) = (x, x)$. Since the relation B is reflexive we have $(x, x) \in B$ for all $x \in A$ and therefore we have $u_r : A \longrightarrow B$. This map is clearly a morphism of relations and acts as a left and right unit for the rule of composition μ_r. We therefore have proved that the relation in our sense,corresponding to a reflexive and transitive relation in the usual sense, is in fact a monoid in the category of relations. We can thus think of monoids in $\mathcal{R}^A(C)$ as generalized reflexive and transitive relations or generalized categories.

3.8 Symmetries for the category of relations

¿From an algebraic point of view we know that commutative monoids is an important and interesting subclass of all monoids. From a categorical point of view the notion of commutativity can not be formulated unless there is a symmetry defined on the category.

Let us therefore consider the notion of a symmetry for the category of relations. In $Sets$ a relation r with domain B is a labelled and directed graph with vertex set A and arrow set B. We have seen that the tensor product of two relations r and s with domains B and E is a new graph on the vertex set A where the set of arrows consists of all composable pairs of arrows from B and E. If (x, y) with $x \in B$ and $y \in E$ is a composable pair of arrows it is clear that in general the pair (y, x) is not a composable pair. It is thus evident that for relations the simple transposition $(x, y) \longrightarrow (y, x)$ is not the right notion for a symmetry. We will develop our theory for the category $\mathcal{S}^A(C)$ and use the isomorphism whenever we need the corresponding structures in the category of relations. Dual properties holds for the categories $\mathcal{S}_A(C)$ and the category of C-corelations.

Before we give the right definition of symmetry for the category of relations we need to introduce a new structure. Let $\delta = \{\delta^l, \delta^r\}$ be a object in $\mathcal{S}^A(C)$. Let $r = \Psi(\delta)$ be the corresponding relation and define $r^* = \sigma_{A,A} \circ r$ and

$$\delta^* = \Phi(r^*)$$

An explicit expression for the new object δ^* is given by the following.

Proposition 51 *Let δ be a object in $\mathcal{S}^A(C)$ with underlying object B. Then we have*

$$\begin{aligned}(\delta^*)^l &= \sigma_{B,A} \circ \delta^r \\ (\delta^*)^r &= \sigma_{A,B} \circ \delta^l\end{aligned}$$

Proof. Let $r = \Psi(\delta)$. Then we have

$$\begin{aligned}
&(\delta^*)^l \\
&= (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (r^* \otimes 1_B) \circ \delta_B \\
&= (\gamma_A \otimes 1_B) \circ ((1_A \otimes \epsilon_A) \otimes 1_B) \circ (\sigma_{A,A} \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B \\
&= (\gamma_A \otimes 1_B) \circ (\sigma_{e,A} \otimes 1_B) \circ ((\epsilon_A \otimes 1_A) \otimes 1_B) \circ (r \otimes 1_B) \circ \delta_B \\
&= (\beta_A \otimes 1_B) \circ ((\epsilon_A \otimes 1_A) \otimes 1_B) \circ (r \otimes 1_B) \circ \sigma_{B,B} \circ \delta_B \\
&= \sigma_{B,A} \circ (1_B \otimes \beta_A) \circ (1_B \otimes (\epsilon_A \otimes 1_A)) \circ (1_B \otimes r) \circ \delta_B \\
&= \sigma_{B,A} \circ \delta^r
\end{aligned}$$

where we have used the commutativity of δ_B. In a similar way we show that $(\delta^*)^r = \sigma_{A,B} \circ \delta^l$. ■

Note that δ and δ^* both have the same underlying object. In order to extend the new operation * to morphisms we need the following lemma

Lemma 52 *Let δ and γ be two objects in $\mathcal{S}^A(C)$ with underlying objects B and E and let $f : \delta \longrightarrow \gamma$ be a morphism. Then the corresponding arrow in C define a morphism in $\mathcal{S}^A(C)$ with domain δ^* and codomain γ^*.*

Proof. We have

$$
\begin{aligned}
&(1_A \otimes f) \circ (\delta^*)^l \\
&= (1_A \otimes f) \circ \sigma_{B,A} \circ \delta^r \\
&= \sigma_{E,A} \circ (f \otimes 1_A) \circ \delta^r \\
&= \sigma_{E,A} \circ \gamma^r \circ f \\
&= (\gamma^*)^l \circ f
\end{aligned}
$$

and in a similar way we prove that $(f \otimes 1_A) \circ (\delta^*)^r = (\gamma^*)^r \circ f$. ■

We define $f^* : \delta^* \longrightarrow \gamma^*$ to be the morphism described in the previous lemma. It is evident that the map $T : \mathcal{S}^A(C) \longrightarrow \mathcal{S}^A(C)$ defined on objects and arrows by $T(\delta) = \delta^*$ and $T(f) = f^*$ is a endofunctor on $\mathcal{S}^A(C)$ and since σ is a symmetry we have $T \circ T = 1_{\mathcal{S}^A(C)}$. This show that the category $\mathcal{S}^A(C)$ has a nontrivial action by the group $S_2 = \langle t \mid t^2 = 1 \rangle$.

The action have at least one fixpoint

Proposition 53 *Let $a = \{\delta_A, \delta_A\}$ be unit the object for the monoidal structure $\otimes^A$ on $\mathcal{S}^A(C)$ Then we have $a^* = a$*

Proof. Recall that $\delta_A : A \longrightarrow A \otimes A$ defines a commutative coalgebra structure on A. But then we have

$$
\begin{aligned}
&a^* \\
&= \{\delta_A, \delta_A\}^* \\
&= \{\sigma_{A,A} \circ \delta_A, \sigma_{A,A} \circ \delta_A\} \\
&= \{\delta_A, \delta_A\} \\
&= a
\end{aligned}
$$

■

The nontrivial group of symmetries must be taken into account when the notion of a symmetry for the product structures $\boxtimes^A$and $\otimes^A$ in $\mathcal{S}^A(C)$ are defined. We have previously shown that for a symmetric monoidal category in the usual sense we have an interpretation of the Yang-Baxter equation and the unit symmetry conditions in terms of invariance with respect to the group $H = \{T_t(\sigma), 1_\otimes\}$. This whole construction was based on a certain choice of action by the group $S_2 = \{1, t\}$ on the category C, C^2 and C^3 generated by the functors $T_1 = 1_C$,$T_2 = \tau$ and $T_3 = (1_C \times \tau) \circ (\tau \times 1_C) \circ (1_C \times \tau)$. What is new for the category of relations is that we have a nontrivial action of S_2. We will generalize this and consider monoidal categories $\langle C, \otimes, K_e, \alpha, \beta, \gamma \rangle$ where we have a nontrivial action of S_2 generated by a functor $T_1 : C \longrightarrow C$. We use this action together with τ to define actions of S_2 on the categories C^2 and C^3 generated by $T_2 = \tau \circ (T_1 \times T_1)$ and $T_3 = (T_1 \times T_2) \circ (T_2 \times T_1) \circ (T_1 \times T_2)$. It is easy to see that $T_2 \circ T_2 = 1_{C^2}$ and $T_3 \circ T_3 = 1_{C^3}$ so that these functors really defines an action of S_2. Note that if $T_1 = 1_C$ we get the action we discussed previously in the section on symmetries and group action. We now lift this action to the functor categories $[C^2, C]$ and $[C^3, C]$ in the usual way. From this point we proceed in a way that is exactly parallel to what we did in the section on symmetries and group action. In general one could imagine that the functor T_1 does not fix the unit so that $T_1(e) \neq e$. In this general situation we would assume the existence of a natural isomorphism $\theta : K_e \longrightarrow tK_e$ in addition to the isomorphism $\sigma : \otimes \longrightarrow t\otimes$. We would thus allow the constant functor K_e to be fixed only up to natural isomorphism. In this paper we will not consider such a possibility. This is because the unit is fixed both for the usual case with trivial action and for the case of the category of relations $S^A(C)$ as proved in proposition 53. Allowing the unit to move would also make all formulas and derivations more complicated. With this out of the way we can now state that all results derived in the section on symmetries and group actions,up to and including corollary 12 ,also holds for the current situation if we substitute the S_2 action from this section in all statements. The proofs of these results are of course different since they must take into account the more general action considered in this section. We do not reproduce these proofs here since they are long and rather similar to the ones already given in the section on symmetries and group action.

We now reverse proposition 12 that characterized symmetric monoidal categories in terms of invariance and is lead to the following definition of symmetries for monoidal categories with a action of the group S_2.

Definition 54 *Let C be a category where there is defined an action of the group S_2. Then $\langle C, \otimes, K_e, \alpha, \beta, \gamma, \sigma\rangle$ is a symmetric monoidal category if $\langle C, \otimes, K_e, \alpha, \beta, \gamma\rangle$ is a monoidal category and $\sigma : \otimes \longrightarrow t\otimes$ is a natural isomorphism such that the following identities holds.*

$$\begin{aligned}
\sigma \circ (1_{1_C} \times \sigma) &= (t\alpha) \cdot (\sigma \circ (\sigma \times 1_{1_C})) \cdot \alpha \\
\beta &= (t\gamma) \cdot (\sigma \circ 1_{K_e \times 1_C}) \\
\gamma &= (t\beta) \cdot (\sigma \circ 1_{1_C \times K_e}) \\
t\sigma &= \sigma^{-1}
\end{aligned}$$

We say that σ is the symmetry for the monoidal category $\langle C, \otimes, K_e, \alpha, \beta, \gamma\rangle$.

The first condition is equivalent to the Yang-Baxter equation if we consider symmetric monoidal categories in the usual sense with trivial action of S_2. We will in all cases call the first condition for the Yang-Baxter equation. Note that even for the case of trivial action our notion of symmetric monoidal category is more general than the standard one. The standard definition of symmetry for a monoidal category implies that the Yang-Baxter equation holds but the fact that the Yang-Baxter equation holds for σ does not necessarily imply that σ is a symmetry in the usual sense.

We say that $\langle C, \otimes, \alpha, \sigma\rangle$ is a symmetric semimonoidal category if $\langle C, \otimes, \alpha\rangle$ is a semimonoidal category and the first and last of the above conditions hold.

We will now apply the definition of symmetry for the case of the category of relations. For this case we denoted the map T_1 by *. Let us first consider the structure $\boxtimes^A$. In terms of objects, the definition of a symmetry for the semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ is as follows.

Definition 55 *A symmetry for the semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ is a isomorphism $S^A_{\delta,\gamma} : \delta \boxtimes^A \gamma \longrightarrow (\gamma^* \boxtimes^A \delta^*)^*$ that is natural in δ and γ and such that the following identities are satisfied for all δ, γ and ρ.*

$$(M^A_{\rho^*,\gamma^*,\delta^*})^* \circ (1^*_\rho \boxtimes^A (S^A_{\delta,\gamma})^*)^* \circ S^A_{\delta \boxtimes^A \gamma,\rho} \circ M^A_{\delta,\gamma,\rho} = ((S^A_{\gamma,\rho})^* \boxtimes^A 1^*_\delta)^* \circ S^A_{\delta,\gamma \boxtimes^A \rho}$$
$$(S^A_{\gamma^*,\delta^*})^* \circ S^A_{\delta,\gamma} = 1_{\delta \boxtimes^A \gamma}$$

In general many symmetric semimonoidal structures may exist for $\mathcal{S}^A(C)$ with the product $\boxtimes^A$. We will now show there is always at least one.

Proposition 56 *Let δ, γ and ρ be objects in $\mathcal{S}^A(C)$ with underlying objects B,E and D. Define*

$$M^A_{\delta,\gamma,\rho} = \alpha_{B,E,D}$$
$$S^A_{\delta,\gamma} = \sigma_{B,E}$$

where α and σ are the associativity constraint and symmetry for the category C. Then $\langle \mathcal{S}^A(C), \boxtimes^A, M^A, S^A \rangle$ is a symmetric semimonoidal category.

Proof. We already know that $\langle \mathcal{S}^A(C), \boxtimes^A, M^A \rangle$ is a semimonoidal category. First we need to prove that $S^A_{\delta,\gamma}$ is a morphism in $\mathcal{S}^A(C)$. Note that the underlying object for $(\gamma^* \boxtimes^A \delta^*)^*$ is $E \otimes B$. If we use the fact that σ is a symmetry in C we have

$$\begin{aligned}
&((\gamma^* \boxtimes^A \delta^*)^*)^l \circ S^A_{\delta,\gamma} \\
&= \sigma_{E\otimes B,A} \circ (\gamma^* \boxtimes^A \delta^*)^r \circ \sigma_{B,E} \\
&= \sigma_{E\otimes B,A} \circ \alpha_{E,B,A} \circ (1_E \otimes (\delta^*)^r) \circ \sigma_{B,E} \\
&= \sigma_{E\otimes B,A} \circ \alpha_{E,B,A} \circ (1_E \otimes \sigma_{A,B} \circ \delta^l) \circ \sigma_{B,E} \\
&= \sigma_{E\otimes B,A} \circ \alpha_{E,B,A} \circ (1_E \otimes \sigma_{A,B}) \circ (1_E \otimes \delta^l) \circ \sigma_{B,E} \\
&= \sigma_{E\otimes B,A} \circ \alpha_{E,B,A} \circ (1_E \otimes \sigma_{A,B}) \circ \sigma_{A\otimes B,E} \circ (\delta^l \otimes 1_E) \\
&= (1_A \otimes \sigma_{B,E}) \circ \alpha^{-1}_{A,B,E} \circ (\delta^l \otimes 1_E) \\
&= (1_A \otimes S^A_{\delta,\gamma}) \circ (\delta \boxtimes^A \gamma)^l
\end{aligned}$$

and this proves that $S^A_{\delta,\gamma}$ is a morphism in $\mathcal{S}^A(C)$. It is clearly a isomorphism and naturality is evident. The condition for S^A to be a symmetry in $\mathcal{S}^A(C)$ is satisfied since it turns into the condition for σ being is a symmetry in the category C. ■

The previous proposition leads us to make the following definition.

