NONDEGENERATE MONGE-AMPÈRE STRUCTURES IN THE DIMENSION 6

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ABSTRACT. We define a nondegenerate Monge-Ampère structure on a 6-dimensional manifold as a pair (Ω,ω), such that Ω is a symplectic form and ω is a 3-differential form which satisfies $\omega\wedge\Omega=0$ and which is nondegenerate in the sense of Hitchin. We associate with such a pair a generalized almost (pseudo) Calabi-Yau structure and we study its integrability from the point of view of Monge-Ampère operators theory. The result we prove appears as an analogue of Lychagin and Roubtsov theorem on integrability of the almost complex or almost product structure associated with an elliptic or hyperbolic Monge-Ampère equation in the dimension 4. We study from this point of view the example of the Stenzel metric on T^*S^3.

Key words: Calabi-Yau manifolds, special lagrangian submanifolds, Monge-Ampère equations, Symplectic forms

AMS classification: 34A26, 58A10, 53D05, 32Q60, 14J32

1. INTRODUCTION

A Monge-Ampère equation is a differential equation which is nonlinear in a very specific way: its nonlinearity is the derminant one. In the dimension 2, such an equation can be written

$$A\frac{\partial^2 f}{\partial q_1^2}+2B\frac{\partial^2 f}{\partial q_1\partial q_2}+C\frac{\partial^2 f}{\partial q_2^2}+D\Big(\frac{\partial^2 f}{\partial q_1^2}\frac{\partial^2 f}{\partial q_2^2}-(\frac{\partial^2 f}{\partial q_1\partial q_2})^2\Big)+E=0, \tag{1}$$

where A,B,C,D and E are smooth functions on the jet space $J^1\mathbb{R}^2$. This equation is said symplectic if A,B,C,D,E are smooth functions on the cotangent bundle $T^*\mathbb{R}^2\subset J^1\mathbb{R}^2$.

The Monge-Ampère operators theory proposed by Lychagin ([L]) associates to each symplectic Monge-Ampère equation on an n-dimensional manifold M a pair of differential forms (Ω,ω) on the cotangent bundle T^*M , where Ω is the canonical symplectic form and $\omega\in\Omega^n(T^*M)$ is an Ω-effective form, i.e. $\omega\wedge\Omega=0$. To be more precise, the symplectic Monge-Ampère equation associated with such a pair (Ω,ω) is the differential equation

$$(df)^*(\omega)=0, \tag{2}$$

where $df:M\to T^*M$ is the natural section defined by a smooth function f on M. For instance, (1) is associated with the symplectic form on $T^*\mathbb{R}^2$

$$\Omega=dq_1\wedge dp_1+dq_2\wedge dp_2,$$

and the differential form

$$\begin{aligned}\omega=&Adp_1\wedge dq_2+B(dq_1\wedge dp_1-dq_2\wedge dp_2)+Cdq_1\wedge dp_2\\&+Ddp_1\wedge dp_2+Edq_1\wedge dq_2.\end{aligned}$$

A generalized solution of the differential equation (2) is a lagrangian submanifold L of (T^*M,Ω) on which ω vanishes. L is locally the graph of a section $df : M \to T^*M$ with f solution of (2) if and only if the projection $L \to M$ is locally a diffeomorphism.

Generalizing this notion, we define a symplectic Monge-Ampère structure on a $2n$-dimensional X as a pair of differential forms (Ω,ω), $\Omega\in\Omega^2(X)$ being symplectic and $\omega\in\Omega^n(X)$ being Ω-effective. In the dimension $n=2$, if ω is nondegenerate (i.e. the pfaffian $\mathrm{pf}(\omega)=\frac{\omega\wedge\omega}{\Omega\wedge\Omega}$ is non zero), the equality

$$\frac{\omega}{\sqrt{|\,\mathrm{pf}(\omega)|}}=\Omega(A_\omega\cdot,\cdot)$$

defines a section $A_\omega : X\to TX\otimes T^*X$ which is either an almost complex structure ($A_\omega^2=-Id$) or an almost product structure ($A_\omega^2=Id$) on X. Lychagin and Roubtsov gave in [LR1] a necessary and sufficient condition for A_ω to be integrable:

Proposition 1.1. *A_ω is integrable if and only if*

$$d\Big(\frac{\omega}{\sqrt{|\,\mathrm{pf}(\omega)|}}\Big)=0.$$

From the point of view of differential equation (i.e. the local point of view), this result can be formulated as follows:

Proposition 1.2. *A symplectic Monge-Ampère equation $\Delta_\omega=0$ on $\mathbb{R}^2$ is symplectically equivalent to one of these two equations*

$$\begin{cases}\Delta f=0, & (\mathrm{pf}(\omega)>0)\\ \Box f=0, & (\mathrm{pf}(\omega)<0)\end{cases}$$

if and only if

$$d\Big(\frac{\omega}{\sqrt{|\,\mathrm{pf}(\omega)|}}\Big)=0.$$

To classify the symplectic Monge-Ampère equations with constant coefficients in the dimension 3, Lychagin and Roubsov introduced a quadratic invariant q_ω associated with each effective 3-form ω ([LR3]). Hitchin has defined a linear invariant K_ω associated with each 3-form ω ([Hi]) in order to study the geometry of 3-dimensional Calabi-Yau manifolds. We show that these two invariants coincide in the effective case (proposition 3.9) . This remark allows us to combine Lychagin, Roubtsov and Hitchin works to demonstrate a result analogous to 1.1 and 1.2 in the dimension 3. Let us introduce some notations to state this result. We suppose that our 3-form $\omega\in\Omega^3(X^6)$ is nondegenerate in the Hitchin sense. The Hitchin pfaffian $\lambda(\omega)$ is then nonzero and ω can be decomposed in an unique way as a sum of two complex decomposable 3-forms: $\omega=\alpha+\beta$. Following Hitchin, we denote by $\hat{\omega}$ the dual form associated with ω. We associate to the pair $(\Omega,\omega)\in\Omega^2(X^6)\times\Omega^3(X^6)$ a geometric structure on X which we call *generalized almost Calabi-Yau structure*. This structure is essentially composed of a (pseudo) metric q_ω, an almost complex structure or almost product structure K_ω which is compatible with q_ω and Ω and two decomposable 3-forms whose associated distributions are distributions of K_ω eigenvectors. We extend Hitchin's results to demonstrate the analogous of 1.1:

Proposition 1.3. *The generalized almost Calabi-Yau structure*

$$(q_\omega,K_\omega,\Omega,\alpha,\beta)$$

is "integrable" if and only if

$$\begin{cases} d\left(\frac{\omega}{\sqrt[4]{|\lambda(\omega)|}}\right) = 0, \\ d\left(\frac{\hat{\omega}}{\sqrt[4]{|\lambda(\omega)|}}\right) = 0. \end{cases}$$

We demonstrate then the local version of this proposition:

Theorem 1.4. *A symplectic Monge-Ampère equation in the dimension* 3 *associated with a nondegenerate Monge-Ampère structure* (Ω, ω) *is symplectically equivalent to one of these three equations*

$$\begin{cases} \mathrm{hess}(f) = 1, \\ \Delta f - \mathrm{hess}(f) = 0, \\ \Box f + \mathrm{hess}(f) = 0, \end{cases}$$

if and only if

$$\begin{cases} d\left(\frac{\omega}{\sqrt[4]{|\lambda(\omega)|}}\right) = 0, \\ d\left(\frac{\hat{\omega}}{\sqrt[4]{|\lambda(\omega)|}}\right) = 0, \\ q_\omega \textit{ is flat.} \end{cases}$$

Our motivation is to generalize the notion of "special lagrangian submanifolds" in the dimension 3. This notion was first introduced by Harvey and Lawson in their famous paper *Calibrated Geometries* ([HL]) as examples of minimal submanifolds. Recall that a p-calibration on a riemannian manifold (Y, g) is a closed differential p-form $\phi \in \Omega^p(Y)$ such that for any point $y \in Y$ and any oriented p-plane V of T_yY, the following inequality

$$\phi_y|_V \leq vol_V$$

holds. Here vol_V is the volume exterior form on V defined by the metric g and the orientation on V. An oriented p-dimensional submanifold L is said ϕ-calibrated if for any $y \in L$

$$\phi_y|_{T_yL} = vol_{T_yL}.$$

Calibrated submanifolds are volume-minimizing in their homology class. The real form $Re(\alpha)$ with $\alpha = dz_1 \wedge \ldots \wedge dz_n$ is an example of n-calibration on $\mathbb{C}^n$ and $Re(\alpha)$-calibrated submanifolds are said special lagrangian. This notion of special lagrangian calibration can be generalized on Calabi-Yau manifolds, i.e. Kähler manifolds endowed with an holomorphic volume form $\alpha \in \Omega^{n,0}(Y)$ such that

$$\frac{\alpha \wedge \overline{\alpha}}{\Omega^n} \text{ is constant,}$$

Ω being the Kähler form. These special lagrangians submanifolds have become the focus of intention of many mathematicians in the last few years after Strominger, Yau and Zaslow proposed a geometric construction of mirror manifolds based on the conjecture of existence of toric special lagrangian fibration ([SYZ]).

Gromov noted in a discussion with Roubtsov that the Monge-Ampère structures can be seen as an analogue of the calibrations. Effective forms correspond to calibrations and lagrangian submanifolds correspond to calibrated submanifolds. Moreover, special lagrangian submanifolds are in the intersection of these two approach. Harvey and Lawson actually showed that special lagrangian submanifolds of $(\mathbb{C}^n, Re(\alpha))$

are the generalized solutions of the differential equation associated with the Monge-Ampère structure $(\Omega, Im(\alpha))$ with

$$\Omega = \frac{i}{2}(dz_1 \wedge d\overline{z_1} + \ldots + dz_n \wedge d\overline{z_n}).$$

In the dimension 3 this equation is

$$\Delta f - \mathrm{hess}(f) = 0. \tag{3}$$

Our aim is to show how the Calabi-Yau structure of $\mathbb{C}^3$ is encoded in this equation and how we can construct some similar structures with some similar differential equations.

