On symplectic classification of effective 3-forms and Monge-Ampère equations.

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Abstract

We complete the list of normal forms for effective 3-forms with constant coefficients with respect to the natural action of symplectomorphisms in $\mathbb{R}^6$. We show that the 3-form which corresponds to the Special Lagrangian equation is among the new members of the classification. The symplectic symmetry algebras and their Cartan prolongations for these forms are computed and a local classification theorem for the corresponding Monge-Ampère equations is proved.

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A classical problem of the geometric theory of differential equations is the problem of local equivalence: when do two differential equations represent the same equation modulo local change of dependent and independent coordinates (a diffeomorphism on the corresponding jet space)? We can consider this problem to be a special case of the general equivalence problem in differential geometry (see [1], chapter 7). This point of view enables us to recognize equivalent structures, objets, etc., by means of a set of so-called "scalar" differential invariants. Generally speaking, a differential invariant of order at most k is a smooth function on the jet space J^k invariant under the diffeomorphisms generated by the local diffeomorphisms of the base manifold.

An important particular case of this problem is the question of linearization: when is a differential equation equivalent to a linear one? The Monge-Ampère equations (MAE hereafter) provide a natural class of nonlinear second order differential equations for this problem. Sophus Lie posed in 1874 this problem of linearization for 2-dimensional MAE with respect to the (pseudo) group

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of contact diffeomorphism Ct and it was studied in the classical works of G.Darboux and E. Goursat as well as in many recent papers.

An adequate description of the general classification problem for the MAE is achieved by computing of a complete system of differential invariants with a complete set of relations between them. Scalar differential invariants of MAE are interpreted as smooth functions on a "diffiety" quotient $J^\infty(MAE)/Ct$, the object of the category of differential equations corresponding to MAE (see [9]). This quotient is quite singular and admit a stratification (like a spaces of orbits, total spaces of general foliations, etc.). Locally the quotient $J^\infty(MAE)/Ct$ has the form $\mathcal{E}_\infty$ of the infinite prolongation of a system of differential equations $\mathcal{E}$, which is defined by the relation between differential invariants. The stratification of $\mathcal{E}_\infty$ is given by the singular loci

$$\mathcal{E}_\infty \supset \mathcal{E}'_\infty = sing(\mathcal{E}_\infty) \supset \mathcal{E}''_\infty = sing(\mathcal{E}'_\infty) \supset \dots$$

which correspond to a reduction of the number of variables. Due to a generalized "Kerr theorem" ([11], ch. 7.2.3), this reduction is provided by a sufficiently large local symmetry group, so the most "symmetric" MAE should correspond to a very "singular strata". We will call such equations "special MAE". We will restrict ourself throughout this paper to the classification problem of these special MAE.

A modern geometric approach to special MAE was proposed by V. Lychagin [10] (after an idea of E. Cartan) and was applied to this classification problem by V. Lychagin and V. Roubtsov in the papers [11],[12], and [13]. From the general point of view, the Monge-Ampère equations should be distinguished by the differential invariants of order at most 2, but the approach of V. Lychagin permits, using the existence of the canonical contact structure on the space J^1, to take into consideration only scalar invariants of contact (under some mild restrictions, symplectic) diffeomorphisms of J^1 (cotangent space).

V.L and V.R established in [11] an almost complete classification of special MAE in the dimensions 2 and 3 and a partial classification in higher dimensions. In the dimension 2, the equivalence problem of two (generic) MAE can be reduced to the equivalence problem of two generic structures (more exactly, to the equivalence of two G-structures). The corresponding scalar differential invariant is given by the Pfaffian. The integrability is established in [12] by the annihilation of the Nijenhuis tensor of an almost complex structure (for an elliptic MAE) or of an almost product structure (for a hyperbolic MAE). In the dimension 3, the corresponding scalar invariant was proposed in [11] and was identified recently with the Hitchin functional on the space of exterior 3-forms (see [7],[2]). In our paper in progress ([2]), we adapt the approach to the equivalence problem of MAE from the viewpoint of equivalence problem of some geometric structures in dimension 3 using the Hitchin-like invariant. The main goal of the present paper is to complete the list of normal forms for the MAE in the dimension 3 with respect to the action of the symplectic group $Sp(3)$.

Within the classification of 3-dimensional Monge-Ampère equations (as it was observed in [11]) a problem of Geometric Invariant Theory arises: to find the list of the normal forms of effective (primitive) trivectors on $\mathbb{R}^6$ with respect to the symplectic group. The corresponding complex classification was obtained by J.I Igusa in [8] and V. Popov in [14] but it does not help much in the real case we are interested in.

A distinguished MAE in dimension 3 became recently a subject of considerable interest after the famous paper of R. Harvey and H.B Lawson on calibrated geometries ([6]). This is the equation

$$\mathrm{hess}(h) - \Delta h = 0$$

(where $\mathrm{hess}(h)$ is the hessian of h and Δ is the Laplace operator), which is a criteria for the graph of ∇h to be special lagrangian. The normal form associated with this equation was missed in [11] and motivates the present paper.

Here we complete the symplectic classification of the effective 3-forms with constant coefficients and study the special lagrangian orbit. We calculate its symplectic invariant, its stabilizer and its Cartan's prolongation. Finally we prove, using the arguments similar to [11], a classification theorem for 3-dimensional MAE.

This paper has the following structure:

In the first section, we recall Lychagin's approach to MAE based on the differential forms on the 1-jet space. We also formulate here the classification problem in an appropriate form.

The second section deals with the symplectic classification of effective 3-forms on $\mathbb{R}^6$. This classification is obtained by means of a recursive formula for 3-forms, which allows us to reduce the problem to 2-forms on $\mathbb{R}^4$. So, this is a little bit technical. The main result of it is the theorem 9.

The stabilizers of the different orbits and their Cartan's prolongations are computed in the third section. We explicitly identify these stabilizers with Lie subalgebras of $Sp(3)$ and we summarize the results in table 2.

These results are used in the last section to establish a local symplectic classification of special MAE. We adapt here V.L and V.R's results ([11]) to our case which is less general.

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1 Formulation of the problem

Let (V, Ω) be a symplectic $2n$-dimensional vector space over $\mathbb{R}$. Denote by $\Gamma : V \to V^*$ the isomorphism determined by Ω. Let $X_\Omega \in \Lambda^2(V)$ be the unique bivector which satisfies $\Gamma^2(X_\Omega) = \Omega$, where $\Gamma^2 : \Lambda^2(V) \to \Lambda^2(V^*)$ is the exterior power of Γ [1].

One can introduce the operators $\perp : \Lambda^k(V^*) \to \Lambda^{k-2}(V^*)$, $\omega \mapsto i_{X_\Omega}\omega$ and $\top : \Lambda^k(V^*) \to \Lambda^{k+2}(V^*)$, $\omega \mapsto \omega \wedge \Omega$ (see [10]). They have the following properties:

$$\begin{cases} [\perp, \top](\omega) = (n-k)\omega \ , \ \forall \omega \in \Lambda^k(V^*); \\ \perp : \Lambda^k(V^*) \to \Lambda^{k-2}(V^*) \text{ is into for } k \geq n+1; \\ \top : \Lambda^k(V^*) \to \Lambda^{k+2}(V^*) \text{ is onto for } k \leq n-1. \end{cases}$$

We will say that a k-form ω is effective if $\perp\omega = 0$ and we will denote by $\Lambda^k_\varepsilon(V^*)$ the vector space of effective k-forms on V. When $k = n$, ω is effective if and only if $\omega \wedge \Omega = 0$.

The next theorem explains the fundamental role played by the effective forms in the theory of Monge-Ampère operators (see [10]):

Theorem 1 (Hodge-Lepage-Lychagin) *(1) Every form $\omega \in \Lambda^k(V^*)$ can be uniquely decomposed into the finite sum*

$$\omega = \omega_0 + \top\omega_1 + \top^2\omega_2 + \dots ,$$

where all ω_i are effective forms.

