Tangent and normal bundles in almost complex geometry

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Abstract

We define and study pseudoholomorphic vector bundles structures, particular cases of which are tangent and normal bundle almost complex structures. As an application we deduce normal forms of 1-jets of almost complex structures along a pseudoholomorphic submanifold. In dimension four we relate these normal forms to the problem of pseudoholomorphic foliation of a neighborhood of a curve and the question of non-deformation and persistence of pseudoholomorphic tori.

Introduction

In this paper we study the differential geometry of tangent and normal bundles in the almost complex category. Let $J : TM \to TM$ be an almost complex structure, $J^2 = -\mathbf{1}$. A submanifold $L \subset M$ is called *pseudoholomorphic* (PH-submanifold) if $TL \subset TM$ is J-invariant.

We introduce two different canonical almost complex structures $\hat{J}$ and $\check{J}$ on each of the total spaces TL and $N_L M$ of tangent and normal bundles such that the projection to L and the zero section embeddings of L are pseudoholomorphic. We find an explicit relation between these two almost complex structures.

Moreover, we define and investigate the theory of abstract pseudoholomorphic (almost holomorphic) vector bundles, partial cases of which are tangent and normal bundles. We describe their normal forms, which produce normal forms of 1-jets of almost complex structures along a PH-submanifolds.

Generically the only PH-submanifolds are PH-curves ([K2]). Local existence of PH-curves was established by Nijenhuis and Woolf ([NW]). The global existence result is due to Gromov, whose paper [Gro] made compact PH-curves an indispensable tool of symplectic geometry.

For a PH-curve L the structure $\hat{J}$ on $N_L M$ is holomorphic, while the structure $\check{J}$ is not, and they both play an important role in the deformation and regularity questions for PH-curves. In particular, we relate Gromov's operator D_u to our normal bundle structures. Consequently, the structure $\check{J}$ appears to be basic for local Gromov-Witten theory.

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In [Mo] Moser constructed a KAM-type theory for a PH-foliation of an almost complex torus T^{2n} by non-compact curves, namely entire PH-lines $\mathbb{C} \to T^{2n}$ with generic slope. He proved that under a small almost complex perturbation of the standard complex structure J_0 many leaves persist. If the perturbation is big, but tame-restricted, then only some of the leaves persist. This was proven by Bangert in [B]. Another proof is given in [KO].

In [A2](1993-25) Arnold asks about almost complex version, in the spirit of Moser's result, for his Floquet-type theory of elliptic curves neighborhoods ([A1]). It will be shown the direct extension fails (there are moduli in normal forms), though we conjecture the right generalization, treatable by Moser's method, is a possibility of foliation of a PH-torus neighborhood by PH-cylinders.

We consider specially the case $\dim M = 4$ and find the condition for a PH-curve neighborhood to admit a PH-foliation of a special kind. We also study problems of persistence and isolation of PH-tori, as posed by Moser. In particular, we obtain a geometric interpretation for his non-deformable example from [Mo]. There Moser announced "a study of the normal bundle", which has not been performed. The present paper fills the gap.

In appendix A we give a new proof of a theorem by Lichnerowicz, essentially used in the main constructions, and consider some applications of the minimal connections. In appendix B we discuss what happens with the normal and tangent bundles for other geometric structures, which demonstrates, in particular, that relations (5)-(6) from §2 is a PH-analog of the Ricci equation.

1. Almost complex tangent bundle

The Nijenhuis tensor of an almost complex structure $J \in C^\infty(T^*M \otimes TM)$ is given by the formula

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad X, Y \in TM.$$

We write $N_J \in C^\infty(\Lambda^2 T^*M \otimes_{\bar{\mathbb{C}}} TM)$ meaning it is skew-symmetric in X, Y and J-antilinear. By the Newlander-Nirenberg theorem [NW] integrability of J can be expressed as $N_J = 0$.

An almost complex connection is a linear connection ∇ that preserves the almost complex structure: $\nabla J = 0$. It is called minimal if its torsion $T_\nabla = \frac{1}{4} N_J$. Such connections always exist due to [L], see appendix A.

Let $\pi : TM \to M$ denote the projection and $\rho : M \to TM$ the zero section.

Theorem 1. *There exists a canonical almost complex structure $\hat{J}$ on the total space of the tangent bundle TM to an almost complex manifold (M, J) such that:*

1. *The maps $\pi : TM \to M$ and $\rho : M \to TM$ are pseudoholomorphic.*
2. *$(TM, \hat{J})$ is integrable iff (M, J) is integrable.*

Proof. Consider a minimal connection ∇. It produces the splitting $T_a(TM) = H_a \oplus V_a$ into horizontal and vertical components, $a \in TM$. We have natural

isomorphisms $\pi_* : H_a \simeq T_xM$ and $V_a \simeq T_xM$, $x = \pi(a)$. Define the structure $\hat{J}$ on $T_a(TM)$ as $J \oplus J$ with respect to the above splitting and isomorphisms.

If we change the minimal connection $\tilde{\nabla} = \nabla + A$ (Theorem 31 of appendix A), then $A \in C^\infty(S^2T^*M \otimes_{\mathbb{C}} TM)$. The new horizontal space is given by $\tilde{H}_a = \mathrm{graph}\{A(a,\cdot) : H_a \to V_a\}$. Since $A(a,\cdot)$ is a complex linear map, the almost complex structure $\hat{J}$ on TM is defined canonically.

The properties of $\hat{J}$ follow directly from the construction. □

Remark 1. *Whenever integrable, $\hat{J}$ defines the standard holomorphic structure.*

Construction of the structure $\hat{J}$ can be generalized to the cotangent and other tensor bundles. The adjoint J^* to the operator J is a fiberwise complex structure on T^*M. The two structures induce a canonical fiberwise complex structure on the complex-linear tensor bundles $T^{(r,s)}_{\mathbb{C}}M$ of contravariant degree r and covariant degree s tensors and also on the subbundles $S^k_{\mathbb{C}}TM$, $\Lambda^k_{\mathbb{C}}TM$. As usual, the tensor product over $\mathbb{C}$ is formed by the equivalence relation $X \otimes JY \sim JX \otimes Y$ (so that $T^{(r,s)}_{\mathbb{C}}M \neq T^{(r,s)}M \otimes \mathbb{C}$ etc).

Theorem 2. *Let E_M be one of the bundles $T^{(r,s)}_{\mathbb{C}}M$, $S^k_{\mathbb{C}}M$, $\Lambda^k_{\mathbb{C}}M$ or their duals and tensor products over $\mathbb{C}$. There exists a canonical almost complex structure $\hat{J}$ on the total space E_M such that:*

1. *The maps $\pi : E_M \to M$ and $\rho : M \to E_M$ are pseudoholomorphic.*
2. *$(E_M, \hat{J})$ is integrable iff (M, J) is integrable.*

Proof. The claim is obtained similarly to Theorem 1 by checking that the admissible gauge transformations $A^E \in \Omega^1(M, \mathrm{end}_{\mathbb{C}} E_M)$ are complex-linear in all arguments. This follows from the explicit formulae: $A^{(1,0)}(X) = A(X)$, $A^{(0,1)}(X) = -A(X)^*$, $A^{(2,0)}(X) = A(X) \otimes \mathbf{1} + \mathbf{1} \otimes A(X)$ etc. □

Remark 2. *It is possible to define an almost complex structure by the above approach on the bundles $T^{(r,s)}M = (TM)^{\otimes r} \otimes (T^*M)^{\otimes s}$, $S^{2i+1}TM$, $\Lambda^{2j+1}TM$ etc (in some different manners), but it won't be canonical (will depend on ∇).*