In the first section we recall the Lychagin's approach of Monge-Ampère equations. We study as an example Chynoweth-Sewell's equations which come from the "semi-geostrophic" model of athmosphere dynamics. We remark that they are all equivalent to the classic Monge-Ampère equation $\mathrm{hess}(f) = 1$. In the second section we adapt Hitchin's works on 3-forms in the effective case. In the last section, we define the notion of generalized almost Calabi-Yau structure and we demonstrate the theorem 1.4. We study as an example the Stenzel metric on T^*S^3.

This article constitutes a part of the author's PhD thesis being prepared at Angers University. I would like to thank my advisor Volodya Roubtsov for suggesting the problem and helpfull discussions. I would like also to thank Oleg Lisovyy for all his help. I am very grateful to professor Benjamin Enriquez for his valuable remarks and suggestions. I would like finally to thank Michèle Audin for the lecture she gave in Barcelona summer school ([A]) which helped me to understand the geometry of lagrangian and special lagrangian submanifolds.

2. Effective forms and Monge-Ampère operators

Let (V,Ω) be a symplectic $2n$-dimensional vector space over $\mathbb{R}$. Denote by $\Gamma : V \to V^*$ the isomorphism determined by Ω. Let $X_\Omega \in \Lambda^2(V)$ be the unique bivector which satisfies $\Gamma^2(X_\Omega) = \Omega$, where $\Gamma^2 : \Lambda^2(V) \to \Lambda^2(V^*)$ is the exterior power of Γ [1].

One can introduce the operators $\perp : \Lambda^k(V^*) \to \Lambda^{k-2}(V^*)$, $\omega \mapsto i_{X_\Omega}\omega$ and $\top : \Lambda^k(V^*) \to \Lambda^{k+2}(V^*)$, $\omega \mapsto \omega \wedge \Omega$ (see [L]). They have the following properties:

$$\begin{cases} [\perp, \top](\omega) = (n-k)\omega\,, \forall \omega \in \Lambda^k(V^*); \\ \perp : \Lambda^k(V^*) \to \Lambda^{k-2}(V^*) \text{ is into for } k \geq n+1; \\ \top : \Lambda^k(V^*) \to \Lambda^{k+2}(V^*) \text{ is into for } k \leq n-1. \end{cases}$$

We will say that a k-form ω is effective if $\perp\omega = 0$ and we will denote by $\Lambda^k_\varepsilon(V^*)$ the vector space of effective k-forms on V. When $k = n$, ω is effective if and only if $\omega \wedge \Omega = 0$.

The next theorem explains the fundamental role played by the effective forms in the theory of Monge-Ampère operators (see [L]):

[1] We denote by $\Lambda^*(V^*)$ the space of exterior forms on a vector space V and by $\Omega^*(X)$ the space of differential forms on a manifold X

Theorem 2.1 (Hodge-Lepage-Lychagin).

1. *Every form $\omega \in \Lambda^k(V^*)$ can be uniquely decomposed into the finite sum*
$$\omega = \omega_0 + \top\omega_1 + \top^2\omega_2 + \dots,$$
where all ω_i are effective forms.
2. *If two effective k-forms vanish on the same k-dimensional isotropic vector subspaces in (V, Ω), they are proportional.*

Let M be an n-dimensional smooth manifold. Denote by J^1M the space of 1-jets of smooth functions on M. Let $j^1(f) : M \to J^1M, x \mapsto [f]^1_x$ be the natural section associated with a smooth function f on M. A Monge-Ampère operator $\Delta_\omega : C^\infty(M) \to \Omega^n(M)$ associated with a differential n-form $\omega \in \Omega^n(J^1M)$ is the differential operator
$$\Delta_\omega(f) = j_1(f)^*(\omega).$$

Let U be the contact 1-form on J^1M and X_1 be the Reeb's vector field. Denote by $C(x)$ the kernel of U_x for $x \in J^1M$. $(C(x), dU_x)$ is a $2n$-dimensional symplectic vector space and
$$T_xJ^1M = C(x) \oplus \mathbb{R}X_{1x}.$$
A generalized solution of the equation $\Delta_\omega = 0$ is a legendrian submanifold L^n of (J^1M, U) such that $\omega|_L = 0$. In each point $x \in L$, T_xL is a lagrangian subspace of $(C(x), dU_x)$. If the projection $\pi : J^1M \to M$ is a local diffeomorphism on L then L is locally the graph of a section $j^1(f)$, where f is a regular solution of the equation $\Delta_\omega(f) = 0$.

We will denote by $\Omega^*(C^*)$ the space of differential forms vanishing on X_1. In each point x, $(\Omega^k(C^*))_x$ can be naturally identified with $\Lambda^k(C(x)^*)$. Let $\Omega^*_\varepsilon(C^*)$ be the space of forms which are effective on $(C(x), dU_x)$ in each point $x \in J^1M$. The first part of the theorem 2.1 means that
$$\Omega^*_\varepsilon(C^*) = \Omega^*(J^1M)/I_C,$$
where I_C is the Cartan ideal generated by U and dU. The second part means that two differential n-forms ω and θ on J^1M determine the same Monge-Ampère operator if and only if $\omega - \theta \in I_C$.

