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0.1 Introduction

The problems that will be investigated in this paper are similar to problems in quantum physics. We apply ideas from quantum physics and non-commutative geometry with the assumption that all objects are finite dimensional.

In quantum mechanics we are concerned with infinite dimensional Hilbert spaces and the logics determining these spaces are quantum logics. Superselection rules give gradings in quantum physics. In the theory that is developed here, the spaces in question are finite dimensional, or essentially finite dimensional spaces, and the logics involved is non-standard logics defined by some finite lattices (topologies).

We investigate the category of graded modules and, to some extent, the category of graded algebras. The objects that are investigated are mostly graded by monoids, but there we have also looked at grading by lattices. The category of graded modules is a monoidal category. We shall use the theory of monoidal categories; functors, associativity constraints, quantizations, braidings, Mac Lane coherence conditions, or in the case of braidings, the Yang-Baxter equation, and so on. We describe how associativity constraints, quantizations and braidings on monoid graded modules can be represented by cohomology groups of monoids.

This theory is applied to algebras graded by the power algebra of finite sets and Post logic, which produce particular mathematical structures. Realizations of such structures produce for example the algebras of complex numbers, quaternions, Clifford algebras and Cayley numbers. Furthermore, this theory gives commutativity and associativity on quaternions and Cayley numbers, or octonions, which are classical examples of non-commutative and non-associative algebras.

Our choice of working with monoids instead of groups is connected with differential equations. The group algebra over the complex numbers is a semisimple algebra, but the monoid algebra is composed of one semisimple part and one radical part. There is a connection between radical algebras and solutions of differential equations.

Chapter 1

Category of monoid graded modules

Throughout this paper let M be a finite commutative monoid and let k be a commutative ring with unit. We shall first define M-graded k-modules, then show that gradings correspond to actions in certain cases and further describe algebras in the category of M-graded k-modules.

1.1 M-graded modules

Definition 1 *Let $\{X_i\}_{i\in M}$ be a family of k-modules indexed by M. Then*

$$X = \oplus_{i\in M} X_i = \sum_{i\in M} X_i$$

is the M-graded module associated with the family $\{X_i\}_{i\in M}$. Here $\oplus$, or equivalently $\sum$, is the direct sum of modules. $(X)_m = X_m$ is the mth component of the graded module X.

M-graded module-morphisms

A *M-graded module-morphism* $f : X' \to X$, where X' and X are M-graded modules, is a homomorphism such that X'_i maps to X_i, $\forall i \in M$. The M-graded morphism, f, can be expressed as a family of homomorphisms

$$f = \{f_i : X'_i \to X_i\}_{i\in M}.$$

Let $\mathfrak{Mod}_{k(M)}$ denote the category of M-graded k-modules, where the objects are M-graded k-modules and the arrows are the M-graded morphisms.

We denote the set of all M-graded morphisms $f : X' \to X$ by $Hom_M(X', X)$. For $f, g \in Hom_M(X', X)$ we have the addition $f + g = \{f_i + g_i : X'_i \to X_i\}_{i \in M}$, where the addition of each f_i and g_i is the usual addition in hom-sets in the category of k-modules. $Hom_M(X', X)$ has the structure of an additive abelian group with bilinearity in the composition of M-graded k-module-morphisms such that $\mathfrak{Mod}_{k(M)}$ is an *Ab-category [2].*

The object X in $\mathfrak{Mod}_{k(M)}$ defined by and $X_i = 0$ for all $i \in M$, is a null object.

Direct sum or binary biproduct of M-graded modules

Let X and X' be M-graded modules. The direct sum of X and X' is

$$X \oplus X' = (\oplus_{i \in M} X_i) \oplus (\oplus_{i \in M} X'_i) = \oplus_{i \in M}(X_i \oplus X'_i)$$

The mth component of the direct sum is $(X \oplus Y)_m = X_m \oplus Y_m$. The direct sum is the *binary biproduct* in the category $\mathfrak{Mod}_{k(M)}$. Clearly $\mathfrak{Mod}_{k(M)}$ has all finite products.

Kernels and cokernels of M-graded morphisms

A *kernel* of a morphism $f = \{f_i : X'_i \to X_i\}_{i \in M}$ in $\mathfrak{Mod}_{k(M)}$ is the direct sum of the usual kernels of each f_i in the category of k-modules, i.e. $\ker f = \oplus_{i \in M} \ker f_i$, and similarly $\operatorname{coker} f = \oplus_{i \in M} \operatorname{coker} f_i$.

Further we define f in $\mathfrak{Mod}_{k(M)}$ to be *monic* if and only if each f_i is monic and hence every monic is a kernel. We define f to be *epi* if and only if each f_i is epi and hence every epi is a cokernel.

We see that $\mathfrak{Mod}_{k(M)}$ is an *abelian category [2].*

M-graded submodules

Definition 2 *Let $Y = \oplus_{i \in M} Y_i$ be a M-graded module and X as above. Y is called a (M-graded) submodule of X if Y_i is a submodule of X_i for all $i \in M$.*

Define a short sequence

$$0 \to X \xrightarrow{i} Y \xrightarrow{r} Z \to 0$$

of M-graded modules to be *exact* if and only if the sequence

$$0 \to X_m \overset{i_m}{\to} Y_m \overset{r_m}{\to} Z_m \to 0$$

of k-modules is exact for all $m \in M$.

M-graded factor modules

Definition 3 *Let X and Y be graded modules as above, Y is a M-graded submodule of X. The factormodule X/Y is the M-graded module with components*

$$(X/Y)_m = X_m/Y_m$$

for all $m \in M$.

The factormodule X/Y is the cokernel of the injection of Y in X in the exact sequence above.

Tensor product of graded modules

Let X and X' be objects in the category $\mathfrak{Mod}_{k(M)}$. Let $X \otimes_k X'$ be the tensor product of X and X' over k. We have

$$X \otimes_k X' = (\oplus_{i\in M} X_i) \otimes_k (\oplus_{j\in M} X'_j) = \oplus_{i\in M} (X_i \otimes_k (\oplus_{j\in M} X'_j)) = \oplus_{i,j\in M} (X_i \otimes_k X'_j).$$

We define the mth component of the tensor product as

$$(X \otimes_k X')_m = \oplus_{i+j=m} (X_i \otimes_k X'_j).$$

Let $f : X \to Y$ and $g : X' \to Y'$ be M-graded morphisms and X, X', Y, Y' be M-graded modules. Let $f \otimes_k g$ be the tensor product of f and g over k. We have

$$f \otimes_k g = \oplus_{i,j\in M} (f_i \otimes_k g_j) : X \otimes_k X' \to Y \otimes_k Y'$$

by the rule

$$x \otimes x' \longmapsto f \otimes g\,(x \otimes x') = f\,(x) \otimes g\,(x') = f_i\,(x) \otimes_k g_j\,(x')$$

for any $x \in X_i$ and $x' \in X'_j$. We define the mth component as follows

$$(f \otimes_k g)_m = \oplus_{\substack{i,j\in M \\ i+j=m}} (f_i \otimes_k g_j).$$

Together with the tensor product, $\mathfrak{Mod}_{k(M)}$ is a monoidal category *[2]*

$$\left\langle \mathfrak{Mod}_{k(M)}, \otimes_k, e, \alpha, \lambda, \rho \right\rangle$$

where $e = \oplus_{i\in M} e_i$, $e_0 = k$ and $e_m = 0$, $m \neq 0$, and the associativity constraint α is a natural isomorphism

$$\alpha = \alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \tilde{\to} (X \otimes Y) \otimes Z \tag{1.1}$$

$\forall X, Y, Z \in \mathfrak{Mod}_{k(M)}$ which satisfies the Mac Lane coherence condition. We have the well known isomorphisms λ and ρ,

$e \otimes_k X = \oplus_{i\in M}(k \otimes_k X_i) \overset{\lambda}{\cong} X \overset{\rho}{\cong} \oplus_{i\in M}(X_i \otimes_k k) = X \otimes_k e.$

1.2 Projectors of M-graded modules

Let $X = \sum_{i\in M} X_i$ be a M-graded k-module.

Let $\{\pi_m\}_{m\in M}$ be a family of projectors

$$\pi_m : X \to X,$$

$$\left(\sum_{n\in M} x_n\right) \longmapsto (0, ..., x_m, ..., 0)\,,$$

defined by the properties

$$\pi_m^2 = \pi_m, \tag{1.2}$$

$$\pi_m \pi_n = \pi_n \pi_m = \begin{cases} 0; n \neq m \\ \pi_m; n = m \end{cases}, \tag{1.3}$$

$$\sum_{m\in M} \pi_m = 1. \tag{1.4}$$

Lemma 4 *Let X be a k-module. If $\{\pi_m : X \to X\}_{m\in M}$ is a family of projectors on X with the properties above, then there is a grading by M on X given by*

$$X_m = \mathrm{Im}\,(\pi_m)\,.$$

If there is a grading by a module M on X, then there is a family of projectors $\{\pi_m : X \to X\}_{m\in M}$ defined by

$$\pi_m(x) = \pi_m\left(\sum_{n\in M} x_n\right) = (0, ..., x_m, ..., 0)\,,$$

$x \in X$.

Let the set of functions $\{\theta_m\}_{m\in M}$ be the basis of the set of all k-valued functions on M, $k(M)$. By noting that, for a M-graded module $X = \sum_{n\in M} X_n$,

$$\pi_m(x) = \theta_m \cdot x = \sum_{n\in M} \theta_m(n)\, x_n = (0, ..., x_m, ..., 0),\ x \in X,$$

we can introduce a $k(M)$-module structure on X. Let $f \in k(M)$. Define an action of f on X as follows

$$f \cdot x = \sum_{m\in M} f(m)\, \theta_m \cdot x = \sum_{m\in M} f(m)\, \pi_m(x) \in X.$$

Theorem 5 *Let X be a k-module. Then X is a $k(M)$-module if and only if it is a M-graded k-module*

Proof. Let X be a $k(M)$-module. Now let $X_m = \theta_m \cdot X$, and we obviously have a grading on X. The opposite is proved above. ■

Denote the dual of the function algebra by $k(M)^* = Hom(k(M), k) = k[M]$. Let $\{\theta_m^*\} = \{\delta_m\}$ be the dual basis for $k(M)^* = k[M]$. Both $k(M)$ and $k[M]$ are k-k-bialgebras. Further can it be shown that $k(M)^* = k[M]$ is the monoid algebra of M over k.

It is a well known fact that if a k-module X is an M-module, then X is a $k[M]$-module, and similarly, if X is a $k[M]$-module, then X is an M-module. This result will be used to picture grading of modules versus actions on modules.

1.3 Gradings and actions

Let G be a finite abelian group, $k = \mathbb{C}$ and let $\hat{G}$ be the dual group

$$\hat{G} = Hom(G, \Pi),$$

where $\Pi = \{z \in \mathbb{C} \mid |z| = 1\}$ is the 1 dimensional torus.

The Fourier transform is an algebra isomorphism

$$F : \mathbb{C}[G] \to \mathbb{C}\left(\hat{G}\right)$$

given by

$$F(\delta_g)(\varkappa) = \varkappa(g)$$

for $g \in G$, $\varkappa \in \hat{G}$, and

$$F^{-1}(h) = \frac{1}{|G|} \sum_{\substack{\varkappa \in \hat{G} \\ g \in G}} \varkappa\left(g^{-1}\right) \cdot \delta_g,\ h = \sum_{\varkappa \in \hat{G}} \lambda_\varkappa \theta_\varkappa \in \mathbb{C}\left(\hat{G}\right)$$

This gives a way to transform G-modules, that is, $\mathbb{C}[G]$-modules, into $\hat{G}$-graded $\mathbb{C}$-modules, that is, $\mathbb{C}\left(\hat{G}\right)$-modules. Moreover, $\mathbb{C}\left(\hat{G}\right)$-modules corresponds to a coaction of $\hat{G}$, as follows.

If X is a G-module, then X is a $\mathbb{C}\left(\hat{G}\right)$-module where

$$h \cdot x = F^{-1}(h) \cdot x,\ x \in X, h \in \mathbb{C}\left(\hat{G}\right).$$

If X is a $\hat{G}$-graded $\mathbb{C}$-module, then X is G-module where

$$g \cdot x = F(g) \cdot x,\ x \in X, g \in G.$$

Remark 6 *When G is a finite abelian group, then $\hat{G} \cong G$ in a non-cannonical way.*

Remark 7 *We shall apply this to the case $\mathfrak{P}(\Omega) \cong (\mathbb{Z}_2, \cdot)^n$ and get the Fourier-Hadamard transform.*

These isomorphisms will be useful in applications of the theory developed in this paper. When we have an action of a finite abelian group on some k-module, we know that we have a grading by this group and, as we will see later, have a description of associativity constraints, quantizations and, to some extent, braidings on the module.