Definition 57 *A symmetric semimonoidal structure $\langle \mathcal{S}^A(C), \boxtimes^A, M^A, S^A \rangle$ on the category $\mathcal{S}^A(C)$ is external if for all objects δ, γ and ρ we have*

$$M^A_{\delta,\gamma,\rho} = \alpha_{B,E,D}$$
$$S^A_{\delta,\gamma} = \sigma_{B,E}$$

where B,E and D are the underlying objects for δ, γ and ρ and where α and σ are the associativity constraint and symmetry for the category C.

The previous proposition then proves that an external symmetries on $\mathcal{S}^A(C)$ with product $\boxtimes^A$ always exists.

We now turn to the definition of symmetries for $\mathcal{S}^A(C)$ with the product $\otimes^A$. In terms of objects the general definition now takes the form

Definition 58 *A symmetry for the monoidal category $\langle \mathcal{S}^A(C), \otimes^A, a, m^A, l^A, r^A\rangle$ is a isomorphism $s^A_{\delta,\gamma} : \delta \otimes^A \gamma \longrightarrow (\gamma^* \otimes^A \delta^*)^*$ that is natural in δ and γ and such that the following identities are satisfied for all δ, γ and ρ.*

$$(m^A_{\rho^*,\gamma^*,\delta^*})^* \circ (1^*_\rho \otimes^A (s^A_{\delta,\gamma})^*)^* \circ s^A_{\delta\otimes^A\gamma,\rho} \circ m^A_{\delta,\gamma,\rho} = ((s^A_{\gamma,\rho})^* \otimes^A 1^*_\delta)^* \circ s^A_{\delta,\gamma\otimes^A\rho}$$

$$(l^A_{\delta^*})^* \circ s^A_{\delta,a} = r^A_\delta$$

$$(r^A_{\delta^*})^* \circ s^A_{a,\delta} = l^A_\delta$$

$$(s^A_{\delta^*,\gamma^*})^* \circ s^A_{\delta,\gamma} = 1_{\delta\otimes^A\gamma}$$

Note that identity two and three are not independent. One can be derived from the other by using identity four and the fact that the neutral object a is fixed by the action of S_2. There are several equivalent formulations of the first symmetry condition

Proposition 59 *Let s^A be a natural isomorphism $s^A_{\delta,\gamma} : \delta \otimes^A \gamma \longrightarrow (\gamma^* \otimes^A \delta^*)$ such that the following identities hold*

$$(l^A_{\delta^*})^* \circ s^A_{\delta,a} = r^A_\delta$$

$$(s^A_{\delta^*,\gamma^*})^* \circ s^A_{\delta,\gamma} = 1_{\delta\otimes^A\gamma}$$

Then the following statements are equivalent

1. *s^A is a symmetry*
2. $(m^A_{\rho^*,\gamma^*,\delta^*})^* \circ (1^*_\rho \otimes^A (s^A_{\delta,\gamma})^*)^* \circ s^A_{\delta\otimes^A\gamma,\rho} \circ m^A_{\delta,\gamma,\rho} = s^A_{\delta,(\rho^*\otimes^A\gamma^*)^*} \circ (1_\delta \otimes^A s^A_{\gamma,\rho})$
3. $(m^A_{\rho^*,\gamma^*,\delta^*})^* \circ s^A_{(\gamma^*\otimes^A\delta^*)^*,\rho} \circ (s^A_{\delta,\gamma} \otimes^A 1_\rho) \circ m^A_{\delta,\gamma,\rho} = ((s^A_{\gamma,\rho})^* \otimes^A 1^*_\delta)^* \circ s^A_{\delta,\gamma\otimes^A\rho}$
4. $(m^A_{\rho^*,\gamma^*,\delta^*})^* \circ s^A_{(\gamma^*\otimes^A\delta^*)^*,\rho} \circ (s^A_{\delta,\gamma} \otimes^A 1_\rho) \circ m^A_{\delta,\gamma,\rho} = s^A_{\delta,(\rho^*\otimes^A\gamma^*)^*} \circ (1_\delta \otimes^A s^A_{\gamma,\rho})$

Proof. By naturality of s^A we have the following two identities

$$((s^A_{\gamma,\rho})^* \otimes^A 1^*_\delta)^* \circ s^A_{\delta,\gamma\otimes^A\rho} = s^A_{\delta,(\rho^*\otimes^A\gamma^*)^*} \circ (1_\delta \otimes^A s^A_{\gamma,\rho})$$

$$(1^*_\rho \otimes^A (s^A_{\delta,\gamma})^*)^* \circ s^A_{\delta\otimes^A\gamma,\rho} = s^A_{(\gamma^*\otimes^A\delta^*)^*,\rho} \circ (s^A_{\delta,\gamma} \otimes^A 1_\rho)$$

The proposition now follows directly from these identities. ∎

Let us now consider the existence of symmetries.

Definition 60 *A symmetry for the monoidal category $\langle \mathcal{S}^A(C), \otimes^A, a, m^A, l^A, r^A\rangle$ is induced by a symmetry S^A of the semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A\rangle$ if for all objects δ, γ in $\mathcal{S}^A(C)$ the following diagram commute*

$$\begin{tikzcd}
\delta \boxtimes^A \gamma \arrow[r, "S^A_{\delta,\gamma}"] & (\gamma^* \boxtimes^A \delta^*)^* \\
\delta \otimes^A \gamma \arrow[u, "\pi^A_{\delta,\gamma}"] \arrow[r, "s^A_{\delta,\gamma}"'] & (\gamma^* \otimes^A \delta^*)^* \arrow[u, "(\pi^A_{\gamma^*,\delta^*})^*"']
\end{tikzcd}$$

We will in the following show that an induced symmetry exists in the external case and is uniquely determined by the symmetry S^A.

Recall that for any pair of objects δ and γ in $\mathcal{S}^A(C)$, the diagram $\mathcal{P}^A_{\delta,\gamma}$ was given by

$$\begin{tikzcd}
\delta \boxtimes^A (a \boxtimes^A \gamma) \arrow[rr, "M^A_{\delta,a,\gamma}"] & & (\delta \boxtimes^A a) \boxtimes \gamma \\
& \delta \boxtimes \gamma \arrow[ul, "1_\delta \boxtimes^A \gamma^l"] \arrow[ur, "\delta^r \boxtimes^A 1_\gamma"'] &
\end{tikzcd}$$

¿From the general theory of categories it is well known that isomorphisms of categories preserve universal cones. By definition $\langle \gamma^* \otimes^A \delta^*, \pi^A_{\gamma^*,\delta^*}\rangle$ is a universal cone on the diagram $\mathcal{P}^A_{\gamma^*,\delta^*}$and therefore $\langle (\gamma^* \otimes^A \delta^*)^*, (\pi^A_{\gamma^*,\delta^*})^*\rangle$ is a universal cone on the diagram $(\mathcal{P}^A_{\gamma^*,\delta^*})^*$. But we have the following result

Lemma 61 *Let the symmetric semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A, S^A\rangle$ be external, then $\langle \delta \otimes^A \gamma, S^A_{\delta,\gamma} \circ \pi^A_{\delta,\gamma}\rangle$ is a universal cone on $(\mathcal{P}^A_{\gamma^*,\delta^*})^*$.*

Proof. Let us first prove that it is a cone. For this we must prove that the following identity holds

$$(M^A_{\gamma^*,a,\delta^*})^* \circ (1_{\gamma^*} \boxtimes^A (\delta^*)^l)^* \circ S^A_{\delta,\gamma} \circ \pi^A_{\delta,\gamma} = ((\gamma^*)^r \boxtimes^A 1_{\delta^*})^*$$

holds in $\mathcal{S}^A(C)$. Since the semimonoidal structure on $\mathcal{S}^A(C)$ is external, the previous identity is for the strict case equivalent to the following identity in C

$$\alpha_{E,A,B} \circ (1_E \otimes (\delta^*)^l) \circ \sigma_{B,E} \circ \pi^A_{\delta,\gamma} = ((\gamma^*)^r \otimes 1_B) \circ \sigma_{B,E} \circ \pi^A_{\delta,\gamma}$$

But this identity follows from the Yang baxter equation and the fact that

$\langle \delta \otimes^A \gamma, \pi^A_{\delta,\gamma} \rangle$ is a cone on $\mathcal{P}^A_{\delta,\gamma}$.

$$= \alpha_{E,A,B} \circ (1_{\gamma^*} \boxtimes^A (\delta^*)^l)^* \circ \sigma_{B,E} \circ \pi^A_{\delta,\gamma}$$
$$= \alpha_{E,A,B} \circ (1_E \otimes \sigma_{B,A}) \circ (1_E \otimes \delta^r) \circ \sigma_{B,E} \circ \pi^A_{\delta,\gamma}$$
$$= \alpha_{E,A,B} \circ (1_E \otimes \sigma_{B,A}) \circ \sigma_{B \otimes A,E} \circ (\delta^r \otimes 1_E) \circ \pi^A_{\delta,\gamma}$$
$$= \alpha_{E,A,B} \circ (1_E \otimes \sigma_{B,A}) \circ \sigma_{B \otimes A,E} \circ \alpha_{B,A,E} \circ (1_B \otimes \gamma^l) \circ \pi^A_{\delta,\gamma}$$
$$= \alpha_{E,A,B} \circ (1_E \otimes \sigma_{B,A}) \circ \sigma_{B \otimes A,E} \circ \alpha_{B,A,E} \circ \sigma_{A \otimes E,B} \circ (\gamma^l \otimes 1_B) \circ \sigma_{B,E} \circ \pi^A_{\delta,\gamma}$$
$$= \alpha_{E,A,B} \circ (1_E \otimes \sigma_{B,A}) \circ \sigma_{B \otimes A,E} \circ \alpha_{B,A,E} \circ \sigma_{A \otimes E,B} \circ (\sigma_{E,A} \otimes 1_B)$$
$$\circ ((\gamma^*)^r \otimes 1_B) \circ \sigma_{B,E} \circ \pi^A_{\delta,\gamma}$$
$$= ((\gamma^*)^r \otimes 1_B) \circ \sigma_{B,E} \circ \pi^A_{\delta,\gamma}$$

Let now $\langle \theta, u \rangle$ be any cone on $(\mathcal{P}^A_{\gamma^*,\delta^*})^*$. The proposition is proved if we can show that the following equations has a unique solution $\varphi : \theta \longrightarrow \delta \otimes^A \gamma$

$$S^A_{\delta,\gamma} \circ \pi^A_{\delta,\gamma} \circ \varphi = u$$

The equation has at most one solution since $S^A_{\delta,\gamma} \circ \pi^A_{\delta,\gamma}$ is a monomorphism. In a calculation very similar to previous one we can prove that $\langle \theta, (S^A_{\delta,\gamma})^{-1} \circ u \rangle$ is a cone on $\mathcal{P}^A_{\delta,\gamma}$. But $\langle \delta \otimes^A \gamma, \pi^A_{\delta,\gamma} \rangle$ is a universal cone on $\mathcal{P}^A_{\delta,\gamma}$ and therefore there exists a morphism $h : \theta \longrightarrow \delta \otimes^A \gamma$ in $\mathcal{S}^A(C)$ such that $\pi^A_{\delta,\gamma} \circ h = (S^A_{\delta,\gamma})^{-1} \circ u$. Composing on both sides with $S^A_{\delta,\gamma}$ show that $S^A_{\delta,\gamma} \circ \pi^A_{\delta,\gamma} \circ \varphi = u$ and the proposition is proved. ■

We can now prove the existence of induced symmetries in the external case.

Theorem 62 *Let the symmetric semimonoidal category $\langle \mathcal{S}^A(C), \boxtimes^A, M^A, S^A \rangle$ be external, then there exists a induced symmetry for the monoidal category $\langle \mathcal{S}^A(C), \otimes^A, a, m^A, l^A, r^A \rangle$.*

Proof. The previous lemma show that both $\langle \delta \otimes^A \gamma, S^A_{\delta,\gamma} \circ \pi^A_{\delta,\gamma} \rangle$ and $\langle (\gamma^* \otimes^A \delta^*)^*, (\pi^A_{\gamma^*,\delta^*})^* \rangle$ are universal cones on $(\mathcal{P}^A_{\gamma^*,\delta^*})^*$. We can therefore conclude that there exists a unique morphism $s^A_{\delta,\gamma} : \delta \otimes^A \gamma \longrightarrow (\gamma^* \otimes^A \delta^*)^*$ such that $(\pi^A_{\gamma^*,\delta^*})^* \circ s^A_{\delta,\gamma} = S^A_{\delta,\gamma} \circ \pi^A_{\delta,\gamma}$. We will show that $s^A_{\delta,\gamma}$ is a symmetry for the monoidal category $\langle \mathcal{S}^A(C), \otimes^A, a, m^A, l^A, r^A \rangle$ on $\mathcal{S}^A(C)$. The first symmetry condition for s^A follows from the first symmetry condition for S^A, the identity $(\pi^A_{\gamma^*,\delta^*})^* \circ s^A_{\delta,\gamma} = S^A_{\delta,\gamma} \circ \pi^A_{\delta,\gamma}$ and the fact that m^A is induced by M^A. For the second symmetry condition we have for the strict case

$$\begin{aligned}
&(l^A_{\delta^*})^* \circ s^A_{\delta,a} \\
&= \beta_B \circ (\epsilon_A \otimes 1_B) \circ \pi^A_{a,\delta^*} \circ s^A_{\delta,a} \\
&= \beta_B \circ (\epsilon_A \otimes 1_B) \circ S^A_{\delta,a} \circ \pi^A_{\delta,a} \\
&= \beta_B \circ (\epsilon_A \otimes 1_B) \circ \sigma_{B,A} \circ \pi^A_{\delta,a} \\
&= \beta_B \circ \sigma_{B,e} \circ (1_B \otimes \epsilon_A) \circ \pi^A_{\delta,a} \\
&= \gamma_B \circ (1_B \otimes \epsilon_A) \circ \pi^A_{\delta,a} \\
&= r^A_\delta
\end{aligned}$$

where we have used the fact that the symmetry S^A is external. The last symmetry condition follows easily from the commutative diagram defining s^A in terms of S^A and from the fact that S^A is a symmetry. ■

3.9 Commutative monoids in the category of relation

We will define the notion of a commutative monoid for categories with an action of S_2 and then apply this definition to the case of relations. Let now $\langle C, \otimes, K_e, \alpha, \beta, \gamma, \sigma\rangle$ be a symmetric monoidal category with an action of S_2 generated by the functor $T_1 : C \longrightarrow C$. The conditions from definition 3 thus holds for α, β, γ and σ.