$Ct(M)$, the pseudo-group of contact diffeomorphisms on J^1M, naturally acts on the set of Monge-Ampère operators in the following way
$$F(\Delta_\omega) = \Delta_{F^*(\omega)},$$
and the corresponding infinitesimal action is
$$X(\Delta_\omega) = \Delta_{L_X(\omega)}.$$
We are interested in a more restrictive class of operators, the class of symplectic operators. These operators satisfy
$$X_1(\Delta_\omega) = \Delta_{L_{X_1}(\omega)} = 0.$$
Let T^*M be the cotangent space and Ω be the canonical symplectic form on it. Let us consider the projection $\beta : J^1M \to T^*M$, defined by the following commutative

diagram:

$$\begin{tikzcd}
\mathbb{R} & J^1M \arrow[l, "\alpha"'] \arrow[r, "\beta"] & T^*M \\
& M \arrow[ul, "f"] \arrow[u, "j^1(f)"] \arrow[ur, "df"'] &
\end{tikzcd}$$

We can naturally identify the space $\{\omega \in \Omega^*_\varepsilon(C^*) : L_{X_1}\omega = 0\}$ with the space of effective forms on (T^*M, Ω) using this projection β. Then, the group acting on these forms is the group of symplectomorphisms of T^*M.

Definition 2.2. *A Monge-Ampère structure on a $2n$-dimensional manifold X is a pair of differential form $(\Omega, \omega) \in \Omega^2(X) \times \Omega^n(X)$ such that Ω is symplectic and ω is Ω-effective i.e. $\Omega \wedge \omega = 0$.*

Note that (X, Ω) can be locally identified with $(T^*\mathbb{R}^n, \Omega_0)$. We can then associate to the pair (Ω, ω) a symplectic Monge-Ampère equation $\Delta_\omega = 0$. Conversely, to any symplectic Monge-Ampere equation $\Delta_\omega = 0$ on a manifold M we can associate a Monge-Ampère structure (Ω, ω) on T^*M.

Example 2.3. *The Chynoweth-Sewell's equations are an example of Monge-Ampère equations with constant coefficients. They come from the "semi-geostrophic model" of Atmosphere Dynamics (*[CS]*):*

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 + \frac{\partial^2 f}{\partial z^2} = \gamma, \ \ \gamma \in \mathbb{R} \tag{4}$$

Note that $F(x,y) - \frac{1}{2}z^2$ is a solution of (4) *for the particular case $\gamma = 0$ when F is a solution of* $\text{hess}(F) = 1$. *For instance,*

$$\frac{1}{3}\sqrt{(x^2+2y)^3} - \frac{1}{2}z^2$$

is a solution of (4) *when $\gamma = 0$.*

The effective form associated with (4) *is*

$$\omega = dp \wedge dq \wedge dz + dx \wedge dy \wedge dh - \gamma dx \wedge dy \wedge dz,$$

where (x,y,z,p,q,h) is the canonical coordinates system of $T^*\mathbb{R}^3$. This form is clearly the sum of two decomposable 3*-forms:*

$$\omega = dp \wedge dq \wedge dz + dx \wedge dy \wedge (dh - \gamma dz).$$

Then $\phi^*(\omega) = dp \wedge dq \wedge dh - dx \wedge dy \wedge dz$ where ϕ is the symplectomorphism

$$\phi(x,y,z,p,q,h) = (x,y,h,p,q,\gamma h - z).$$

In other words, Chynoweth-Sewell's equations are symplectically equivalent to the equation

$$\text{hess}(f) = 1. \tag{5}$$

It is easy to check that

$$f(x,y,z) = \int_a^{\sqrt{xy+yz+zx}} (b + 4\xi^3)^{1/3} d\xi$$

is a regular solution of (5)*. Therefore,*

$$L = \Big\{ (x, y, (x+y)\alpha, (y+z)\alpha, (z+x)\alpha, \gamma(x+y)\alpha - z) \Big\}$$

is an example of generalized solution of (4) *with*

$$\alpha = \frac{1}{2}\left(\frac{b}{(xy+yz+zx)^{\frac{3}{2}}} + 4\right)^{\frac{1}{3}}.$$

3. The geometry of effective 3-forms in the dimension 6

3.1. **The action of $SL(6,\mathbb{R})$ on $\Lambda^3(\mathbb{R}^6)$.** We recall first Hitchin's results on the geometry of 3-forms. Let V be a 6-dimensional real vector space. We denote by $A : \Lambda^5(V^*) \to V \otimes \Lambda^6(V^*)$ the isomorphism induced by the exterior product and we fix a volume form θ on V. The linear map $K^\theta_\omega : V \to V$ associated with $\omega \in \Lambda^3(V^*)$ is defined by

$$K^\theta_\omega(X)\theta = A(i_X(\omega) \wedge \omega).$$

Definition 3.1. *The Hitchin pfaffian of a* 3*-form* $\omega \in \Lambda^3(V^*)$ *is*

$$\lambda_\theta(\omega) = \frac{1}{6}\mathrm{Tr}(K^\theta_\omega \circ K^\theta_\omega).$$