(2) If two effective k-forms vanish on the same k-dimensional isotropic vector subspaces in (V, Ω), they are proportional.

Let M be an n-dimensional smooth manifold. Denote by J^1M the space of 1-jets of smooth functions on M. Let $j^1(f) : M \to J^1M, x \mapsto [f]^1_x$ be the natural section associated with the smooth function f on M. The Monge-Ampère operator $\Delta_\omega : C^\infty(M) \to \Omega^n(M)$ associated to a differential n-form $\omega \in \Omega^n(J^1M)$ is the differential operator defined by

$$\Delta_\omega(f) = j_1(f)^*(\omega)$$

[1] We denote by $\Lambda^*(V^*)$ the space of exterior forms on a vector space V and by $\Omega^*(X)$ the space of differential forms on a manifold X

Let U be the contact form on J^1M and X_1 be the Reeb's vector field. Denote $C(x) = Ker(U_x)$ for $x \in J^1M$. Note that $(C(x), dU_x)$ is a $2n$-dimensional symplectic vector space and

$$T_xJ^1M = C(x) \oplus \mathbb{R}X_{1x}.$$

A solution of the equation $\Delta_\omega = 0$ is a legendrian submanifold L^{2n} of (J^1M, U) such that $\omega|_L = 0$. Note that in each point $x \in L$, T_xL is a lagrangian subspace of $(C(x), dU_x)$. If the projection $\pi : J^1M \to M$ is a local diffeomorphism on L then L is locally the graph of a section $j^1(f)$, where f is a regular solution of the equation $\Delta_\omega(f) = 0$.

We will denote by $\Omega^*(C^*)$ the space of differential forms vanishing on X_1. In each point x, $(\Omega^k(C^*))_x$ can be naturally identified with $\Lambda^k(C(x)^*)$. Let $\Omega^*_\varepsilon(C^*)$ be the space of forms which are effective on $(C(x), dU_x)$ in each point $x \in J^1M$. The first part of the theorem 1 means that

$$\Omega^*_\varepsilon(C^*) = \Omega^*(J^1M)/I_C,$$

where I_C is the Cartan ideal generated by U and dU. The second part means that two differential n-forms ω and θ on J^1M determine the same Monge-Ampère operator if and only if $\omega - \theta \in I_C$.

$Ct(M)$, the pseudo-group of contact diffeomorphisms on J^1M, naturally acts on the set of Monge-Ampère operators in the following way

$$F(\Delta_\omega) = \Delta_{F^*(\omega)},$$

and the corresponding infinitesimal action is

$$X(\Delta_\omega) = \Delta_{L_X(\omega)}.$$

We are interested in the problem of local classification of Monge-Ampère operators in the dimension 3 with respect to the action of the contact group. More precisely, we are interested in the symplectic operators, i.e., operators which satisfy

$$X_1(\Delta_\omega) = \Delta_{L_{X_1}(\omega)} = 0.$$

Let T^*M be the cotangent space and Ω be the canonical symplectic form on it. Let us consider the projection $\beta : J^1M \to T^*M$, defined by the following commutative diagram:

$$\begin{tikzcd}
\mathbb{R} & J^1M \arrow[l, "\alpha"'] \arrow[r, "\beta"] & T^*M \\
& M \arrow[ul, "f"] \arrow[u, "j^1(f)"] \arrow[ur, "df"'] &
\end{tikzcd}$$

Using β, we can naturally identify the space $\{\omega \in \Omega^*_\varepsilon(C^*) : L_{X_1}\omega = 0\}$ with the space of effective forms on (T^*M, Ω). Therefore, to classify the different orbits of symplectic Monge-Ampère operators on a smooth manifold M^n with respect to the contact group, it is sufficient to classify the different orbits of the effective n-forms on the cotangent space T^*M with respect to the symplectic group.

2 Symplectic classification of effective 3-forms in the dimension 6

Let (V, Ω) be a $2n$-dimensional symplectic vector space over $\mathbb{R}$. We will denote the vectors of V by block letters and their images by Γ by small letters: if $A \in V$ then $a \in V^*$ is defined by

$$a(B) = \Omega(A, B).$$

A basis $(A_1, \ldots, A_n, B_1, \ldots, B_n)$ of V will be called symplectic if

$$\Omega = a_1 \wedge b_1 + \ldots + a_n \wedge b_n.$$

2.1 A recursive formula

Let $\omega \in \Lambda^n_\varepsilon(V^*)$. Choose two vectors A and B such that $\Omega(A, B) = 1$ and let W be the skew orthogonal to $\mathbb{R}A \oplus \mathbb{R}B$ with respect to Ω. Denote by Ω' the restriction of Ω on W and by $\top'$ and $\perp'$ the corresponding operators.

The form ω can be uniquely decomposed in the following way

$$\omega = \omega_0 \wedge a \wedge b + \omega_1 \wedge a + \omega_2 \wedge b + \omega_3,$$

with $\omega_i \in \Lambda^*(W^*)$. Moreover, ω is effective, therefore we obtain:

(1) $\perp'\omega_0 = \perp'\omega_1 = \perp'\omega_2 = 0$
(2) $\omega_3 = -\top'\omega_0 = -\omega_0 \wedge \Omega'$

Proposition 2 *In the symplectic decomposition $V = W \oplus (\mathbb{R}A \oplus \mathbb{R}B)$, ω can be expressed in a unique way:*

$$\omega = \omega_0 \wedge (a \wedge b - \Omega') + \omega_1 \wedge a + \omega_2 \wedge b,$$

where ω_0, ω_1 and ω_2 are effective on (W, Ω').

From this proposition it follows that

$$\Lambda^n_\varepsilon(\mathbb{R}^{2n}) = \Lambda^{n-2}_\varepsilon(\mathbb{R}^{2(n-1)}) \oplus \Lambda^{n-1}_\varepsilon(\mathbb{R}^{2(n-1)}) \oplus \Lambda^{n-1}_\varepsilon(\mathbb{R}^{2(n-1)}),$$

and in particular,

$$\Lambda^3_\varepsilon(\mathbb{R}^6) = \mathbb{R}^4 \oplus \Lambda^2_\varepsilon(\mathbb{R}^4) \oplus \Lambda^2_\varepsilon(\mathbb{R}^4).$$

Consequently, the dimension of $\Lambda^3_\varepsilon(\mathbb{R}^6)$ is $4+5+5=14$. It is worth mentioning that the space $\Lambda^2_\varepsilon(\mathbb{R}^6)$ is also 14-dimensional, since it is the euclidean orthogonal to Ω in the Lie algebra $so(6)$ with respect to the canonical scalar product.

2.2 Effective 2-forms in the dimension 4

The symplectic classification of the effective 2-forms in the dimension 4 is well-known (see for instance [12]):

Proposition 3 *Let (V,Ω) be a 4-dimensional symplectic vector space and $\omega \in \Lambda^2_\varepsilon(V^*)$. If $\omega \neq 0$ then there exists a symplectic basis (A_1, A_2, B_1, B_2) of V such that:*

(1) $\omega = \mu(a_1 \wedge a_2 - b_1 \wedge b_2)$, $\mu \in \mathbb{R}^*$, *if ω is elliptic, i.e.,* $\mathrm{pf}(\omega) > 0$;
(2) $\omega = \mu(a_1 \wedge a_2 + b_1 \wedge b_2)$, $\mu \in \mathbb{R}^*$, *if ω is hyperbolic, i.e.,* $\mathrm{pf}(\omega) < 0$;
(3) $\omega = a_1 \wedge a_2$, *if ω is parabolic, i.e.,* $\mathrm{pf}(\omega) = 0$.

Here the pfaffian $\mathrm{pf}(\omega)$ *is the symplectic invariant defined by*

$$\omega \wedge \omega = \mathrm{pf}(\omega)\Omega \wedge \Omega.$$

Note that this proposition can be immediately deduced from (2).