For two almost complex manifolds (L, J_L) and (M, J_M) a canonical almost complex structure $\hat{J}$ on the space of PH-1-jets

$$J^1_{PH}(L, M) = \{(x, y, \Phi) \,|\, x \in L, y \in M, \Phi \in T^*_xM \otimes T_yM : \ \Phi J_L = J_M \Phi\}$$

was introduced in [Gau]. In particular, we get almost complex structures on $J^1_{PH}(\mathbb{C}, M) = \mathbb{C} \times TM$ and $J^1_{PH}(M, \mathbb{C}) = T^*M \times \mathbb{C}$. They are $\mathbb{C}$-translations invariant and thus yield almost complex structures on TM and T^*M. Also the restriction of $\hat{J}$ defines a canonical almost complex structure on $T^{(1,1)}_{\mathbb{C}}M = \pi^{-1}_{1,0}(\Delta(M))$, where $\Delta(M) \subset M \times M = J^0(M, M)$ is the diagonal PH-submanifold ($\pi_{1,0} : J^1_{PH} \to J^0$ is the canonical projection). It can be shown that the derived structures on TM and T^*M coincide with the ones introduced above.

On the other hand, $J^1_{PH}(L,M) \subset T^{(1,1)}_{\mathbb{C}}(L\times M)$ is a PH-submanifold, whence the canonical structure $\hat{J}$ is a generalization of that one from [Gau].

Note however that the higher PH-jet spaces $J^k_{PH}(L,M)$, $k>1$, bear no structure in general ([K1]) and usually are even non-smooth.

Another canonical almost complex structure. An interesting issue is the paper [LS]. An almost complex structure on TM, which we denote $\check{J}$, is constructed there via the deformation theory approach. It is not however new, for it was introduced long before in [YK] via the complete lift operation $J \rightsquigarrow J^c$ (this fact was not noticed in [LS]). To see the coincidence $\check{J} = J^c$, note that in local coordinates (x^i, y^i) on TM both structures have the form $\begin{pmatrix} J & 0 \\ \partial J & J \end{pmatrix}$, where $\partial = \sum y^i \partial_{x^i}$. It follows from [YK, YI] that the structure $\check{J}$ enjoys the same properties as the structure $\hat{J}$ in theorem 1.

The structures $\hat{J}$ and $\check{J}$ differ because if we let Z denote multiplication by the complex number $z = a+ib \in \mathbb{C}$ along the fibers of TM, $Z(x,y) = (x, ay+bJy)$, we get $[\check{J}, Z_*] = \begin{pmatrix} 0 & 0 \\ bN_J(\partial,\cdot) & 0 \end{pmatrix}$, while from the very construction $\hat{J}\circ Z_* = Z_*\circ\hat{J}$. Thus $\hat{J} \neq \check{J}$ unless J is integrable.

We can also obtain $\hat{J} \not\equiv J^c$ from [YI], where they provide a construction of almost complex structure J^H on TM via horizontal lift of the connection $\bar{\nabla}_X Y = \nabla_Y X + [X,Y]$ (equivalently $\bar{\Gamma}^k_{ij} = \Gamma^k_{ji}$ in terms of Christoffel symbols). By the construction $\hat{J} = J^H$ iff $\bar{\nabla}$ is minimal and by the results in §2.4 of [YI] $J^c = J^H$ iff $\nabla J = 0$ (beware, without one of these specifications the horizontal lift J^H is connection-dependent). But if ∇ is an almost complex connection, then $N_J(X,\cdot) = \bar{\nabla}_{JX}J - J\bar{\nabla}_X J$, whence $J^c = J^H \neq \hat{J}$ unless $N_J = 0$.

The argumentation in [LS] that $[\check{J}, Z_*] \neq 0$ is indirect and based on the fact that kernel of the Gromov operator D_u is not J-invariant. In §6 we describe this operator in terms of a canonical almost complex structure $\check{J}$ on the normal bundle to a PH-curve (in fact, as notation suggests, there is a relation between introduced canonical structures on tangent and normal bundles).

Remark 3. *In [YI] various lifts to tangent and cotangent bundles are discussed. The complete lift of J to the cotangent bundle is not almost complex, but this is amended [Sa] via the calibration $J^c - \frac{1}{2}\gamma(JN_J)$. The transformation is surprisingly similar to our formula (11) below, though we observe no precise relations.*

Let us call TB-I and TB-II the (total space of) tangent bundle TM equipped with the almost complex structures $\hat{J}$ or $\check{J}$ respectively.

2. Almost complex normal bundle

Topologically the normal bundle $N_L M$ of a submanifold $L \subset M$ is defined by the exact sequence:

$$0 \to TL \to TM|_L \to N_L M \to 0. \tag{1}$$

If L is a complex submanifold of a complex manifold M, then N_LM is a holomorphic vector bundle over L (the total space and the projection are holomorphic, as well as fiberwise addition and multiplication by complex numbers). In almost complex case this is no longer so.

Let $\pi : N_LM \to L$ denote the projection and $\rho : L \to N_LM$ the zero section.

NB-I structure. Here we apply the construction of §1 to get a canonical almost complex structure $\hat{J}$ on N_LM, called NB-I in what follows:

Theorem 3. *There exists a canonical almost complex structure $\hat{J}$ on the total space of the normal bundle N_LM to a PH-submanifold $L \subset M$ such that:*

1. *The maps $\pi : N_LM \to L$ and $\rho : L \to N_LM$ are pseudoholomorphic.*
2. *The structure $\hat{J}$ is integrable iff $J|_L$ is integrable and the J-antilinear by each argument part of the curvature vanishes, $R^{--}_{\nabla}(X,Y) = 0$, $\forall X, Y \in TL$, for some minimal connection ∇ totally geodesic and flat on L.*

Remark 4. *If $J|_L$ is integrable, the specified connection always exists locally (the above integrability criterion is indeed local) and then $R^{--}_{\nabla}(X,Y)$ does not depend on its choice (see appendix A). Moreover, $R^{--}_{\nabla}(X,Y) = 0$ $\forall X, Y \in TL$, whenever J is integrable along L to the second order: $N_J(x) = 0$ $\forall x \in L$.*

Proof. Let ∇ be a minimal connection on M. It can be chosen so that L is totally geodesic. In fact, one chooses any linear connection for which parallel transports along L preserve TL and note that the procedures of making the connection almost complex and then minimal (see appendix A) do not destroy the property of L to be totally geodesic.

We define a connection $\hat{\nabla}$ on the bundle N_LM via parallel transports as follows. Let $v = [\theta] \in (N_LM)_x$ be the class of $\theta \in T_xM$ and let $\gamma(t) \subset L$ be a curve, $\gamma(0) = x$. Calculate the parallel transport $\theta(t)$ of θ along $\gamma(t)$. Then define $v(t) = [\theta(t)]$ to be the parallel transport of v along $\gamma(t)$. Since L is totally geodesic, the definition is correct ($\hat{\nabla}$-parallel transport of 0 is 0). Moreover the connection $\hat{\nabla}$ is $\mathbb{R}$-linear. So as usual in the theory of generalized connections we conclude that $\hat{\nabla}$ is a linear connection.

Let $T_a(N_LM) = H_a \oplus V_a$ be the splitting into the horizontal and vertical components induced by $\hat{\nabla}$, $a \in N_LM$. The first space $H_a \overset{\pi_*}{\simeq} T_xL$ has a canonical complex structure J_1 induced from $J|_L$ by π_*, $x = \pi(a)$, and the second $V_a \simeq T_xM/T_xL$ inherits a canonical complex structure J_2 from J as the quotient. So we obtain the structure $\hat{J} = J_1 \oplus J_2$ on $T_a(N_LM)$ for each a.