1.4 The Fourier-Hadamard transform for $(\mathbb{Z}_2)^n$

We shall now look at a special case, $k = \mathbb{R}$ or $\mathbb{C}$ and $M = (\mathbb{Z}_2, \cdot) = \mathbb{Z}_2$ and establish an isomorphism F between $\mathbb{R}(\mathbb{Z}_2)$ and $\mathbb{R}[\mathbb{Z}_2]$ which is the Fourier-Hadamard transform.

Let $M = \mathbb{Z}_2 = \{e, \varepsilon\}, \varepsilon^2 = e$ and let $k = \mathbb{R}$.

Projectors

Let X be a $\mathbb{Z}_2$-space, that is, X is a $\mathbb{R}$-module X together with a representation $\rho : \mathbb{Z}_2 \to Aut_{\mathbb{R}}(X)$ of $\mathbb{Z}_2$ on X, such that

$$\begin{aligned} e &\to \rho(e) = \mathbf{1} = id : X \to X \\ \varepsilon &\to \rho(\varepsilon) = A_\varepsilon = A : X \to X \end{aligned}$$

and $A^2 = \mathbf{1}$.

Clearly

$$p = \frac{(\mathbf{1} - A)}{2}$$

and

$$q = \mathbf{1} - p = \frac{(\mathbf{1} + A)}{2}$$

have the properties of projectors such that $X = pX \oplus qX = X_{-1} \oplus X_{+1}$ and X is a $\mathbb{Z}_2$-graded $\mathbb{R}$-module.

Similarly, if one has a grading on X, which implies that there are projectors, p and q, on X, then one can define an $\mathbb{Z}_2$-action on X by $A = A_\varepsilon = \mathbf{1} - 2p$, such that A satisfies $A^2 = \mathbf{1}$. Hence, X is a $\mathbb{R}[\mathbb{Z}_2]$-module. So a $\mathbb{R}$-module X is an $\mathbb{R}(\mathbb{Z}_2)$-module if and only if X is an $\mathbb{R}[\mathbb{Z}_2]$-module.

The Fourier-Hadamard transform

Let's describe the isomorphism between $\mathbb{R}(\mathbb{Z}_2)$ and $\mathbb{R}[\mathbb{Z}_2]$,

$$F : \mathbb{R}(\mathbb{Z}_2) \to \mathbb{R}[\mathbb{Z}_2].$$

$\mathbb{R}(\mathbb{Z}_2)$ has the basis $\{\theta_e, \theta_\varepsilon\}$ as a vector space over $\mathbb{R}$, $\mathbb{R}[\mathbb{Z}_2]$ has the basis $\{\delta_e, \delta_\varepsilon\}$ and for a representation ρ of $\mathbb{Z}_2$ in X we can set $A_m(X) = \delta_m \cdot X$, for all $m \in M = \mathbb{Z}_2$. We know from above that $\theta_m \cdot X = \pi_m(X)$, $m \in M$, where π_m is a projector on the M-graded set X. Hence

$$\theta_e \cdot X = q(X) = \frac{(\mathbf{1} + A)}{2}(X) = \frac{(\delta_e + \delta_\varepsilon)}{2} \cdot X$$

and

$$\theta_\varepsilon \cdot X = p(X) = \frac{(\mathbf{1} - A)}{2}(X) = \frac{(\delta_e - \delta_\varepsilon)}{2} \cdot X.$$

We have an algebra isomorphism F given by

$$\theta_e \longmapsto \frac{(\delta_e + \delta_\varepsilon)}{2} = \Delta_1 \tag{1.5}$$

$$\theta_\varepsilon \longmapsto \frac{(\delta_e - \delta_\varepsilon)}{2} = \Delta_2. \tag{1.6}$$

We have $\delta_e = \Delta_1 + \Delta_2$ and $\delta_\varepsilon = -\Delta_1 + \Delta_2$, and one sees immediately that $\mathbb{R}[\mathbb{Z}_2]$ is generated by Δ_1 and Δ_2. This is the Fourier transform for $\mathbb{Z}_2$.

Let $M = (\mathbb{Z}_2)^n$. Then

$$\mathbb{R}(M) = \underbrace{\mathbb{R}(\mathbb{Z}_2) \otimes \cdots \otimes \mathbb{R}(\mathbb{Z}_2)}_{n-times}$$

is isomorphic to

$$\mathbb{R}[M] = \underbrace{\mathbb{R}[\mathbb{Z}_2] \otimes \cdots \otimes \mathbb{R}[\mathbb{Z}_2]}_{n-times},$$

and $\underbrace{F \otimes \cdots \otimes F}_{n-times}$, the Fourier-Hadamard transform, is an algebra-isomorphism, between these two.

Let Ω be a set consisting of n elements labeled form 1 to n; $\Omega = \{a_1, \ldots, a_n\}$. Let $\mathfrak{P}(\Omega)$ be the power algebra of Ω. The operation $(A, B) \longmapsto A \square B = A \cup B \setminus A \cap B$ introduces an group structure on $\mathfrak{P}(\Omega)$. The characteristic functions $\chi_A : \Omega \to \mathbb{Z}_2^+$, $A \in \mathfrak{P}(\Omega)$ give isomorphisms of groups,

$$(\mathfrak{P}(\Omega), \square) \cong \left(\mathfrak{F}\left(\Omega, \mathbb{Z}_2^+\right), +\right) \cong (\mathbb{Z}_2, +)^n = (\mathbb{Z}_2, +)^\Omega$$

given by $A \longmapsto \chi_A(\Omega)$, where $\left(\mathfrak{F}\left(\Omega, \mathbb{Z}_2^+\right), +\right)$ is the set of functions on Ω with values in $\mathbb{Z}_2^+$. Hence we have an isomorphism

$$\underbrace{F \otimes \cdots \otimes F}_{n-times} : \mathbb{R}(\mathfrak{P}(\Omega)) \to \mathbb{R}[\mathfrak{P}(\Omega)].$$

1.5 M-graded k-algebras

Definition 8 *Let $\langle A, \mu, \eta\rangle$ be an algebra or monoid [2] in the monoidal category $\mathfrak{Mod}_{k(M)}$ where μ and η are morphisms $\mu : A \otimes A \to A$ and $\eta : e \to A$ such that μ is associative and unit preserving. We call A an M-graded k-algebra.*

$(A)_m = A_m$ is the mth component of the graded algebra A. $A_\mathbf{0}$ is a subring of A. The elements of A_r, $r \in M$, are called the *homogeneous elements of degree* r.

M-graded algebra-morphism

An M-graded k-algebra-morphism, or a morphism of algebras in $\mathfrak{Mod}_{k(M)}$,

$$f : A' \to A,$$

where A' and A are M-graded k-algebras, is an M-graded module-morphism such that A'_i maps to A_i, $\forall i \in M$, and such that the diagrams

$$\begin{array}{ccc} A' \otimes A' & \xrightarrow{\mu'} & A' \\ {\scriptstyle f \otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}, \qquad \begin{array}{ccc} e & \xrightarrow{\eta'} & A' \\ || & & \downarrow{\scriptstyle f} \\ e & \xrightarrow{\eta} & A \end{array}$$

commute.

M-graded subalgebras

Let $B = \oplus_{i \in M} B_i$ be a M-graded k-algebra and A as above. B is called a (M-graded) subalgebra of A if B is a submodule of the M-graded module A, B is closed under the multiplication μ and $1_A \in B$.

M-graded ideals

Let $I = \oplus_{i \in M} I_i$ be a M-graded submodule of A. I is a graded ideal of A if μ maps $A_r \otimes I_s$ into I_{r+s} and $I_s \otimes A_r$ into I_{s+r}, $\forall r, s \in M$.

M-graded factoralgebras

Let A be a graded algebra as above, let B be a graded ideal of the M-graded algebra A. We define the M-graded factoralgebra A/B to be the factormodule

$$A/B = \oplus_{i \in M} A_i/B_i$$

equipped with the induced product.

$(A/B)_m = A_m/B_m$ is the mth component of the M-graded algebra A/B.

Direct sum of M-graded algebras

Let $A = \langle A, \mu_A, \eta_A \rangle$ and $A' = \langle A', \mu_{A'}, \eta_{A'} \rangle$ be M-graded k-algebras. The direct sum of A and A' is the direct sum taken as M-graded modules

$$A \oplus A' = (\oplus_{i \in M} A_i) \oplus (\oplus_{i \in M} A'_i) = \oplus_{i \in M} (A_i \oplus A'_i).$$

with the monoid structure

$$\mu : (A \oplus A') \otimes (A \oplus A') = (A \otimes A) \oplus (A' \otimes A') \to A \oplus A'$$

given by

$$(a_i \oplus a_j) \otimes (a'_i \oplus a'_j) = (a_i \otimes a_j) \oplus (a'_i \otimes a'_j) \longmapsto \mu_A (a_i \otimes a_j) \oplus \mu_{A'} (a'_i \otimes a'_j)$$

and

$$\eta : e \to A \oplus A'$$

given by

$$r \longmapsto \eta_A (r) \oplus \eta_{A'} (r) .$$

The mth component of the direct sum is $(A \oplus A')_m = A_m \oplus A'_m$.

1.6 M-graded modules over M-graded algebras

1.6.1 M-graded A-modules

Definition 9 *Let $\langle A, \mu, \eta \rangle$ be an algebra in the monoidal category $\mathfrak{Mod}_{k(M)}$ and let $\nu : A \otimes X \to X$ be a left action [2] of A on the object X in $\mathfrak{Mod}_{k(M)}$. X together with the morphism ν is called a M-graded A-module.*

M-graded A-module-morphisms

A morphism of M-graded A-modules $f : \langle X', \nu' \rangle \to \langle X, \nu \rangle$ is a morphism $f : X' \to X$ of M-graded k-modules such that the diagram

$$\begin{array}{ccc} A \otimes X' & \xrightarrow{\nu'} & X' \\ {\scriptstyle 1 \otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ A \otimes X & \xrightarrow{\nu} & X \end{array}$$

commutes.

M-graded A-submodules

Let $\langle X, \nu \rangle$ be a M-graded A-module. A M-graded k-submodule of X, Y, is a M-graded A-submodule of X if ν map $A \otimes Y$ into Y.

M-graded A-factormodules

Let X be a M-graded A-module and let Y be a M-graded A-submodule of X. We define the factormodule X/Y as

$$X/Y = \oplus_{i \in M} X_i/Y_i,$$

where X_i/Y_i is the factor module of X_i over Y_i. $(X/Y)_m = X_m/Y_m$ is the mth component of the graded module X/Y.

Note: $\mu : A_r \otimes X_s/Y_s \to X_{r+s}/Y_{r+s}$, $\forall r, s \in M$. .

Direct sum of M-graded A-modules

Let $\langle X, \nu \rangle$ and $\langle X', \nu' \rangle$ be M-graded A-modules. The direct sum of X and X' is

$$X \oplus X' = (\oplus_{i \in M} X_i) \oplus (\oplus_{i \in M} X'_i) = \oplus_{i \in M} (X_i \oplus X'_i)$$

and the action of A is preserved as $\nu \oplus \nu' : A \otimes X \oplus X' \to X \oplus X'$ is given by

$$a_i \otimes (x_j \oplus x'_l) = a_i \otimes x_j \oplus a_i \otimes x'_l \longmapsto \nu (a_i \otimes x_j) \oplus \nu' (a_i \otimes x'_l) .$$

The mth component of the direct sum is $(X \oplus Y)_m = X_m \oplus Y_m$.

The tensor product of M-graded A-modules

Let $\langle X, \nu \rangle$ and $\langle X', \nu' \rangle$ be M-graded A-modules. The tensor product, $X \otimes_k X'$, of X and X'over k is defined as for M-graded k-modules where the action of A is preserved as $\nu \otimes \nu' : A \otimes X \otimes X' \to X \otimes X'$ is given by

$$a_i \otimes (x_j \otimes x'_l) \longmapsto \nu (a_i \otimes x_j) \otimes x'_l = x_j \otimes \nu' (a_i \otimes x'_l) .$$

The mth component of the tensor product is $(X \otimes X')_m = \oplus_{i+j=m} (X_i \otimes X'_j)$.