2.3 Effective 3-forms in the dimension 6

Now let $\omega \in \Lambda^3_\varepsilon(V^6)$ be an effective 3-form in the dimension 6.

Following V. Lychagin and V. Roubtsov ([11]), let us associate to ω the quadratic form $q_\omega \in S^2(V^*)$:

$$q_\omega(X) = -\frac{1}{4}\perp^2 \omega_X^2,$$

with $\omega_X = i_X\omega \in \Lambda^2(V^*)$.

In fact, this form gives us the roots of the characteristic polynom of ω:

$$P_{\omega_X}(\lambda) = \frac{(\omega_X - \lambda\Omega)^3}{\Omega^3} = \lambda(\lambda - \sqrt{q_\omega(X)})(\lambda + \sqrt{q_\omega(X)}).$$

Moreover, if $F \in Sp(3)$ then one has

$$q_{F^*\omega}(X) = q_\omega(F^{-1}X).$$

This invariant provides us with some information about the decomposition of ω (proposition 2). More precisely, if in the symplectic decomposition $V = W \oplus (\mathbb{R}A \oplus \mathbb{R}B)$

$$\omega = \omega_0 \wedge (a \wedge b - \Omega') + \omega_1 \wedge a + \omega_2 \wedge b$$

holds, then

$$\begin{cases} q_\omega(A) = -\frac{1}{4}\bot^2(\omega_2 \wedge \omega_2) \\ q_\omega(A,B) = \frac{1}{4}\bot^2(\omega_1 \wedge \omega_2) \end{cases}$$

(we denote the symmetric bilinear form associated with q_ω also by q_ω).

Therefore, one obtains:

$$\begin{cases} q_\omega(A) = 0 \Leftrightarrow \ \omega_2 \text{ is degenerate on } W \\ q_\omega(A,B) = 0 \Leftrightarrow \omega_1 \wedge \omega_2 = 0 \end{cases} \tag{1}$$

Now let us consider two distinct cases: $q_\omega = 0$ and $q_\omega \neq 0$.

2.3.1 $q_\omega = 0$

Lemma 4 *If $\omega \in \Lambda^3_\varepsilon(V^6)$ satisfies $\omega \wedge \omega_X = 0$ for all $X \in V$, then there exists $X \in V - \{0\}$ such that $\omega_X = 0$.*

Proof. Let $A \in V - \{0\}$: $\omega \wedge \omega_A = 0$. This leads to $\omega_A \wedge \omega_A = 0$ and, therefore, ω_A must have the form $\omega_A = \theta_1 \wedge \theta_2$. Consequently, $dim\{B : \omega_{A\wedge B} = 0\} \geq 4$ and there exists $B \in V$ such that $\Omega(A,B) = 1$ and $\omega_{A\wedge B} = 0$. Therefore, in the symplectic decomposition $V = W \oplus (\mathbb{R}A \oplus \mathbb{R}B)$ we obtain

$$\omega = \omega_1 \wedge a + \omega_2 \wedge b.$$

Moreover, $q_\omega(B) = 0$, so $\omega_1 \wedge \omega_1 = 0$ and then $\omega_1 = a_1 \wedge a_2$ with $A_1, A_2 \in W$. Similarly, $\omega_2 = b_1 \wedge b_2$ with $B_1, B_2 \in W$. Since both ω_1 and ω_2 are effective, $\Omega(A_1,A_2) = \Omega(B_1,B_2) = 0$. We can suppose, for example, that $A_1 \neq 0$. If $\omega_{A_1} = 0$ then the proof is finished. If not, since $\omega_{A_1} = \Omega(B_1,A_1)b_2 \wedge b - \Omega(B_2,A_1)b_1 \wedge b$, we can suppose that $\Omega(B_1,A_1) \neq 0$. Denote then:

$$B_2' = B_2 - \frac{\Omega(B_2,A_1)}{\Omega(B_1,A_1)}B_1$$

Since $\Omega(B_2', A_1) = 0$, $\omega_{A_1} = \Omega(B_1, A_1) b_2' \wedge b$. Then necessarily $B_2' \neq 0$. Recall that $\omega \wedge \omega_{A_1} = 0$ so we have $a_1 \wedge a_2 \wedge b_2' = 0$, i.e. $B_2' = \alpha_1 A_1 + \alpha_2 A_2$ and then,

$$\Omega(B_2', A_1) = \Omega(B_2', A_2) = \Omega(B_2', B_1) = 0.$$

This implies that $\omega_{B_2'} = 0$. $\Box$

Proposition 5 *Let $\omega \in \Lambda^3_\varepsilon(V^6)$ with $q_\omega = 0$ and $\omega \neq 0$. Then there exists a symplectic basis $(A_1, A_2, A_3, B_1, B_2, B_3)$ of (V, Ω) in which*

$$\omega = a_1 \wedge a_2 \wedge a_3.$$

Proof.

For all X and Y, $\perp^2(\omega_X \wedge \omega_Y) = 0$. Consequently, $\forall X, Y \in V$, $\perp^2 \top(\omega_X \wedge \omega_Y) = \top \perp^2 \omega_X \wedge \omega_Y = 0$. Recall that $\perp : \Lambda^{3+i}(V^6) \to \Lambda^{1+i}(V^6)$ is into. This leads to the relation

$$\Omega \wedge \omega_X \wedge \omega_Y = 0,$$

which holds for all $X, Y \in V$. Note that $\Omega \wedge \omega_X = -\Omega_X \wedge \omega$ since ω is effective. Therefore, $\Omega_X \wedge \omega \wedge \omega_Y = 0$, for all $X, Y \in V$ and $\omega \wedge \omega_Y = 0$ holds for all $Y \in V$. Applying lemma 4, we can deduce that there exists $A \in V - \{0\}$ such that $\omega_A = 0$. Let then $B \in V$ be such that $\Omega(A, B) = 1$. In the symplectic decomposition $W \oplus (\mathbb{R}A \oplus \mathbb{R}B)$ we obtain

$$\omega = \omega_1 \wedge a,$$

where ω_1 is an effective form on W. Moreover, ω_1 is parabolic on W, since $q_\omega(B) = 0$. From 3 one can conclude that there exists a symplectic basis (A_1, A_2, B_1, B_2) of W in which

$$\omega_1 = a_1 \wedge a_2.$$

$\Box$

2.3.2 $q_\omega \neq 0$

Consider a non-zero vector $A \in V$ and denote $E_A = Ker(\omega_A : V \to V^*)$.

Lemma 6 *If $q_\omega(A) \neq 0$ then $dim(E_A) = 2$.*

Proof. ω_A is non-degenerate on any subspace Z of V such that $V = Z \oplus E_A$. Therefore, $dim(E_A)$ must be even. Moreover, $\omega_A \wedge \omega_A \in \Lambda^4(Z^*)$ is not zero, so $dim(E_A) \leq 2$. Since $A \in E_A$, $dim(E_A) = 2$. $\Box$

Lemma 7 *If $q_\omega(A) \neq 0$ then E_A is not isotropic.*

Proof. Let B be a vector satisfying $\Omega(A,B)=1$ and let $\omega = \omega_0 \wedge (a \wedge b - \Omega') + \omega_1 \wedge a + \omega_2 \wedge b$ be the symplectic decomposition of ω in $W \oplus (\mathbb{R}A \oplus \mathbb{R}B)$. If $C \in E_A$ with $A \wedge B \wedge C \neq 0$, we can write $C = D + \alpha A + \beta B$, $D \in W - \{0\}$. Then, if we assume that $\beta = 0$ for such a C, it is straightforward to obtain

$$0 = i_{A\wedge C}\omega = -i_C(\omega_0 \wedge a - \omega_2) = -\omega_0(D)a - i_D\omega_2,$$

and therefore, $i_D\omega_2 = 0$. However, since $q_\omega(A) \neq 0$, ω_2 is non-degenerate on W and then $D = 0$. It is impossible, so we can conclude that $\Omega(A,C) = \beta \neq 0$. $\square$

Choose then $B_0 \in E_A$ such that $\Omega(A,B)=1$ and denote $B = B_0 + tA$, with $t = -\frac{q_\omega(A,B_0)}{q_\omega(A)}$. In the symplectic decomposition $W \oplus (\mathbb{R}A \oplus \mathbb{R}B)$ we have

$$\omega = \omega_1 \wedge a + \omega_2 \wedge b,$$

where ω_1 and ω_2 are effective on W and ω_1 is non-degenerate. At last, $q_\omega(A,B)$ equals to zero and so does $\omega_1 \wedge \omega_2$. All this can be summarized to the following

Proposition 8 *Let $\omega \in \Lambda^3_\varepsilon(V^6)$ such that $q_\omega \neq 0$. There exists a symplectic decomposition $V = W \oplus (\mathbb{R}A \oplus \mathbb{R}B)$ in which ω can be written as*

$$\omega = \omega_1 \wedge a + \omega_2 \wedge b$$

Moreover, ω_1 and ω_2 are effective on the symplectic space (W,Ω'), while ω_2 and Ω' are effective on the symplectic space (W,ω_1).