The same arguments as in Theorem 1 show that the almost complex structure $\hat{J}$ on N_LM does not depend on the choice of a minimal connection ∇, preserving TL. The first property of $\hat{J}$ is obvious. For the other one we use

Lemma 4. *If a vector $Y \in T_a(N_LM)$ is vertical, then $N_{\hat{J}}(\cdot, Y) = 0$.*

Actually, the fiber is integrable, so it is enough to consider the pairing $N_{\hat{J}}(\hat{X}, Y)$, where $\hat{X}$ is the $\hat{\nabla}$-lift of $X \in TL$. Recall ([KN]) that $\hat{\nabla}_X Y$ coincides with the

Lie derivative $L_{\hat{X}}\tilde{Y}$ of the section Y extended by translations to a vertical vector field $\tilde{Y}$ on N_LM ($\hat{X}$ is the $\hat{\nabla}$-lift of any vector field extending X; the result will not depend on an extension). Thus $\hat{\nabla}_X Y = [\hat{X}, \tilde{Y}]$ and we have (see also the remark after proposition 13):

$$N_{\hat{J}}(\hat{X}, Y) = \hat{\nabla}_{\hat{J}X}\hat{J}Y - \hat{J}\hat{\nabla}_{\hat{J}X}Y - \hat{J}\hat{\nabla}_X\hat{J}Y - \hat{\nabla}_X Y = (\hat{\nabla}_{\hat{J}X}\hat{J})Y + (\hat{\nabla}_X\hat{J})\hat{J}Y = 0.$$

Now since the curvature of $\hat{\nabla}$ is $R_{\hat{\nabla}}(X,Y)a = \widehat{[X,Y]}_a - [\hat{X}, \hat{Y}]_a$, we get:

$$N_{\hat{J}}(\hat{X}, \hat{Y})_a = \widehat{N_J(X,Y)}_a + 4R^{--}_{\hat{\nabla}}(X,Y)a, \qquad X, Y \in TL. \tag{2}$$

For an integrable $J|_L$ we can choose minimal ∇ to be flat on L and preserving TL, whence we get $R^{--}_{\hat{\nabla}}(X,Y) = R^{--}_{\nabla}(X,Y)$ and the claim follows. $\square$

NB-II structure. From the integrability condition of Theorem 3 we read off that some features of 1-jet of the almost complex structure J along L are lost in $\hat{J}$ on N_LM. It is however possible to keep most of them with another definition of the normal bundle structure $\check{J}$, which we call NB-II.

Theorem 5. *There exists a canonical almost complex structure $\check{J}$ on the total space of the normal bundle N_LM to a PH-submanifold $L \subset M$ such that:*

1. *The maps $\pi : N_LM \to L$ and $\rho : L \to N_LM$ are pseudoholomorphic.*
2. *The structure $\check{J}$ is integrable iff the following 3 conditions hold:*
   - *$J|_L$ is integrable,*
   - *(M, J) is normally integrable along L, i. e. $N_J(TL, TM|_L) \subset TL$,*
   - *The normal component $N_J^{\perp} = \chi \circ N_J$ vanishes on TL to the second order, where $\chi : TM|_L \to N_LM$ is the natural projection.*

Proof. We describe the structure $\check{J}$ on the germ of zero section in N_LM, which then uniquely determines it on the whole total space. Let $\mathcal{O}_L^M$ be a tubular neighborhood of $L \subset M$. Fix a J-invariant subbundle $F \subset TM|_L$ such that $TL \oplus F = TM|_L$ (the totality of all such subspaces F forms a bundle over L with contractible fibers). We identify $F = TM|_L/TL \simeq N_LM$.

Let us fix some minimal connection ∇ on M with L being totally geodesic. Denote by $N_LM \supset \mathcal{O}_L^N \xrightarrow{\varphi} \mathcal{O}_L^M$ the ∇-exponential map that associates to the vector $v \in F_x$, $x \in L$, the value $\gamma(1)$ along the ∇-geodesic γ with initial conditions $(\gamma(0), \dot{\gamma}(0)) = (x, v)$.

Denote by R^t the t-times dilatation $v \mapsto tv$ along the fibers of F. We define:

$$J^{\varphi} = \varphi_*^{-1} J \varphi_*, \quad J_t = \mathrm{ad}_{R^t}(J^{\varphi}) = R_*^{1/t} J^{\varphi} R_*^t \quad \text{and} \quad \check{J} = \lim_{t \to 0} J_t. \tag{3}$$

Consider local split coordinates (x, y) on N_LM such that $L = \{y = 0\}$ and the fibers of F equal $\{x = \text{const}\}$. In terms of these coordinates the limit process transforms the matrix of J^{φ} as follows:

$$J^{\varphi} = \begin{pmatrix} A(x,y) & C(x,y) \\ B(x,y) & D(x,y) \end{pmatrix} \mapsto \check{J} = \begin{pmatrix} A(x,0) & 0 \\ d_F B(x,y) & D(x,0), \end{pmatrix}$$

where $d_F B(x,y) = \lim_{t\to 0} B(x,ty)/t$ (notice that $B(x,0)=0$ because TL is J-invariant).

Let us check independence of $\check{J}$ on ∇ and F. When we change the connection or the J-invariant subbundle, it is equivalent to changing the map φ to $\tilde{\varphi}$. In the above split coordinates (x^i, y^j) on $N_L M$ we have (assuming the standard rule of summation by repeated indices)

$$\varphi^{-1}\tilde{\varphi} : (x^i, y^j) \mapsto (x^i + \alpha^i_k(x) y^k, y^j) + o(|y|)$$

(choice of the norm in $o(|y|)$ is not essential). Thus writing the matrix of J^φ in block form we observe that the transformation $J^\varphi \mapsto J^{\tilde{\varphi}}$ has the following matrix form:

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \mapsto \Delta^{-1} \cdot \begin{pmatrix} A & C \\ B & D \end{pmatrix} \cdot \Delta = \begin{pmatrix} \tilde{A} & \tilde{C} \\ \tilde{B} & \tilde{D} \end{pmatrix}, \tag{4}$$

where

$$\Delta = d\big(\varphi^{-1}\tilde{\varphi}\big) = \mathbf{1} + \begin{pmatrix} U & V \\ 0 & W \end{pmatrix} + o(|y|),$$

and $U, W = o(1)$ have to vanish on L, but V needs not to.

Since $B(x,0)=0$ we deduce from (4): $\tilde{A}(x,0) = A(x,0)$ and $\tilde{D}(x,0) = D(x,0)$. The transformation of C is inessential and B changes to $\tilde{B}(x,y) = B(x,y)(\mathbf{1}+o(1))$. Thus $d_F\tilde{B}(x,y) = d_F B(x,y)$ and we see that the limit process (3) gives a well-defined result.

In addition we observe that the structure $\check{J}$ has affine behavior w.r.t. y and thus its restriction to $\mathcal{O}^N_L$ determines the structure on the whole $N_L M$.

To prove integrability criterion we note that $N_{\check{J}} = \lim_{t\to\infty} N_{J_t} = \lim_{t\to\infty} \mathrm{ad}_{R^t}(N_J)$. Consider (x^i, y^j) as coordinates on both $\mathcal{O}^N_L$ and $\mathcal{O}^M_L$ using the identification φ.