Our definition of a commutative monoid is a natural extension and categorization of the notion of a commutative monoids in algebra. Let $\langle M, \cdot, e\rangle$ be a monoid in the usual algebraic sense, so that M is a set and $\cdot$ is a associative product on M with unit element e. Define a new associative product on M by $x * y = y \cdot x$. Then $\langle M, *, e\rangle$ is a new monoid on the same underlying set. The monoid M is said to be commutative if $\langle M, *, e\rangle$ is the same monoid as $\langle M, \cdot, e\rangle$ and this is equivalent to the condition $x \cdot y = y \cdot x$ for all x and y in M. The previous condition is really too strict since in algebra we consider isomorphic monoids to be essentially the same. Thus it would be more natural to require that the two monoids $\langle M, \cdot, e\rangle$ and $\langle M, *, e\rangle$ are isomorphic. From a categorical point of view the last condition is the only one that really makes sense since the relation of equality exists only between arrows and not between objects. If we now recall that the symmetry σ is the categorization of the idea of changing order in the category C we arrive at our definition of commutativity.

Let X be a monoid in the category C with product $\mu : X \otimes X \longrightarrow X$ and unit $u : e \longrightarrow X$. Define morphisms $\mu^\sigma : T_1(X) \otimes T_1(X) \longrightarrow T_1(X)$ and $u^s : e \longrightarrow T_1(X)$ by

$$\begin{aligned}
\mu^\sigma &= T_1(\mu) \circ \sigma_{T_1(X),T_1(X)} \\
u^s &= T_1(u)
\end{aligned}$$

Proposition 63 $\langle T_1(X), \mu^\sigma, u^s\rangle$ *is a monoid in* C

Proof. The Yang-Baxter equation and the naturality of σ implies when evaluated on $(T_1(X), T_1(X), T_1(X))$ the following relation

$$\begin{aligned}\sigma_{T_1(X),T_1(X\otimes X)} \circ (1_{T_1(X)} \otimes \sigma_{T_1(X),T_1(X)}) = T_1(\alpha_{X,X,X}) \circ \sigma_{T_1(X\otimes X),T_1(X)} \\ \circ (\sigma_{T_1(X),T_1(X)} \otimes 1_{T_1(X)})\end{aligned}$$

Using this relation we have

$$\begin{aligned}
&\mu^\sigma \circ (1_{T_1(X)} \otimes \mu^\sigma) \\
&= T_1(\mu) \circ \sigma_{T_1(X),T_1(X)} \circ (1_{T_1(X)} \otimes (T_1(\mu) \circ \sigma_{T_1(X),T_1(X)})) \\
&= T_1(\mu) \circ \sigma_{T_1(X),T_1(X)} \circ (1_{T_1(X)} \otimes T_1(\mu)) \circ (1_{T_1(X)} \otimes \sigma_{T_1(X),T_1(X)}) \\
&= T_1(\mu) \circ T_1(\mu \otimes 1_X) \circ \sigma_{T_1(X),T_1(X\otimes X)} \circ (1_{T_1(X)} \otimes \sigma_{T_1(X),T_1(X)}) \\
&= T_1(\mu \circ (\mu \otimes 1_X)) \circ T_1(\alpha_{X,X,X}) \circ \sigma_{T_1(X\otimes X),T_1(X)} \circ (\sigma_{T_1(X),T_1(X)} \otimes 1_{T_1(X)}) \\
&= T_1(\mu) \circ T_1(1_X \otimes \mu) \circ \sigma_{T_1(X\otimes X),T_1(X)} \circ (\sigma_{T_1(X),T_1(X)} \otimes 1_{T_1(X)}) \\
&= T_1(\mu) \circ \sigma_{T_1(X),T_1(X)} \circ ((T_1(\mu) \circ \sigma_{T_1(X),T_1(X)}) \otimes 1_{T_1(X)}) \\
&= \mu^\sigma \circ (\mu^\sigma \otimes 1_{T_1(X)})
\end{aligned}$$

so the morphism μ^σ is associative. The first unit condition evaluated at the pair of objects $(e, T_1(X))$ given the identity

$$\beta_{e,T_1(X)} = T_1(\gamma_{X,e}) \circ \sigma_{e,T_1(X)}$$

¿From the naturality of σ and the fact that $\langle X, \mu, u\rangle$ is a monoid we have

$$\begin{aligned}
&\mu^\sigma \circ (u^s \otimes 1_{T_1(X)}) \\
&= T_1(\mu) \circ \sigma_{T_1(X),T_1(X)} \circ (T_1(u) \otimes 1_{T_1(X)}) \\
&= T_1(\mu) \circ T_1(1_X \otimes u) \circ \sigma_{e,T_1(X)} \\
&= T_1(\mu \circ (1_X \otimes u)) \circ \sigma_{e,T_1(X)} \\
&= T_1(\gamma_{X,e}) \circ \sigma_{e,T_1(X)} \\
&= \beta_{e,T_1(X)}
\end{aligned}$$

and this is the left condition on the unit. The proof for the right condition is similar. ■

Recall that $\varphi : X \longrightarrow Y$ is a morphism of monoids $\varphi : \langle X, \mu, u\rangle \longrightarrow \langle Y, \mu', u'\rangle$ if the following two diagrams commute

$$\begin{tikzcd} X \otimes X \arrow[r, "\phi \otimes \phi"] \arrow[d, "\mu"'] & Y \otimes Y \arrow[d, "\mu'"] \\ X \arrow[r, "\phi"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\phi"] & & Y \\ & e \arrow[ul, "u"] \arrow[ur, "u'"'] & \end{tikzcd}$$

We are now ready to define the notion of a commutative monoid in the symmetric monoidal category C.

Definition 64 *Let $\langle C, \otimes, K_e, \alpha, \beta, \gamma, \sigma\rangle$ be a symmetric monoidal category. A monoid $\langle X, \mu, u\rangle$ in C is commutative if there exists an isomorphism of monoids*

$$\varphi : \langle X, \mu, u\rangle \longrightarrow \langle T_1(X), \mu^\sigma, u^s\rangle$$

We will now apply these definitions to the *Sets*. For this case there is only one possible choice that makes *Sets* into a C-category. Let $r(x) = (f(x), g(x))$ and $s(y) = (h(y), k(y))$ be two relations with domains B and E. Then $r^*(x) = (g(x), f(x))$ and $s^*(y) = (k(y), h(y))$. If X and Y are the underlying sets for $r \otimes^A s$ and $(s^* \otimes^A r^*)^*$ we have

$$X = \{(x, y) \mid g(x) = h(y)\}$$
$$Y = \{(y, x) \mid h(y) = g(x)\}$$

and the relations are given by

$$(r \otimes^A s)(x, y) = (f(x), k(y))$$
$$(s^* \otimes^A r^*)^*(y, x) = (f(x), k(y))$$

Define $s^A_{r,s}(x, y) = (y, x)$. Then clearly we have $s^A_{r,s} : X \longrightarrow Y$ and also

$$\begin{aligned}&((s^* \otimes^A r^*)^* \circ s^A_{r,s})(x, y)\\ &= (s^* \otimes^A r^*)^*(y, x)\\ &= (f(x), k(y))\\ &= (r \otimes^A s)(x, y)\end{aligned}$$

so that we have a morphism in $s^A_{r,s} : r \otimes^A s \longrightarrow (s^* \otimes^A r^*)^*$. It is straight forward to prove that s^A is a symmetry on the category of relations. It is in fact induced by the symmetry of the external category *Sets*. Since a relation $r : B \longrightarrow A \times A$ is a directed labelled graph it is clear that we get the relation $r^* : B \longrightarrow A \times A$ by reversing all arrows in the relation r. We have seen that r is a monoid if there exists a associative rule of composition for composable arrows in r such that for each object $x \in A$ there exists an arrow with source and target given by x and that acts as right and left unit for the composition. Let b and b' be two objects in B. Then the rule of composition for the relation $r^*$is defined by first reversing both arrows, then composing them as arrows in r and then reversing the result to get a arrow in r^*. Now a isomorphism $\varphi : r \longrightarrow r^*$ is a bijective map with domain and codomain given by B and such that

$$g(\varphi(x)) = f(x)$$
$$f(\varphi(x)) = g(x)$$

for all $x \in B$. If φ is also a isomorphism of the monoids $\langle r, \mu, u\rangle$ and $\langle r^*, \mu^s, u^s\rangle$ we must have

$$\varphi(\mu((x, y))) = \mu((\varphi(y), \varphi(x)))$$

for all objects $(x, y) \in A \times A$ such that $g(x) = f(y)$. These conditions are in general impossible to satisfy for the identity map $\varphi = 1_B$. Let $B \subset A \times A$ be a relation in the usual sense. Then B corresponds to a relation in our sense if we define $r : B \longrightarrow A \times A$ by $r((x, y)) = (x, y)$ so that $f((x, y)) = x$ and $g((x, y)) = y$. We know that if r is a monoid in the category of relations then B is a reflexive and transitive relation in the usual sense. Assume that B is also symmetric so that it is in fact a equivalence relation. Thus we have $(x, y) \in B$ if and only if $(y, x) \in B$. Then we can define a map $\varphi : B \longrightarrow B$ by $\varphi((x, y)) = (y, x)$. For this map we have

$$g(\varphi((x, y))) = g((y, x)) = x = f((x, y))$$
$$f(\varphi((x, y))) = f((y, x)) = y = g((x, y))$$

so that $\varphi : r \longrightarrow r^*$. We have seen that the rule of composition and unit maps for r are given by

$$\mu_r(((x, y), (y, z))) = (x, z)$$
$$u_r(x) = (x, x)$$

but then the composition and unit maps for r^* must be given by

$$\mu^s(((y, x), (z, y))) = (z, x)$$
$$u^s(x) = (x, x)$$

It is evident that φ preserve that unit and the following computation show that $\varphi : \langle r, \mu, u\rangle \longrightarrow \langle r^*, \mu^s, u^s\rangle$ also preserve the product

$$\begin{aligned}
&\varphi(\mu(((x, y), (y, z)))) \\
&= \varphi((x, z)) \\
&= (z, x) \\
&= \mu^s(((y, x), (z, y)))
\end{aligned}$$

We have thus proved the following result

Proposition 65 *Let $B \subset A \times A$ be a equivalence relation. Define $r : B \longrightarrow A \times A$ by $r((x, y)) = (x, y)$. Then r is a commutative monoid in the category of relations with respect to the symmetry in $\mathcal{R}^A(C)$ induced by the symmetry $\sigma(x, y) = (y, x)$ in Sets.*

Note that this result show that relations that are not equivalence relations in the usual sense might correspond to commutative monoids with respect to a different symmetry than the standard one used in the proposition. Such a class of relations would corresponds to a extension of the notion of equivalence that might be of interest.

4 Quantization of relations

In this section we apply our ideas of quantization as properties of functors in categories of representations of constraints. The constraints here are the system of functors and natural transformations defining a symmetric monoidal category where we have a action of the group S_2. Morphisms in this category of representations are what we call quantized functors. These are determined by a functor and a triple of natural isomorphisms that satisfy certain conditions that ensure that the functors behave in a natural way with respect to the representations. Properties of relations are coded in terms of commutative diagrams of arrows in the category of relations. Equivalence relations appears as commutative associative algebras with unit. In the last section we show how we can quantize relations by mapping them with quantized functors.

4.1 Quantized functors

Quantization has in our view its most natural formulation as a property of functors between categories. We will define quantization in the context of symmetric monoidal categories with an action of the group S_2. The symmetries are supposed to be symmetries in our modifies sense, they are natural isomorphisms that satisfy the conditions given in definition 54 .

Let now $\langle C_i, \otimes_i, P_{e_i}, \alpha_i, \beta_i, \gamma_i, \sigma_i \rangle$ for $i = 1, 2$ be two symmetric monoidal categories and let $F : C_1 \longrightarrow C_2$ be a functor.