Since Ω' is non-degenerate, Ω' is hyperbolic or elliptic on (W,ω_1). ω_2 can be hyperbolic, elliptic or parabolic. A careful study of these six cases allows us to obtain all possible orbits. This study is a little bit long and tiresome, so we will only report the details of one case, which is most exemplary [2]: namely, the case of elliptic ω_2 and Ω'.

If ω_2 is elliptic, then there exists a symplectic basis (A_1,A_2,B_1,B_2) of (W,ω_1) in which

$$\omega_2 = \lambda(a_1 \wedge b_2 - a_2 \wedge b_1),\ \lambda \neq 0.$$

After observing that $\Omega' \wedge \omega_1 = \Omega' \wedge \omega_2 = 0$, we can conclude that

$$\Omega' = pa_1 \wedge a_2 + qb_1 \wedge b_2 + r(a_1 \wedge b_2 + a_2 \wedge b_1) + s(a_1 \wedge b_1 - a_2 \wedge b_2).$$

[2] See ([11]) for the others

Note that $pq+r^2+s^2<0$ since Ω' is elliptic. In particular, q cannot be equal to zero. Let A_t and B_t be the transformations that depend on the real parameter t

$$A_t = \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B_t = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & -t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(A_t and B_t are written in the basis (A_1, A_2, B_1, B_2)). A_t and B_t fix ω_1, ω_2 and act on Ω in the following way:

$$\begin{cases} (p,q,r,s) \xrightarrow{A_t} (p-qt^2-2st, q, r, s+qt) \\ (p,q,r,s) \xrightarrow{B_t} (p-qt^2+2rt, q, r, r-qt) \end{cases}$$

Let us apply the transformation B_u and then the transformation A_v with $u=-\frac{r}{q}$ and $v=-\frac{s}{q}$. In the new basis we will obtain:

$$\begin{cases} \omega_1 = a_1\wedge b_1 + a_2\wedge b_2, \\ \omega_2 = \lambda(a_1\wedge b_2 - a_2\wedge b_1), \quad \lambda\neq 0, \\ \Omega' = pa_1\wedge a_2 + qb_1\wedge b_2, \qquad pq<0. \end{cases}$$

After applying the next transformation

$$F = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix},$$

with $e^{4t} = -\frac{q}{p} > 0$, we will get the following expression for Ω':

$$\Omega' = \mu(a_1\wedge a_2 - b_1\wedge b_2)$$

(note that F does not change ω_1 and ω_2). In the symplectic basis $(A'_1 = A, A'_2 = \mu A_1, A'_3 = \mu B_1, B'_1 = B, B'_2 = A_2, B'_3 = -B_2)$ of (V,Ω) one obtains

$$\omega = \frac{1}{\mu^2}a'_1\wedge a'_2\wedge a'_3 - a'_1\wedge b'_2\wedge b'_3 - \frac{\lambda}{\mu}b'_1\wedge a'_2\wedge b'_3 - \frac{\lambda}{\mu}b'_1\wedge b'_2\wedge a'_3.$$

In the symplectic basis $(A_1, A_2, A_3, B_1, B_2, B_3)$, where

$$\begin{cases} A_1 = \mu A'_1, & B_1 = \frac{1}{\mu}B'_1, \\ A_2 = \mu\nu A'_2, & B_2 = \frac{1}{\mu\nu}B'_2, \\ A_3 = \nu B'_3, & B_3 = -\frac{1}{\nu}A'_3, \end{cases}$$

with $\frac{\mu}{\lambda} = \varepsilon\nu^2$, $\varepsilon = \pm 1$, we have

$$\omega = b_3 \wedge a_1 \wedge a_2 - b_2 \wedge a_1 \wedge a_3 + \varepsilon b_1 \wedge a_2 \wedge a_3 - \varepsilon\xi^2 b_1 \wedge b_2 \wedge b_3.$$

Similar results concerning other cases lead us to the

Theorem 9 *Let $(e_1, e_2, e_3, f_1, f_2, f_3)$ be a symplectic basis of a 6 dimensional symplectic vector space V. Every effective 3-form on V is $Sp(3)$-equivalent to one and only one form of the table 1.*

N^o	Form	Invariant
1.	$e_1^* \wedge e_2^* \wedge e_3^* + \gamma f_1^* \wedge f_2^* \wedge f_3^*, \gamma \neq 0$	$2q_\omega = \gamma(e_1^* f_1^* + e_2^* f_2^* + e_3^* f_3^*)$
2.	$f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_3^*$ $+ f_3^* \wedge e_1^* \wedge e_2^* + \nu^2 f_1^* \wedge f_2^* \wedge f_3^*, \nu \neq 0$	$q_\omega = (e_1^*)^2 - (e_2^*)^2 + (e_3^*)^2$ $+\nu^2((f_1^*)^2 - (f_2^*)^2 + (f_3^*)^2)$
3.	$f_1^* \wedge e_2^* \wedge e_3^* - f_2^* \wedge e_1^* \wedge e_3^*$ $+ f_3^* \wedge e_1^* \wedge e_2^* - \nu^2 f_1^* \wedge f_2^* \wedge f_3^*, \nu \neq 0$	$q_\omega = -(e_1^*)^2 - (e_2^*)^2 - (e_3^*)^2$ $+\nu^2(-(f_1^*)^2 - (f_2^*)^2 - (f_3^*)^2)$
4.	$f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_3^* + f_3^* \wedge e_1^* \wedge e_2^*$	$q_\omega = (e_1^*)^2 - (e_2^*)^2 + (e_3^*)^2$
5.	$f_1^* \wedge e_2^* \wedge e_3^* - f_2^* \wedge e_1^* \wedge e_3^* + f_3^* \wedge e_1^* \wedge e_2^*$	$q_\omega = -(e_1^*)^2 - (e_2^*)^2 - (e_3^*)^2$
6.	$f_3^* \wedge e_1^* \wedge e_2^* + f_2^* \wedge e_1^* \wedge e_3^*$	$q_\omega = (e_1^*)^2$
7.	$f_3^* \wedge e_1^* \wedge e_2^* - f_2^* \wedge e_1^* \wedge e_3^*$	$q_\omega = -(e_1^*)^2$
8.	$e_1^* \wedge e_2^* \wedge e_3^*$	$q_\omega = 0$
9.	0	$q_\omega = 0$

Table 1
Classification of the effective 3-forms in the dimension 6

Note that we have abandoned the notation *block letters* $\xrightarrow{\Gamma}$ *small letters*. In what follows, it is more convenient to use the dual basis $(e_1^*, e_2^*, e_3^*, f_1^*, f_2^*, f_3^*)$. There is an obvious correspondence between these two notations:

$$\begin{cases} \Gamma(e_i) = f_i^* \\ \Gamma(f_i) = -e_i^* \end{cases}$$

3 The stabilizers of orbits and their prolongation

3.1 Stabilizers of the orbits

Let us denote $sp(V) = sp(6)$ the Lie algebra of $Sp(V)$ and consider its action on $\Lambda^*(V^*)$, induced by the action of $Sp(V)$:

$$X.\omega = L_X \omega$$

We compute here the stabilizers of the most significant orbits. Recall that the stabilizer of a form ω is:

$$\mathcal{J}_\omega = \{X \in sp(V) : L_X \omega = 0\}.$$

Our forms have constant coefficients, therefore $L_X\omega = d(i_X\omega)$. Straightforward computation (made with help of Maple V) shows the following