Denote by $N^\perp_J$ the y-component of the value of N_J. Note that $N^\perp_J$ is well-defined along L and whenever $J|_L$ is integrable, i.e. $N_J|_{TL} \equiv 0$, its 1-jet is well-defined. Then we calculate:

$$N_{\check{J}}(\partial_{x^i}, \partial_{x^j}) = N_J(\partial_{x^i}, \partial_{x^j})|_{y=0} + y^k\big(\partial_{y^k} N^\perp_J(\partial_{x^i}, \partial_{x^j})|_{y=0}\big) \tag{5}$$

and

$$N_{\check{J}}(\partial_{x^i}, \partial_{y^j}) = N^\perp_J(\partial_{x^i}, \partial_{y^j})|_{y=0}, \quad N_{\check{J}}(\partial_{y^i}, \partial_{y^j}) = 0, \tag{6}$$

The claim follows. □

If $\mathrm{codim}_{\mathbb{C}} L = 1$, then the connection ∇ can be chosen so that the exponential image of the vertical foliation $\varphi(F)$ is J-holomorphic. This follows from

Proposition 6. *Small neighborhood $\mathcal{O}_L$ of a PH-submanifold $L^{2n-2} \subset M^{2n}$ can be foliated by transversal PH-disks D^2.*

Proof. This follows from Nijenhuis-Woolf theorem [NW] on the existence of a small PH-disk in a given direction, smoothly depending on it. □

Remark 5. *For $n = 2$ a construction of certain structure $\bar{J}$ on $\mathcal{O}_L^M$, using the dilatation R^t and based on the idea of Proposition 6, was used in [M2].*

Denote by $N_L^I M$ and $N_L^{II} M$ the normal bundle equipped with the NB-I structure $\hat{J}$ or with the NB-II structure $\check{J}$ respectively. The tangent bundle structures TB-I and TB-II can be deduced from the normal ones via the diagonal embedding $\Delta : M \hookrightarrow M \times M$ because $N_{\Delta(M)}(M \times M) \simeq TM$.

We are going to relate the concept of NB-II with the deformation theory. The following statement will be used in §6.

Proposition 7. *Let $\phi_t : (\mathcal{C}, J_{\mathcal{C}}^t) \to (M, J_M)$ with $\phi_0(\mathcal{C}) \subset L$ be a family of J-holomorphic embeddings. Then $\phi_t'|_{t=0} : (\mathcal{C}, J_{\mathcal{C}}^0) \hookrightarrow N_L^{II} M$ is a PH-embedding. In particular, deformations of $\mathcal{C} = L$ lead to PH-sections $\phi_t'|_{t=0} : (L, J_L^0) \hookrightarrow N_L^{II} M$.*

Notice that by virtue of the relation between NB-I and NB-II from the next section the embedding $\phi_t'|_{t=0}$ of L into $N_L^I M$ is not pseudoholomorphic.

Proof. We have $J_M\, d\phi_t = d\phi_t J_{\mathcal{C}}^t$, whence

$$(R_*^{1/t} J_M R_*^t)(R_*^{1/t} d\phi_t) = (R_*^{1/t} d\phi_t) J_{\mathcal{C}}^t.$$

In the limit $t \to 0$ we get: $\check{J}\, d\phi_0' = d\phi_0'\, J_{\mathcal{C}}^0$. □

This proposition leads to an equivalent definition of the NB-II structure $\check{J}$.

Consider $x \in L$, $v \in (N_L M)_x$ and $\zeta \in T_v(N_L M)$. Let $w \in T_x M$ represent v, $v = [w]$. Consider a curve $\gamma(t)$ in M with $\gamma(0) = x$, $\dot{\gamma}(0) = w$ and a vector field along the curve $\xi_t \in T_{\gamma(t)} M$ that represents ζ. Then $\eta_t = J_M \xi_t \in T_{\gamma(t)} M$ represents $\varsigma = \check{J}\zeta \in T_v(N_L M)$.

In fact, there exists a family of PH-disks $\phi_t : (D_\varepsilon^2, J_0) \to (M, J_M)$ with $\phi_t(0) = \gamma(t)$, $d_0\phi_t(1) = \xi_t$. Then $d_0\phi_t(i) = \eta_t \in T_{\gamma(t)} M$, where $1, i \in T_0 D_\varepsilon^2$.

From this alternative definition we obtain

Lemma 8. *Let $L \subset M$ be a PH-submanifold w.r.t. two almost complex structures J_1 and J_2 with equal normal bundles $N_L^{II} M$. Then $\nabla_Y (J_1 - J_2)(X) = 0$ for all $X \in TL$ and $Y \in TM|_L$ (the choice of connection is inessential).*

Proof. Let $\gamma(t) \subset M$ be a curve with $\gamma(0) = x \in L$, $\dot{\gamma}(0) = Y$. Consider two family of PH-disks $\phi_t : (D_\varepsilon^2, J_0) \to (M, J_1)$ and $\psi_t : (D_\varepsilon^2, J_0) \to (M, J_2)$ with $\phi_t(0) = \psi_t(0) = \gamma(t)$ and $d_0\phi_0(1) = d_0\psi_0(1) = X$. We can suppose that they induce the same map $\phi_0' = \psi_0' : (D_\varepsilon^2, J_0) \to N_L^{II} M$. Then:

$$\nabla_Y (J_1 - J_2)(X) = \tfrac{d}{dt}\big|_{t=0} \big(J_1 d_0\phi_t(1) - J_2 d_0\psi_t(1)\big) = d_0\phi_0'(i) - d_0\psi_0'(i) = 0.$$

□

3. Pseudoholomorphic vector bundles

Consider a real vector bundle $\pi : (E, \hat{J}) \xrightarrow{F} (L, J)$ with almost complex total space, base and projection: $\pi_* \hat{J} = J \pi_*$. The following statement is obvious:

Proposition 9. *The Nijenhuis tensor $N_{\hat{J}}$ is projectible: $\pi_* N_{\hat{J}} = N_J \circ \Lambda^2 \pi_*$.* $\square$

Corollary 10. *Let (L, J) be integrable (for example $\dim_{\mathbb{C}} L = 1$). Then we have: $\mathrm{Im}(N_{\hat{J}}) \subset TF$.* $\square$

Definition 1. *Call π a* almost holomorphic vector bundle *(we write PH – pseudoholomorphic), if the restrictions $\hat{J}|_{F_x}$ are constant coefficients complex structures on the fibers and there exists a linear (not necessary J-linear) connection $\hat{\nabla}$ on π such that the $\hat{\nabla}$-lift $\mathbb{C}$-splits the exact PH-sequences*

$$0 \to F_x \to T_a E \overset{\leftarrow\cdots}{\longrightarrow} T_x L \to 0, \qquad x = \pi(a),$$

In this case the zero section $L \subset E$ is a $\hat{J}$-holomorphic submanifold.

Proposition 11. *The canonical almost complex structures $\hat{J}, \check{J}$ on TM and $\hat{J}, \check{J}$ on $N_L M$ are PH vector bundle structures.*

Proof. For TB-I and NB-I structures $\hat{J}$ the claim is implied directly by the construction. For TB-II and NB-II structures $\check{J}$ this follows from the explicit formulas and the affine behavior by the fiber coordinates. $\square$

Consider an arbitrary splitting $TE = H \oplus V$ into horizontal and vertical components. Restricting the first argument of the Nijenhuis tensor to H and the second to $V = TF$ we obtain a tensor $N'_{\hat{J}} : \pi^* TL \otimes TF \to TF$.