If $\lambda_\theta(\omega)$ *is nonzero then* ω *is said nondegenerate.*

Proposition 3.2 (Hitchin)**.** *Let* $\omega \in \Lambda^3(V^*)$ *be nondegenerate. Then,*

1. $K^\theta_\omega \circ K^\theta_\omega = \lambda_\theta(\omega) Id$.
2. $\lambda_\theta(\omega) > 0$ *if and only if* $\omega = \alpha + \beta$ *where* α *and* β *are real decomposable* 3*-forms on* V*. Moreover, if we impose* $\frac{\alpha\wedge\beta}{\theta} > 0$ *then* α *and* β *are unique:*
$$\begin{cases} 2\alpha = \omega + |\lambda_\theta(\omega)|^{-\frac{3}{2}}(K^\theta_\omega)^*(\omega) \\ 2\beta = \omega - |\lambda_\theta(\omega)|^{-\frac{3}{2}}(K^\theta_\omega)^*(\omega) \end{cases}$$
3. $\lambda_\theta(\omega) < 0$ *if and only if* $\omega = \alpha + \overline{\alpha}$ *where* α *is a complexe decomposable* 3*-form on* V*. Moreover, if we impose* $\frac{\alpha\wedge\overline{\alpha}}{i\theta} > 0$ *then* α *is unique:*
$$\alpha = \omega + i|\lambda_\theta(\omega)|^{-\frac{3}{2}}K^*_\omega(\omega).$$

Remark 3.3. *Let* $(e_1, \dots, e_6)$ *be a basis of* V *and fix* $\theta = e_1^* \wedge \dots \wedge e_6^*$.

1. $\lambda_\theta(\omega) > 0$ *if and only if* ω *is in the* $GL(6)$*-orbit of*
$$e_1^* \wedge e_2^* \wedge e_3^* + e_4^* \wedge e_5^* \wedge e_6^*.$$
2. $\lambda_\theta(\omega) < 0$ *if and only if* ω *is in the* $GL(6)$*-orbit of*
$$(e_1^* + ie_4^*) \wedge (e_2^* + ie_5^*) \wedge (e_3^* + ie_6^*) + (e_1^* - ie_4^*) \wedge (e_2^* - ie_5^*) \wedge (e_3^* - ie_6^*).$$

Therefore, the action of $GL(6)$ *on* $\Lambda^3(V^*)$ *has two open orbits separated by the quartic hypersurface* $\lambda_\theta = 0$*. This explains this notion of nondegenerate* 3*-form.*

Definition 3.4 (Hitchin)**.** *Let* ω *be a nondegenerate* 3*-form on* V*. The dual form* $\hat{\omega}$ *is*

1. $\hat{\omega} = \alpha - \beta$ *if* $\omega = \alpha + \beta$,
2. $\hat{\omega} = i(\overline{\alpha} - \alpha)$ *if* $\omega = \alpha + \overline{\alpha}$.

To conclude we remark that the exterior product defines a symplectic form on $\Lambda^3(V^*)$

$$\Theta_\theta(\omega,\omega') = \frac{\omega\wedge\omega'}{\theta}$$

and that the action of $SL(6)$ is hamiltonian:

Proposition 3.5 (Hitchin)**.** *The action of $SL(6)$ on $(\Lambda^3(V^*),\Theta_\theta)$ is hamiltonian with moment map $K^\theta : \Lambda^3(V^*) \to sl(6)$.*

Example 3.6. *This invariant K can be used to construct some almost complex or almost product structures on 6-dimensional manifolds. We study in this example the restriction of the famous associative 3-form to the sphere S^6. Let $(\mathbb{O},<,>)$ be the octonions normed algebra and denote by E_7 the 7-dimensional subspace of imaginary octonions. The associative form $\phi \in \Lambda^3(E_7^*)$ is defined by*

$$\phi(x,y,z) = < x, yz > .$$

(see for instance for instance [HL]*). Let us see the sphere S^6 as a submanifold of E_7 and let us consider the form $\omega \in \Omega^3(S^6)$ defined by*

$$\omega = \frac{1}{\sqrt{2}}\phi|_{S^6}.$$

Let θ be the volume form defined by the metric induced on S^6. A straightforward computation shows that

$$\lambda_\theta(\omega) = -1.$$

Therefore K^θ_ω is an almost complex structure on S^6. This almost complex structure is in fact already known. It actually coincides with the almost complex structure

$$I_x(Y) = x.Y,$$

with $x \in S^6$ and $Y \in T_xS^6 = \left\{Y \in E_7 : < x, Y >= 0\right\}$. This almost complex structure is not integrable ([HC]*).*

3.2. **The action of $SP(3)$ on $\Lambda^3_\varepsilon(V^*)$.** We assume now that V is a 6-dimensional symplectic vector space. We fix $\theta = -\frac{1}{6}\Omega^3$ with Ω the symplectic form on V. We denote $\lambda = \lambda_\theta$, $K = K^\theta$ and $\Theta = \Theta_\theta$. $\omega \in \Lambda^3(V^*)$ is said effective if $\Omega\wedge\omega = 0$. We denote by $\Lambda^3_\varepsilon(V^*)$ the space of effective 3-forms. $(\Lambda^3_\varepsilon(V^*),\Theta)$ is a symplectic subspace of $(\Lambda^3(V^*),\Theta)$ since, according to the Hodge-Lepage-Lychagin theorem, any 3-form ω admits the decomposition

$$\omega = \omega_0 + \Omega\wedge\omega_1,$$

with ω_0 effective.