Proposition 10 *The stabilizers of forms* 1 *to* 5 *listed in table* 1 *are:*

(1) $\omega = e_1^* \wedge e_2^* \wedge e_3^* + \gamma f_1^* \wedge f_2^* \wedge f_3^*,\ \gamma \neq 0$

$$\mathcal{J}_\omega = \{ \begin{pmatrix} B & 0 \\ 0 & -B^t \end{pmatrix} : B \in sl(3,\mathbb{R})\};$$

(2) $\omega = f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_3^* + f_3^* \wedge e_1^* \wedge e_2^* + \nu^2 f_1^* \wedge f_2^* \wedge f_3^*,\ \nu \neq 0$

$$\mathcal{J}_\omega = \{ \begin{pmatrix} 0 & \alpha & \beta & \lambda_1 & \xi_1 & \xi_2 \\ \alpha & 0 & \gamma & \xi_1 & \lambda_2 & \xi_3 \\ -\beta & \gamma & 0 & \xi_2 & \xi_3 & \lambda_3 \\ -\nu^2\lambda_1 & \nu^2\xi_1 & -\nu^2\xi_2 & 0 & -\alpha & \beta \\ \nu^2\xi_1 & -\nu^2\lambda_2 & \nu^2\xi_3 & -\alpha & 0 & -\gamma \\ -\nu^2\xi_2 & \nu^2\xi_3 & -\nu^2\lambda_3 & -\beta & -\gamma & 0 \end{pmatrix} : \lambda_1 - \lambda_2 + \lambda_3 = 0\};$$

(3) $\omega = f_1^* \wedge e_2^* \wedge e_3^* - f_2^* \wedge e_1^* \wedge e_3^* + f_3^* \wedge e_1^* \wedge e_2^* - \nu^2 f_1^* \wedge f_2^* \wedge f_3^*,\ \nu \neq 0$

$$\mathcal{J}_\omega = \{ \begin{pmatrix} 0 & \alpha & \beta & \lambda_1 & \xi_1 & \xi_2 \\ -\alpha & 0 & \gamma & \xi_1 & \lambda_2 & \xi_3 \\ -\beta & -\gamma & 0 & \xi_2 & \xi_3 & \lambda_3 \\ -\nu^2\lambda_1 & -\nu^2\xi_1 & -\nu^2\xi_2 & 0 & \alpha & \beta \\ -\nu^2\xi_1 & -\nu^2\lambda_2 & -\nu^2\xi_3 & -\alpha & 0 & \gamma \\ -\nu^2\xi_2 & -\nu^2\xi_3 & -\nu^2\lambda_3 & -\beta & -\gamma & 0 \end{pmatrix} : \lambda_1 + \lambda_2 + \lambda_3 = 0\};$$

(4) $\omega = f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_3^* + f_3^* \wedge e_1^* \wedge e_2^*$

$$\mathcal{J}_\omega = \{ \begin{pmatrix} 0 & \alpha & \beta & \lambda_1 & \xi_1 & \xi_2 \\ \alpha & 0 & \gamma & \xi_1 & \lambda_2 & \xi_3 \\ -\beta & \gamma & 0 & \xi_2 & \xi_3 & \lambda_3 \\ 0 & 0 & 0 & 0 & -\alpha & \beta \\ 0 & 0 & 0 & -\alpha & 0 & -\gamma \\ 0 & 0 & 0 & -\beta & -\gamma & 0 \end{pmatrix} : \lambda_1 - \lambda_2 + \lambda_3 = 0\};$$

(5) $\omega = f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_3^* - f_3^* \wedge e_1^* \wedge e_2^*$

$$\mathcal{J}_\omega = \{ \begin{pmatrix} 0 & \alpha & \beta & \lambda_1 & \xi_1 & \xi_2 \\ \alpha & 0 & \gamma & \xi_1 & \lambda_2 & \xi_3 \\ \beta & -\gamma & 0 & \xi_2 & \xi_3 & \lambda_3 \\ 0 & 0 & 0 & 0 & -\alpha & -\beta \\ 0 & 0 & 0 & -\alpha & 0 & \gamma \\ 0 & 0 & 0 & -\beta & -\gamma & 0 \end{pmatrix} : \lambda_1 - \lambda_2 - \lambda_3 = 0\};$$

3.2 Prolongation of stabilizers

We are interested now in the prolongation of these stabilizers. The prolongation of a linear subspace $\mathcal{J}$ of $Hom(V, W)$ is ([3]):

$$\mathcal{J}^{(1)} = \{T \in Hom(V, \mathcal{J}) : Tuv = Tvu \ \forall u, v \in V\} = (W \otimes S^2(V^*)) \cap (\mathcal{J} \otimes V^*).$$

In our case $V = W$ and $\mathcal{J} = \mathcal{J}_\omega$ is a subspace of $sp(V)$. An element $\theta \in \mathcal{J}_\omega \otimes V^* \subset sp(V) \otimes V^* \subset V \otimes V^* \otimes V^*$ can be described as follows:

$$\theta = \sum_{i,j,k=1}^{3} b_{ij}^k e_i \otimes e_j^* \otimes e_k^* - a_{ij}^k e_i \otimes f_j^* \otimes e_k^* + c_{ij}^k f_i \otimes e_j^* \otimes e_k^* - b_{ji}^k f_i \otimes f_j^* \otimes e_k^*$$
$$+ b_{ij}^{k+3} e_i \otimes e_j^* \otimes f_k^* - a_{ij}^{k+3} e_i \otimes f_j^* \otimes f_k^* + c_{ij}^{k+3} f_i \otimes e_j^* \otimes f_k^* - b_{ji}^{k+3} f_i \otimes f_j^* \otimes f_k^*$$

with $\begin{pmatrix} B_k & -A_k \\ C_k & -B_k^t \end{pmatrix} \in \mathcal{J}_\omega \subset sp(V)$ for $k = 1 \ldots 6$. Note that $\theta \in \mathcal{J}_\omega^{(1)}$ if and only if

$$\begin{cases} \theta(e_j, e_k) = \theta(e_k, e_j), \\ \theta(f_j, f_k) = \theta(f_k, f_j), \\ \theta(e_j, f_k) = \theta(f_k, e_j). \end{cases}$$

If we take into account the next four relations

$$\begin{cases} \theta(e_j, e_k) = \sum\limits_{i=1}^{3} b_{ij}^{k} e_i + c_{ij}^{k} f_i, \\ \theta(f_j, f_k) = -\sum\limits_{i=1}^{3} a_{ij}^{k+3} e_i + b_{ji}^{k+3} f_i, \\ \theta(e_j, f_k) = \sum\limits_{i=1}^{3} b_{ij}^{k+3} e_i + c_{ij}^{k+3} f_i, \\ \theta(f_k, e_j) = -\sum\limits_{i=1}^{3} a_{ik}^{j} e_i + b_{ki}^{j} f_i, \end{cases}$$

we can conclude that $\theta \in \mathcal{J}_{\omega}^{(1)}$ if and only if for all $i, j, k = 1 \dots 3$ the equalities

$$\begin{aligned} b_{ij}^{k} &= b_{ik}^{j}, \\ c_{ij}^{k} &= c_{ik}^{j}, \\ a_{ij}^{k+3} &= a_{ik}^{j+3}, \\ b_{ji}^{k+3} &= b_{ki}^{j+3}, \\ b_{ij}^{k+3} &= -a_{ik}^{j}, \\ b_{ki}^{j} &= -c_{ij}^{k+3}, \end{aligned}$$

are satisfied. This allows us to check that $\mathcal{J}_{\omega}^{(1)} = 0$ for the five first forms $\omega_1, \dots, \omega_5$ listed in the table 1. These results are summed up in the table 2.