Let $f : X \to Y$ and $g : X' \to Y'$ be M-graded A-morphisms and $\langle X, \nu_X \rangle$, $\langle X', \nu'_X \rangle$, $\langle Y, \nu_Y \rangle$ and $\langle Y', \nu'_Y \rangle$ M-graded A-modules. Let $f \otimes_k g$ denote the

tensor product of f and g. The tensor product $f \otimes_k g$ is the tensor product of f and g as morphisms of M-graded k-modules such that the diagram

$$\begin{array}{ccc} A \otimes X \otimes X' & \xrightarrow{\nu_X \otimes \nu_{X'}} & X \otimes X' \\ \Big\downarrow{\scriptstyle 1 \otimes (f \otimes g)} & & \Big\downarrow{\scriptstyle f \otimes g} \\ A \otimes Y \otimes Y' & \xrightarrow{\nu_Y \otimes \nu_{Y'}} & Y \otimes Y' \end{array}$$

commutes.

1.7 M-graded algebras over M-graded algebras

1.7.1 M-graded A-algebra

Definition 10 *Let $\langle A, \mu_A, \eta_A \rangle$ and $\langle B, \mu, \eta \rangle$ be M-graded k-algebras in the monoidal category $\mathfrak{Mod}_{k(M)}$ and let $\nu : A \otimes B \to B$ be a left action of A on B. B together with the morphism ν is called a M-graded A-algebra.*

That is, B is *the M-graded A-algebra indexed by the family* $\{B_i\}_{i \in M}$ if there exists a morphism $\nu : A \otimes B \to B$ satisfying unit preserving properties that is associative .

An M-graded A-algebra can also be described by a *comma construction* [2]. The subcategory $\mathfrak{Alg}_{A(M)}$ of $\mathfrak{Mod}_{k(M)}$ consisting of all M-graded A-algebras is equivalent to the comma category $\left(A \downarrow \mathfrak{Alg}_{k(M)}\right)$ which is the set of all pairs (B, ν_B) where $B = \langle B, \mu, \eta \rangle$ is an M-graded k-algebra and ν_B is a map from A to B. The left action ν of A on B we now define to be $\nu = \mu \circ (\nu_B \otimes \mathbf{1}_B)$. We see easily that ν satisfies the conditions for left actions. Similarly we define a right action ν^r of A on B by $\nu^r = \mu \circ (\mathbf{1}_B \otimes \nu_B)$. With both these actions B is a bialgebra, or more precisely an A-A-algebra.

Morphisms

A morphism $f : \langle B, \mu, \eta, \nu \rangle \to \langle B', \mu', \eta', \nu' \rangle$ is a monoid morphism

$$f : \langle B', \mu', \eta' \rangle \to \langle B, \mu, \eta \rangle$$

such that

$$\begin{array}{ccc} A \otimes B' & \xrightarrow{\nu'} & B' \\ {\scriptstyle 1 \otimes f}\downarrow & & \downarrow{\scriptstyle f}\,. \\ A \otimes B & \xrightarrow{\nu} & B \end{array} \tag{1.7}$$

Sub-, factoralgebras, ideals, the direct sum are similar to those for M-graded k-algebras.

1.8 Quantizations of monoidal categories

A quantization [1] of a monoidal category C is a natural isomorphism of the bifunctor

$$q \;:\; \otimes \to \otimes, \tag{1.8}$$

$$q_{X,Y} \;:\; X \otimes Y \to X \otimes Y, \tag{1.9}$$

$X, Y \in Ob\,(C)$, which preserves the unit, $q_{X,e} = q_{e,X} = \mathbf{1}_X$, and such that the diagram

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & & \xrightarrow{\alpha} & & (X \otimes Y) \otimes Z \\ {\scriptstyle 1 \otimes q_{Y,Z}}\downarrow & & & & \downarrow{\scriptstyle q_{X,Y} \otimes 1} \\ X \otimes (Y \otimes Z) & & & & (X \otimes Y) \otimes Z \\ \downarrow{\scriptstyle q_{X,Y \otimes Z}} & & & & \downarrow{\scriptstyle q_{X \otimes Y,Z}} \\ X \otimes (Y \otimes Z) & & \xrightarrow{\alpha} & & (X \otimes Y) \otimes Z \end{array} \tag{1.10}$$

commutes for all $X, Y, Z \in Ob\,(C)$.

Quantizations of algebras

Definition 11 *Let A be an algebra in C. We shall define a quantization A_q of an algebra A, given a quantization q in the category C, as the same object A equipped with a new multiplication $\mu_q = \mu \circ q_{A,A} : A \otimes A \to A$.*

Theorem 12 *(A, μ_q, η) is an algebra. [1]*

Quantizations of modules

Definition 13 *Let there be a left action of $A \in Ob(C)$ on $X \in Ob(C)$. By a quantization X_q of the module X we mean the same object X equipped with a new morphism*

$$\nu_q = \nu \circ q_{A,X} : A \otimes X \to A.$$

Theorem 14 *(X, ν_q) is a left A-module in C. [1]*

1.9 Braidings in monoidal categories

Now equip the category with a braiding [2], that is, a natural isomorphism

$$\sigma = \sigma_{X,Y} : X \otimes Y \tilde{\to} Y \otimes X \tag{1.11}$$

which preserves the unit, $\sigma_{X,e} = \sigma_{e,X} = \mathbf{1}_X$, and such that the diagrams

$$\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\alpha} & (X \otimes Y) \otimes Z \\
\downarrow{\scriptstyle 1 \otimes \sigma} & & \downarrow{\scriptstyle \sigma} \\
X \otimes (Z \otimes Y) & & Z \otimes (X \otimes Y), \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
(X \otimes Z) \otimes Y & \xrightarrow{\sigma \otimes 1} & (Z \otimes X) \otimes Y
\end{array}
\qquad
\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\alpha} & (X \otimes Y) \otimes Z \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma \otimes 1} \\
(Y \otimes Z) \otimes X & & (Y \otimes X) \otimes Z \\
\downarrow{\scriptstyle \alpha^{(-1)}} & & \downarrow{\scriptstyle \alpha^{(-1)}} \\
Y \otimes (Z \otimes X) & \xrightarrow{1 \otimes \sigma} & Y \otimes (X \otimes Z)
\end{array} \tag{1.12}$$

commute for all $X, Y, Z \in Ob(C)$. This can, of course be chosen to be the twist, $\otimes \circ \tau : X \times Y \to Y \otimes X$, where $\tau : X \times Y \to Y \times X$.

1.9.1 σ-commutative algebras

Let A be an algebra in C. We define a *braiding* σ of the algebra A as the composition

$$\mu_\sigma = \mu \circ \sigma_{A,A} : A \otimes A \to A. \tag{1.13}$$

Definition 15 *We say that A is σ-commutative if $\mu = \mu \circ \sigma$. [6]*

Chapter 2

Associativity constraints, quantizations and braidings.

Let M be a finite commutative monoid, written additively, and let k be a commutative ring with unit.

2.1 Representations of natural isomorphisms

The category of M-graded k-modules is equipped with an associativity constraint, which is a natural isomorphism

$$\alpha = \alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$$

for objects X, Y, Z in $\mathfrak{Mod}_{k(M)}$. We shall show that by naturality the associativity constraint only depends on the grading, and hence that it can be written as a function of 3 variables defined on the monoid M, with values in the group of unit elements in k.

Let X, X', Y, Y', Z, Z' be M-graded k-modules and let

$f : X' \to X$, $g : Y' \to Y$ and $h : Z' \to Z$ be M-graded k-morphisms. Let $\alpha_{X,Y,Z}$ be an unit preserving associativity constraint. Then the diagram

$$\begin{array}{ccc} X \otimes (Y \otimes Z) & \xrightarrow{\alpha_{X,Y,Z}} & (X \otimes Y) \otimes Z \\ \uparrow{\scriptstyle f \otimes (g \otimes h)} & & \uparrow{\scriptstyle (f \otimes g) \otimes h} \\ X' \otimes (Y' \otimes Z') & \xrightarrow{\alpha_{X',Y',Z'}} & (X' \otimes Y') \otimes Z' \end{array}$$

commutes for all f, g, h. In other words $\alpha_{X,Y,Z}$ is a natural isomorphism from the functor $\otimes \circ (\mathbf{1} \times \otimes)$ to $\otimes \circ (\otimes \times \mathbf{1})$.

We want to know what happens on the level of elements. Choose elements $x_i \in X_i \subset X$, $y_j \in Y_j \subset Y$ and $z_l \in Z_l \subset Z$.

We know that the ring k itself can be viewed as an M-graded k-module $k_{\mathbf{0}}$ with $(k_0)_{\mathbf{0}} = k$ and $(k_0)_i = 0$ for $i \neq 0$. Generally, for each $i \in M$, let k_i be an M-graded k-module such that

$$\begin{aligned} (k_i)_m &= k, \text{ if } m = i \\ (k_i)_m &= 0, \text{ if } m \neq i \end{aligned}$$

We can write $k_i = (0, \ldots, k, \ldots, 0)$ where k appears in the ith place.

For x_i let $\phi_{x_i} : k_i = (0, \ldots, k, \ldots, 0) \to X$ be a map such that

$$\phi_{x_i} : (0, \ldots, 1, \ldots, 0) \mapsto (0, \ldots, x_i, \ldots, 0)\,.$$

Similarly for y_j and z_l construct graded modules k_j and k_l with k in the jth and lth component respectively and construct maps ϕ_{y_j} and ϕ_{z_l} as for x_i. Now $\phi_{x_i} : k_i \to X$, $\phi_{y_j} : k_j \to Y$ and $\phi_{z_l} : k_l \to Z$ are M-graded morphisms. Hence, the diagram

$$\begin{array}{ccc} X \otimes (Y \otimes Z) & \xrightarrow{\alpha_{X,Y,Z}} & (X \otimes Y) \otimes Z \\ \uparrow{\scriptstyle \phi_{x_i} \otimes (\phi_{y_j} \otimes \phi_{z_l})} & & \uparrow{\scriptstyle (\phi_{x_i} \otimes \phi_{y_j}) \otimes \phi_{z_l}} \\ k_i \otimes (k_j \otimes k_l) & \xrightarrow{\alpha_{k_i,k_j,k_l}} & (k_i \otimes k_j) \otimes k_l \end{array}$$

commutes.

The element $1_i \otimes (1_j \otimes 1_l)$ is the unit element in $k_i \otimes (k_j \otimes k_l)$ so

$$\alpha : 1_i \otimes (1_j \otimes 1_l) \longmapsto \alpha\,(i, j, l)\,(1_i \otimes 1_j) \otimes 1_l$$

where $\alpha\,(i, j, l) \in U\,(k)$, $U\,(k)$ is the group of units in k. By construction will $(\phi_{x_i} \otimes \phi_{y_j}) \otimes \phi_{z_l}$ map $\alpha\,(i, j, l)\,(1_i \otimes 1_j) \otimes 1_l$ to $\alpha\,(i, j, l)\,(x_i \otimes y_j) \otimes z_l$. By commutativity we get

$$\alpha : x_i \otimes (y_j \otimes z_l) \longmapsto \alpha\,(i, j, l)\,(x_i \otimes y_j) \otimes z_l.$$

This is independent of the choice of x_i, y_j and z_l. We see that α only depends on the grading and can be represented as a map $\alpha : M \times M \times M \to U\,(k)$.

Theorem 16 *All natural isomorphisms* $\alpha : \otimes \circ (\mathbf{1} \times \otimes) \to \otimes \circ (\otimes \times \mathbf{1})$ *in the category of* M*-graded* k*-modules have the form*

$$\alpha : x_r \otimes (y_s \otimes z_t) \longmapsto \alpha(r,s,t)(x_r \otimes y_s) \otimes z_t,$$

where $\alpha : M \times M \times M \to U(k)$.

The category $\mathfrak{Mod}_{k(M)}$ is also equipped with natural isomorphisms λ and ρ

$$\begin{aligned} \lambda &: e \otimes X \tilde{\to} X, \\ \rho &: X \otimes e \tilde{\to} X \end{aligned}$$

$\forall X \in \mathfrak{Mod}_{k(M)}$, such that the triangular diagram

$$\begin{array}{ccc} X \otimes (e \otimes Z) & \xrightarrow{\alpha} & (X \otimes e) \otimes Z \\ {\scriptstyle 1 \otimes \lambda}\downarrow & & \downarrow{\scriptstyle \rho \otimes 1} \\ X \otimes Z & = & X \otimes Z \end{array}$$

commutes $\forall X, Z$ in $\mathfrak{Mod}_{k(M)}$. By the Coherence theorem, [2], we know that every diagram involving α, ρ and λ will commute, hence for any associativity

$$\alpha : x_r \otimes (y_s \otimes z_t) \longmapsto \alpha(r,s,t)(x_r \otimes y_s) \otimes z_t$$

for which any one of r, s or t is equal to $0 \in M$, one gets $\alpha(r,s,t) = 1 \in k$. Hence α satisfies the normalization condition [7].