Definition 66 *A quantization of the functor F is a triple of natural isomorphisms $\langle \lambda, \mu, \eta \rangle$*

$$\begin{aligned} \lambda &: \otimes_2 \circ (F \times F) \longrightarrow F \circ \otimes_1 \\ \mu &: F \longrightarrow tF \\ \eta &: K_{e_2} \longrightarrow F \circ K_{e_1} \end{aligned}$$

such that the following relations hold

$$\begin{aligned} \alpha_2 \circ 1_{F \times F \times F} &= (1_{\otimes_2} \circ (\lambda^{-1} \times 1_F)) \cdot (\lambda^{-1} \circ 1_{\otimes_1 \times 1_{C_1}}) \cdot (1_F \circ \alpha_1) \\ &\quad \cdot (\lambda \circ 1_{1_{C_1} \times \otimes_1}) \cdot (1_{\otimes_2} \circ (1_F \times \lambda)) \\ \beta_2 \circ 1_{F \times F} &= (1_F \circ \beta_1) \cdot (\lambda \circ 1_{K_{e_1} \times 1_{C_1}}) \cdot (1_{\otimes_2} \circ (\eta \times 1_F)) \\ \gamma_2 \circ 1_{F \times F} &= (1_F \circ \gamma_1) \cdot (\lambda \circ 1_{1_{C_1} \times K_{e_1}}) \cdot (1_{\otimes_2} \circ (1_F \times \eta)) \\ \sigma_2 \circ 1_{F \times F} &= (1_{t\otimes_2} \circ (\mu^{-1} \times \mu^{-1})) \cdot (t\lambda^{-1}) \cdot (\mu \circ \sigma_1) \cdot \lambda \\ t\mu &= \mu^{-1} \end{aligned}$$

The only true justification of this definition, as for any mathematical definition, lies in the importance and depth of its consequences. We will now start investigating some of those consequences. We will first show that quantized functors are composable.

Proposition 67 *Let $F : C_1 \longrightarrow C_2$ and $G : C_2 \longrightarrow C_3$ be quantized functors with quantizations $\langle \lambda_F, \mu_F, \eta_F \rangle$ and $\langle \lambda_G, \mu_G, \eta_G \rangle$. Then $G \circ F$ is a quantized functor with quantization $\langle \lambda_{G \circ F}, \mu_{G \circ F}, \eta_{G \circ F} \rangle$ where*

$$
\begin{aligned}
\lambda_{G \circ F} &= (1_G \circ \lambda_F) \cdot (\lambda_G \circ 1_{F \times F}) \\
\mu_{G \circ F} &= \mu_G \circ \mu_F \\
\eta_{G \circ F} &= (1_G \circ \eta_F) \cdot \eta_G
\end{aligned}
$$

Proof. For the first condition we have

$$
\begin{aligned}
&\alpha_3 \circ 1_{G\circ F\times G\circ F\times G\circ F} \\
&= \alpha_3 \circ 1_{G\times G\times G} \circ 1_{F\times F\times F} \\
&= [(1_{\otimes_3} \circ (\lambda_G^{-1} \times 1_G) \circ 1_{F\times F\times F}) \cdot (\lambda_G^{-1} \circ 1_{\otimes_2 \times 1_{C_2}} \circ 1_{F\times F\times F}) \\
&\cdot (1_G \circ \alpha_2 \circ 1_{F\times F\times F}) \cdot (\lambda_G \circ 1_{C_2 \times \otimes_2 \circ 1_{F\times F\times F}}) \\
&\cdot (1_{\otimes_3} \circ (1_G \times \lambda_G) \circ 1_{F\times F\times F})] \\
&= (1_{\otimes_3} \circ (\lambda_G^{-1} \circ 1_{F\times F} \times 1_{G\circ F})) \cdot (\lambda_G^{-1} \circ (1_{\otimes_2 \circ (F\times F)} \times 1_F)) \\
&\cdot (1_G \circ 1_{\otimes_2} \circ (\lambda_F^{-1} \times 1_F)) \cdot (1_G \circ \lambda_F^{-1} \circ 1_{\otimes_1 \times 1_{C_1}}) \\
&\cdot (1_G \circ 1_F \circ \alpha_1) \cdot (1_G \circ \lambda_F \circ 1_{C_1 \times \otimes_1}) \\
&\cdot (1_G \circ 1_{\otimes_2} \circ (1_F \times \lambda_F)) \cdot (\lambda_G \circ (1_F \times 1_{\otimes_2 \circ (F\times F)})) \\
&\cdot (1_{\otimes_3} \circ (1_{G\circ F} \times (\lambda_G \circ 1_{F\times F}))) \\
&= (1_{\otimes_3} \circ ((\lambda_G^{-1} \circ 1_{F\times F}) \times 1_{G\circ F})) \cdot (\lambda_G^{-1} \circ (1_{\otimes_2 \circ (F\times F)} \times 1_F)) \\
&\cdot (1_{G\circ \otimes_2} \circ (\lambda_F^{-1} \times 1_F)) \cdot (1_G \circ \lambda_F^{-1} \circ (1_{\otimes_1} \times 1_{1_{C_1}})) \\
&\cdot (1_{G\circ F} \circ \alpha_1) \cdot (1_G \circ \lambda_F \circ (1_{1_{C_1}} \times 1_{\otimes_1})) \cdot (1_{G\circ \otimes_2} \circ (1_F \times \lambda_F)) \\
&\cdot (\lambda_G \circ (1_F \times 1_{\otimes_2 \circ (F\times F)})) \cdot (1_{\otimes_3} \circ (1_{G\circ F} \times (\lambda_G \circ 1_{F\times F}))) \\
&= (1_{\otimes_3} \circ ((\lambda_G^{-1} \circ 1_{F\times F}) \times 1_{G\circ F})) \cdot (\lambda_G^{-1} \circ (\lambda_F^{-1} \times 1_F)) \\
&\cdot (1_G \circ \lambda_F^{-1} \circ (1_{\otimes_1} \times 1_{1_{C_1}})) \cdot (1_{G\circ F} \circ \alpha_1) \\
&\cdot (1_G \circ \lambda_F \circ (1_{C_1} \times 1_{\otimes_1})) \cdot (\lambda_G \circ (1_F \times \lambda_F)) \\
&\cdot (1_{\otimes_3} \circ (1_{G\circ F} \times (\lambda_G \circ 1_{F\times F})) \\
&= (1_{\otimes_3} \circ ((\lambda_G^{-1} \circ 1_{F\times F}) \times 1_{G\circ F})) \cdot (1_{\otimes_3} \circ 1_{G\times G} \circ (\lambda_F^{-1} \times 1_F)) \\
&\cdot (\lambda_G^{-1} \circ (1_{F\circ \otimes_1} \times 1_F)) \cdot (1_G \circ \lambda_F^{-1} \circ (1_{\otimes_1} \times 1_{1_{C_1}})) \\
&\cdot (1_{G\circ F} \circ \alpha_1) \cdot (1_G \circ \lambda_F \circ (1_{1_{C_1}} \times 1_{\otimes_1})) \\
&\cdot (\lambda_G \circ (1_F \times 1_{F\circ \otimes_1})) \cdot (1_{\otimes_3} \circ 1_{G\times G} \circ (1_F \times \lambda_F)) \\
&\cdot (1_{\otimes_3} \circ (1_{G\circ F} \times (\lambda_G \circ 1_{F\times F}))) \\
&= (1_{\otimes_3} \circ ([(\lambda_G^{-1} \circ 1_{F\times F}) \cdot (1_G \circ \lambda_F^{-1})] \times 1_{G\circ F}) \\
&\cdot ([(\lambda_G^{-1} \circ 1_{F\times F}) \cdot (1_G \circ \lambda_F^{-1})] \circ (1_{\otimes_1} \times 1_{1_{C_1}})) \\
&\cdot (1_{G\circ F} \circ \alpha_1) \cdot ([(1_G \circ \lambda_F) \cdot (\lambda_G \circ 1_{F\times F})] \circ (1_{1_{C_1}} \times 1_{\otimes_1})) \\
&\cdot (1_{\otimes_3} \circ (1_{G\circ F} \times [(1_G \circ \lambda_F) \cdot (\lambda_G \circ 1_{F\times F})]) \\
&= (1_{\otimes_3} \circ (\lambda_{G\circ F}^{-1} \times 1_{G\circ F})) \cdot (\lambda_{G\circ F}^{-1} \circ 1_{\otimes_1 \times 1_{C_1}}) \\
&\cdot (1_{G\circ F} \circ \alpha_1) \cdot (\lambda_{G\circ F} \circ 1_{1_{C_1} \times \otimes_1}) \cdot (1_{\otimes_3} \circ (1_{G\circ F} \times \lambda_{G\circ F}))
\end{aligned}
$$

The proof for the second and third conditions are the similar and we only show the proof for the third condition

$$
\begin{aligned}
&\gamma_3 \circ 1_{G \circ F \times 1_T} \\
&= \gamma_3 \circ 1_{G \times 1_T} \circ 1_{F \times 1_T} \\
&= [(1_G \circ \gamma_2) \cdot (\lambda_G \circ 1_{1_{C_2} \times K_{e_2}}) \cdot (1_{\otimes_3} \circ (1_G \times \eta_G))] \circ 1_{F \times 1_T} \\
&= (1_G \circ \gamma_2 \circ 1_{F \times 1_T}) \cdot (\lambda_G \circ 1_{1_{C_2} \times K_{e_2}} \circ 1_{F \times 1_T}) \cdot (1_{\otimes_3} \circ (1_G \times \eta_G) \circ 1_{F \times 1_T}) \\
&= (1_G \circ [(1_F \circ \gamma_1) \cdot (\lambda_F \circ 1_{1_{C_1} \times K_{e_1}}) \cdot (1_{\otimes_2} \circ (1_F \times \eta_F))]) \\
&\cdot (\lambda_G \circ 1_{1_{C_2} \times K_{e_2}} \circ 1_{F \times 1_T}) \cdot (1_{\otimes_3} \circ (1_G \times \eta_G) \circ 1_{F \times 1_T}) \\
&= (1_G \circ 1_F \circ \gamma_1) \cdot (1_G \circ \lambda_F \circ 1_{1_{C_1} \times K_{e_1}}) \cdot (1_G \circ 1_{\otimes_2} \circ (1_F \times \eta_F)) \\
&\cdot (\lambda_G \circ 1_{1_{C_2} \times K_{e_2}} \circ 1_{F \times 1_T}) \cdot (1_{\otimes_3} \circ (1_G \times \eta_G) \circ 1_{F \times 1_T}) \\
&= (1_{G \circ F} \circ \gamma_1) \cdot (1_G \circ \lambda_F \circ 1_{1_{C_1} \times K_{e_1}}) \\
&\cdot (1_{G \circ \otimes_2} \circ (1_F \times \eta_F)) \circ (\lambda_G \circ (1_F \times 1_{K_{e_2}})) \cdot (1_{\otimes_3} \circ (1_{G \circ F} \times \eta_G)) \\
&= (1_{G \circ F} \circ \gamma_1) \cdot (1_G \circ \lambda_F \circ 1_{1_{C_1} \times K_{e_1}}) \cdot (\lambda_G \circ (1_F \times \eta_F)) \cdot (1_{\otimes_3} \circ (1_{G \circ F} \times \eta_G)) \\
&= (1_{G \circ F} \circ \gamma_1) \cdot (1_G \circ \lambda_F \circ 1_{1_{C_1} \times K_{e_1}}) \cdot (\lambda_G \circ (1_F \times \eta_F)) \\
&\cdot (1_{\otimes_3} \circ (1_{G \circ F} \times 1_{G \circ K_{e_2}})) \cdot (1_{\otimes_3} \circ (1_{G \circ F} \times \eta_G)) \\
&= (1_{G \circ F} \circ \gamma_1) \cdot (1_G \circ \lambda_F \circ 1_{1_{C_1} \times K_{e_1}}) \cdot (\lambda_G \circ (1_F \times \eta_F)) \\
&\cdot (1_{\otimes_3} \circ (1_{G \circ F} \times \eta_G)) \\
&= (1_{G \circ F} \circ \gamma_1) \cdot (1_G \circ \lambda_F \circ 1_{1_{C_1} \times K_{e_1}}) \cdot (\lambda_G \circ (1_F \times 1_{F \circ K_{e_1}})) \\
&\cdot (1_{\otimes_3} \circ (1_G \times 1_G) \circ (1_F \times \eta_F)) \cdot (1_{\otimes_3 \circ (1_{G \circ F}} \times \eta_G)) \\
&= (1_{G \circ F} \circ \gamma_1) \cdot (1_G \circ \lambda_F \circ 1_{1_{C_1} \times K_{e_1}}) \cdot (\lambda_G \circ 1_{F \times F} \circ (1_{1_{C_1}} \times 1_{K_{e_1}})) \\
&\cdot (1_{\otimes_3} \circ (1_{G \circ F} \times (1_G \circ \eta_F))) \cdot (1_{\otimes_3} \circ (1_{G \circ F} \times \eta_G)) \\
&= (1_{G \circ F} \circ \gamma_1) \cdot ([(1_G \circ \lambda_F) \cdot (\lambda_G \circ 1_{F \times F})] \circ 1_{1_{C_1} \times K_{e_1}}) \\
&\cdot (1_{\otimes_3} \circ (1_{G \circ F} \times [(1_G \circ \eta_F) \cdot \eta_G])) \\
&= (1_{G \circ F} \circ \gamma_1) \cdot (\lambda_{G \circ F} \circ 1_{1_{C_1} \times K_{e_1}}) \cdot (1_{\otimes_3} \circ (1_{G \circ F} \times \eta_{G \circ F}))
\end{aligned}
$$