N^o	$\mathcal{J}_{\omega}$	$\mathcal{J}_{\omega}^{(1)}$
1.	$sl(3, \mathbb{R})$	0
2.	$su(2, 1)$	0
3.	$su(3)$	0
4.	$H_2(2, 1) \rtimes_{\alpha} so(2, 1)$	0
5.	$H_2(1, 2) \rtimes_{\alpha} so(1, 2)$	0

Table 2
Stabilizers of the effective 3-forms in the dimension 6 and their prolongation

4 Local symplectic classification of special Monge-Ampère equations

Now we are going to establish the conditions under which an effective n-form $\omega_2 \in \Omega_{\varepsilon}^{n}(T^*M)$ with non-constant coefficients is in the same orbit as an effective n-form $\omega_1 \in \Omega^n_\varepsilon(T^*M)$ with constant coefficients.

This problem is equivalent to the resolution of the differential equation $\Sigma \subset J^1(2n,2n)$ defined by

$$\Sigma = \{[F]^1_q : [F^*\omega_1 - \omega_2]^0_q = 0 \text{ and } [F^*\Omega - \Omega]^0_q = 0\}.$$

Recall that $J^1(m,m)$ is the space of 1-jets of smooth maps $\mathbb{R}^m \to \mathbb{R}^m$ with the canonical system of coordinates (q,u,p):

$$\begin{cases} q_i([F]^1_q) = q_i & i = 1 \dots m; \\ u_i([F]^1_q) = F_i(q) & i = 1 \dots m; \\ p_{ij}([F]^1_q) = \frac{\partial F_i}{\partial q_j}(q) & i,j = 1 \dots m; \end{cases}$$

4.1 Symbol of Σ

To simplify the notations, we put $m = 2n$. One can write

$$\Sigma = \{L_1 = \dots = L_r = L_{r+1} = \dots L_{r+s} = 0\} \subset J^1(m,m),$$

with

$$\begin{cases} [F^*\omega_1 - \omega_2]^0_q = \sum\limits_{i=1}^{r} L_i([F]^1_q)\alpha_i, & \{\alpha_i\}_{i=1\dots r} \text{ basis of } \Lambda^n(\mathbb{R}^m) \\ [F^*\Omega - \Omega]^0_q = \sum\limits_{i=1}^{s} L_{r+i}([F]^1_q)\beta_i, & \{\beta_i\}_{i=1\dots s} \text{ basis of } \Lambda^2(\mathbb{R}^m) \end{cases}$$

Let $\theta = [F]^1_{q_0} \in \Sigma$. The symbol $g(\theta)$ of Σ in θ is the set of $h = (h_{ij}) \in \mathbb{R}^m \otimes \mathbb{R}^m$, satisfying the relation

$$\sum_{i,j=1}^{m} h_{ij} \frac{\partial L_k}{\partial p_{ij}}(\theta) = 0 \tag{2}$$

for all $k = 1 \dots r+s$. Let $h \in g(\theta)$. Define $H : \mathbb{R}^m \to \mathbb{R}^m$ by

$$H(q) = (\sum_{j=1}^{m} h_{1j}(q_j - q^0_j), \dots, \sum_{j=1}^{m} h_{mj}(q_j - q^0_j))$$

and put $\phi_t([G]^1_q) = [G + tH]^1_q$, $\forall t \in \mathbb{R}$. From (2) one can deduce the relations

$$\frac{d}{dt} L_k \circ \phi_t(\theta)|_{t=0} = 0,$$

which hold for $k = 1 \dots r$. Taking into account that $F_t = F + tH$ and $q_1 = F(q_0)$, we obtain

$$\begin{cases} \lim\limits_{t\to 0} \frac{(T_{q_0}F_t)^*\omega_{1,q_1} - \omega_{2,q_0}}{t} = 0 \\ \lim\limits_{t\to 0} \frac{(T_{q_0}F_t)^*\Omega_{q_1} - \Omega_{q_0}}{t} = 0 \end{cases}$$

In other words,

$$\begin{cases} \lim\limits_{t\to 0} \frac{\psi_t^*\omega_1 - \psi_0^*\omega_1}{t} = 0 \\ \lim\limits_{t\to 0} \frac{\psi_t^*\Omega - \psi_0^*\Omega}{t} = 0 \end{cases}$$

where the linear map $\psi_t : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$\psi_{ij}^t = \frac{\partial F_i}{\partial q_j}(q_0) + th_{ij}.$$

Therefore, $L_X\omega_1 = L_X\Omega = 0$ with $X = (h_{ij})$. This proves the following

Proposition 11 *For all $\theta \in \Sigma$ the symbol of Σ in θ can be naturally identified with the stabilizer of ω_1:*

$$g(\theta) = \mathcal{J}_{\omega_1}$$

Remark 12 *Note that the prolongation of the symbol coincides with the prolongation of the Lie subalgebra $\mathcal{J}_{\omega_1}$:*

$$g^{(k)}(\theta) = \mathcal{J}_{\omega_1}^{(k)},\ \forall k \in \mathbb{N}.$$

4.2 The bundle $\Sigma^{(k)} \to \Sigma^{(k-1)}$

Let us find the obstructions for the map $\pi_{k+1,k} : \Sigma^{(k)} \to \Sigma^{(k-1)}$ to be surjective. In our case $\Sigma^{(k)}$ is the differential equation

$$\Sigma^{(k)} = \{[F]_q^{k+1} : [F^*\omega_1 - \omega_2]_q^k = 0 \text{ and } [F^*\Omega - \Omega]_q^k = 0\}.$$

Let $\theta = [F]_{q_0}^k \in \Sigma^{(k-1)}$. If we introduce the notation

$$\sigma_k(F) = [\omega_1 - F^{-1*}\omega_2]_{q_1}^k$$

and

$$\sigma_k(\theta) = \sigma_k(F) \text{ modulo } Im(c_k),$$

where $c_k : S^{k+2}(\mathbb{R}^m) \to \Omega^n(\mathbb{R}^m)$ is defined by

$$c_k(h) = L_{X_h}(\omega_1),$$

we can formulate the next

Proposition 13 *Let* $\theta = [F]^k_{q_0} \in \Sigma^{(k-1)}$*. The intersection* $\pi^{-1}_{k+1,k}(\theta) \cap \Sigma^{(k)}$ *is not empty if and only if* $\sigma_k(\theta) = 0$*.*

Proof. We can assume that F is a polynomial map of degree less than k. Denote $q_1 = F(q_0)$. We assume first that there exists $\theta' = [G]^{k+1}_{q_0} \in \Sigma^{(k)}$ such that $[G]^k_{q_0} = [F]^k_{q_0}$. Since $(T_{q_0}G)^*\Omega = \Omega$, $T_{q_0}G$ is an isomorphism and then $G : (\mathbb{R}^m, q_0) \to (\mathbb{R}^m, q_1)$ is a local diffeomorphism. Let η be defined by

$$\eta = F \circ G^{-1}.$$

Denote $P = [\eta]^{k+1}_{q_1} - id$. P is a homogeneous polynom of degree $k+1$. It is easy to check that for any form ω we have

$$[\eta^*\omega]^k_{q_1} = [\omega]^k_{q_1} + L_X[\omega]^0_{q_1}, \tag{3}$$

where $X = \sum\limits_{i=1}^{m} P_i \frac{\partial}{\partial q_i}$. Then, one gets

$$\begin{aligned} 0 = [\omega_1 - G^{-1*}\omega_2]^k_{q_1} &= [\omega_1 - \eta^*\omega_1]^k_{q_1} + [\eta^*(\omega_1 - F^{-1*}\omega_2)]^k_{q_1} = \\ &= -L_X(\omega_1) + [\omega_1 - F^{-1*}\omega_2]^k_{q_1} + L_X([\omega_1 - F^{-1*}\omega_2]^0_{q_1}). \end{aligned}$$