Denote by $sp(3)$ the Lie algebra of $SP(3) = SP(\Omega)$. A straightforward comuptation shows the following lemma.

Lemma 3.7. *Let ω be a 3-form on V. ω is effective if and only if $K(\omega) \in sp(3)$.*

Corollary 3.8. *The action of $SP(3)$ on $(\Lambda^3_\varepsilon(V^*),\Theta)$ is hamiltonian with moment map $K : \Lambda^3_\varepsilon(V^*) \to sp(3)$.*

Lychagin and Roubtsov have defined an other invariant $q_\omega \in S^2(V^*)$ associated with one effective 3-form ω ([LR3]). It is natural to ask what is the link between these two invariants. The quadratic form q_ω is defined by

$$q_\omega(X) = -\frac{1}{4}\perp^2(i_X\omega \wedge i_X\omega).$$

In fact, this invariant gives us the roots of the characteristic polynom of $i_X\omega$:

$$(i_X\omega - \xi\Omega)^3 = -\xi(\xi - \sqrt{q_\omega(X)})(\xi + \sqrt{q_\omega(X)})\Omega^3.$$

Using this invariant, Lychagin and Roubtsov have listed the different orbits of the action of $SP(3)$ on $\Lambda^3_\varepsilon(V^*)$. This list has been completed by the author in [Ba1] and is summed up in table 1.

Computing q_ω and K_ω for each normal form and using their invariant properties, we can check the following:

Proposition 3.9. *Let ω be an effective 3-form on V. Then*

$$q_\omega(X) = \Omega(K_\omega X, X),$$

for all $X \in V$.

Remark 3.10. *The Lie algebra $(sp(3), [\ ,\ ])$ can be naturally identified with the Lie algebra $(S^2(V^*), \{\ ,\ \})$ where $\{\ ,\ \}$ is the Poisson bracket associated with Ω. $q : \Lambda^3_\varepsilon(V^*) \to S^2(V^*)$ is then the moment map of the hamiltonian action of $SP(3)$ on $(\Lambda^3_\varepsilon(V^*), \Theta)$.*

	ω	q_ω	$\lambda(\omega)$
1.	$e_1^* \wedge e_2^* \wedge e_3^* + \gamma f_1^* \wedge f_2^* \wedge f_3^*, \gamma \neq 0$	$\frac{\gamma}{2}(e_1^* f_1^* + e_2^* f_2^* + e_3^* f_3^*)$	γ^4
2.	$f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_3^*$ $+ f_3^* \wedge e_1^* \wedge e_2^* + \gamma^2 f_1^* \wedge f_2^* \wedge f_3^*, \gamma \neq 0$	$(e_1^*)^2 - (e_2^*)^2 + (e_3^*)^2$ $+\gamma^2((f_1^*)^2 - (f_2^*)^2 + (f_3^*)^2)$	$-4\gamma^4$
3.	$f_1^* \wedge e_2^* \wedge e_3^* - f_2^* \wedge e_1^* \wedge e_3^*$ $+ f_3^* \wedge e_1^* \wedge e_2^* - \gamma^2 f_1^* \wedge f_2^* \wedge f_3^*, \gamma \neq 0$	$-(e_1^*)^2 - (e_2^*)^2 - (e_3^*)^2$ $+\gamma^2(-(f_1^*)^2 - (f_2^*)^2 - (f_3^*)^2)$	$-4\gamma^4$
4.	$f_1^* \wedge e_2^* \wedge e_3^* + f_2^* \wedge e_1^* \wedge e_3^* + f_3^* \wedge e_1^* \wedge e_2^*$	$(e_1^*)^2 - (e_2^*)^2 + (e_3^*)^2$	0
5.	$f_1^* \wedge e_2^* \wedge e_3^* - f_2^* \wedge e_1^* \wedge e_3^* + f_3^* \wedge e_1^* \wedge e_2^*$	$-(e_1^*)^2 - (e_2^*)^2 - (e_3^*)^2$	0
6.	$f_3^* \wedge e_1^* \wedge e_2^* + f_2^* \wedge e_1^* \wedge e_3^*$	$(e_1^*)^2$	0
7.	$f_3^* \wedge e_1^* \wedge e_2^* - f_2^* \wedge e_1^* \wedge e_3^*$	$-(e_1^*)^2$	0
8.	$e_1^* \wedge e_2^* \wedge e_3^*$	0	0
9.	0	0	0