For the fifth condition we have

$$\begin{aligned}
&\sigma_3 \circ 1_{G\circ F\times G\circ F}\\
&= \sigma_3 \circ 1_{G\times G} \circ 1_{F\times F}\\
&= [(1_{t\otimes_3} \circ (\mu_G^{-1} \times \mu_G^{-1})) \cdot (t\lambda_G^{-1}) \cdot (\mu_G \circ \sigma_2) \cdot \lambda_G] \circ 1_{F\times F}\\
&= (1_{t\otimes_3} \circ (\mu_G^{-1} \times \mu_G^{-1}) \circ 1_{F\times F}) \cdot (t\lambda_G^{-1} \circ 1_{F\times F}) \cdot (\mu_G \circ \sigma_2 \circ 1_{F\times F})\\
&\quad \cdot (\lambda_G \circ 1_{F\times F})\\
&= (1_{t\otimes_3} \circ (\mu_G^{-1} \times \mu_G^{-1}) \circ 1_{F\times F}) \cdot (t\lambda_G^{-1} \circ 1_{F\times F})\\
&\quad \cdot (\mu_G \circ [(1_{t\otimes_2} \circ (\mu_F^{-1} \times \mu_F^{-1})) \cdot (t\lambda_F^{-1}) \cdot (\mu_F \circ \sigma_1) \cdot \lambda_F]) \cdot (\lambda_G \circ 1_{F\times F})\\
&= (1_{t\otimes_3} \circ (\mu_G^{-1} \times \mu_G^{-1}) \circ 1_{F\times F}) \cdot (t\lambda_G^{-1} \circ 1_{F\times F}) \cdot (1_{t(G\circ\otimes_2)} \circ (\mu_F^{-1} \times \mu_F^{-1}))\\
&\quad \cdot (1_{tG} \circ (t\lambda_F^{-1})) \cdot (\mu_G \circ \mu_F \circ \sigma_1) \cdot (1_G \circ \lambda_F) \cdot (\lambda_G \circ 1_{F\times F})\\
&= (1_{t\otimes_3} \circ (\mu_G^{-1} \times \mu_G^{-1}) \circ 1_{F\times F}) \cdot ((t\lambda_G^{-1}) \circ (\mu_F^{-1} \times \mu_F^{-1})) \cdot (1_{tG} \circ (t\lambda_F^{-1}))\\
&\quad \cdot (\mu_G \circ \mu_F \circ \sigma_1) \cdot (1_G \circ \lambda_F) \cdot (\lambda_G \circ 1_{F\times F})\\
&= (1_{t\otimes_3} \circ (\mu_G^{-1} \times \mu_G^{-1}) \circ 1_{F\times F}) \cdot (1_{t\otimes_3} \circ (1_{tG} \times 1_{tG}) \circ (\mu_F^{-1} \times \mu_F^{-1}))\\
&\quad \cdot ((t\lambda_G^{-1}) \circ (1_{tF} \times 1_{tF})) \cdot (1_{tG} \circ (t\lambda_F^{-1})) \cdot (\mu_G \circ \mu_F \circ \sigma_1)\\
&\quad \cdot (1_G \circ \lambda_F) \cdot (\lambda_G \circ 1_{F\times F})\\
&= (1_{t\otimes_3} \circ ((\mu_G \circ \mu_F)^{-1} \times (\mu_G \circ \mu_F)^{-1})) \cdot (t[(1_G \circ \lambda_F) \cdot (\lambda_G \circ 1_{F\times F})]^{-1})\\
&\quad \cdot (\mu_G \circ \mu_F \circ \sigma_1) \cdot [(1_G \circ \lambda_F) \cdot (\lambda_G \circ 1_{F\times F})]\\
&= (1_{t\otimes_3} \circ (\mu_{G\circ F}^{-1} \times \mu_{G\circ F}^{-1})) \cdot (t\lambda_{G\circ F}^{-1}) \cdot (\mu_{G\circ F} \circ \sigma_1) \circ \lambda_{G\circ F}
\end{aligned}$$

The last condition is clearly satisfied because action by t pass through horizontal composition. ■

As a consequence of this proposition the class of symmetric monoidal categories form a category where arrows are four tuples $\langle F, \lambda_F, \mu_F, \eta_F\rangle$ and where composition of four tuples is defined using the previous proposition.

$$\langle G, \lambda_G, \mu_G, \eta_G\rangle \circ \langle F, \lambda_F, \mu_F, \eta_F\rangle = \langle G\circ F, \lambda_{G\circ F}, \mu_{G\circ F}, \eta_{G\circ F}\rangle$$

A given category C with a product bifunctor $\otimes$ and unit functor K_e is a symmetric monoidal category if the conditions on $\alpha, \beta, \gamma, \sigma$ and θ stated in definition 54 are satisfied. These conditions are equations that may have none or many solutions depending on the category C and the choice of functors $\otimes$ and K_e. We thus in general have a set of solutions. Let this set be denoted by S. We will now show that there is a group acting on S. The definition of this group action is derived from the formulas defining a quantized functor. Let G be the following group of natural isomorphisms

$$G = \{(\lambda, \mu, \eta) \mid \lambda : \otimes \longrightarrow \otimes, \mu : 1_C \longrightarrow 1_C, \eta : K_e \longrightarrow K_e\}$$

where the product is taken componentwise.The size of this group depends on the category C and functors $\otimes$ and P_e. Let now (λ, μ, η) be any element of the group G and define a mapping $F_{\lambda,\mu,\eta}$ on S by

$$F_{\lambda,\mu,\eta}(\alpha, \beta, \gamma, \sigma) = (\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\sigma})$$

where

$$
\begin{aligned}
\widehat{\alpha} &= (\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})) \cdot \alpha \cdot (\lambda \circ (1_{1_C} \times \lambda)) \\
\widehat{\beta} &= \beta \cdot (\lambda \circ (\eta \times 1_{1_C}) \\
\widehat{\gamma} &= \gamma \cdot (\lambda \circ (1_{1_C} \times \eta)) \\
\widehat{\sigma} &= (1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot (t\lambda^{-1}) \cdot (\mu \circ \sigma) \cdot \lambda
\end{aligned}
$$

Let H be the subgroup of G defined by the relations

$$
\begin{aligned}
t\mu &= \mu^{-1} \\
\mu \circ 1_{\otimes} \circ (\mu^{-1} \times 1_{K_e}) &= 1_{\otimes \circ (1_C \times K_e)} \\
\mu \circ 1_{\otimes} \circ (1_{K_e} \times \mu^{-1}) &= 1_{\otimes \circ (K_e \times 1_C)} \\
t\eta &= (\mu \circ 1_{K_e}) \cdot \eta
\end{aligned}
$$

Then we have the following important result.

Theorem 68 *$F_{\lambda,\mu,\eta} : S \longrightarrow S$ and defines a action of the group H on the set S.*

Proof. In order to prove that $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\sigma}) \in S$ we must show that $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\sigma})$ defines a symmetric monoidal structure. There are eight such conditions. For the first condition we have (this is also a proof that the map $T_1(\sigma)$ from section 2.2 maps associativity constraints to associativity constraints)

$$
\begin{aligned}
&(\widehat{\alpha} \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (\widehat{\alpha} \circ 1_{1_C \times 1_C \times \otimes}) \\
&= (\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C}) \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (\alpha \circ 1_{\otimes \times 1_C \times 1_C}) \\
&\cdot (\lambda \circ (1_C \times \lambda) \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (\lambda^{-1} \circ (\lambda^{-1} \times 1_C) \circ 1_{1_C \times 1_C \times \otimes}) \\
&\cdot (\alpha \circ 1_{1_C \times 1_C \times \otimes}) \cdot (\lambda \circ (1_C \times \lambda) \circ 1_{1_C \times 1_C \times \otimes}) \\
&= (\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C}) \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (\lambda \circ (1_{\otimes} \times \lambda)) \cdot (\lambda^{-1} \circ (\lambda^{-1} \times 1_{\otimes})) \\
&\cdot (\alpha \circ 1_C \times 1_C \times \otimes) \cdot (\lambda \circ (1_C \times \lambda) \circ 1_{1_C \times 1_C \times \otimes}) \\
&= (\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C}) \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (1_{\otimes} \circ (\lambda^{-1} \times \lambda)) \cdot (\alpha \circ 1_{1_C \times 1_C \times \otimes}) \\
&\cdot (\lambda \circ (1_C \times \lambda) \circ 1_{1_C \times 1_C \times \otimes}) \\
&= (\lambda^{-1} \circ ((\lambda^{-1} \circ (1_{\otimes} \times 1_{1_C})) \times 1_{1_C})) \cdot (\alpha \circ 1_{\otimes \times 1_C \times 1_C}) \cdot (1_{\otimes} \circ (\lambda^{-1} \times \lambda)) \\
&\cdot (\alpha \circ 1_{1_C \times 1_C \times \otimes}) \cdot (\lambda \circ (1_{1_C} \times (\lambda \circ (1_{1_C} \times 1_{\otimes})))) \\
&= ([(\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})) \cdot \alpha] \circ (1_{\otimes} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot ((1_{\otimes} \circ (1_{1_C} \times 1_{\otimes})) \circ (\lambda^{-1} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot ((1_{\otimes} \circ (1_{\otimes} \times 1_{1_C})) \circ (1_{1_C} \times 1_{1_C} \times \lambda)) \\
&\cdot ([\alpha \cdot (\lambda \circ (1_{1_C} \times \lambda))] \circ (1_{1_C} \times 1_{1_C} \times 1_{\otimes})) \\
&= ([(\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})) \cdot \alpha] \circ (\lambda^{-1} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot ([\alpha \cdot (\lambda \circ (1_{1_C} \times \lambda))] \circ (1_{1_C} \times 1_{1_C} \times \lambda)) \\
&= (\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C}) \circ (\lambda^{-1} \times 1_{1_C} \times 1_{1_C})) \cdot (\alpha \circ (1_{\otimes} \times 1_{1_C} \times 1_{1_C})) \\
&\cdot (\alpha \circ (1_{1_C} \times 1_{1_C} \times 1_{\otimes})) \cdot (\lambda \circ (1_{1_C} \times \lambda) \circ (1_{1_C} \times 1_{1_C} \times \lambda)) \\
&= (\lambda^{-1} \circ ((\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})) \times 1_{1_C})) \cdot (\alpha \circ 1_{\otimes \times 1_C \times 1_C}) \\
&\cdot (\alpha \circ 1_{1_C \times 1_C \times \otimes}) \cdot (\lambda \circ (1_{1_C} \times (\lambda \circ (1_{1_C} \times \lambda)))) \\
&= (\lambda^{-1} \circ ((\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})) \times 1_{1_C})) \cdot (1_{\otimes} \circ (\alpha \times 1_{1_C})) \cdot (\alpha \circ 1_{1_C \times \otimes \times 1_C}) \\
&\cdot (1_{\otimes} \circ (1_{1_C} \times \alpha)) \cdot (\lambda \circ (1_{1_C} \times (\lambda \circ (1_{1_C} \times \lambda)))) \\
&= (\lambda^{-1} \circ ((\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})) \times 1_{1_C})) \cdot (1_{\otimes} \circ (\alpha \times 1_{1_C})) \\
&\cdot (1_{\otimes} \circ (1_{\otimes} \times 1_{1_C}) \circ (1_{1_C} \times \lambda \times 1_{1_C})) \cdot (\alpha \circ (1_{1_C} \times 1_{\otimes} \times 1_{1_C})) \\
&\cdot (1_{\otimes} \circ (1_{1_C} \times 1_{\otimes}) \circ (1_{1_C} \times \lambda^{-1} \times 1_{1_C})) \cdot (1_{\otimes} \circ (1_{1_C} \times \alpha)) \\
&\cdot (\lambda \circ (1_{1_C} \times (\lambda \circ (1_{1_C} \times \lambda)))) \\
&= (1_{\otimes} \circ ((\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})) \times 1_{1_C})) \cdot (\lambda^{-1} \circ ((1_{\otimes} \circ (1_{\otimes} \times 1_{1_C})) \times 1_{1_C})) \\
&\cdot (1_{\otimes} \circ (\alpha \times 1_{1_C})) \cdot (1_{\otimes} \circ ((1_{\otimes} \circ (1_{1_C} \times \lambda)) \times 1_{1_C})) \cdot (\alpha \circ 1_{1_C \times \otimes \times 1_C}) \\
&\cdot (\lambda \circ (1_{1_C} \times (1_{\otimes} \circ (\lambda^{-1} \times 1_{1_C})))) \cdot (1_{\otimes} \circ (1_{1_C} \times \alpha)) \\
&\cdot (1_{\otimes} \circ (1_{1_C} \times (\lambda \circ (1_{1_C} \times \lambda)))) \\
&= (1_{\otimes} \circ ((\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})) \times 1_{1_C})) \cdot (1_{\otimes} \circ (\alpha \times 1_{1_C})) \\
&\cdot (\lambda^{-1} \circ (1_{\otimes} \times 1_{1_C}) \circ (1_{1_C} \times \lambda \times 1_{1_C})) \cdot (\alpha \circ 1_{1_C \times \otimes \times 1_C}) \\
&\cdot (\lambda \circ (1_{1_C} \times 1_{\otimes}) \circ (1_{1_C} \times \lambda^{-1} \times 1_{1_C})) \cdot (1_{\otimes} \circ (1_{1_C} \times \alpha)) \\
&\cdot (1_{\otimes} \circ (1_{1_C} \times (\lambda \circ (1_{1_C} \times \lambda))))
\end{aligned}
$$

$$
\begin{aligned}
&= (1_{\otimes} \circ ((\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})) \times 1_{1_C})) \cdot (1_{\otimes} \circ (\alpha \times 1_{1_C})) \\
&\quad \cdot (\lambda^{-1} \circ ((1_{\otimes} \circ (1_{1_C} \times \lambda)) \times 1_{1_C})) \cdot (\alpha \circ 1_{1_C \times \otimes \times 1_C}) \\
&\quad \cdot (\lambda \circ (1_{1_C} \times (1_{\otimes} \circ (\lambda^{-1} \times 1_{1_C})))) \cdot (1_{\otimes} \circ (1_{1_C} \times \alpha)) \\
&\quad \cdot (1_{\otimes} \circ (1_{1_C} \times (\lambda \circ (1_{1_C} \times \lambda)))) \\
&= (1_{\otimes} \circ ((\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})) \times 1_{1_C})) \cdot (1_{\otimes} \circ (\alpha \times 1_{1_C})) \\
&\quad \cdot (1_{\otimes} \circ ((\lambda \circ (1_{1_C} \times \lambda)) \times 1_{1_C})) \cdot (\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C}) \circ 1_{1_C \times \otimes \times 1_C}) \\
&\quad \cdot (\alpha \circ 1_{1_C \times \otimes \times 1_{1_C}}) \cdot (\lambda \circ (1_{1_C} \times \lambda) \circ 1_{1_C \times \otimes 1_C}) \\
&\quad \cdot (1_{\otimes} \circ (1_{1_C} \times (\lambda^{-1} \circ (\lambda^{-1} \times 1_{1_C})))) \cdot (1_{\otimes} \circ (1_{1_C} \times \alpha)) \\
&\quad \cdot (1_{\otimes} \circ (1_{1_C} \times (\lambda \circ (1_{1_C} \times \lambda)))) \\
&= (1_{\otimes} \circ (\widehat{\alpha} \times 1_{1_C})) \cdot (\widehat{\alpha} \circ 1_{1_C \times \otimes \times 1_C}) \cdot (1_{\otimes} \circ (1_{1_C} \times \widehat{\alpha}))
\end{aligned}
$$