Proposition 12. *The tensor $N'_{\hat{J}}$ does not depend on a choice of horizontal component H (not necessary $\hat{J}$-lift) and is constant along the fibers. So it is lifted from a canonical tensor (we will use the same notation) $N'_{\hat{J}} : TL \otimes F \to F$ with $\hat{J}$-invariant image $\Pi'_{\hat{J}} = N_{\hat{J}}(H, V) \subset F$.*

Proof. Independence of H follows from Proposition 9. Let us prove constancy along the fibers F. Let $\hat{\nabla}$ be a connection from the definition.

Denote $\#j = j - (-1)^j$. There are local coordinates (x^i, y^j) on $\pi^{-1}(U) = U \times F$, with x a base coordinate and y a linear fiber coordinate, such that the structure $\hat{J}|_F$ has constant coefficients w.r.t. y:

$$\hat{J}\partial_{y^j} = (-1)^{j-1} \partial_{y^{\#j}}. \tag{7}$$

Let $\hat{\nabla}_{\partial_{x^i}} \partial_{y^j} = \Gamma^k_{ij}(x) \partial_{y^k}$. The $\hat{\nabla}$-lift of ∂_{x^i} is: $\hat{\partial}_{x^i} = \partial_{x^i} - \Gamma^s_{ij}(x) y^j \partial_{y^s}$.

Let $J\partial_{x^i} = a^k_i(x) \partial_{x^k}$ on the base. Then $\hat{J}\hat{\partial}_{x^i} = a^k_i(x) \hat{\partial}_{x^k}$ and we get:

$$\hat{J}\partial_{x^i} = a^k_i \partial_{x^k} + \big((-1)^s \Gamma^{\#s}_{ij} - a^k_i \Gamma^s_{kj}\big) y^j \partial_{y^s}. \tag{8}$$

Thus $N_{\hat{J}}(\partial_{x^i}, \partial_{y^j}) = \gamma^s_{ij}(x) \partial_{y^s}$ is expressed via the Christoffel symbols as

$$\gamma^s_{ij} = (-1)^{s+j} \Gamma^{\#s}_{i,\#j} - (-1)^s a^k_i \Gamma^{\#s}_{kj} - (-1)^j a^k_i \Gamma^s_{k,\#j} - \Gamma^s_{ij}, \tag{9}$$

so it is constant along the fibers. Note that $\mathrm{rk}(\Pi'_{\hat{J}})$ can vary with $x \in L$. $\square$

Definition 2. *Let us call a PH-bundle almost complex structure $\hat{J}$ on (E,π)* normally integrable *if $N'_{\hat{J}}=0$.*

For such a structure integrability is equivalent to integrability of (L,J) and vanishing of $R^{--}_{\hat{\nabla}}$ (cf. proofs of Theorems 3, 5 and formula (2)). In particular:

Proposition 13. *Normally integrable PH bundles over holomorphic curves are holomorphic.* □

If $\hat{\nabla}$ is obtained from a minimal connection ∇, as for the structures $\hat{J}$ of §1-2, then it additionally preserves $\hat{J}|_F$, meaning $\Gamma^{\#s}_{i,\#j}=(-1)^{s+j}\Gamma^s_{ij}$. So (9) implies $\gamma^s_{ij}=0$ and $N'_{\hat{J}}=0$. In particular, the NB-I structures $\hat{J}$ over a PH-curve is normally integrable (while the NB-II structure $\check{J}$ is usually not). To describe such structures in general notice that formula (2) implies the following:

Proposition 14. *If a PH bundle structure $\hat{J}$ is normally integrable, then restriction of the Nijenhuis tensor to both horizontal components determines a canonical tensor $N''_{\hat{J}}:\pi^*\Lambda^2TL\to TE$ with the image $\Pi''_{\hat{J}}=N_{\hat{J}}(H,H)\subset TE$ being a $\hat{J}$-invariant differential system. This tensor projects to the tensor N_J on the base and is affine-linear along the fiber.* □

Let $a\in E$ and $x=\pi(a)\in L$ be its projection. Denote by $r=r_a\in F_x\subset T_aE$ the radius-vector $\vec{xa}$.

Theorem 15. *Let $(E,\hat{J},\pi)$ be a pseudoholomorphic vector bundle over an almost complex manifold (L,J). Then $\hat{J}$ can be expressed via some normally integrable PH vector bundle structure J_0 and the tensor $N_{\hat{J}}$ by the formula:*

$$\hat{J}=J_0+\frac{1}{2}J_0N_{\hat{J}}(r,\cdot). \tag{10}$$

Proof. Let us define the structure by the formula

$$J_0=\hat{J}-\frac{1}{2}\hat{J}N_{\hat{J}}(r,\cdot). \tag{11}$$

Since $N_{\hat{J}}|_F\equiv 0$ this structure $J_0|_F=\hat{J}|_F$ is a constant complex structure on the fibers F, proving formula (10) for $\hat{J}$.

To show that the structure J_0 is almost complex, we note that $N_{\hat{J}}(r,Y)\in F$ for any Y and $N_{\hat{J}}(r,Y)=0$ for $Y\in F$. Therefore

$$J_0^2=\hat{J}^2-\frac{1}{2}\hat{J}^2N_{\hat{J}}(r,\cdot)-\frac{1}{2}\hat{J}N_{\hat{J}}(r,\hat{J}\cdot)+\frac{1}{4}\hat{J}N_{\hat{J}}(r,\hat{J}N_{\hat{J}}(r,\cdot))=\hat{J}^2=-\mathbf{1}.$$

To obtain $N'_{J_0}=0$ we use (11) and the coordinates of proposition 12:

$$\begin{aligned}
&N_{J_0}(\partial_{x^i},\partial_{y^j})=\\
&=N_{\hat{J}}(\partial_{x^i},\partial_{y^j})-[\tfrac{1}{2}\hat{J}N_{\hat{J}}(y^s\partial_{y^s},\partial_{x^i}),(-1)^{j-1}\partial_{y^{\#j}}]+\hat{J}[\tfrac{1}{2}\hat{J}N_{\hat{J}}(y^s\partial_{y^s},\partial_{x^i}),\partial_{y^j}]\\
&=N_{\hat{J}}(\partial_{x^i},\partial_{y^j})+\tfrac{1}{2}(-1)^j\hat{J}N_{\hat{J}}(\partial_{x^i},\partial_{y^{\#j}})+\tfrac{1}{2}\hat{J}^2N_{\hat{J}}(\partial_{x^i},\partial_{y^j})=0,
\end{aligned}$$

where we expressed $r=y^s\partial_{y^s}$. The claim follows. □

Corollary 16. *If the base is a PH-curve,* $\dim_{\mathbb{C}} L = 1$*, then the structure* J_0 *in formula (10) is complex analytic, making* π *into a holomorphic vector bundle.* $\square$

Definition 3. *Let us call the structure* J_0 *of theorem 15 the normally integrable form* (n.i.f.) *of the PH-bundle structure* $\hat{J}$*.*

Certainly normally integrable form of a normally integrable structure (e.g. TB-I or NB-I) $\hat{J}$ is this structure itself. Now we will describe a relation between NB-I and NB-II structures (implying a similar relation for TB-I and TB-II). We consider the latter as a general pseudoholomorphic vector bundle.