TABLE 1. Classification of effective 3-forms in dimension 6

4. GEOMETRICAL STRUCTURES ASSOCIATED WITH NONDEGENERATE MONGE-AMPÈRE EQUATIONS

Definition 4.1. *A Monge-Ampère structure (Ω, ω) on a 6-dimensional manifold X is called*

1. *nondegenerate if $\lambda(\omega)$ never vanishes,*
2. *elliptic if $\lambda(\omega) < 0$ everywhere,*
3. *hyperbolic if $\lambda(\omega) > 0$ everywhere.*

A nondegenerate Monge-Ampère structure (Ω, ω) is said

1. closed if

$$\begin{cases} d\left(\frac{\omega}{\sqrt[4]{|\lambda(\omega)|}}\right) = 0, \\ d\left(\frac{\hat{\omega}}{\sqrt[4]{|\lambda(\omega)|}}\right) = 0. \end{cases}$$

2. locally constant if there exists a Darboux coordinates system of (X,Ω) in which ω has constant coefficients.

4.1. **Generalized Calabi-Yau structures.**

Definition 4.2. *A generalized almost Calabi-Yau structure on a 6-dimensional manifold X is a 5-uple $(g,\Omega,K,\alpha,\beta)$ where*

1. g is a (pseudo) metric on X,
2. Ω is a symplectic on X,
3. K is a smooth section $X \to TX \otimes T^*X$ such that $K^2 = \pm Id$ and such that

$$g(U,V) = \Omega(KU,V)$$

for all tangent vectors U,V,
4. α and β are (eventually complex) decomposable 3-forms whose associated distributions are the distributions of K eigenvectors and such that

$$\frac{\alpha\wedge\beta}{\Omega^3} \text{ is constant.}$$

A generalized Calabi-Yau structure $(g,\Omega,K,\alpha,\beta)$ is said integrable if α and β are closed.

Note that a generalized Calabi-Yau structure is a Calabi-Yau structure if and only if the metric is definite positive and K is a complex structure.

Remark 4.3. *The condition $d\alpha = d\beta = 0$ implies the integrability (in the Frobenius sense) of the distributions defined by the almost complex structure or almost product structure K. Therefore, according to the Newlander-Nirenberg theorem, it implies its integrability. For instance, when K is an almost complex structure and g is definite positive , the almost Calabi-Yau structure $(g,\Omega,K,\alpha,\overline{\alpha})$ is integrable if and only if K is a complex structure and α is holomorphic.*

Example 4.4. *Each nondegenerate Monge-Ampère structure (Ω,ω_0) defines the generalized almost Calabi-Yau structure $(q_\omega,\Omega,K_\omega,\alpha,\beta)$ with*

$$\omega = \frac{\omega_0}{\sqrt[4]{|\lambda(\omega_0)|}}.$$

For instance, on $\mathbb{R}^6$, the generalized Calabi-Yau structure associated with the equation

$$\Delta(f) - \text{hess}(f) = 0$$

is the canonical Calabi-Yau structure of $\mathbb{C}^3$

$$\begin{cases} g = -\sum_{j=1}^{3} dx_j.dx_j + dy_j.dy_j \\ K = \sum_{j=1}^{3} \frac{\partial}{\partial y_j} \otimes dx_j - \frac{\partial}{\partial x_j} \otimes dy_j \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \\ \beta = \overline{\alpha} \end{cases}$$

The generalized Calabi-Yau associated with the equation

$$\Box(f) - \mathrm{hess}(f) = 0$$

is the pseudo Calabi-Yau structure

$$\begin{cases} q = dx_1.dx_1 - dx_2.dx_2 + dx_3.dx_3 + dy_1.dy_1 - dy_2.dy_2 + dx_3.dx_3 \\ K = \frac{\partial}{\partial x_1} \otimes dy_1 - \frac{\partial}{\partial y_1} \otimes dx_1 + \frac{\partial}{\partial y_2} \otimes dx_2 - \frac{\partial}{\partial x_2} \otimes dy_2 - \frac{\partial}{\partial y_3} \otimes dx_3 + \frac{\partial}{\partial x_3} \otimes dy_3 \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dz_1 \wedge dz_2 \wedge dz_3 \\ \beta = \overline{\alpha} \end{cases}$$

The generalized Calabi-Yau structure associated with the equation

$$\mathrm{hess}(f) = 1$$

is the "real" Calabi-Yau structure

$$\begin{cases} g = \sum_{j=1}^{3} dx_j.dy_j \\ K = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \otimes dx_j - \frac{\partial}{\partial y_j} \otimes dy_j \\ \Omega = \sum_{j=1}^{3} dx_j \wedge dy_j \\ \alpha = dx_1 \wedge dx_2 \wedge dx_3 \\ \beta = dy_1 \wedge dy_2 \wedge dy_3 \end{cases}$$

A manifold endowed with a "real" Calabi-Yau structure is the analogue of a "Monge-Ampère manifold" in the Kontsevich and Soibelman sense ([KS]*). A Monge-Ampere manifold is an affine riemannian manifold* (M, g) *such that locally*

$$g = \sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j} dx_i.dx_j,$$

F *being a smooth function satisfying*

$$\det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right) = \textit{ constant.}$$