For any natural isomorphism γ in the category of M-graded k-modules we can make a similar argument as for α.

Theorem 17 *Let* $C = \mathfrak{Mod}_{k(M)}$. *Any unit preserving natural isomorphism* γ *in the category of* M*-graded* k*-modules from one* n*-functor* $F : C \times \cdots \times C \to C$ *to another* n*-functor* $G : C \times \cdots \times C \to C$ *does only depend on the grading and has the following form* $F(x_{i_1}, \ldots, x_{i_n}) \cong \gamma(i_1, \ldots, i_n) G(x_{i_1}, \ldots, x_{i_n})$ *where* x_{i_j} *is an element of degree* i_j *in a* M*-graded* k*-module* X^j *and* γ *is a function* $\gamma : \underbrace{M \times \cdots \times M}_{n-times} \to U(k)$. *Furthermore, if* F *and* G *satisfies unit preserving conditions, then* γ *can be shown to be normalized.*

Hence, the natural isomorphism γ can be represented as a normalized n-cochain of M over $U(k)$ [7].

2.2 Associativity constraints in the category of M-graded modules

Let α be an associativity constraint in the monoidal category $\mathfrak{Mod}_{k(M)}$ of M-graded k-modules. For any objects X, Y, Z and W in $\mathfrak{Mod}_{k(M)}$ does α satisfy the Mac Lane coherence condition, that is, the diagram

$$\begin{array}{ccccc} X\otimes(Y\otimes(Z\otimes W)) & \xrightarrow{\alpha} & (X\otimes Y)\otimes(Z\otimes W) & \xrightarrow{\alpha} & ((X\otimes Y)\otimes Z)\otimes W \\ \downarrow{\scriptstyle 1\otimes\alpha} & & & & \downarrow{\scriptstyle \alpha\otimes 1} \\ X\otimes((Y\otimes Z)\otimes W) & & \xrightarrow{\qquad\alpha\qquad} & & (X\otimes(Y\otimes Z))\otimes W \end{array}$$

commutes. By Theorem 17 has the associativity constraint the form of a 3-cochain

$$\alpha : M\times M\times M\to U(k)$$

and the natural isomorphism has the form

$$\alpha : x_i\otimes(y_j\otimes z_k)\to \alpha(i,j,k)(x_i\otimes y_j)\otimes z_k$$

for $x_i\in X_i\subset X$, $y_j\in Y_j\subset Y$ and $z_k\in Z_k\subset Z$.

The coherence condition gives the following

$$\begin{aligned} & \alpha(i+j,k,l)\,\alpha(i,j,k+l)\,((x_i\otimes y_j)\otimes z_k)\otimes w_l \\ = \;& \alpha(j,k,l)\,\alpha(i,j,k)\,\alpha(i,j+k,l)\,((x_i\otimes y_j)\otimes z_k)\otimes w_l. \end{aligned}$$

So the coherence condition is satisfied if and only if

$$\alpha(i+j,k,l)\,\alpha(i,j,k+l)=\alpha(j,k,l)\,\alpha(i,j,k)\,\alpha(i,j+k,l),$$

that is, if and only if the coboundary of α is equal to 1.

Let $Z^3(M,U(k))$ denote the set of all 3-cocycles of M over$U(k)$.

Theorem 18 *The set of associativity constraints α represented by normalized 3-cochains is in correspondence with the group of 3-cocycles, $Z^3(M,U(k))$.*

Factorization by 3-coboundaries

Let p be a natural isomorphism of the tensor product functor in $\mathfrak{Mod}_{k(M)}$,

$$p_{X,Y} : X \otimes Y \to X \otimes Y \tag{2.1}$$

for objects X and Y. By Theorem 17, p has the form

$$p_{X,Y} : x_i \otimes y_j \to p(i,j)\, x_i \otimes y_j \tag{2.2}$$

and can be represented by a normalized 2-cochain of M with coefficients in $U(k)$. We define a new associativity constraint $p(\alpha)$ to be a natural isomorphism such that the diagram

$$\begin{array}{ccc} X \otimes (Y \otimes Z) & \xrightarrow{\alpha} & (X \otimes Y) \otimes Z \\ \Big\downarrow{\scriptstyle 1 \otimes p_{Y,Z}} & & \Big\downarrow{\scriptstyle p_{X,Y} \otimes 1} \\ X \otimes (Y \otimes Z) & & (X \otimes Y) \otimes Z \\ \Big\downarrow{\scriptstyle p_{X,Y \otimes Z}} & & \Big\downarrow{\scriptstyle p_{X \otimes Y,Z}} \\ X \otimes (Y \otimes Z) & \xrightarrow{p(\alpha)} & (X \otimes Y) \otimes Z \end{array} \tag{2.3}$$

commutes for all objects X, Y and Z in $\mathfrak{Mod}_{k(M)}$. Hence

$$\begin{aligned} & \alpha(i,j,k)\, p(i,j+k)\, p(j,k)\, x_i \otimes (y_j \otimes z_k) \\ = \;& p(i,j+k)\, p(j,k)\, p(\alpha)(i,j,k)\, (x_i \otimes y_j) \otimes z_k, \end{aligned}$$

so

$$\begin{aligned} p(\alpha)(i,j,k) &= \alpha(i,j,k)\, p(j,k)\, p^{-1}(i+j,k)\, p(i,j+k)\, p^{-1}(i,j) \\ &= \alpha(i,j,k)\, \delta(p)(i,j,k). \end{aligned}$$

We get the following theorem.

Theorem 19 *The set of all associativity constraints α on the category $\mathfrak{Mod}_{k(M)}$ up to natural isomorphisms of $\otimes$, is in one to one correspondence with the 3rd cohomology group $H^3(M, U(k))$.*

2.3 Quantizations of M-graded k-modules

Let q, $q_{X,Y} : X \otimes Y \to X \otimes Y$, be a quantization of the category of M-graded k-modules and let α be an associativity constraint. A quantization is a natural isomorphism of the tensor product functor, hence, by Theorem 17, q is represented by a 2-cochain of M with coefficients in $U(k)$ and has the form

$$q_{X,Y} : x_i \otimes y_j \longmapsto q(i,j)\, x_i \otimes y_j$$

for all $x_i \in X_i \subset X$ and $y_j \in Y_j \subset Y$. Associativity constraints are preserved under quantizations. For all quantizations does diagram

$$\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\alpha} & (X \otimes Y) \otimes Z \\
\Big\downarrow{\scriptstyle 1 \otimes q_{Y,Z}} & & \Big\downarrow{\scriptstyle q_{X,Y} \otimes 1} \\
X \otimes (Y \otimes Z) & & (X \otimes Y) \otimes Z \\
\Big\downarrow{\scriptstyle q_{X,Y \otimes Z}} & & \Big\downarrow{\scriptstyle q_{X \otimes Y,Z}} \\
X \otimes (Y \otimes Z) & \xrightarrow{\alpha} & (X \otimes Y) \otimes Z
\end{array}$$

hence

$$\alpha(i,j,k) = q(j,k)\, q^{-1}(i+j,k)\, q(i,j+k)\, q^{-1}(i,j)\, \alpha(i,j,k)$$

and $\delta(q) = 1$, that is, q is a 2-cocycle.

Theorem 20 *The set of quantizations on the category of M-graded k-modules is in correspondence with the group of 2-cocycles, $Z^2(M, U(k))$.*

Factorization by 2-coboundaries

Let

$$\begin{aligned} \lambda &: Id \to Id \\ \lambda &= \lambda_X : X \tilde{\to} X \end{aligned}$$

be a natural isomorphism in $\mathfrak{Mod}_{k(M)}$, where Id is the identity functor. We define a new quantization $\lambda(q)$ to be a natural isomorphism such that the following

diagram commutes

$$\begin{array}{ccc} X\otimes Y & \xrightarrow{q_{X,Y}} & X\otimes Y \\ {\scriptstyle \lambda_X\otimes\lambda_Y}\downarrow & & \downarrow{\scriptstyle \lambda_{X\otimes Y}} \\ X\otimes Y & \xrightarrow{\lambda(q_{X,Y})} & X\otimes Y \end{array} \tag{2.4}$$

for all X, Y in $\mathfrak{Mod}_{k(M)}$. We get

$$q(i,j) = \lambda(i)\,\lambda^{-1}(i+j)\,\lambda(j)\,q(i,j) = \delta(\lambda)(i,j)\,q(i,j).$$

Theorem 21 *The set of quantizations on the category of M-graded k-modules up to natural isomorphisms on the identity functor is in one to one correspondence with the 2nd cohomology group, $H^2(M, U(k))$.*

2.4 Braidings of M-graded k-modules

Let us equip our category with a braiding, i.e. a natural isomorphism

$$\sigma = \sigma_{X,Y} : X\otimes Y \cong Y\otimes X$$

such that the diagrams

$$\begin{array}{ccccc} X\otimes(Y\otimes Z) & \xrightarrow{\alpha} & (X\otimes Y)\otimes Z & \qquad X\otimes(Y\otimes Z) & \xrightarrow{\alpha} & (X\otimes Y)\otimes Z \\ {\scriptstyle 1\otimes\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} & {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma\otimes 1} \\ X\otimes(Z\otimes Y) & & Z\otimes(X\otimes Y), & (Y\otimes Z)\otimes X & & (Y\otimes X)\otimes Z \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} & {\scriptstyle \alpha^{(-1)}}\downarrow & & \downarrow{\scriptstyle \alpha^{(-1)}} \\ (X\otimes Z)\otimes Y & \xrightarrow{\sigma\otimes 1} & (Z\otimes X)\otimes Y & Y\otimes(Z\otimes X) & \xrightarrow{1\otimes\sigma} & Y\otimes(X\otimes Z) \end{array} \tag{2.5}$$

and

$$\begin{array}{ccc} X\otimes e & \xrightarrow{\sigma} & e\otimes X \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \lambda} \\ X & = & X \end{array} \tag{2.6}$$

commute. We know that for elements $x_i \in X_i \subset X$ and $y_j \in Y_j \subset Y$, σ has the form

$$\sigma : x_i \otimes y_j \to \sigma(i,j)\, y_j \otimes x_i$$

where σ is a normalized 2-cochain of M with coefficients in $U(k)$. By commutativity the diagrams 2.5

$$\begin{aligned} \alpha(i,j,k)\,\sigma(i+j,k)\,\alpha(k,i,j) &= \sigma(i,k)\,\alpha(i,k,j)\,\sigma(j,k), && (2.7)\\ \alpha(i,j,k)\,\sigma(i,j)\,\alpha^{-1}(j,i,k) &= \sigma(i,j+k)\,\alpha^{-1}(j,k,i)\,\sigma(k,i). && (2.8) \end{aligned}$$

Let $j = 0$, then equation 2.8 gives us

$$\sigma(i,k)\,\sigma(k,i) = 1$$

such that any braiding is a symmetry. We get that for any braiding σ in $\mathfrak{Mod}_{k(M)}$ the equations

$$\begin{aligned} \sigma(i+j,k) &= \frac{\alpha(i,k,j)}{\alpha(i,j,k)\,\alpha(k,i,j)}\sigma(i,k)\,\sigma(j,k), && (2.9)\\ \sigma(i,j+k) &= \frac{\alpha(i,j,k)\,\alpha(j,k,i)}{\alpha(j,i,k)}\sigma(i,j)\,\sigma(i,k) && (2.10) \end{aligned}$$

are satisfied. If α is trivial, then we get bihomomorphism conditions for σ,

$$\begin{aligned} \sigma(i+j,k) &= \sigma(i,k)\,\sigma(j,k), && (2.11)\\ \sigma(i,j+k) &= \sigma(i,j)\,\sigma(i,k). && (2.12) \end{aligned}$$

Theorem 22 *Any braiding in the category of M-graded k-modules is a symmetry.*

Theorem 23 *A braiding σ is a normalized 2-cochain such that 2.9 and 2.10 are satisfied.*

Any braiding σ together with the twist τ, $\sigma' = \sigma \circ \tau$, satisfies the coherence condition of quantizations, that is, $\sigma' \circ (\sigma' \otimes 1) \circ \alpha = (\sigma' \otimes 1) \circ \sigma' \circ \alpha$, hence $\sigma' = \sigma \circ \tau$ is a quantization and is represented by a symmetric 2-cocycle.