This proves the first condition. For the second condition we have

$$
\begin{aligned}
&(1_\otimes \circ (\widehat{\gamma} \times 1_{1_C})) \cdot (\widehat{\alpha} \circ 1_{1_{C \times K_e \times 1_C}}) \\
&= (1_\otimes \circ ([\gamma \cdot (\lambda \circ 1_{1_{C \times K_e}}) \cdot (1_\otimes \circ (1_{1_C} \times \eta))] \times 1_{1_C})) \cdot ([1_\otimes \circ (\lambda^{-1} \times 1_{1_C})) \\
&\cdot (\lambda^{-1} \circ 1_{\otimes \times 1_C}) \cdot \alpha \cdot (\lambda \circ 1_{1_{C \times \otimes}}) \cdot (1_\otimes \circ (1_{1_C} \times \lambda))] \circ 1_{1_{C \times K_e \times 1_C}}) \\
&= (1_\otimes \circ ((\gamma \times 1_{1_C}) \cdot ((\lambda \circ 1_{1_{C \times K_e}}) \times 1_{1_C}) \cdot ((1_\otimes \circ (1_{1_C} \times \eta)) \times 1_{1_C}))) \cdot \\
&(1_\otimes \circ (\lambda^{-1} \times 1_{1_C}) \circ 1_{1_{C \times K_e \times 1_C}}) \cdot (\lambda^{-1} \circ 1_{\otimes \times 1_C} \circ 1_{1_{C \times K_e \times 1_C}}) \\
&\cdot (\alpha \circ 1_{1_{C \times K_e \times 1_C}}) \cdot (\lambda \circ 1_{1_{C \times \otimes}} \circ 1_{1_{C \times K_e \times 1_C}}) \cdot (1_\otimes \circ (1_{1_C} \times \lambda) \circ 1_{1_{C \times K_e \times 1_C}}) \\
&= (1_\otimes \circ (\gamma \times 1_{1_C})) \cdot (1_\otimes \circ ((\lambda \circ 1_{1_{C \times K_e}}) \times 1_{1_C})) \cdot (1_\otimes \circ ((1_\otimes \circ (1_{1_C} \times \eta)) \times 1_{1_C})) \\
&\cdot (1_\otimes \circ (\lambda^{-1} \times 1_{1_C}) \circ 1_{1_{C \times K_e \times 1_C}}) \cdot (\lambda^{-1} \circ 1_{\otimes \times 1_C} \circ 1_{1_{C \times K_e \times 1_C}}) \\
&\cdot (\alpha \circ 1_{1_{C \times K_e \times 1_C}}) \cdot (\lambda \circ 1_{1_{C \times \otimes}} \circ 1_{1_{C \times K_e \times 1_C}}) \cdot (1_\otimes \circ (1_{1_C} \times \lambda) \circ 1_{1_{C \times K_e \times 1_C}}) \\
&= (1_\otimes \circ (\gamma \times 1_{1_C})) \cdot (1_\otimes \circ ((\lambda \circ (1_{1_C} \times 1_{K_e})) \times 1_{1_C})) \\
&\cdot (1_\otimes \circ ((\lambda^{-1} \circ (1_{1_C} \times \eta)) \times 1_{1_C})) \cdot (\lambda^{-1} \circ 1_{\otimes \times 1_C} \circ 1_{1_{C \times K_e \times 1_C}}) \\
&\cdot (\alpha \circ 1_{1_{C \times K_e \times 1_C}}) \cdot (\lambda \circ 1_{1_{C \times \otimes}} \circ 1_{1_{C \times K_e \times 1_C}}) \cdot (1_\otimes \circ (1_{1_C} \times \lambda) \circ 1_{1_{C \times K_e \times 1_C}}) \\
&= (1_\otimes \circ (\gamma \times 1_{1_C})) \cdot (1_\otimes \circ ((1_\otimes \circ (1_{1_C} \times \eta)) \times 1_{1_C})) \\
&\cdot (\lambda^{-1} \circ ((1_\otimes \circ (1_{1_C} \times 1_{K_e})) \times 1_{1_C})) \cdot (\lambda \circ 1_{1_{C \times \otimes}} \circ 1_{1_{C \times K_e \times 1_C}}) \\
&\cdot (1_\otimes \circ (1_{1_C} \times \lambda) \circ 1_{1_{C \times K_e \times 1_C}}) \\
&= (1_\otimes \circ (\gamma \times 1_{1_C})) \cdot (1_\otimes \circ ((1_\otimes \circ (1_{1_C} \times \eta)) \times 1_{1_C})) \\
&\cdot (\lambda^{-1} \circ ((1_\otimes \circ (1_{1_C} \times 1_{K_e})) \times 1_{1_C})) \cdot (\lambda \circ (1_{1_C} \times (\lambda \circ 1_{K_e \times 1_C}))) \\
&= (\lambda^{-1} \circ (\gamma \times 1_{1_C})) \cdot (1_\otimes \circ ((1_\otimes \circ (1_{1_C} \times \eta)) \times 1_{1_C})) \\
&\cdot (\alpha \circ 1_{1_{C \times K_e \times 1_C}}) \cdot (\lambda \circ (1_{1_C} \times (\lambda \circ 1_{K_e \times 1_C}))) \\
&= (\lambda^{-1} \circ (1_G \times 1_{1_C})) \cdot (1_\otimes \circ (\gamma \times 1_{1_C})) \\
&\cdot (1_\otimes \circ (1_\otimes \times 1_{1_C}) \circ (1_{1_C} \times \eta \times 1_{1_C})) \cdot (\alpha \circ 1_{1_{C \times K_e \times 1_C}}) \\
&\cdot (\lambda \circ (1_{1_C} \times (\lambda \circ 1_{K_e \times 1_C}))) \\
&= (\lambda^{-1} \circ (1_G \times 1_{1_C})) \circ (1_\otimes \circ (\gamma \times 1_{1_C})) \\
&\cdot (1_\otimes \circ (1_\otimes \times 1_{1_C}) \circ (1_{1_C} \times \eta \times 1_{1_C})) \cdot (\alpha \circ (1_{1_C} \times 1_{K_e} \times 1_{1_C})) \\
&\cdot (\lambda \circ (1_{1_C} \times (\lambda \circ 1_{K_e \times 1_C}))) \\
&= (\lambda^{-1} \circ (1_G \times 1_{1_C})) \cdot (1_\otimes \circ (\gamma \times 1_{1_C})) \\
&\cdot (\alpha \circ (1_{1_C} \times 1_{K_e} \times 1_{1_C}) \circ (1_{1_C} \times \eta \times 1_{1_C})) \\
&\cdot (\lambda \circ (1_{1_C} \times (\lambda \circ 1_{K_e} c))) \\
&= (\lambda^{-1} \circ (1_G \times 1_{1_C})) \cdot (1_\otimes \circ (\gamma \times 1_{1_C})) \\
&\cdot (\alpha \circ 1_{1_{C \times K_e \times 1_C}} \circ (1_{1_C} \times \eta 1_{1_C})) \\
&\cdot (\lambda \circ (1_{1_C} \times \lambda) \circ (1_{1_C} \times 1_{K_e} \times 1_{1_C})) \\
&= (\lambda^{-1} \circ (1_G \times 1_{1_C})) \cdot (1_\otimes \circ (\gamma \times 1_{1_C})) \cdot ([(\alpha \circ 1_{1_{C \times K_e \times 1_C}}) \\
&\cdot (\lambda \circ (1_{1_C} \times \lambda))] \circ (1_{1_C} \times \eta \times 1_{1_C})) \\
&= (\lambda^{-1} \circ (1_{1_C} \times 1_B)) \cdot (1_\otimes \circ (\gamma \times 1_{1_C})) \cdot (\alpha \circ 1_{1_{C \times K_e \times 1_C}}) \\
&\cdot ((\lambda \circ (1_{1_C} \times \lambda)) \circ (1_{1_C} \times \eta \times 1_{1_C}))
\end{aligned}
$$

$$
\begin{aligned}
&= (\lambda^{-1} \circ (1_{1_C} \times 1_B)) \cdot (1_\otimes \circ (1_{1_C} \times \beta)) \\
&\quad \cdot(\lambda \circ (1_{1_C} \times \lambda) \circ (1_{1_C} \times \eta \times 1_{1_C})) \\
&= (\lambda^{-1} \circ (1_{1_C} \times \beta)) \cdot (\lambda \circ (1_{1_C} \times \lambda) \circ (1_{1_C} \times \eta \times 1_{1_C})) \\
&= (\lambda^{-1} \circ (1_{1_C} \times \beta)) \cdot (\lambda \circ (1_{1_C} \times \lambda \circ (\eta \times 1_{1_C}))) \\
&= (1_\otimes \circ (1_{1_C} \times (\beta \cdot (\lambda \circ (\eta \times 1_{1_C}))))) \\
&= 1_\otimes \circ (1_{1_C} \times (\beta \cdot (\lambda \circ 1_{K_e \times 1_C}) \cdot (1_\otimes \circ (\eta \times 1_{1_C})))) \\
&= 1_\otimes \circ (1_{1_C} \times \widehat{\beta})
\end{aligned}
$$

This proves the second condition. For the third condition we have

$$
\begin{aligned}
&\widehat{\beta} \circ 1_{1_C \times K_e} \\
&= [\beta \cdot (\lambda \circ 1_{K_e \times 1_C} \cdot (1_\otimes \circ (\eta \times 1_{1_C}))] \circ 1_{1_C \times K} \\
&= (\beta \circ 1_{1_C \times K}) \cdot (\lambda \circ (1_{K_e} \times 1_{1_C}) \circ 1_{1_C \times K}) \cdot (1_\otimes \circ (\eta \times 1_{1_C}) \circ 1_{1_C \times K}) \\
&= (\gamma \circ 1_{K \times 1_C}) \cdot (\lambda \circ (1_{K_e} \times 1_{K_e})) \cdot (1_\otimes \circ (\eta \times 1_{K_e})) \\
&= (\gamma \circ 1_{K \times 1_C}) \cdot (\lambda \circ (1_{1_C} \times 1_{K_e}) \circ (1_{K_e} \times 1_{1_C})) \cdot (1_\otimes \circ (1_{K_e} \times \eta)) \\
&= (\gamma \circ 1_{K \times 1_C}) \cdot (\lambda \circ (1_{1_C} \times 1_{K_e}) \circ 1_{K \times 1_C}) \cdot (1_\otimes \circ (1_{1_C} \times \eta) \circ 1_{K \times 1_C}) \\
&= [(\gamma \cdot (\lambda \circ 1_{1_C \times K_e}) \cdot (1_\otimes \circ (1_{1_C} \times \eta))] \circ 1_{K \times 1_C} \\
&= \widehat{\gamma} \circ 1_{K_e \times 1_C}
\end{aligned}
$$

This proves the third condition. The proof of the fourth condition is very technical. In the proof we will use the following symbols

$$
\begin{aligned}
L_1 &= (1_{t\otimes} \circ (1_{1_C} \times t\lambda^{-1}) \cdot (t\lambda^{-1} \circ 1_{t\otimes \times 1_C}) \cdot (t\alpha) \cdot (t\lambda \circ 1_{t\otimes \times 1_C}) \\
L_2 &= (1_{t\otimes} \circ (1_{1_C} \times t\lambda^{-1}) \cdot (t\lambda^{-1} \circ 1_{t\otimes \times 1_C}) \cdot (t\alpha) \\
L_3 &= (1_{t\otimes} \circ (1_{1_C} \times t\lambda^{-1}) \cdot (t\lambda^{-1} \circ 1_{t\otimes \times 1_C}) \\
L_4 &= (1_{t\otimes} \circ (1_{1_C} \times t\lambda^{-1}) \\
R &= ((t\lambda^{-1}) \circ (t\lambda^{-1} \times 1_{1_C})) \cdot ((\mu \circ \sigma) \circ ((\mu \circ \sigma) \times 1_{1_C})) \cdot (\lambda \circ (\lambda \times 1_{1_C})) \\
R_1 &= ((\mu \circ \sigma) \circ ((\mu \circ \sigma) \times 1_{1_C})) \cdot (\lambda \circ (\lambda \times 1_{1_C})) \\
R_2 &= (\lambda \circ (\lambda \times 1_{1_C})) \\
S_1 &= \alpha^{-1} \cdot (\lambda \circ 1_{\otimes \times 1_C}) \cdot (1_\otimes \circ (\lambda \times 1_{1_C})) \\
S_2 &= (\lambda^{-1} \circ 1_{1_C \times \otimes}) \cdot \alpha^{-1} \cdot (\lambda \circ 1_{\otimes \times 1_C}) \cdot (1_\otimes \circ (\lambda \times 1_{1_C}))
\end{aligned}
$$