Theorem 17. *Let* (L, J) *be the zero section of a PH vector bundle* $(E, \hat{J}, \pi)$*. Then its NB-I structure coincides with the n.i.f.* J_0 *of the structure* $\hat{J}$ *as in (10).*

Proof. We use formulae (7) and (8) for the almost complex structure. Consider a linear connection ∇, given by the relations

$$\nabla_{\partial_{x^i}} \partial_{x^j} = 0,\ \nabla_{\partial_{x^i}} \partial_{y^j} = \Gamma^k_{ij}(x) \partial_{y^k},\ \nabla_{\partial_{y^i}} \partial_{x^j} = 0,\ \nabla_{\partial_{y^i}} \partial_{y^j} = 0.$$

Calculate by it a minimal connection $\tilde{\nabla}$ by the algorithm of appendix A. It in turn produces the following connection on the normal bundle $N_L E \simeq E$:

$$\bar{\nabla}_{\partial_{x^i}} \partial_{y^j} = \Big(\frac{3}{8} \Gamma^s_{ij} + \frac{1}{8} (-1)^s a^k_i \Gamma^{\#s}_{kj} - \frac{1}{8} (-1)^j a^k_i \Gamma^s_{k,\#j} + \frac{3}{8} (-1)^{s+j} \Gamma^{\#s}_{i,\#j} \Big) \partial_{y^s}.$$

Using the relation $\bar{J} \bar{\partial}_{x^i} = a^k_i \bar{\partial}_{x^k}$ we get the formula

$$\bar{J} \partial_{x^i} = a^k_i \partial_{x^k} + \frac{1}{2} \Big((-1)^s \Gamma^{\#s}_{ij} - (-1)^j \Gamma^s_{i,\#j} - a^k_i \Gamma^s_{kj} - (-1)^{s+j} a^k_i \Gamma^{\#s}_{k,\#j} \Big) y^j \partial_{y^s},$$

which together with the formula $\bar{J}|_F = \hat{J}|_F$ (7) describes the NB-I structure $(E, \bar{J})$ of the zero section.

But substitution of formulae (8) and (9) into (11) yields the same expressions for J_0, proving the claim: $\bar{J} = J_0$. $\square$

Thus the two PH-bundles $N^I_L M$ and $N^{II}_L M$ are related as follows:

$$\text{NB-II} \xrightarrow{\text{n.i.f.}} \text{NB-I} \tag{12}$$

Relation to other generalizations of holomorphic bundles. Our PH-vector bundle structures differ from "bundle almost complex structures" of Bartolomeis and Tian [BT], because (see §1) the multiplication morphism $\mu : \mathbb{C} \times E \to E$ is not pseudoholomorphic in general (though its restriction $\mu : \mathbb{R} \times E \to E$ is). But they satisfy the requirements of "almost holomorphic vector bundles" by Lempert and Szöke [LS]. Actually our definitions are equivalent:

Proposition 18. (E, M, π) *is a PH vector bundle structure iff the fiber-wise addition* $\alpha : E \times_M E \to E$ *is a PH-map.*

Proof. The almost complex structure $\hat{J}$ on $E \times_M E$ is induced from the natural product structure on $E \times E$, since the former is the preimage of the diagonal $\Delta(M) \subset M \times M$ (which is pseudoholomorphic).

In local coordinates (x^i, y^j) the structure $\hat{J}$ on E is given by formulae (7)-(8). Then the structure on $E \times_M E$ is given in local coordinates (x^i, z^j, w^k) as follows (we do not specify coefficients b^s_{ij} via the Christoffel coefficients):

$$\hat{J}\partial_{x^i} = a^k_i(x)\partial_{x^k} + b^s_{ij}(x)z^j\partial_{z^s} + b^s_{ij}(x)w^j\partial_{w^s},$$
$$\hat{J}\partial_{z^j} = (-1)^{j-1}\partial_{z^{\#j}}, \quad \hat{J}\partial_{w^j} = (-1)^{j-1}\partial_{w^{\#j}}.$$

The map α_* maps both ∂_{z^j} and ∂_{w^j} to ∂_{y^j}. It is enough to check that it is a PH-map only on the basic vectors. Consider a point $(x, z, w) \overset{\alpha}{\mapsto} (x, y = z + w)$. For ∂_{z^j} and ∂_{w^j} we have: $\alpha_*\hat{J} = \hat{J}\,\alpha_*$. And for the horizontal vectors:

$$\hat{J}_{(x,y)}\alpha_*(\partial_{x^i}) - \alpha_*(\hat{J}_{(x,z,w)}\partial_{x^i}) = b^s_{ij}(x)(y^j\partial_{y^s} - z^j\partial_{y^s} - w^j\partial_{y^s}) = 0.$$

Thus if $(E, \hat{J})$ is a PH bundle, the map α is pseudoholomorphic.

On the other hand if α is a PH-map, then the above arguments show local existence of a connection $\hat{\nabla}$, satisfying the requirement of definition 1. The space of such connections is contractible, whence the global existence. $\square$

4. Normal form of 1-jet of J along a submanifold

Consider the ideal of $\mathbb{R}$-valued functions corresponding to a submanifold L:

$$\mu_L = \{f \in C^\infty(M) \,|\, f(L) = 0\}.$$

Its degrees determine the filtration μ^k on every $C^\infty(M)$-module, in particular we can talk about jets of tensor fields along L: $J^k(\mathcal{T}) = C^\infty(\mathcal{T})/\mu_L^{k+1}C^\infty(\mathcal{T})$.

Theorem 19. *Let $L \subset M$ be a PH-submanifold with respect to two almost complex structures J_1 and J_2. Assume that the following holds:*

1. *For every point $x \in L$: $J_1(x) = J_2(x)$, $N_{J_1}(x) = N_{J_2}(x)$.*
2. *The normal bundles $N^{II}_L M$ w.r.t. the structures J_1 and J_2 coincide.*

Then J_1 and J_2 are 1-jet equivalent along L: There exists a diffeomorphism φ of a neighborhood $\mathcal{O}(L)$, such that $\varphi|_L = \mathrm{Id}$, $d_x\varphi = \mathbf{1}$ for all $x \in L$ and

$$J_2 = \varphi^* J_1 \bmod \mu_L^2.$$

Notice that the required conditions are necessary for 1-equivalence.

Remark 6. *When J_i are integrable and defined on different manifolds M_i, but with the same normal bundle N, there is the Nirenberg-Spencer cohomology obstruction $\mathrm{ns}_0(J_1, J_2) \in H^1(L; TL \otimes N^*)$ ([NS, MR]) for the 1st order equivalence.*

It equals the difference of obstructions to splitting the normal bundle sequence (1). In particular, if the sequences are isomorphic, then $\mathrm{ns}_0(J_1, J_2) = 0$.

In our case $M_1 = M_2$ *and the class* ns_0 *vanishes by condition 2. However if we want to formulate the equivalence of 1-jets of* J_1 *and* J_2 *on different manifolds, we should require* $\mathrm{ns}_0(J_1, J_2) = 0$*, where the latter will be determined via NB-I structure (common for* J_1 *and* J_2*) and sequence (1).*

In the calculations below we denote by $\doteq$ the equivalence modulo μ_L (equality of 0-jets) and by $\ddot{=}$ the equivalence modulo μ_L^2 (equality of 1-jets).