Chapter 3

Cohomology of the Sierpinsky topology

In this chapter we shall calculate the zero, first, second and third cohomology groups of the Sierpinsky Topology, also called Post logic. We will use these groups to find possible quantizations and associativity constraints on objects graded by the Sierpinsky topology.

The Sierpinsky topology, τ, is the topology of the set $\Omega = \{a, b\}$ where a is open and b is closed, hence $\tau = \{\emptyset, \{a\}, \Omega\}$. Denote $\emptyset$ by 0, $\{a\}$ by a and Ω by 1.

We define an addition on τ as follows,

$$0 + t = t + 0 = t, \forall t \in \tau \tag{3.1}$$

$$1 + t = t + 1 = 1, \forall t \in \tau \tag{3.2}$$

$$a + a = a. \tag{3.3}$$

The set τ is a monoid w.r.t. $+$. This addition corresponds to intersection.

We are now working on τ graded objects. Let A be a τ-graded algebra, i.e.

$$A = A_{\mathbf{0}} \oplus A_a \oplus A_1.$$

By the addition rules defined above we get that $A_{\mathbf{0}}$ is a k-algebra with a multiplication

$$A_{\mathbf{0}} \times A_{\mathbf{0}} \to A_{\mathbf{0}},$$

A_a and A_1 are $A_{\mathbf{0}}$-$A_{\mathbf{0}}$-algebras with bilinear maps

$$A_a \times A_a \to A_a \tag{3.4}$$

$$A_1 \times A_1 \to A_1 \tag{3.5}$$

and furthermore, A_1 is an A_a-A_a-algebra.

We shall first calculate the cohomology groups of the Sierpinsky topology with coefficients in an abelian group G and then move on to calculating the cohomology groups of $\tau \times \tau$ with coefficients in $\mathbb{C} \setminus \{0\}$.

3.1 $H^0(\tau, G)$

Let G be a multiplicative abelian group.

$C^0(\tau, G)$ is the set of all functions $f : \tau^0 \to G$. Now τ^0 is just a point so

$$C^0(\tau, G) = G.$$

We define the action of τ on S^1 to be the trivial one. Then $\delta(g) = 1$, $\forall g \in C^0(\tau, G)$, and

$$Z^0(\tau, G) = C^0(\tau, G) = G.$$

$B^0(\tau, G)$ is trivial by construction, so

$$H^0(\tau, G) = G \tag{3.6}$$

for any abelian group G.

3.2 $H^1(\tau, G)$

We have

$$\begin{aligned} f &\in Z^1(\tau, G) \\ &\Leftrightarrow \delta(f)(i,j) = f(i) f^{-1}(i+j) f(j) = 1 \\ &\Leftrightarrow f(i) f(j) = f(i+j) \end{aligned}$$

if and only if $Z^1(\tau, G)$ is the group of monoid homomorphisms, $Hom(\tau, G)$. Furthermore, $Z^0(\tau, G) = C^0(\tau, G)$, so $B^1(\tau, G)$ is trivial. Hence

$$H^1(\tau, G) = Z^1(\tau, G) = Hom(\tau, G).$$

Let $f \in Hom(\tau, G)$. For i or j equal 0, by normalization, the equation

$$f(i) f(j) = f(i+j)$$

is satisfied. Now let $i,j \in \tau \setminus \{0\}$. We require

$$\begin{aligned} f(1) f(1) &= f(1) \\ f(1) f(a) &= f(1) \\ f(a) f(1) &= f(1) \\ f(a) f(a) &= f(a). \end{aligned}$$

This implies $f(1) = 1$ and $f(a) = 1$. Hence, $f(x) = 1$, $\forall x \in \tau$. So

$$H^1(\tau, G) \text{ is trivial.} \tag{3.7}$$

3.3 $H^2(\tau, G)$

We have

$$\begin{aligned} f &\in Z^2(\tau, G) \\ &\Leftrightarrow \delta(f)(i,j,l) = f(j,l) f(i+j,l)^{-1} f(i,j+l) f(i,j)^{-1} = 1 \\ &\Leftrightarrow f(j,l) f(i,j+l) = f(i+j,l) f(i,j). \end{aligned}$$

By normalization this cocycle condition for f is satisfied when at least one of i,j,l are equal to 0. Now let's look at when $i,j,l \in \tau \setminus \{0\}$. We get the conditions

$$\begin{aligned} \delta(f)(1,a,1) &= 1 \Leftrightarrow \\ f(a,1) f(1,1) &= f(1,a) f(1,1) \Leftrightarrow f(a,1) = f(1,a) \\ \delta(f)(a,a,1) &= 1 \Leftrightarrow \\ f(a,1) f(a,1) &= f(a,a) f(a,1) \Leftrightarrow f(a,1) = f(a,a) \\ \delta(f)(1,a,a) &= 1 \Leftrightarrow \\ f(a,a) f(1,a) &= f(1,a) f(1,a) \Leftrightarrow f(a,a) = f(a,1) \end{aligned}$$

which give

$$Z^2(\tau, G) = \left\{ f \in C^2(\tau, G) \mid f(a,1) = f(1,a) = f(a,a) \right\}.$$

Now

$$g \in B^2(\tau, G) \Leftrightarrow g(i,j) = h(i) h^{-1}(i+j) h(j)$$

for $h \in C^1(\tau, G)$. Let's look at what happens when $i,j \in \tau \setminus \{0\}$,

$$\begin{aligned} g(1,1) &= h(1) h^{-1}(1) h(1) = h(1) \\ g(1,a) &= h(1) h^{-1}(1) h(a) = h(a) = g(a,1) = g(a,a). \end{aligned}$$

So

$$B^2(\tau,G)=\left\{\begin{array}{c} f\in Z^2(\tau,G)\mid f(a,1)=f(1,a)=f(a,a)=h(a),\\ f(1,1)=h(1),h\in C^1(\tau,G)\end{array}\right\}.$$

We get that the image of any $f\in Z^2(\tau,G)$ on τ is three elements $\{1,f(a,a),f(1,1)\}$. We can find $h\in C^1(\tau,G)$ such that $h(a)=f(a,a)=f(a,1)=f(1,a)$ and $h(1)=f(1,1)$. Then $\delta(h)=f\in B^2(\tau,G)$. Hence $Z^2(\tau,G)=B^2(\tau,G)$ and

$$H^2(\tau,G) \text{ is trivial.} \tag{3.8}$$

3.4 $H^3(\tau,G)$

We have

$$\begin{aligned} f &\in Z^3(\tau,G)\\ &\Leftrightarrow \delta(f)(i,j,l,m)=f(j,l,m)f^{-1}(i+j,l,m)f(i,j+l,m)f^{-1}(i,j,l+m)f(i,j,l)=1\\ &\Leftrightarrow f(j,l,m)f(i,j+l,m)f(i,j,l)=f(i+j,l,m)f(i,j,l+m).\end{aligned}$$

This is satisfied when at least one of i,j,l,m are 0. Let $i,j,l,m\in\tau\setminus\{0\}$, then we get the conditions

$$\begin{aligned} \delta(f)(1,1,1,1) &= 1\Leftrightarrow\\ f(1,1,1)f(1,1,1)f(1,1,1) &= f(1,1,1)f(1,1,1)\Leftrightarrow f(1,1,1)=1,\\ \delta(f)(a,1,1,1) &= 1\Leftrightarrow\\ f(1,1,1)f(a,1,1)f(a,1,1) &= f(1,1,1)f(a,1,1)\Leftrightarrow f(a,1,1)=1,\\ \delta(f)(1,1,a,1) &= 1\Leftrightarrow\\ f(1,a,1)f(1,1,1)f(1,1,a) &= f(1,a,1)f(1,1,1)\Leftrightarrow f(1,1,a)=1,\\ \delta(f)(a,1,a,1) &= 1\Leftrightarrow\\ f(1,a,1)f(a,1,1)f(a,1,a) &= f(1,a,1)f(a,1,1)\Leftrightarrow f(a,1,a)=1,\\ \delta(f)(1,a,a,1) &= 1\Leftrightarrow\\ f(a,a,1)f(1,a,1)f(1,a,a) &= f(1,a,1)f(1,a,1)\Leftrightarrow f(a,a,1)f(1,a,a)=f(1,a,1),\\ \delta(f)(a,a,a,a) &= 1\Leftrightarrow\\ f(a,a,a)f(a,a,a)f(a,a,a) &= f(a,a,a)f(a,a,a)\Leftrightarrow f(a,a,a)=1.\end{aligned}$$

So

$$Z^3(\tau,G)=\left\{\begin{array}{c} f\in C^3(\tau,G)\mid f(a,a,1)f(1,a,a)=f(1,a,1),\\ f(1,1,1)=f(a,1,1)=f(1,1,a)=f(a,a,a)=f(a,1,a)=1\end{array}\right\}.$$

For the coboundaries we have

$$g \in B^3(\tau, G) \Leftrightarrow g(i,j,l) = h(j,l)\, h(i+j,l)^{-1}\, h(i,j+l)\, h(i,j)^{-1}.$$

We get

$$\begin{aligned} g(a,a,1) &= h(a,1)\, h(a,1)^{-1}\, h(a,1)\, h(a,a)^{-1} = h(a,1)\, h(a,a)^{-1} \\ g(1,a,a) &= h(a,a)\, h(1,a)^{-1}\, h(1,a)\, h(1,a)^{-1} = h(a,a)\, h(a,1)^{-1} \\ &\Rightarrow g(a,a,1)\, g(1,a,a) = 1 \\ &\Rightarrow g(1,a,1) = 1, \end{aligned}$$

so

$$B^3(\tau, G) = \left\{ g \in Z^3(\tau, G) \mid g(a,a,1)\, g(1,a,a) = 1, g(1,a,1) = 1 \right\}.$$

So whether $f \in Z^3(\tau, G)$ is a coboundary depends on the image of $f(1,a,1)$. If $f(1,a,1) = 1$ then f is a coboundary, otherwise it is not. Hence

$$H^3(\tau, G) = G. \tag{3.9}$$

In summary

$$H^0(\tau, G) = G, \tag{3.10}$$
$$H^1(\tau, G) = 1, \tag{3.11}$$
$$H^2(\tau, G) = 1, \tag{3.12}$$
$$H^3(\tau, G) = G \tag{3.13}$$

for any abelian group G.

Conclusion 24 *For the category of τ-graded G-modules there exists no non-trivial quantizations, but there are $(\#(G) - 1)$ non-trivial associativity constraints.*

For any $g \in G$, let α_g be the associativity constraint on the category of τ-graded G-modules associated with g. Then, for $i, j, k \in \tau$,

$$\alpha_g(i,j,l) = g^{p(i,j)} g^{q(j,k)}$$

where $p, q : \tau \times \tau \to \tau$, $p(a,1) = q(1,a) = 1$, $p, q = 0$ otherwise, and $g^0 = 1, g^1 = g$. There is no straightforward representation of α_g.

3.5 $H^*\left(\tau \times \tau, S^1\right)$

We shall now calculate the cohomology groups of $\tau\times\tau$ with coefficients in $\mathbb{C}\backslash\{0\}$. Since $\tau \times \tau$ is finite, this is equivalent to computing the cohomology group with coefficients in S^1. We shall use the isomorphism $S^1 \cong \mathbb{R}/\mathbb{Z}$ when computing the cohomologies.