In the "real" Calabi-Yau case we have such a potential F:

$$g = \sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial y_j} dx_i . dy_j,$$

and $\det\left(\frac{\partial^2 F}{\partial x_i \partial y_j}\right) =$ *constant. (see* [Ba2] *for more details).*

Let (Ω, ω) be a Monge-Ampère structure with $\lambda(\omega) = \pm 1$. Since $d\omega = d\hat{\omega} = 0$ if and only if $d\alpha = d\beta = 0$, we have the obvious proposition:

Proposition 4.5. *A generalized almost Calabi-Yau structure associated with a nondegenerate Monge-Ampere structure is integrable if and only this Monge-Ampère structure is closed.*

4.2. **Nondegenerate Monge-Ampère equations.** Let us come back now to the differential equation associated with a nondegenerate Monge-Ampère structure (Ω, ω) on a 6-dimensional manifold X. It is natural to ask if this equation is locally symplectically equivalent to one of these:

$$\begin{cases} \mathrm{hess}(f) = 1 \\ \Delta(f) - \mathrm{hess}(f) = 0 \\ \Box(f) + \mathrm{hess}(f) = 0 \end{cases}$$

According to table 1, it will be the case if and only if (Ω, ω) is locally constant. The following theorem gives a criterion using the generalized Calabi-Yau structure associated.

Theorem 4.6. *A Monge-Ampère equation associated with a nondegenerate Monge-Ampère structure can be reduced by a symplectic change of coordinates to one of the following equations*

$$\begin{cases} \mathrm{hess}(f) = 1 \\ \Delta(f) - \mathrm{hess}(f) = 0 \\ \Box(f) + \mathrm{hess}(f) = 0 \end{cases}$$

if and only if the generalized Calabi-Yau structure associated is integrable and flat.

We refer to [Ba2] for the proof. The idea is that the integrability condition implies the existence of a "generalized" Kähler potential and the flat condition allows us to choose a Darboux coordinates system in which this potential has a nice expression.

Lychagin and Roubtsov have proved an equivalent theorem in [LR3] using technics of formal integrability. Theorem 4.6 is more restrictive since it only concerns nondegenerate Monge-Ampère equations but it is worth mentioning that its statement and its proof are much more simple and that it has a nice geometric meaning.

We sum up in table 2 the correspondance between (pseudo) Calabi-Yau structures and ellipic Monge-Ampère structures.

Example 4.7. *There are very few explicit examples of Calabi-Yau metrics. One of these is the Stenzel metric on T^*S^n (see for instance* [A],[St]*). This metric is not flat, therefore the special lagrangian equation associated with is not the classical one.*

$T^*S^n = \left\{ (u,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : ||u|| = 1, < u,v >= 0 \right\}$ *can be seen as the complex manifold* $Q^n = \left\{ z \in C^{n+1} : z_1^2 + \ldots + z_{n+1}^2 = 1 \right\}$ *using the isomorphism*

$$\xi(x+iy) = (\frac{x}{\sqrt{1+||y||^2}}, y).$$

The holomorphic form is then

$$\alpha_z(Z_1, \ldots, Z_n) = det_{\mathbb{C}}(z, Z_1, \ldots, Z_n).$$

and the Kähler form is $\Omega = i\partial\bar{\partial}\phi$ *with* $\phi = f(\tau)$ *where* τ *is the restriction to* Q^n *of* $|z_1|^2 + \ldots + |z_n^2$ *and* f *is a solution of the ordinary differential equation*

$$x(f')^n + f''(f')^{n-1}(x^2-1) = c > 0.$$

To write the special lagrangian equation we have to find some Darboux coordinates. Using the relations

$$\begin{cases} \sum_{k=1}^{4} u_k du_k + v_k dv_k = 0 \\ \sum_{k=1}^{4} u_k dv_k + v_k du_k = 0 \end{cases}$$

on T^*S^3, *we see that on the chart* $u_4 \neq 0$,

$$\Omega = \sum_{k=1}^{3} dw_k \wedge du_k$$

with

$$w_k = 2\frac{f'(2+2||v||^2)\sqrt{1+||v||^2}}{u_4}(u_k v_4 - v_k u_4).$$

Denote by ψ *the map* $(u,w) \mapsto (x+iy)$. *The special lagrangian equation on* T^*S^3 *is then*

$$(\psi \circ df)^*(Im(\alpha)) = 0.$$

Note that it is difficult to explicit this equation and it doesn't seem possible to write it in a simple way.

To resume, we have defined the generalized Calabi-Yau structures in order to study an equivalence problem for Monge-Ampère equations. The author hopes that their construction is enough "natural" to have a physical meaning. It is actually natural to ask if the generalized Calabi-Yau manifolds have also mirror partners and if this mirror conjecture can be formulated in terms of Monge-Ampère equations.

almost (pseudo) CY	elliptic MA
(pseudo) CY	closed elliptic MA
flat (pseudo) CY	locally constant elliptic MA

TABLE 2. (pseudo) Calabi-Yau and elliptic Monge-Ampère structures

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