However, $[F]^k_{q_0} \in \Sigma^{(k-1)}$, so $[\omega_1 - F^{-1*}\omega_2]^0_{q_1}$ must be zero. This leads to

$$\sigma_k(F) = L_X\omega_1.$$

Moreover, we have:

$$L_X\Omega = [\eta^*\Omega - \Omega]^k_{q_1} = [G^{-1*}F^*\Omega - \Omega]^k_{q_1} = [[G^{-1}]^{k+1*}_{q_1}[F^*\Omega]^k_{q_0}]^k_{q_1} - \Omega.$$

Recall that F is a polynomial map with $\deg(F) \leq k$. Therefore,

$$[F^*\Omega]^k_{q_0} = [[F]^{k+1*}_{q_0}[\Omega]^k_{q_1}]^k_{q_0} = [[F]^{k*}_{q_0}\Omega]^k_{q_0} = [F^*\Omega]^{k-1}_{q_0} = \Omega.$$

Since $[G]^{k+1}_{q_0} \in \Sigma^{(k)}$, it is easy to check that

$$L_X\Omega = [G^{-1*}\Omega]^k_{q_1} = 0.$$

X is a hamiltonian vector field with the homogeneous coefficents of degree $k+1$. Consequently, there exists $h \in S^{k+2}(\mathbb{R}^m)$ for which $X = X_h$.

Conversely, let us assume that there exists $X = X_h$ with h in $S^{k+2}(\mathbb{R}^m)$ such that $\sigma_k(F) = L_X\omega_1$. Put $\eta = id + P$ with $X = \sum\limits_{i=1}^{m} P_i \frac{\partial}{\partial q_i}$ and $G = \eta^{-1} \circ F$. Similar considerations lead to

$$\begin{cases} [\omega_1 - G^{-1*}\omega_2]^k_{q_1} = -L_X(\omega_1) + \sigma_k(F) = 0 \\ [\Omega - G^{-1*}\Omega]^k_{q_1} = -L_X\Omega = 0. \end{cases}$$

Consequently, one can conclude that $[G]^{k+1}_{q_0} \in \Sigma^{(k)}$ and $[G]^k_{q_0} = [F]^k_{q_0}$. $\square$

Corollary 14 *If $\mathcal{J}^{(k)}_{\omega_1} = 0$ and $\sigma_k(\theta) = 0$ then $\pi_{k+1,k} : \Sigma^{(k)} \to \Sigma^{(k-1)}$ is a local diffeomorphism in a neighbourhood of $\theta_0 \in \Sigma^{(k-1)}$.*

Proof. $\pi_{k+1,k}$ is surjective. Moreover,

$$Ker(T_{\theta_0}\pi_{k+1,k}) = g^{(k)}(\theta) = \mathcal{J}^{(k)}_{\omega_1} = 0.$$

Therefore, $\pi_{k+1,k}$ is a local diffeomorphism. $\square$

4.3 Integrability of Σ

Let $\omega_1, \omega_2 \in \Omega^3_\varepsilon(\mathbb{R}^6)$ and suppose that ω_1 has constant coefficients and satisfies the equation $\mathcal{J}^{(1)}_{\omega_1} = 0$. Besides, suppose that:

(1) for all q in a neighbourhood of q_0, $\omega_2(q)$ is in the same orbit as ω_1.
(2) for all θ in a neighbourhood of $\theta_0 = [F]^1_{q_0} \in \Sigma$, $\sigma_1(\theta) = 0$.
(3) for all θ' in a neighbourhood of $\theta'_0 = [F]^2_{q_0} \in \Sigma^{(1)}$, $\sigma_2(\theta) = 0$.

It follows from 14 that $\pi_{3,2} : (\Sigma^{(2)}, \theta''_0) \to (\Sigma^{(1)}, \theta'_0)$ and $\pi_{2,1} : (\Sigma^{(1)}, \theta'_0) \to (\Sigma, \theta_0)$ are local diffeomorphisms.

Let $D : \theta \mapsto C(\theta) \cap T_\theta\Sigma^{(1)}$ be the restriction of the Cartan distribution on $\Sigma^{(1)}$. Recall that $C(\theta)$ is the vector space generated by the $T_\theta j_2 G$ where $[G]^2_q = \theta$. Let $\theta = [F]^2_q \in \Sigma^{(1)}$ be in a neighbourhood of θ'_0. Choose F such that $[F]^3_q \in \Sigma^{(2)}$: $T_\theta j_2 F \subset C(\theta)$. Moreover, $[F]^3_q \in \Sigma^{(2)}$ if and only if $T_\theta j_2 F \subset T_\theta\Sigma^{(1)}$. Therefore,

$$T_\theta j_2 F \subset D(\theta).$$

Let G be a map satisfying $[G]^2_q = \theta$. For any $X \in T_\theta j_2 G \cap T_\theta\Sigma^{(1)}$ there exists $X_F \in T_\theta j_2 F$ such that $X - X_F \in Ker(T_\theta\pi_{2,1})$. However, since $X, X_F \in T_\theta\Sigma^{(1)}$, we obtain

$$X - X_F \in T_\theta\Sigma^{(1)} \cap Ker(T_\theta\pi_{2,1}) = g^{(1)}(\theta) = \mathcal{J}^{(1)}_{\omega_1} = 0,$$

and then $X \in T_\theta j_2 F$.

Finally, for all θ in a neighbourhood of θ'_0 there exists F such that $D(\theta) = T_\theta j_2(F)$. Therefore, D is completely integrable on $J^2(6,6)$ and, according to the Frobenius theorem, it is completely integrable on $\Sigma^{(1)}$.

Consequently, there exists a submanifold $L_0 \subset \Sigma^{(1)}$ which is an integral submanifold of D containing θ'_0. Locally $L_0 = j_2 G$ if and only if $\pi_{2,0} : L_0 \to \mathbb{R}^6$ is a local diffeomorphism. However, there exists F_0 such that $T_{\theta'_0} L_0 = D(\theta'_0) = T_{\theta'_0} j_2 F_0$ and $T_{\theta'_0}\pi_{2,0} : T_{\theta'_0} L_0 \to \mathbb{R}^6$ is an isomorphism: $L_0 = j_2 G$ locally.

Proposition 15 *Let $\omega_2 \in \Omega^3_\varepsilon(\mathbb{R}^6)$ be a form with the following local properties:*

(1) for every q, $\omega_2(q) \in \Lambda^3_\varepsilon(\mathbb{R}^6)$ belongs to the $Sp(3)$-orbit of a form ω_1 with constant coefficients, satisfying $\mathcal{J}^{(1)}_{\omega_1} = 0$;
(2) $\sigma_1 = \sigma_2 = 0$.

Then, locally, ω_2 belongs to the orbit of ω_1.

4.4 Another expression for σ_1 and σ_2

Let $\theta = [F]^1_{q_0} \in \Sigma$. We assume that $F : (\mathbb{R}^6, q_0) \to (\mathbb{R}^6, q_1)$ is affine. Denote

$$[\omega_2]^1_{q_0} = \omega_2^0 + \omega_2^1,$$

where $\omega_2^0 = \omega_2(q_0)$ has constant coefficients and $\omega_2^1 = [\omega_2]^1 - \omega_2(q_0)$ has linear coefficients.

Lemma 16 *$\sigma_1(\theta) = 0$ if and only if there exists $h \in S^3(\mathbb{R}^6)$ such that*

$$\omega_2^1 = L_{X_h}\omega_2^0$$

Proof.