We have the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1 \to 0 \tag{3.14}$$

which yields the long exact sequence of cohomology groups

$$\begin{aligned} 0 \quad &\to \quad H^0(X,\mathbb{Z}) \to H^0(X,\mathbb{R}) \to H^0\left(X,S^1\right) \to H^1(X,\mathbb{Z}) \to \cdots \to \\ &\to \quad H^r(X,\mathbb{Z}) \to H^r(X,\mathbb{R}) \to H^r\left(X,S^1\right) \to H^{r+1}(X,\mathbb{Z}) \to \cdots. \end{aligned}$$

Let's calculate $H^r(\tau \times \tau, \mathbb{Z})$, $r = 0, 1, 2, 3$. Let $\tau \times \tau$ act on $\mathbb{Z}$ trivially. Then

$$H^0(\tau \times \tau, \mathbb{Z}) = C^0(\tau \times \tau, \mathbb{Z}) = \mathbb{Z}.$$

From the Künneth formula we have the sequence

$$\begin{aligned} 0 \quad &\to \quad \sum_{i+j=r} H^i(M,k) \otimes_k H^j(N,k) \to H^r(M \times N, k) \\ &\to \quad \sum_{i+j=r-1} Tor_1^k\left(H^i(M,k), H^j(N,k)\right) \to 0. \end{aligned}$$

We can use this to find the cohomology groups to $\tau \times \tau$ with coefficients in $\mathbb{Z}$.

$$0 \quad \to \quad \sum_{i+j=r} H^i(\tau,\mathbb{Z}) \otimes_k H^j(\tau,\mathbb{Z}) \to H^r(\tau \times \tau, \mathbb{Z}) \tag{3.15}$$

$$\to \quad \sum_{i+j=r-1} Tor_1^k\left(H^i(\tau,\mathbb{Z}), H^j(\tau,\mathbb{Z})\right) \to 0 \tag{3.16}$$

and for $r = 1$ this gives

$$0 \to \mathbb{Z} \otimes_k \{0\} \oplus \{0\} \otimes_k \mathbb{Z} \to H^1(\tau \times \tau, \mathbb{Z}) \to Tor_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \to 0$$

$Tor_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$ because $\mathbb{Z}$ is free and $\mathbb{Z} \otimes_k \{0\} = \{0\} \otimes_k \mathbb{Z} = \{0\}$ so

$$H^1(\tau \times \tau, \mathbb{Z}) \cong \{0\}. \tag{3.17}$$

Let $r = 2$. Then

$$\begin{aligned} 0 &\to \mathbb{Z} \otimes_k \{0\} \oplus \{0\} \otimes_k \mathbb{Z} \oplus \{0\} \otimes_k \{0\} \to H^2(\tau \times \tau, \mathbb{Z}) \\ &\to Tor_1^{\mathbb{Z}}(\{0\}, \mathbb{Z}) \oplus Tor_1^{\mathbb{Z}}(\mathbb{Z}, \{0\}) \to 0, \end{aligned}$$

and $\{0\} \otimes_k \{0\} = 0$ and $Tor_1^{\mathbb{Z}}(\mathbb{Z}, \{0\}) = 0$, so

$$H^2(\tau \times \tau, \mathbb{Z}) \cong \{0\}. \tag{3.18}$$

Further, for $r = 3$,

$$\begin{aligned} 0 &\to \mathbb{Z} \otimes_k \mathbb{Z} \oplus \mathbb{Z} \otimes_k \mathbb{Z} \to H^3(\tau \times \tau, \mathbb{Z}) \\ &\to Tor_1^{\mathbb{Z}}(\{0\}, \mathbb{Z}) \oplus Tor_1^{\mathbb{Z}}(\mathbb{Z}, \{0\}) \oplus Tor_1^{\mathbb{Z}}(\{0\}, \{0\}) \to 0 \end{aligned}$$

and $Tor_1^{\mathbb{Z}}(\{0\}, \mathbb{Z}) = Tor_1^{\mathbb{Z}}(\mathbb{Z}, \{0\}) = Tor_1^{\mathbb{Z}}(\{0\}, \{0\}) = 0$, so

$$H^3(\tau \times \tau, \mathbb{Z}) \cong \mathbb{Z} \otimes_k \mathbb{Z} \oplus \mathbb{Z} \otimes_k \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}. \tag{3.19}$$

We shall now calculate $H^*(\tau \times \tau, \mathbb{R})$.

Since $\mathbb{R}$ is a field can we use the Künneth theorem when calculating the cohomology group of $\tau \times \tau$ with coefficients in $\mathbb{R}$. The Künneth formula gives the isomorphism

$$H^*(M \times N, k) \cong H^*(M, k) \otimes_k H^*(N, k).$$

Let M and N both be τ and $k = \mathbb{R}$. From 3.10, 3.11, 3.12 and 3.13 we have

$$\begin{aligned} H^0(\tau, \mathbb{R}) &= H^3(\tau, \mathbb{R}) = \mathbb{R}, \\ H^1(\tau, \mathbb{R}) &= H^2(\tau, \mathbb{R}) = 0. \end{aligned}$$

$$\begin{aligned} H^0(\tau \times \tau, \mathbb{R}) &\cong H^0(\tau, \mathbb{R}) \otimes_{\mathbb{R}} H^0(\tau, \mathbb{R}) = \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} = \mathbb{R} \\ H^1(\tau \times \tau, \mathbb{R}) &\cong H^0(\tau, \mathbb{R}) \otimes_{\mathbb{R}} H^1(\tau, \mathbb{R}) \oplus H^1(\tau, \mathbb{R}) \otimes_{\mathbb{R}} H^0(\tau, \mathbb{R}) \\ &= \mathbb{R} \otimes_{\mathbb{R}} \{e\} \oplus \{e\} \otimes_{\mathbb{R}} \mathbb{R} = 0 \\ H^2(\tau \times \tau, \mathbb{R}) &\cong H^2(\tau, \mathbb{R}) \otimes_{\mathbb{R}} H^0(\tau, \mathbb{R}) \oplus H^0(\tau, \mathbb{R}) \otimes_{\mathbb{R}} H^2(\tau, \mathbb{R}) \\ &= \mathbb{R} \otimes_{\mathbb{R}} \{e\} \oplus \mathbb{R} \otimes_{\mathbb{R}} \{e\} = 0 \\ H^3(\tau \times \tau, \mathbb{R}) &\cong H^0(\tau, \mathbb{R}) \otimes_{\mathbb{R}} H^3(\tau, \mathbb{R}) \oplus H^3(\tau, \mathbb{R}) \otimes_{\mathbb{R}} H^0(\tau, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R} \end{aligned}$$

We have the long exact sequence

$$\begin{aligned} 0 \quad &\to \quad H^0(\tau\times\tau,\mathbb{Z}) \to H^0(\tau\times\tau,\mathbb{R}) \to H^0\left(\tau\times\tau,S^1\right) \to H^1(\tau\times\tau,\mathbb{Z}) \to \cdots \to \\ &\to \quad H^k(\tau\times\tau,\mathbb{Z}) \to H^k(\tau\times\tau,\mathbb{R}) \to H^k\left(\tau\times\tau,S^1\right) \to H^{k+1}(\tau\times\tau,\mathbb{Z}) \to \cdots \end{aligned}$$

and substituting, we get

$$\begin{aligned} 0 \quad &\to \quad \mathbb{Z}\to\mathbb{R}\to\mathbb{R}/\mathbb{Z}\to 0\to 0\to H^1\left(\tau\times\tau,S^1\right)=0\to \\ &\to \quad 0\to 0\to H^2\left(\tau\times\tau,S^1\right)=0\to\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{R}\oplus\mathbb{R}\to H^3\left(\tau\times\tau,S^1\right)\to 0\to\cdots \end{aligned}$$

So

$$H^0\left(\tau\times\tau,S^1\right) = S^1 \tag{3.20}$$

$$H^1\left(\tau\times\tau,S^1\right) = \{e\} \tag{3.21}$$

$$H^2\left(\tau\times\tau,S^1\right) = \{e\} \tag{3.22}$$

We see that there are no non-trivial quantizations for $\mathbb{C}$-modules graded by $\tau\times\tau$. Similarly can the cohomology groups be computed for $M=(\tau)^n$ for any $n\in\mathbb{N}$.

Chapter 4

Cohomology of $(\mathbb{Z}_2, +)^n$ with coefficients in $U(\mathbb{R})$

4.1 Cohomology of $(\mathbb{Z}_2, +)$ with coefficients in $U(\mathbb{R})$

Let $M = (\mathbb{Z}_2, +) = (\{0, 1\}, +)$, $k = \mathbb{R}$ and $U(k) = \mathbb{R} \setminus \{0\}$. Since M is finite, it is only needed to calculate the cohomology group with coefficients in

$$(\mathbb{Z}_2, \cdot) = (\{\pm 1\}, \cdot).$$

4.1.1 The zero cohomology group of $(\mathbb{Z}_2, +)$ with coefficients in $U(\mathbb{R})$

$C^0\left(\mathbb{Z}_2^+, \mathbb{Z}_2^{\cdot}\right)$ is the set of all functions $f : M^0 \to \mathbb{Z}_2^{\cdot}$. M^0 is just a point so

$$C^0\left(\mathbb{Z}_2^+, \mathbb{Z}_2^{\cdot}\right) = (\mathbb{Z}_2, \cdot).$$

We choose to work with the trivial action, so that

$$Z^0\left(\mathbb{Z}_2^+, \mathbb{Z}_2^{\cdot}\right) = C^0\left(\mathbb{Z}_2^+, \mathbb{Z}_2^{\cdot}\right) = \mathbb{Z}_2^{\cdot}$$

and $B^0(\tau, S^1)$ is trivial by construction, so

$$H^0\left(\mathbb{Z}_2^+, \mathbb{Z}_2^{\cdot}\right) = (\mathbb{Z}_2, \cdot). \tag{4.1}$$

4.1.2 The rth cohomology group of $(\mathbb{Z}_2,+)$ with coefficients in $U(\mathbb{R})$

For any r do we by normalization condition get only one non-trivial r-cochain, hence $C^r(M,U(k)) = C^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right) = \mathbb{Z}_2$. Let $f \in Z^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right)$ be a non-trivial cocycle, that is, $f(1,\ldots,1) = -1$. Now f is required to satisfy

$$\begin{aligned}\delta(f)(x_1,...,x_{r+1}) &= f(x_2,...,x_{r+1})\, f^{-1}(x_1+x_2,...,x_{r+1})\, f(x_1,x_2+x_3,...,x_{r+1})\cdots \\ &\quad \cdots f^{(-1)^r}(x_1,...,x_r+x_{r+1})\, f^{(-1)^{r+1}}(x_1,...,x_r) = 1.\end{aligned}$$

It is only necessary to check this for $x_1 = \cdots = x_{r+1} = 1$ and certainly

$$\delta(f)(1,...,1) = f(1,...,1)\, f^{(-1)^{r+1}}(1,...,1) = f(1,...,1)\, f(1,...,1) = (-1)(-1) = 1$$

for any r. Hence, there exist a non-trivial r-cocycle such that $Z^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right) = C^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right)$ for all $r \geq 0$. Moreover $B^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right)$ is trivial.

Theorem 25 *For all* $r \geq 0$

$$H^r\left(\mathbb{Z}_2^+,U(\mathbb{R})\right) = H^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right) = (\mathbb{Z}_2,\cdot) \tag{4.2}$$

Hence for any r there is one trivial ($f = 1$) and one non-trivial cohomology class.

4.1.3 Representatives of the cohomology classes

The trivial cohomology class is represented by $f \in Z^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right)$ for which

$f(1,1,\ldots,1) = 1$, that is, $f = 1$.

The non-trivial cohomology class can be represented by $f \in Z^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right)$ where

$f(x_1,\ldots,x_r) = (-1)^{x_1\cdots x_r}$.

In the case $r = 0$ the trivial cohomology class is represented by $f = 1$ and the non-trivial by $f = -1$.

4.2 Cohomology of $(\mathbb{Z}_2,+)^n$ with coefficients in $U(\mathbb{R})$

We shall find the cohomology groups of $M = (\mathbb{Z}_2,+)^n$ with coefficients in $(\mathbb{Z}_2,\cdot)$.

Isomorphism from $H^r(M,\mathbb{Z}_2)$ to $H^r\left(M,\mathbb{Z}_2^+\right)$

Let $f\in Z^r(M,\mathbb{Z}_2)$ be non-trivial, $f:\underbrace{M\times\cdots\times M}_{n-times}\to(\mathbb{Z}_2,\cdot)$. Any such f can be written as $f(x_1,\ldots,x_r)=(-1)^{P(x_1,\ldots,x_r)}$ where P is a function

$$P:M\times\cdots\times M\to(\mathbb{Z}_2,+)$$

satisfying the normalization condition, that is, $P\in C^r\left(M,\mathbb{Z}_2^+\right)$. Furthermore does f satisfy the r-cocycle condition,

$$\begin{aligned}\delta(f)(x_1,...,x_{r+1}) &= f(x_2,...,x_{r+1})\,f^{-1}(x_1+x_2,...,x_{r+1})\,f(x_1,x_2+x_3,...,x_{r+1})\cdots\\ &\quad\cdots f^{(-1)^r}(x_1,...,x_r+x_{r+1})\,f^{(-1)^{r+1}}(x_1,...,x_r)=1.\end{aligned}$$

so that

$$\begin{aligned}&P(x_2,...,x_{r+1})-P(x_1+x_2,...,x_{r+1})+P(x_1,x_2+x_3,...,x_{r+1})+\cdots\\ &\cdots+(-1)^r(x_1,...,x_r+x_{r+1})+(-1)^{r+1}P(x_1,...,x_r)=0\end{aligned}$$

and $P\in Z^r\left(M,\mathbb{Z}_2^+\right)$.