Using these symbols we have for the fourth condition

$$
\begin{aligned}
&(t\widehat{\alpha}) \cdot (\widehat{\sigma} \circ (\widehat{\sigma} \times 1_{1_C}))\\
&= (t\widehat{\alpha}) \cdot ([(1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot (t\lambda^{-1}) \cdot (\mu \circ \sigma) \cdot \lambda] \circ ([(1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1}))\\
&\quad \cdot (t\lambda^{-1}) \cdot (\mu \circ \sigma) \cdot \lambda] \times 1_{1_C}))\\
&= L_1 \cdot (1_{t\otimes} \circ (t\lambda \times 1_{1_C}))\\
&\quad \cdot ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \circ ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \times 1_{1_C})) \cdot R\\
&= L_1 \cdot ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \circ ((t\lambda \circ (\mu^{-1} \times \mu^{-1})) \times 1_{1_C})) \cdot R\\
&= L_1 \cdot ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \circ ((t\lambda \circ (\mu^{-1} \times \mu^{-1})) \times 1_{1_C}))\\
&\quad \cdot (t\lambda^{-1} \circ (t\lambda^{-1} \times 1_{1_C})) \cdot R_1\\
&= L_1 \cdot ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \circ ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \times 1_{1_C})) \cdot R_1\\
&= L_2 \cdot (t\lambda \circ (1_{t\otimes} \times 1_{1_C}))\\
&\quad \cdot ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \circ ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \times 1_{1_C})) \cdot R_1\\
&= L_2 \cdot ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \circ ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \times 1_{1_C})) \cdot R_1\\
&= L_2 \cdot ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \circ ((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \times 1_{1_C}))\\
&\quad \cdot ((\mu \circ \sigma) \circ ((\mu \circ \sigma) \times 1_{1_C})) \cdot R_2\\
&= L_2 \cdot ((1_{1_C} \circ 1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \circ ((1_{1_C} \circ 1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \times 1_{1_C}))\\
&\quad \cdot ((\mu \circ \sigma \circ (1_{1_C} \times 1_{1_C})) \circ ((\mu \circ \sigma \circ (1_{1_C} \times 1_{1_C})) \times 1_{1_C})) \cdot R_2\\
&= L_2 \cdot ((\mu \circ \sigma \circ (\mu^{-1} \times \mu^{-1})) \circ ((\mu \circ \sigma \circ (\mu^{-1} \times \mu^{-1})) \times 1_{1_C})) \cdot R_2\\
&= L_2 \cdot ((\mu \circ \sigma \circ (\mu^{-1} \times \mu^{-1})) \circ ((\mu \circ \sigma \circ (\mu^{-1} \times \mu^{-1})) \times 1_{1_C})) \cdot R_2\\
&= L_2 \cdot ((\mu \circ \sigma \circ (\mu^{-1} \times \mu^{-1})) \circ ((\mu \circ \sigma \circ (\mu^{-1} \times \mu^{-1})) \times (\mu \circ 1_{1_C} \circ \mu^{-1}))) \cdot R_2\\
&= L_2 \cdot ((1\mu \circ \sigma \circ (\mu^{-1} \times \mu^{-1})) \circ (\mu \times \mu) \circ (\sigma \times 1_{1_C}) \circ (\mu^{-1} \times \mu^{-1} \times \mu^{-1})) \cdot R_2\\
&= L_2 \cdot (\mu \circ \sigma \circ ((\sigma \circ (\mu^{-1} \times \mu^{-1})) \times \mu^{-1})) \cdot R_2\\
&= L_3 \cdot (1_{1_C} \circ t\alpha) \cdot (\mu \circ \sigma \circ (\sigma \times 1_{1_C})) \cdot (1_{\otimes} \circ ((1_{\otimes} \circ (\mu^{-1} \times \mu^{-1})) \times \mu^{-1}))\\
&\quad \cdot (\lambda \circ (\lambda \times 1_{1_C}))\\
&= L_3 \cdot (\mu \circ (t\alpha \cdot (\sigma \circ (\sigma \times 1_{1_C})))) \cdot (\lambda \circ ((\lambda \circ (\mu^{-1} \times \mu^{-1})) \times \mu^{-1})\\
&= L_3 \cdot (\mu \circ ((\sigma \circ (1_{1_C} \times \sigma)) \cdot \alpha^{-1})) \cdot (\lambda \circ ((\lambda \circ (\mu^{-1} \times \mu^{-1})) \times \mu^{-1})\\
&= L_3 \cdot (\mu \circ ((\sigma \circ (1_{1_C} \times \sigma)) \cdot \alpha^{-1})) \cdot (\lambda \circ (\lambda \times 1_{1_C}) \circ (\mu^{-1} \times \mu^{-1} \times \mu^{-1}))\\
&= L_3 \cdot (\mu \circ \sigma \circ (1_{1_C} \times \sigma)) \cdot \alpha^{-1} \cdot (\lambda \circ (\lambda \times 1_{1_C}) \circ (\mu^{-1} \times \mu^{-1} \times \mu^{-1}))\\
&= L_3 \cdot (\mu \circ \sigma \circ (1_{1_C} \times \sigma)) \cdot ((\alpha^{-1} \cdot (\lambda \circ (\lambda \times 1_{1_C}))) \circ (\mu^{-1} \times \mu^{-1} \times \mu^{-1}))\\
&= L_3 \cdot (\mu \circ \sigma \circ (1_{1_C} \times \sigma)) \cdot ((1_{\otimes} \circ (1_{1_C} \times 1_{\otimes})) \circ (\mu^{-1} \times \mu^{-1} \times \mu^{-1}))\\
&\quad \cdot (\alpha^{-1} \cdot (\lambda \circ (\lambda \times 1_{1_C})))\\
&= L_3 \cdot (\mu \circ \sigma \circ (1_{1_C} \times \sigma)) \cdot (1_{\otimes} \circ (\mu^{-1} \times (1_{\otimes} \circ (\mu^{-1} \times \mu^{-1})))) \cdot S_1\\
&= L_3 \cdot (\mu \circ \sigma \circ (\mu^{-1} \times (\sigma \circ (\mu^{-1} \times \mu^{-1})))) \cdot S_1\\
&= L_3 \cdot (((1_{1_C} \circ 1_{\otimes}) \cdot (\mu \circ \sigma)) \circ ((\mu^{-1} \times (\sigma \circ (\mu^{-1} \times \mu^{-1})))\\
&\quad \cdot ((1_{1_C} \times (1_{t\otimes} \circ (1_{1_C} \times 1_{1_C})))))) \cdot S_1
\end{aligned}
$$

$$
\begin{aligned}
&= L_3 \cdot ((1_{1_C} \circ 1_{t\otimes}) \circ (\mu^{-1} \times (\sigma \circ (\mu^{-1} \times \mu^{-1})))) \\
&\quad \cdot ((\mu \circ \sigma) \circ (1_{1_C} \times (1_{t\otimes} \circ (1_{1_C} \times 1_{1_C})))) \cdot S_1 \\
&= L_3 \cdot (1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1}) \circ (1_{1_C} \times (\mu \circ \sigma \circ (\mu^{-1} \times \mu^{-1})))) \\
&\quad \cdot ((\mu \circ \sigma) \circ (1_{1_C} \times (1_{t\otimes} \circ (1_{1_C} \times 1_{1_C})))) \cdot S_1 \\
&= L_3 \cdot (1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1}) \circ (1_{1_C} \times (\mu \circ \sigma \circ (\mu^{-1} \times \mu^{-1})))) \\
&\quad \cdot ((\mu \circ \sigma) \circ (1_{1_C} \times 1_{t\otimes})) \cdot S_1 \\
&= L_3 \cdot (1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1}) \circ (1_{1_C} \times [(1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot (\mu \circ \sigma)])) \\
&\quad \cdot ((\mu \circ \sigma) \circ 1_{1_C \times \otimes}) \cdot S_1 \\
&= L_4 \cdot (t\lambda^{-1} \circ (1_{1_C} \times 1_{t\otimes})) \cdot (1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1}) \circ (1_{1_C} \times [(1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \\
&\quad \cdot (\mu \circ \sigma)])) \cdot ((\mu \circ \sigma) \circ 1_{1_C \times \otimes}) \cdot S_1 \\
&= L_4 \cdot (t\lambda^{-1} \circ (\mu^{-1} \times \mu^{-1}) \circ (1_{1_C} \times [(1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot (\mu \circ \sigma)])) \\
&\quad \cdot ((\mu \circ \sigma) \circ 1_{1_C \times \otimes}) \cdot S_1 \\
&= L_4 \cdot (t\lambda^{-1} \circ (\mu^{-1} \times \mu^{-1}) \circ (1_{1_C} \times [(1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot (\mu \circ \sigma)])) \\
&\quad \cdot ([(\mu \circ \sigma) \cdot \lambda] \circ 1_{1_C \times \otimes}) \cdot S_2 \\
&= (1_{t\otimes} \circ (1_{1_C} \times t\lambda^{-1})) \cdot (t\lambda^{-1} \circ (\mu^{-1} \times \mu^{-1}) \circ (1_{1_C} \times [(1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \\
&\quad \cdot (\mu \circ \sigma)])) \cdot ([(\mu \circ \sigma) \cdot \lambda] \circ 1_{1_C \times \otimes}) \cdot S_2 \\
&= (t\lambda^{-1} \circ (\mu^{-1} \times \mu^{-1}) \circ (1_{1_C} \times [t\lambda^{-1} \cdot (1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot (\mu \circ \sigma)])) \\
&\quad \cdot ([(\mu \circ \sigma) \cdot \lambda] \circ 1_{1_C \times \otimes}) \cdot S_2 \\
&= ([(1_{t\otimes} \cdot t\lambda^{-1}) \circ ((\mu^{-1} \times \mu^{-1}) \cdot (1_{1_C} \times 1_{1_C}))] \circ (1_{1_C} \times [(t\lambda^{-1} \circ (\mu^{-1} \times \mu^{-1})) \\
&\quad \cdot (\mu \circ \sigma)])) \cdot ([(\mu \circ \sigma) \cdot \lambda] \circ 1_{1_C \times \otimes}) \cdot S_2 \\
&= ([(1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot t\lambda^{-1}] \circ (1_{1_C} \times [(1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot t\lambda^{-1} \cdot (\mu \circ \sigma)])) \\
&\quad \cdot ([(\mu \circ \sigma) \cdot \lambda] \circ 1_{1_C \times \otimes}) \cdot S_2 \\
&= (\widehat{\sigma} \circ (1_{1_C} \times [(1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot t\lambda^{-1} \cdot (\mu \circ \sigma)])) \\
&\quad \cdot ([(\mu \circ \sigma) \cdot \lambda] \circ 1_{1_C \times \otimes}) \cdot S_2 \\
&= (\widehat{\sigma} \circ (1_{1_C} \times \widehat{\sigma})) \cdot (1_{\otimes} \circ (1_{1_C} \times \lambda^{-1})) \cdot S_2 \\
&= (\widehat{\sigma} \circ (1_{1_C} \times \widehat{\sigma})) \cdot \widehat{\alpha}^{-1}
\end{aligned}
$$

This proves the fourth condition. The fifth and sixth condition is proved in a similar way and we only prove the sixth.

$$
\begin{aligned}
& (t\widehat{\beta}) \cdot (\widehat{\sigma} \circ 1_{1_C \times K_e}) \\
= {} & t\beta \cdot (t\lambda \circ (1_{1_C} \times t\eta)) \cdot (((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot (t\lambda^{-1}) \cdot (\mu \circ \sigma) \cdot \lambda) \circ 1_{1_C \times K_e}) \\
= {} & \gamma \cdot (t\sigma \circ 1_{1_C \times K_e}) \cdot (t\lambda \circ (1_{1_C} \times t\eta)) \cdot (1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1}) \circ 1_{1_C \times K_e}) \\
& \cdot (t\lambda^{-1} \circ 1_{1_C \times K_e}) \cdot (\mu \circ \sigma \circ 1_{1_C \times K_e}) \cdot (\lambda \circ 1_{1_C \times K_e}) \\
= {} & \gamma \cdot (\mu \circ \lambda \circ (\mu^{-1} \times (\mu^{-1} \circ t\eta))) \\
= {} & \gamma \cdot (\mu \circ \lambda \circ (\mu^{-1} \times \eta)) \\
= {} & \gamma \cdot (\lambda \circ (1_{1_C} \times \eta)) \cdot (\mu \circ 1_{\otimes} \circ (\mu^{-1} \times 1_{K_e})) \\
= {} & \gamma \cdot (\lambda \circ (1_{1_C} \times \eta)) \\
= {} & \widehat{\gamma}
\end{aligned}
$$

For the seventh condition we have

$$
\begin{aligned}
& t\widehat{\sigma} \\
= {} & t((1_{t\otimes} \circ (\mu^{-1} \times \mu^{-1})) \cdot t\lambda^{-1} \cdot (\mu \circ \sigma) \cdot \lambda) \\
= {} & (1_{\otimes} \circ (\mu \times \mu)) \cdot \lambda^{-1} \cdot (\mu^{-1} \circ \sigma^{-1}) \cdot t\lambda \\
= {} & (1_{\otimes} \circ (\mu \times \mu)) \cdot (\lambda^{-1} \circ (1_{1_C} \times 1_{1_C})) \cdot ((\mu^{-1} \circ \sigma^{-1}) \circ (1_{1_C} \times 1_{1_C})) \\
& \cdot (t\lambda \circ (1_{1_C} \times 1_{1_C})) \\
= {} & (\lambda^{-1} \cdot (\mu^{-1} \circ \sigma^{-1}) \cdot t\lambda \cdot 1_{t\otimes}) \circ (1_{1_C \times 1_C} \cdot (\mu \times \mu)) \\
= {} & \lambda^{-1} \cdot (\mu^{-1} \circ \sigma^{-1}) \cdot t\lambda \cdot (1_{t\otimes} \circ (\mu \times \mu)) \\
= {} & \widehat{\sigma}^{-1}
\end{aligned}
$$

■

¿From this point of view the quantizations of the identity functor on a symmetric monoidal category $\langle C, \otimes, K_e, \alpha, \beta, \gamma, \sigma \rangle$ is exactly equal to the subgroup of H that fix the point $(\alpha, \beta, \gamma, \sigma)$.