Proof. Let us choose a minimal connection ∇ near L with L being totally geodesic. We wish to find $\varphi : \mathcal{O}_L \to \mathcal{O}_L$ with $d\varphi \circ J_1 \ddot{=} J_2 \circ d\varphi$. This implies

$$d\varphi \circ N_{J_1} \doteq N_{J_2} \circ \Lambda^2 d\varphi. \tag{13}$$

Thus the tensor $\nabla d\varphi$ is symmetric along L. Indeed, we have: $(\nabla_X d\varphi)(Y) = \nabla_{d\varphi(X)}(d\varphi(Y)) - d\varphi(\nabla_X Y)$ and so

$$\begin{aligned}(\nabla_X d\varphi)(Y) - (\nabla_Y d\varphi)(X)\\ = T_\nabla(d\varphi(X), d\varphi(Y)) + [d\varphi(X), d\varphi(Y)] - d\varphi\, T_\nabla(X,Y) - d\varphi[X,Y]\\ = \tfrac{1}{4}(N_{J_2} \circ \Lambda^2 d\varphi - d\varphi \circ N_{J_1})(X,Y) \doteq 0.\end{aligned}$$

Denote $\Phi^{(2)} = \nabla d\varphi \in C^\infty(S^2T^*M \otimes TM|_L)$. In terms of this tensor, the condition $d\varphi \circ J_1 \ddot{=} J_2 \circ d\varphi$ holds iff for all $X, Y \in TM|_L$ we have:

$$\Phi^{(2)}(X, J_1Y) + d\varphi\,(\nabla_X J_1)(Y) = J_2\Phi^{(2)}(X,Y) + (\nabla_{d\varphi\,(X)} J_2)(d\varphi\,(Y)). \tag{14}$$

Denote

$$P(X,Y) = (\nabla_{d\varphi\,(X)} J_2)(d\varphi\,(Y)) - d\varphi\,(\nabla_X J_1)(Y). \tag{15}$$

This yields the followings property along L:

$$P(X, J_1Y) = -J_2 \circ P(X,Y),$$

which implies that $P(X,Y) = J_2B(X,Y) - B(X, J_1Y)$ for some $(2,1)$-tensor B. Conditions (15) and (13) yield (with $J = J_1 = J_2$ along L):

$$P(X,Y) - P(Y,X) = P(JX, JY) - P(JY, JX).$$

From this we obtain a solution (similarly to Theorem 1 of [K1])

$$\begin{aligned}\Phi^{(2)}(X,Y) = -\frac{1}{2}[B(X,Y) + B(Y,X)]\\ + \frac{J}{4}[B(JX,Y) + B(JY,X) - B(X,JY) - B(Y,JX)]\end{aligned}$$

of the equation $P(X,Y) = \Phi^{(2)}(X, J_1Y) - J_2\Phi^{(2)}(X,Y)$ and hence of (14).

We want to construct a map with $d\varphi \doteq \mathbf{1}$. This requirement, equation (15) and assumptions of the theorem imply that $P(X,Y) = 0$ along L if $X \in TL$ or

$Y \in TL$ (see lemma 8). Thus we can choose B with the same property and get $\Phi^{(2)}(X,Y) = 0$ if at leat one of X, Y belongs to TL.

Now we integrate the symbols $\Phi^{(2)}$ to get the 2-jet of φ along L, using the Taylor-Maclaurin decomposition by the normal coordinate y along a complimentary to TL J-invariant subbundle F, as in the proof of theorem 5. $\square$

A combination of Theorems 15 and 19 yields normal forms of 1-jets of almost complex structures J along a PH-submanifold $L \subset (M, J)$.

Let us choose a J-invariant subbundle F transversal to L, as in the proof of Theorem 5. Consider the radial vector field r, which equals $\vec{xa}$ at the point $a \in F_x$, $x \in L$ (as in Theorem 15). Let A be an automorphism of $TM|_{\mathcal{O}_L}$, which equals $A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}$ along L in the decomposition $TM|_L = TL \oplus F$.

Theorem 20. *Let $L \subset (M, J)$ be a PH-submanifold and $N_J \in C^\infty(\Lambda^2 T^*M \otimes_{\bar{\mathbb{C}}} TM|_L)$ be the field of Nijenhuis tensors of J along it. Then there exist a normally integrable almost complex structure J_0 in a neighborhood $\mathcal{O}_L \subset M$ and a diffeomorphism φ of $\mathcal{O}_L$ such that $J_0 = J$ along L, $d\varphi = \mathbf{1}$ along L and we have:*

$$\varphi^* J = J_0 + J_0 N_J(r, A\cdot) \bmod \mu_L^2. \tag{16}$$

In particular, when L is a PH-curve, the structure J_0 can be chosen complex.

Proof. Define $J' = J - JN_J(r, A\cdot)$. This is an almost complex structure $\bmod \mu_L^2$ (see [K1] about such jets). In fact, $J' \doteq J$ and $AJ \doteq JA$, so that

$$J'^2 \ddot{=} J^2 - J^2 N_J(r, A\cdot) - JN_J(r, AJ\cdot) \ddot{=} J^2 = -\mathbf{1}.$$

Notice that we get $J \ddot{=} J' + J' N_J(r, A\cdot)$.

Let $J_0 = \check{J}'$ be the corresponding NB-II structure (it is already a genuine almost complex structure). Then $\tilde{J} = J_0 + J_0 N_J(r, A\cdot)$ is an almost complex structure $\bmod \mu_L^2$ and it has the same NB-II structure as the structure J.

Now we want to check the second part of assumption 1 in Theorem 19 for the structures $J, \tilde{J}$ (we obviously have $\tilde{J} \doteq J$).

Let $X^\perp$ denote the F-component of $X \in TM|_L$. Then we get $[X, N_J(r, Y)] \doteq N_J(X^\perp, Y)$ (compare with the proof of Theorem 15, where $r = y^i \partial_{y^i}$ in local coordinates). And so we calculate:

$$\begin{aligned} N_{J'}(X,Y) &\doteq N_J(X,Y) - [JX, JN_J(r, AY)] - [JN_J(r, AX), JY] \\ &\qquad + J[X, JN_J(r, AY)] + J[JN_J(r, AX), Y] \\ &\doteq N_J(X,Y) - JN_J(JX^\perp, AY) - JN_J(AX, JY^\perp) \\ &\qquad - N_J(X^\perp, AY) - N_J(AX, Y^\perp) \\ &= N_J(X,Y) - 2N_J(X^\perp, AY) - 2N_J(AX, Y^\perp). \end{aligned}$$

Thus if $X, Y \in TL$, then $N_{J'}(X,Y) = N_J(X,Y)$. If $X \in TL$, $Y \in F$, then $N_{J'}(X,Y) = N_J(X,Y) - 2N_J(AX, Y) = 0$. And if $X, Y \in F$, then $N_{J'}(X,Y) = N_J(X,Y) - 2N_J(AX, Y) - 2N_J(X, AY) = 0$.

Therefore, $N_{J'}$ vanishes for vertical vectors and J_0 is normally integrable. In particular, J_0 is the NB-I structure of the structure J', see (12).

By a calculation, similar to the above one, we obtain along L:

$$N_{\tilde{J}}(X,Y) = N_{J_0}(X,Y) + 2N_J(X^\perp, AY) + 2N_J(AX, Y^\perp).$$

Since $N_{J_0}(X,Y) = 0$ if X or Y belongs to F and $N_{J_0}|_{TL} = N_J|_{TL}$, we conclude that $N_{\tilde{J}}(X,Y) = N_J(X,Y)$ for all $X, Y \in TM|_L$.

Thus from Theorem 19 we get a local diffeomorphism φ identical up to the first order on L and such that $\tilde{J} \stackrel{..}{=} \varphi^* J$. □

Remark 7. *When L is a point, the structure J_0 can also be chosen complex. Moreover in this case $A = 1/4$ and formula (16) looks especially simple. We write it in local coordinates (x^i) centered at the given point $x_0 \in M$:*

$$J_i^k = (-1)^k \delta_i^{\#k} - (-1)^k \tfrac{1}{4} N_{ij}^{\#k}(0) x^j + o(|x|).$$

A general way to obtain similar formulae for jets at a point is related to the structural function (Weyl tensor) of the corresponding geometric structure ([KL]).