Now suppose f is a coboundary, $f=\delta(g)$, $g\in C^{r-1}(M,\mathbb{Z}_2)$. Explicitly

$$\begin{aligned}f(x_1,\ldots,x_r) &= g(x_2,...,x_r)\,g^{-1}(x_1+x_2,...,x_r)\,g(x_1,x_2+x_3,...,x_r)\cdots\\ &\quad\cdots g^{(-1)^{r-1}}(x_1,...,x_{r-1}+x_r)\,g^{(-1)^r}(x_1,...,x_{r-1})\\ &= (-1)^{Q(x_2,...,x_r)}(-1)^{-Q(x_1+x_2,...,x_r)}(-1)^{Q(x_1,x_2+x_3,...,x_r)}\cdots\\ &\quad\cdots(-1)^{(-1)^{r-1}Q(x_1,...,x_{r-1}+x_r)}(-1)^{(-1)^rQ(x_1,...,x_{r-1})}\\ &= (-1)^{P(x_1,...,x_r)}\end{aligned}$$

with $Q\in C^{r-1}\left(M,\mathbb{Z}_2^+\right)$ and P is a coboundary of M over $\mathbb{Z}_2^+$. Similarly, it easy to prove that if $P\in Z^r\left(M,\mathbb{Z}_2^+\right)$ (resp. $P\in B^r\left(M,\mathbb{Z}_2^+\right)$), then $f\in C^r(M,\mathbb{Z}_2)$ defined by $f=(-1)^P$ is a cocycle (resp. a coboundary).

Lemma 26 *The correspondence $P\in C^r\left(M,\mathbb{Z}_2^+\right)\longmapsto(-1)^P\in C^r(M,\mathbb{Z}_2)$ establishes an isomorphism between $H^r\left(M,\mathbb{Z}_2^+\right)$ and $H^r(M,\mathbb{Z}_2)$.*

The cohomology of $\left(\mathbb{Z}_2^+\right)^n$ with coefficients in $\mathbb{Z}_2$

Because of the isomorphism $H^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right)\cong H^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2^+\right)$, the Künneth formula can be used on $H^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2^+\right)$ to find $H^r\left(M,\mathbb{Z}_2\right)$ for $M=$, $n>1$. We have

$$H^*\left(\left(\mathbb{Z}_2^+\right)^n,\mathbb{Z}_2\right)\cong H^*\left(\left(\mathbb{Z}_2^+\right)^n,\mathbb{Z}_2^+\right)\cong \underbrace{H^*\left(\mathbb{Z}_2^+,\mathbb{Z}_2^+\right)\otimes_{\mathbb{Z}_2^+}\ldots\otimes_{\mathbb{Z}_2^+}H^*\left(\mathbb{Z}_2^+,\mathbb{Z}_2^+\right)}_{n-times}. \tag{4.3}$$

Let's describe $H^*\left(\mathbb{Z}_2^+,\mathbb{Z}_2^+\right)$. If $f\in Z^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right)$ is a representative of the non-trivial cohomology class, then $f\left(x_1,\ldots,x_r\right)=(-1)^{P(x_1,\ldots,x_r)}$, where

$$P\left(x_1,\ldots,x_r\right)=x_1\cdots x_r,$$

that is, P is a homogeneous polynomial of degree r in $\mathbb{Z}_2^+\left[x_1,\ldots,x_r\right]$.

Now let $P\left(x_1,\ldots,x_r\right)=l\left(x_1\right)\cdots l\left(x_r\right)$, where $l:\mathbb{Z}_2^+\to\mathbb{Z}_2^+$ is a non-trivial representative of the first non-trivial cohomology group, $H^1\left(\mathbb{Z}_2^+,\mathbb{Z}_2^+\right)$. The multiplication of the cocycles is the cup product, $\cup$. Hence, any non-trivial class in the cohomology group $H^r\left((\mathbb{Z}_2,+),\mathbb{Z}_2\right)$ can be represented by a homogeneous polynomial in one variable, l, of degree r with coefficients in $(\mathbb{Z}_2,+)$, and $l:(\mathbb{Z}_2,+)\to(\mathbb{Z}_2,+)$ is the identity. The polynomial, which necessarily must be $\underbrace{l\cup\cdots\cup l}_{r-times}=l^r$, is applied to the string of elements $\left(x_1,\ldots,x_r\right)$ and gives us the required function

$$f\left(x_1,\ldots,x_r\right)=(-1)^{l(x_1)\cdots l(x_r)}=(-1)^{x_1\cdots x_r}.$$

We get the algebra isomorphism

$$H^*\left(\mathbb{Z}_2^+,\mathbb{Z}_2^+\right)\cong H^*\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right)\cong\mathbb{Z}_2^+\left[l\right] \tag{4.4}$$

and

$$H^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2^+\right)\cong H^r\left(\mathbb{Z}_2^+,\mathbb{Z}_2\right)\cong\mathbb{Z}_2^{r+}\left[l\right]=\left\{e,l^r\right\}$$

where $\mathbb{Z}_2^{r+}\left[l\right]$ is the set of homogeneous polynomials of degree r.

Let's look at $H^1\left(\left(\mathbb{Z}_2^+\right)^n,\mathbb{Z}_2^+\right)$. $\left(\mathbb{Z}_2^+\right)^n$ is a vector space over $\mathbb{Z}_2^+$. By construction is $B^1\left(\left(\mathbb{Z}_2^+\right)^n,\mathbb{Z}_2^+\right)$ trivial. Let $f\in Z^1\left(\left(\mathbb{Z}_2^+\right)^n,\mathbb{Z}_2^+\right)$. Then

$$f\left(x\right)+f\left(y\right)=f\left(x+y\right),$$

so $Z^1\left(\left(\mathbb{Z}_2^+\right)^n,\mathbb{Z}_2^+\right)$ is isomorphic to the space of all linear maps from the vector space $\left(\mathbb{Z}_2^+\right)^n$ to $\mathbb{Z}_2^+$ which is generated by the coordinate functions, denoted by $l_1,l_2,\ldots,l_n$.

Now, from the Künneth formula,

$$H^*\left(\left(\mathbb{Z}_2^+\right)^n,\mathbb{Z}_2\right)\cong\mathbb{Z}_2^+\left[l_1\right]\otimes\ldots\otimes\mathbb{Z}_2^+\left[l_n\right]\cong\mathbb{Z}_2^+\left[l_1,\ldots,l_n\right],$$

where each l_i is the ith coordinate function from $(\mathbb{Z}_2,+)^n$ to $(\mathbb{Z}_2,+,\cdot)$. Hence

$$H^r\left(M,\mathbb{Z}_2\right)\cong\mathbb{Z}_2^{r+}\left[l_1,\ldots,l_n\right]$$

which is the subspace of the polynomial algebra consisting of all homogeneous polynomials of degree r.

We get

$$H^r\left(M,\mathbb{Z}_2\right)=\left\{\begin{array}{c}f\in C^r\left(M,\mathbb{Z}_2\right)\mid\\ f\left(x_1,\ldots,x_r\right)=(-1)^{P(l_1,\ldots,l_n)(x_1,\ldots,x_r)},\\ P\left(l_1,\ldots,l_n\right)\left(x_1,\ldots,x_r\right)=\\ \sum_{i_1,\ldots i_r=1}^n a_{i_1\ldots i_r}l_{i_1}\left(x_1\right)\cdots l_{i_r}\left(x_r\right),\ a_{i_1\ldots i_r}\in(\mathbb{Z}_2,+,\cdot)\end{array}\right\}\quad(4.5)$$

where $P\left(l_1,\ldots,l_n\right)$ is a homogeneous polynomial of degree r. We shall sometimes write P for $P\left(l_1,\ldots,l_n\right)$.

Chapter 5

$Aut\,(\Omega)$-invariant cohomologies

Let Ω be a set consisting of n elements labeled form 1 to n; $\Omega = \{a_1, \ldots, a_n\}$. Let $\mathfrak{P}(\Omega)$ be the power algebra of Ω. By the characteristic functions $\chi_A : \Omega \to \mathbb{Z}_2^+$, $A \in \mathfrak{P}(\Omega)$ we have the isomorphisms of groups,

$$(\mathfrak{P}(\Omega), \square) \cong \left(\mathfrak{F}\left(\Omega, \mathbb{Z}_2^+\right), +\right) \cong (\mathbb{Z}_2, +)^n = (\mathbb{Z}_2, +)^\Omega .$$

Denote by $Aut\,(\Omega) \cong S(n)$ the group of all automorphisms of Ω. Let

$$\sigma \in Aut\,(\Omega) .$$

This introduces an automorphism on the set of functions $\left(\mathfrak{F}\left(\Omega, \mathbb{Z}_2^+\right), +\right)$ by $\chi_A \longmapsto \chi_A \circ \sigma^{-1}$, hence we get an automorphism σ_Ω on $\left(\mathbb{Z}_2^+\right)^\Omega$,

$$\sigma_\Omega : (\mathbb{Z}_2, +, \cdot)^\Omega \to (\mathbb{Z}_2, +, \cdot)^\Omega$$

$$\chi_A(\Omega) \longmapsto \chi_A \circ \sigma^{-1}(\Omega) .$$

5.1 $Aut\,(\Omega)$-invariant cohomologies of $\mathfrak{P}(\Omega)$ with coefficients in $U(\mathbb{R})$

Let's look at the cohomologies of $(\mathbb{Z}_2, +)^\Omega$ with coefficients in $U(\mathbb{R})$, where Ω consists of n elements.

We want to look at cohomology groups that are independent of the labelling of the elements of Ω, that is, that only depend on Ω.

Let $f \in Z^r\left(\left(\mathbb{Z}_2,+\right)^{\Omega},\mathbb{Z}_2\right)$ and $f=(-1)^P$, $P \in Z^r\left(\left(\mathbb{Z}_2,+\right)^{\Omega},\mathbb{Z}_2^+\right)$. The cocycle f is $Aut(\Omega)$-*invariant* or *symmetric,* that is,

$$\sigma f = f$$

$\forall \sigma \in Aut(\Omega)$, if and only if P is symmetric. Then $P(l_1,\ldots,l_n)=P'(s_1,\ldots,s_n)$, where the weight of $P' \leq \deg P$ and s_i, $i=1,\ldots,n$, are the elementary symmetric polynomials. P is homogeneous of degree r, hence P' must also be homogeneous of degree r. Each s_i is homogeneous of degree i, so possible P' are

$$P'=\sum_{p_1 i_1+\cdots+p_s i_s=r} a_{i_1\ldots i_s} s_{i_1}^{p_1}\cdots s_{i_s}^{p_s}, a_{i_1\ldots i_s} \in (\mathbb{Z}_2,+,\cdot).$$

We shall investigate quantizations and associativity constraints on $(\mathbb{Z}_2,+)^{\Omega}$-graded algebras represented by $Aut(\Omega)$-invariant cohomology classes, that is, find the second and third $Aut(\Omega)$-invariant cohomology groups of $(\mathbb{Z}_2,+)^{\Omega}$ with coefficients in $U(\mathbb{R})$.

5.2 $Aut(\Omega)$-invariant second cohomologies of $\mathfrak{P}(\Omega)$ with coefficients in $U(\mathbb{R})$

Let a representative $f=(-1)^P \in Z^2\left(\left(\mathbb{Z}_2^+\right)^{\Omega},\mathbb{Z}_2\right)$ of a non-trivial cohomology class be $Aut(\Omega)$-invariant. Then

$$P=P'(s_1,s_2) \in \left\{s_1^2, s_2, s_1^2+s_2\right\}.$$

5.2.1 Quantizations of the two-point algebra

Now look at the case $n=1$. Then Ω has only one point and $\mathbb{Z}_2^{\Omega}=(\mathbb{Z}_2,+)$. In this case, we only have one symmetric quantization, that is, $q=(-1)^{s_1^2}=(-1)^{l^2}$.

We shall investigate quantizations on the monoid algebra $A=\mathbb{R}\left[\mathbb{Z}_2^+\right]$. The algebra $\mathbb{R}[\mathbb{Z}_2]$ can be pictured as two points, $\{\bar{0},\bar{1}\}$, with the real line passing through each of these points. We call $\mathbb{R}[\mathbb{Z}_2]$ the *two-point algebra.* A is of rank 2 over $\mathbb{R}$ and has the basis $\{\mathbf{1},\varepsilon\}$, with the property $\varepsilon^2=\mathbf{1}$.