4.2 Quantization of algebraic structures in symmetric monoidal categories

Let $\langle C_i, \otimes_i, P_{e_i}, \alpha_i, \beta_i, \gamma_i, \sigma_i \rangle$ be symmetric monoidal categories for $i = 1, 2$ and let $F : C_1 \longrightarrow C_2$ be a quantized functor with quantization (λ, μ, η). Let the S_2 action on C_1 and C_2 be generated by the functors $T_1 : C_1 \longrightarrow C_1$ and $T_2 : C_2 \longrightarrow C_2$. In this section we will work with objects and need the object formulation of the conditions defining a symmetric monoidal category and quantized functors. We collect these conditions in the following proposition whose proof consists of applying the definition of vertical composition and horizontal composition.

Proposition 69

$$(\alpha_2)_{F(X),F(Y),F(Z)} = (\lambda^{-1}_{X,Y} \otimes_2 1_{F(Z)}) \circ \lambda^{-1}_{X \otimes_1 Y, Z} \circ F((\alpha_1)_{X,Y,Z}) \circ \lambda_{X, Y \otimes_1 Z} \circ (1_{F(X)} \otimes_2 \lambda_{Y,Z})$$

$$(\beta_2)_{F(X),F(Y)} = F(\beta_{X;Y}) \circ \lambda_{e_1,Y} \circ (\eta_X \otimes_2 1_{F(Y)})$$

$$(\gamma_2)_{F(X),F(Y)} = F((\gamma_1)_{X,Y}) \circ \lambda_{X,e_1} \circ (1_{F(X)} \otimes_2 \eta_Y)$$

$$(\sigma_2)_{F(X),F(Y)} = T_2(T_2(\mu^{-1}_Y) \otimes_2 T_2(\mu^{-1}_X)) \circ T_2(\lambda^{-1}_{T_1(Y),T_1(X)}) \circ \mu_{T_1(T_1(Y) \otimes_1 T_1(X))} \circ F(\sigma_{X;Y}) \circ \lambda_{X,Y}$$

Quantized functors preserve algebraic structures. Let $\langle X, \nu, u \rangle$ be a monoid in the symmetric monoidal category C_1 and Define arrows in C_2

$$\nu^\lambda : F(X) \otimes_2 F(X) \longrightarrow F(X)$$
$$u^\eta : e_2 \longrightarrow F(X)$$

by $\nu^\lambda = F(\nu) \circ \lambda_{X,X}$ and $u^\eta = F(u) \circ \eta$. In

Proposition 70 *$\langle F(X), \nu^\lambda, u^\eta \rangle$ is a monoid in C_2.*

Proof. Since $\langle X, \nu, u \rangle$ is a monoid in C_1 we have the identities

$$\nu \circ (1_X \otimes_1 \nu) = \nu \circ (\nu \otimes_1 1_X) \circ (\alpha_1)_{X,X,X}$$
$$\nu \circ (u \otimes_1 1_X) = (\beta_1)_{X,X}$$
$$\nu \circ (1_X \otimes_1 \nu) = (\gamma_1)_{X,X}$$

If we use these identities and the relations from proposition 69 we have

$$\begin{aligned}
&\nu^\lambda \circ (1_{F(X)} \otimes_2 \nu^\lambda) \\
&= F(\nu) \circ \lambda_{X,X} \circ (1_{F(X)} \otimes_2 F(\nu)) \circ (1_{F(X)} \otimes_2 \lambda_{X,X}) \\
&= F(\nu) \circ F(1_X \otimes_1 \nu) \circ \lambda_{X, X \otimes_1 X} \circ (1_{F(X)} \otimes_2 \lambda_{X,X}) \\
&= F(\nu \circ (1_X \otimes_1 \nu)) \circ \lambda_{X, X \otimes_1 X} \circ (1_{F(X)} \otimes_2 \lambda_{X,X}) \\
&= F(\nu) \circ F(\nu \otimes_1 1_X) \circ F((\alpha_1)_{X,X,X}) \circ \lambda_{X, X \otimes_1 X} \circ (1_{F(X)} \otimes_2 \lambda_{X,X}) \\
&= F(\nu) \circ F(\nu \otimes_1 1_X) \circ \lambda_{X \otimes_1 X, X} \circ (\lambda_{X,X} \otimes_2 1_{F(X)}) \circ (\alpha_2)_{F(X),F(X),F(X)} \\
&= F(\nu) \circ \lambda_{X,X} \circ (F(\nu) \otimes_2 1_{F(X)}) \circ (\lambda_{X,X} \otimes_2 1_{F(X)}) \circ (\alpha_2)_{F(X),F(X),F(X)} \\
&= \nu^\lambda \circ (\nu^\lambda \otimes_2 1_{F(X)}) \circ (\alpha_2)_{F(X),F(X),F(X)}
\end{aligned}$$

and

$$\begin{aligned}
&\nu^\lambda \circ (u^\lambda \otimes_2 1_{F(X)}) \\
&= F(\nu) \circ \lambda_{X,X} \circ (F(u) \otimes_2 F(1_X)) \circ (\eta_X \otimes_2 1_{F(X)}) \\
&= F(\nu) \circ F(u \otimes_1 1_X) \circ \lambda_{e_1,X} \circ (\eta_X \otimes_2 1_{F(X)}) \\
&= F(\nu \circ (u \otimes 1_X)) \circ \lambda_{e_1,X} \circ (\eta_X \otimes_2 1_{F(X)}) \\
&= F((\beta_1)_{X,X}) \circ \lambda_{e_1,X} \circ (\eta_X \otimes_2 1_{F(X)}) \\
&= (\beta_2)_{F(X),F(X)}
\end{aligned}$$

■

We call the monoid $\langle F(X), \nu^\lambda, u^\eta \rangle$ a quantization of the monoid $\langle X, \nu, u \rangle$ in C_1. Quantization of comonoids is defined by duality. Let us assume that the monoid $\langle X, \nu, u \rangle$ is commutative. This property is preserved by quantization.

Proposition 71 *Let $\langle X, \nu, u \rangle$ be a commutative monoid in C_1. Then $\langle F(X), \nu^\lambda, u^\eta \rangle$ is a commutative monoid in C_2.*

Proof. Using the exchange identity for horizontal and vertical composition of natural transformations, the two last conditions in the definition of quantized functors 66 and the symmetry conditions $t\sigma_i = \sigma_i^{-1}$, $i = 1, 2$ we get the following identity

$$t(1_F \circ \sigma_1) \cdot (t\lambda) \cdot (1_{t\otimes_2} \circ (\mu \times \mu)) = (\mu \circ 1_\otimes) \cdot \lambda \cdot (t\sigma_2 \circ 1_{F\times F})$$

The (X, X, X) component of this identity is gives after application of the functor T_2 the following relation

$$\begin{aligned} &F((\sigma_1)_{T_1(X),T_1(X)}) \circ \lambda_{T_1(X),T_1(X)} \circ (T_2(\mu_X) \otimes_2 T_2(\mu_X)) \\ &\qquad = T_2(\mu_{X\otimes_1 X}) \circ T_2(\lambda_{X,X}) \circ (\sigma_2)_{T_2(F(X)),T_2(F(X))} \end{aligned}$$

But then we have

$$\begin{aligned} &T_2(\mu_X) \circ (\nu^\lambda)^{\sigma_2} \\ &= T_2(\mu_X) \circ T_2(F(\nu)) \circ T_2(\lambda_{X,X}) \circ (\sigma_2)_{T_2(F(X)),T_2(F(X))} \\ &= F(T_1(\nu)) \circ T_2(\mu_{X\otimes_1 X}) \circ T_2(\lambda_{X,X}) \circ (\sigma_2)_{T_2(F(X)),T_2(F(X))} \\ &= F(T_1(\nu)) \circ F((\sigma_1)_{T_1(X),T_1(X)}) \circ \lambda_{T_1(X),T_1(X)} \circ (T_2(\mu_X) \otimes_2 T_2(\mu_X)) \\ &= (\nu^{\sigma_1})^\lambda \circ (T_2(\mu_X) \otimes_2 T_2(\mu_X)) \end{aligned}$$

Since $\langle X, \nu, u \rangle$ is commutative in C_1 there exists a isomorphism $\varphi : T_1(X) \longrightarrow X$ such that the following identity holds

$$\varphi \circ \nu^{\sigma_1} = \nu \circ (\varphi \otimes_1 \varphi)$$

Let the isomorphism $\widehat{\varphi} : T_2(F(X)) \longrightarrow F(X)$ be defined by $\widehat{\varphi} = F(\varphi) \circ T_2(\mu_X)$. For this isomorphism in C_2 we have

$$\begin{aligned} &\widehat{\varphi} \circ (\nu^\lambda)^{\sigma_2} \\ &= F(\varphi) \circ T_2(\mu_X) \circ (\nu^\lambda)^{\sigma_2} \\ &= F(\varphi) \circ (\nu^{\sigma_1})^\lambda \circ (T_2(\mu_X) \otimes_2 T_2(\mu_X)) \\ &= F(\varphi) \circ F(\nu^{\sigma_1}) \circ \lambda_{T_1(X),T_1(X)} \circ (T_2(\mu_X) \otimes_2 T_2(\mu_X)) \\ &= F(\nu) \circ F(\varphi \otimes_1 \varphi) \circ \lambda_{T_1(X),T_1(X)} \circ (T_2(\mu_X) \otimes_2 T_2(\mu_X)) \\ &= F(\nu) \circ \lambda_{X,X} \circ (F(\varphi) \otimes_2 F(\varphi)) \circ (T_2(\mu_X) \otimes_2 T_2(\mu_X)) \\ &= \nu^l \circ (\widehat{\varphi} \otimes_2 \widehat{\varphi}) \end{aligned}$$

and this proves that $\langle F(X), \nu^\lambda, u^\lambda\rangle$ is a commutative monoid. ■

Commutative comonoids will by duality also be preserved by quantization. Similar results holds for other algebraic structures like modules and condoles. As a special case of the above constructions let $F = I_C$ and let $\langle X, \rho, u\rangle$ be a commutative monoid in C. Then any element (λ, μ, η) in the group H described in the previous section defines a quantization $\langle X, \rho^\lambda, u^\eta\rangle$ of the given monoid. We thus get a whole family of quantized product and unit structures on the object X. Each such quantized product and unit does not define a commutative monoid with respect to the original structure $\langle \alpha, \beta, \gamma, \sigma\rangle$, but with respect to the structure $\langle \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\sigma}\rangle$.

References

[1] Fronsdal C. Lichnerowicz A. Bayen F., Flato M. and Sternheimer D. Deformation theory and quantization i, II. *Ann. Phys. (NY)* , 111:61–110,111–151, 1978.

[2] Louise De Broglie. Ondes et quanta. *Comptes rendus de l'Academie des Sciences*, 177:507–510, September 1923.

[3] Louise De Broglie. *Recherches sur la Theorie Des Quanta* . PhD thesis, University of Paris, 1924.

[4] Louise De Broglie. Phase wave of louise deBroglie. *A. J. Phys.* , 40(9):1315–1320, September 1972.

[5] Albert Einstein. Zur theorie der lichterzeugung and lichtabsorption. *Annalen der Physik* , page 1, 1906.

[6] W. Heisenberg. Uber quantentheoretische umdentung kinematischer und mechanischer beziehungen. *Zeitschrift fur physik* , 33:879–893, Juli 1925.

[7] Woodhouse N. M. J. *Geometric Quantization* . Oxford Mathematical Monographs. Oxford Science Publications, 2 edition, 1992.

[8] Saunders Mac Lane. *Categories for the Working Mathematician* , volume 5 of *Graduate Texts in Mathematics.* springer, 1998.

[9] V. V. Lychagin. Colour calculus and colour quantizations. geometric and algebraic structures in differential equations. *Acta Appl. Math.* , 41(1-3):193 – 226, 1995.

[10] V. V. Lychagin. Calculus and quantizations over hopf algebras. *Acta Appl. Math.* , 51(3):303–352, 1998.

[11] V. V. Lychagin. Quantum mechanics on manifolds. geometrical aspects of nonlineardifferential equations. *Acta Appl. Math.* , 56(2-3):231–251, 1999.

[12] M. Planck. Zur theorie des gesetzes der energieverteilung im normalspectrum. *Verhandl. Dtsch. phys. Ges.* , 2:237, 1900.

[13] Jakub Rembielinski, editor. *Deformation Quantization: Twenty Years After.* Amer. Inst. Phys., Woodbury, Ny., 1998.

[14] B. L. Van Der Waerden. *Sources of Quantum Mechanics* . Dover, 1967.