Since $\theta \in \Sigma$, F satisfies:

$$\begin{cases} F^*\omega_1 = \omega_2^0 \\ F^*\Omega = \Omega \end{cases}$$

Therefore,

$$\sigma_1(F) = [\omega_1 - F^{-1*}\omega_2]^1_{q_1} = \omega_1 - [[F^{-1}]^{2*}_{q_1}[\omega_2]^1_{q_0}]^1_{q_1} = \omega_1 - F^{-1*}(\omega_2^0 + \omega_2^1) = -F^{-1*}\omega_2^1$$

and

$$\omega_2^1 = -F^*(\sigma_1(F)).$$

After observing that for any form ω the relation

$$F^* L_Y \omega = L_X F^* \omega$$

must hold ($TF(X) = Y \circ F$), we get the result. $\square$

Now we can study σ_2. Let $\theta \in \Sigma^{(1)}$. Let us choose a linear map $F : (\mathbb{R}^6, q_0) \to (\mathbb{R}^6, q_1)$ for which $[F]^1_{q_0} = \theta$, and consider the map G such that $[G]^2_{q_0} = \theta$ and G^{-1} is polynomial of degree less than 2:

$$G^{-1} = \eta \circ F^{-1},$$

where $\eta = id + Q$, Q is a homogeneous polynom of degree 2. Let us denote

$$V_i(q) = \frac{1}{6} \sum_{j,k,l=1}^{6} a^i_{jkl}(q_j - q^0_j)(q_k - q^0_k)(q_l - q^0_l)$$

for $i = 1 \dots 6$, with

$$a^i_{jkl} = \sum_{m=1}^{6} \frac{\partial^2 Q_i}{\partial q_m \partial q_k} \frac{\partial^2 Q_m}{\partial q_j \partial q_l} + \frac{\partial^2 Q_i}{\partial q_m \partial q_j} \frac{\partial^2 Q_m}{\partial q_k \partial q_l} + \frac{\partial^2 Q_i}{\partial q_m \partial q_l} \frac{\partial^2 Q_m}{\partial q_j \partial q_k}$$

and put $U = \sum\limits_{i=1}^{6} Q_i \frac{\partial}{\partial q_i}$, $V = \sum\limits_{i=1}^{6} V_i \frac{\partial}{\partial q_i}$. It is not difficult to check that for any form ω_2

$$[\eta^* \omega_2]^2_{q_0} - [\eta^* \omega_2]^1_{q_0} = \omega^2_2 + L_U \omega^1_2 + \frac{1}{2}(L_U L_U \omega^0_2 - L_V \omega^0_2). \tag{4}$$

Since $[G]^2_{q_0} \in \Sigma^{(1)}$, $[\omega^1 - G^{-1*}\omega_2]^1_{q_0} = [\Omega - G^{-1*}\Omega] = 0$, i.e., according to (3):

$$0 = \omega_1 - F^{-1*}([\eta^*\omega_2]^1_{q_0}) = \omega_1 - F^{-1*}([\omega]^1_{q_0} + L_U \omega^0_2) = -F^{-1} * (\omega^1_2 + L_U \omega^0_2)$$

and then $\omega^1_2 = -L_U \omega^0_2$. In a similar way we can check that $L_U \Omega = 0$: there exists $h \in S^3(\mathbb{R}^6)$ such that $U = X_h$. Moreover, $\deg(G^{-1}) \leq 2$, so

$$[\Omega - G^{-1*}\Omega]^2_{q_1} = \Omega - [[G^{-1}]^{3*}_{q_1}[\Omega]^2_{q_0}]^2_{q_1} = \Omega - G^{-1*}\Omega = [\Omega - G^{-1*}\Omega]^2_{q_1} = 0.$$

Therefore, since the following equality holds,

$$[\Omega - G^{-1*}\Omega]^2_{q_1} = \Omega - F^{-1*}[\eta^*\Omega]^2_{q_0} = \Omega - F^{-1*}(\Omega - \frac{1}{2}(L_U L_U \Omega - L_V \Omega)) = -F^{-1*} L_V \Omega$$

one obtains $L_V \Omega = 0$, i.e. there exists $k \in S^4(\mathbb{R}^6)$ such that $V = X_k$. At last, we have checked that

$$\sigma_2(G) = -F^{-1*}(\omega^2_2 - \frac{1}{2}(L_U L_U \omega^0_2 + L_V \omega^0_2)).$$

Proposition 17 $\sigma_1([G]^1_{q_0}) = \sigma_2([G]^2_{q_0}) = 0$ *if and only if there exist* $h \in S^3$ *and* $k \in S^4$ *for which*

$$\begin{cases} \omega^1_2 = L_{X_h} \omega^0_2 \\ \omega^2_2 = \frac{1}{2}(L_{X_h} \omega^1_2 + L_{X_k} \omega^0_2) \end{cases}$$

Proof.

Suppose first that $\sigma_2([G]^2_{q_0}) = 0$. Then there exists $W = X_k$ with $k \in S^4(\mathbb{R}^6)$ such that $\sigma_2(G) = L_W\omega_1$. So, one has

$$L_W\omega_0 = -F^{-1*}(\omega_2^2 - \frac{1}{2}(L_U L_U \omega_2^0 + L_V \omega_2^0)).$$

Therefore,

$$\omega_2^2 = \frac{1}{2}(L_{U_0}\omega_2^1 + L_{V_0}\omega_2^0)$$

with $U_0 = X_h$, $h \in S^3$, $V_0 = X_k$, $k \in S^4$ and $L_{U_0}\omega_2^0 = \omega_2^1$.

Conversely, if there exist $h \in S^3$ and $k \in S^4$ such that

$$\begin{cases} \omega_2^1 = L_{U_0}\omega_2^0 \\ \omega_2^2 = \frac{1}{2}(L_{U_0}\omega_2^1 + L_{V_0}\omega_2^0) \end{cases}$$

with $U_0 = X_h$ and $V_0 = X_k$, then $L_U\omega_2^0 = -L_{U_0}\omega_2^0$. It means that $U + U_0 \in \mathcal{J}^{(1)}_{\omega_2^0} = 0$ and, therefore,

$$\sigma_2(G) = -F^{-1*}(\omega_2^2 - \frac{1}{2}(L_U L_U \omega_2^0 + L_V \omega_2^0)) = -F^{-1*} L_{\frac{V-V_0}{2}}\omega_2^0 = L_W\omega_1$$

with $W = X_K$, $K \in S^4$. $\square$

Theorem 18 *Consider a Monge-Ampère equation in the dimension 3, corresponding to an effective 3-form ω, such that locally:*

(1) for all q, $\omega(q)$ belongs to one of the five first orbits listed in table 1
(2) for all q, the exterior form $[\omega]^2_q = \omega^0 + \omega^1 + \omega^2$ satisfies

$$\begin{cases} \omega^1 = L_{X_h}\omega^0 \\ \omega^2 = \frac{1}{2}(L_{X_h}\omega^1 + L_{X_k}\omega^0) \end{cases}$$

with $h \in S^3(\mathbb{R}^6)$ and $k \in S^4(\mathbb{R}^6)$.

Then this differential equation is locally Sp-equivalent to one of the following equations:

$$\lambda + \mathrm{hess}(h) = 0, \lambda \neq 0$$
$$\frac{\partial^2 h}{\partial q_1^2} - \frac{\partial^2 h}{\partial q_2^2} + \frac{\partial^2 h}{\partial q_3^2} + \nu^2\,\mathrm{hess}(h) = 0, \nu \neq 0$$
$$\frac{\partial^2 h}{\partial q_1^2} + \frac{\partial^2 h}{\partial q_2^2} + \frac{\partial^2 h}{\partial q_3^2} - \nu^2\,\mathrm{hess}(h) = 0, \nu \neq 0$$
$$\frac{\partial^2 h}{\partial q_1^2} - \frac{\partial^2 h}{\partial q_2^2} + \frac{\partial^2 h}{\partial q_3^2} = 0$$

$$\frac{\partial^2 h}{\partial q_1^2} + \frac{\partial^2 h}{\partial q_2^2} + \frac{\partial^2 h}{\partial q_3^2} = 0$$

where hess(h) *is the hessian of* h.

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