We'll quantize by introducing a new multiplication, $*_P$, on A. For $\varepsilon \in A_1$,

$$\varepsilon *_P \varepsilon=(-1)^{s_1^2}\varepsilon\varepsilon=(-1)^{l(1)l(1)}\cdot 1=(-1)^{1\cdot 1}=-1.$$

The new $\mathbb{R}$-algebra A_P produced by quantization, has as the basis $\{\mathbf{1}, \varepsilon\}$ with the property $\varepsilon^2 = -\mathbf{1}$, hence is $A_P = \mathbb{C}$.

If we now again quantize A_P by P we are back at our original two-point algebra A.

5.2.2 Quantizations of the four-point algebra

For all $n \geq 2$ we have three non-trivial possibilities for P as described above. For the case $n = 2$, Ω consists of two points so $(\mathbb{Z}_2, +)^{\Omega} = \{(0,0), (1,0), (0,1), (1,1)\}$.

Let $A = \mathbb{R}\left[\left(\mathbb{Z}_2^+\right)^2\right]$ be the four-point algebra. A has rank 4 over $\mathbb{R}$ with the basis $\{1, e_1, e_2, e_1e_2\}$ where e_1 corresponds to the grading by $(1,0)$ and e_2 to $(0,1)$. These basis elements are closed under a multiplication with the properties $e_1^2 = 1 = e_2^2$ and $e_1e_2 = e_2e_1$. A quantization will change the multiplication with respect to these basis elements.

We will quantize the four-point algebra A by defining a new multiplication

$$a *_P a' = a * a' = (-1)^P aa', a, a' \in A. \tag{5.1}$$

The matrix algebra, $Mat_2(\mathbb{R})$

Choose $P = s_2 = l_1l_2$. Then

$$\begin{aligned}
e_1 * e_1 &= (-1)^{l_1((1,0))l_2((1,0))} e_1^2 = 1 \\
e_2 * e_2 &= (-1)^{l_1((0,1))l_2((0,1))} e_2^2 = 1 \\
e_1 * e_2 &= (-1)^{l_1((1,0))l_2((0,1))} e_1e_2 = e_1e_2 \\
e_2 * e_1 &= (-1)^{l_1((0,1))l_2((1,0))} e_2e_1 = -e_2e_1,
\end{aligned}$$

hence

$$e_1 * e_1 = 1 \tag{5.2}$$

$$e_2 * e_2 = 1 \tag{5.3}$$

$$e_1 * e_2 = -e_2 * e_1. \tag{5.4}$$

We have a real skew-symmetric algebra A_{s_2}, generated by four elements, which is isomorphic to $Mat_2(\mathbb{R})$ with

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, e_1e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$e_2e_1 = -e_1e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

If we set $E_{11} = \frac{1+e_1}{2}, E_{12} = \frac{-e_2-e_1e_2}{2}, E_{21} = \frac{-e_2+e_1e_2}{2}, E_{22} = \frac{1-e_1}{2}$, then $Mat_2(\mathbb{R})$ obviously is generated by the elements E_{ij} with the multiplication determined by the rule

$$\begin{aligned} E_{ij}E_{kl} &= 0, \text{ if } j \neq k \\ E_{ij}E_{jl} &= E_{il}. \end{aligned}$$

The tensor algebra, $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$

Now choose $P = s_1^2 = l_1^2 + l_2^2$. Then

$$\begin{aligned} e_1 * e_1 &= (-1)^{1\cdot 1+0\cdot 0} e_1^2 = -1 \\ e_2 * e_2 &= (-1)^{0\cdot 0+1\cdot 1} e_2^2 = -1 \\ e_1 * e_2 &= (-1)^{1\cdot 0+0\cdot 1} e_1e_2 = e_1e_2 \\ e_2 * e_1 &= (-1)^{0\cdot 1+1\cdot 0} e_2e_1 = e_2e_1, \end{aligned}$$

hence

$$e_1 * e_1 = -1 \tag{5.5}$$
$$e_2 * e_2 = -1 \tag{5.6}$$
$$e_1 * e_2 = e_2 * e_1. \tag{5.7}$$

We have a complex symmetric algebra, generated by four elements, which is isomorphic to $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ with the basis $1 = 1\otimes 1, e_1 = 1\otimes i, e_2 = i\otimes 1, e_1e_2 = i\otimes i$.

Quaternions, $\mathbb{H}$

Further choose $P = s_1^2 + s_2 = l_1l_2 + l_1^2 + l_2^2$. Then

$$e_1 * e_1 = -1 \tag{5.8}$$
$$e_2 * e_2 = -1 \tag{5.9}$$
$$e_1 * e_2 + e_2 * e_1 = 0. \tag{5.10}$$

We have a complex skew-symmetric algebra, generated by four elements. This algebra is isomorphic to $\mathbb{H}$ which is the skew commutative algebra over $\mathbb{R}$. $\mathbb{H}$

has rank 4 over $\mathbb{R}$ with the basis $1, i, j, k$, having the multiplicative law

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j.$$

Writing $e_1 = i, e_2 = j$ the isomorphism follows at once.

Remark 27 *These 4 algebras we have described make out the complete list of the semisimple $\mathbb{R}$-algebras of rank 4.*

5.2.3 Quantization of the n-point algebra

For the case $n \geq 2$ we similarly get for $P = s_2$ a new algebra A_{s_2} with a basis consisting of all combinations

$$e_{i_1} * \cdots * e_{i_r}$$

with $i_1 < \cdots < i_r, 1 \leq r \leq n$ having the following relations

$$\begin{aligned} e_1 * e_1 &= 1 \\ e_2 * e_2 &= 1 \\ &\cdots \\ e_n * e_n &= 1 \\ e_i * e_j + e_j * e_i &= 0, \forall i, j = 1, \ldots, n. \end{aligned}$$

This algebra is isomorphic to the Clifford algebra prime of $\mathbb{R}^n$, C'_n.

For $P = s_1^2$ the algebra $A_{s_1^2}$ has the basis consisting of all combinations $e_{i_1} * \cdots * e_{i_r}$, $i_1 < \cdots < i_r, 1 \leq r \leq n$ with the relations

$$\begin{aligned} e_1 * e_1 &= -1 \\ e_2 * e_2 &= -1 \\ &\cdots \\ e_n * e_n &= -1 \\ e_i * e_j &= e_j * e_i, \forall i, j = 1, \ldots, n. \end{aligned}$$

This algebra $A_{s_1^2}$ is isomorphic to $\underbrace{\mathbb{C} \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathbb{C}}_{n-times}$.

Further, for $P = s_2 + s_1^2$ we get the algebra $A_{s_2+s_1^2}$ with a basis consisting of all combinations

$$e_{i_1} * \cdots * e_{i_r},$$

$i_1 < \cdots < i_r, 1 \leq r \leq n$. The following relations are satisfied

$$\begin{aligned} e_1 * e_1 &= -1 \\ e_2 * e_2 &= -1 \\ &\cdots \\ e_n * e_n &= -1 \\ e_i * e_j + e_j * e_i &= 0, \forall i, j = 1, \ldots, n \end{aligned}$$

This algebra is isomorphic to the Clifford algebra of $\mathbb{R}^n$, C_n.

5.3 $Aut(\Omega)$-invariant third cohomologies of $\mathfrak{P}(\Omega)$ with coefficients in $U(\mathbb{R})$

Let $\alpha = (-1)^P \in Z^3\left((\mathbb{Z}_2, +)^\Omega, \mathbb{Z}_2\right)$ be $Aut(\Omega)$-invariant. Then there are 7 possibilities for $Aut(\Omega)$-invariant P's which are

$$P = P'(s_1, s_2, s_3) \in \left\{ s_1^3, s_2 s_1, s_3, s_1^3 + s_2 s_1, s_1^3 + s_3, s_2 s_1 + s_3, s_1^3 + s_2 s_1 + s_3, \right\}.$$

The cohomologies of degree 3 are associativity constraints on graded algebras $A = \sum_{i \in M} A_i$ such that

$$a_i (a_j a_k) = \alpha(i, j, k) (a_i a_j) a_k.$$

Let $i = j = k$. Then

$$a_i (a_i a_i) = \alpha(i, i, i) (a_i a_i) a_i$$

and $(a_i a_i) = b_0$ is an element in A_0. So

$$a_i b_0 = \alpha(i, i, i) b_0 a_i = \alpha(i, i, i) a_i b_0.$$

We require $\alpha(i, i, i) = (-1)^{P(i,i,i)} = 1$, that is,

$$P(i, i, i) = 0. \tag{5.11}$$

Case $(\mathbb{Z}_2, +)^\Omega \cong (\mathbb{Z}_2, +)$

Now look at the case $n = 1$. Then we only have one possible symmetric associativity, that is, $P = s_1^3 = l^3$, but $P = l^3$ does not satisfy $P(i, i, i) = 0$.

Case $(\mathbb{Z}_2,+)^{\Omega} \cong (\mathbb{Z}_2,+)^2$

Let Ω consist of two points. Let $A = \mathbb{R}\left[\left(\mathbb{Z}_2^+\right)^{\Omega}\right]$ be the four-point algebra, graded by $(\mathbb{Z}_2,+)^{\Omega}$.

We now have 3 possibilities for P, that is, s_1^3, $s_1 s_2$ and $s_1^3 + s_1 s_2$. Only $P = s_1 s_2$ does satisfy the condition $P\,(i,i,i) = 0$, but $s_1 s_2 = 0$ as a function over $(\mathbb{Z}_2,+)$, so, in this case, we only have the trivial associativity constraint.

5.3.1 The Cayley algebra or octonions

Now let Ω consist of three points, say. Let $A = \mathbb{R}\left[\left(\mathbb{Z}_2^+\right)^{\Omega}\right]$ be the eight-point algebra constructed in a similar manner as the two- and four-point algebras.

We have 7 possibilities for P, that is, s_1^3, $s_1 s_2$ and s_3 and all $\mathbb{Z}_2^+$-linear combinations of these, but only $P = s_1 s_2 + s_3$ satisfies the property $P\,(i,i,i) = 0$.

Let $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the basis of the Cayley numbers, $\mathfrak{C}$, that have the multiplication table [4]

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_1	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_2	u_2	$-u_1$	$-u_4$	u_3	$-u_6$	u_5	u_8	$-u_7$
u_3	u_3	u_4	$-u_1$	$-u_2$	$-u_7$	$-u_8$	u_5	u_6
u_4	u_4	$-u_3$	u_2	$-u_1$	$-u_8$	u_7	$-u_6$	u_5
u_5	u_5	u_6	u_7	u_8	$-u_1$	$-u_2$	$-u_3$	$-u_4$
u_6	u_6	$-u_5$	u_8	$-u_7$	u_2	$-u_1$	u_4	$-u_3$
u_7	u_7	$-u_8$	$-u_5$	u_6	u_3	$-u_4$	$-u_1$	u_2
u_8	u_8	u_7	$-u_6$	$-u_5$	u_4	u_3	$-u_2$	$-u_1$

.

$\mathfrak{C}$ is an alternative algebra of rank 8 over $\mathbb{R}$ and is, in classical points of view, non-associative and non-commutative. Let u_1, u_2, u_3 and u_5 have gradings corresponding to $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ respectively. Then we see that the associativity on the Cayley numbers is given by the associativity constraint

$$\alpha = (-1)^P = (-1)^{s_1 s_2 + s_3}\,.$$

That is, we can say that $\mathfrak{C}$ is α-associative. Furthermore are the octonions σ-commutative w.r.t. the braiding

$$\sigma = (-1)^{l_1 l_2 + l_2 l_1 + l_1 l_3 + l_3 l_1 + l_2 l_3 + l_3 l_2 + \delta(g)}$$

where $g((1,1,1)) = 1$ and $g(x) = 0$ for $x \neq (1,1,1)$.

If we now look at quantizations on $\mathfrak{C}$, will the three non-trivial quantizations, $q = (-1)^{s_1^2+s_2}$, $q = (-1)^{s_1^2}$ and $q = (-1)^{s_2}$, give new commutativities on $\mathfrak{C}$ and produce new so called non-associative algebras, or α-associative algebras, with commutativities different from the commutativity of $\mathfrak{C}$, as when we got the Clifford algebras when we quantized $A = \mathbb{R}\left[\left(\mathbb{Z}_2^+\right)^\Omega\right]$ in the last section.

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