5. Four-dimensional case and Arnold's question

In this and next sections we consider the special case $\dim M = 4$. Proper PH-submanifolds are PH-curves $L^2 \subset (M^4, J)$. So $N_L^I M = (N_L M, \hat{J})$ is a holomorphic line bundle, while $N_L^{II} M = (N_L M, \check{J})$ is a PH-line bundle.

Nijenhuis tensor characteristic distribution $\Pi = \mathrm{Im}(N_J) \subset TM^4$ ([K3]) is J-invariant and has rank 2 in the domain of non-integrability for J, $N_J \neq 0$.

Proposition 21. *At the points $x \in L$, where the Nijenhuis tensor characteristic distribution Π is transversal to L, the same happens to the NB-II characteristic distribution $\check{\Pi}$. But $N_{\check{J}}(x) = 0$ at the points x, where $\Pi \subset TL$.*

Proof. This follows from formulae (6). □

Corollary 22. *If the Nijenhuis tensor characteristic distribution Π is tangent to L, then the NB-I and NB-II structures coincide and are holomorphic.* □

Holomorphic line bundles over a genus g curve $L = \Sigma_g$ are parameterized by g complex parameters. Line bundles over rational curves $L = \bar{\mathbb{C}} \simeq S^2$ are determined by the topological type, i.e. by the self-intersection number $L \cdot L$ of the zero section. But for other curves the holomorphic and differentiable types of holomorphic bundles are different.

A holomorphic line bundle over an elliptic curve $L = \mathbb{C}/\mathbb{Z}^2(2\pi, \omega) \simeq T^2$ $(g = 1)$, $\omega \in \mathbb{C} \setminus \mathbb{R}$, depends on one parameter $\lambda \in \mathbb{C} \setminus \{0\}$. If the zero section has self-intersection number p, the bundle is: $E \to T^2$, $(z, w) \mapsto z$, $J_0 = i$, with

$$E = \mathbb{C}^2/(z,w) \sim (z + 2\pi, w) \sim (z + \omega, \lambda e^{-ipz} w). \tag{17}$$

The pair (ω, λ) can be chosen to satisfy: $|\omega| \geq 2\pi$, $-\pi < |\operatorname{Re}\omega| \leq \pi$, $\operatorname{Im}\omega > 0$, $e^{-\operatorname{Im}\omega} < |\lambda| \leq 1$. The number ω is defined by the restriction $J_0|_{T^2}$ and the number λ is defined by 1-jet of the structure J_0 on T^2.

A PH-line bundle $(N_L M, \check{J})$ over a genus g curve $L = \Sigma_g^2$ is parametrized by g complex parameters (for NB-I structure $J_0 = \hat{J}$), a cohomology class $\mathrm{ns}_0 \in H^1(L; TL \otimes N^*)$, see remark 6, and a smooth 1-form $N_{\check{J}} \in \Omega^1(L; \mathrm{aut}_{\bar{\mathbb{C}}}(N_L M))$.

Consider an elliptic curve $L = T^2$ in a complex surface (M^4, J_0) with the normal bundle $N_L^I M$ given by (17). For $p < 0$ ([Gra]) or $p = 0$ and generic pair (ω, λ) ([A1]) a small neighborhood of the torus in M^4 is biholomorphically equivalent to a neighborhood of the zero section in $N_L M$. In [A2] Arnold asks about non-integrable version of this result.

Proposition 23. *Codimension of the set of almost complex structures, the germs of which on the PH-curve $L \subset (M, J)$ are isomorphic to these of the normal bundle $L \subset N_L M$, in the set of all almost complex structures is infinity.*

Proof. For existence of such an isomorphism two conditions must fulfil. First, by Corollary 10, the Nijenhuis tensor characteristic distribution Π^2 should be integrable and transversal to L whenever non-zero. Second, by Proposition 12, the Nijenhuis tensor N_J should be constant along the leaves of Π^2. Both conditions are of codim $= \infty$. $\square$

The two mentioned conditions are necessary, but not sufficient.

Example. Let $M^4 = L^2 \times D^2$ have coordinates $(z = x + iy, w = s + it)$. Equip it with the almost complex structure

$$J\partial_x = a_1\partial_x + (1 + a_2)\partial_y + b_1\partial_s + b_2\partial_t, \quad J\partial_s = \partial_t. \tag{18}$$

Then $L \times \{0\}$ is a PH-curve, if $b_i = 0$ on it. Moreover, one can achieve $a_i|_L = 0$.

The integrability condition $\Pi^2 = T\mathcal{F}$, $\mathcal{F}_c = \{z = c\}$, and the requirement of the tensor N_J constancy along $\mathcal{F}$ write as follows ($c_i = c_i(x, y)$):

$$\begin{cases} \dfrac{\partial a_1}{\partial t} = a_1 \dfrac{\partial a_1}{\partial s} - \dfrac{1 + a_1^2}{1 + a_2} \dfrac{\partial a_2}{\partial s}, & \dfrac{\partial b_1}{\partial t} = -\dfrac{\partial b_2}{\partial s} + b_1 \dfrac{\partial a_1}{\partial s} + \dfrac{b_2 - b_1 a_1}{1 + a_2} \dfrac{\partial a_2}{\partial s} + c_1, \\ \dfrac{\partial a_2}{\partial t} = (1 + a_2) \dfrac{\partial a_1}{\partial s} - a_1 \dfrac{\partial a_2}{\partial s}, & \dfrac{\partial b_2}{\partial t} = \dfrac{\partial b_1}{\partial s} + b_2 \dfrac{\partial a_1}{\partial s} - \dfrac{b_1 + b_2 a_1}{1 + a_2} \dfrac{\partial a_2}{\partial s} + c_2. \end{cases}$$

This is a Cauchy-Kovalevskaya type system, so any analytical initial condition $(a_i, b_i)|_{t=0} = (\alpha_i^0(s), \beta_i^0(s))$ determines uniquely the solution. PH bundle structures correspond to $\alpha_i^0 = \lambda_i(x, y)$, $\beta_i^0 = \mu_i(x, y) + \nu_i(x, y)s$. There are however different solutions, for example: $a_1 = -b_1 = -\frac{s}{1+t}$, $a_2 = -b_2 = -\frac{t}{1+t}$.

Thus the answer to Arnold's question is negative. A generalization of his theory should look differently. It will concern existence of a PH-foliation of a T^2-neighborhood by cylinders. In holomorphic situation there exists a foliation by holomorphic cylinders, given in the representation (17) as $\{w = \mathrm{const}\}$. Does it persist if we perturb the structure J to an almost complex one?

We discuss this question in [K3]. Note however that in the complex situation transport along the leaves of the foliation is holomorphic. When does a PH-foliation exist with pseudoholomorphic transports?

By transports here we mean the following. Let D^2_z be a foliation by transversal PH-disks as in proposition 6. Let $\mathcal{H}$ be a PH-foliation with L as a leaf. A path between two points $z_1, z_2 \in L$ determines a map $D^2_{z_1} \to D^2_{z_2}$ of shifts along $\mathcal{H}$, called the transport. Homotopically non-trivial loops yield the monodromy (for a PH-foliation $\mathcal{H}$ by cylinders, one cycle has a trivial monodromy).

The requirement of PH-transports is independent of the choice of transversal disks family. For a generic almost complex structure the monodromy and transports are non PH-maps of the disks D^2_z.

Proposition 24. *Let $L \subset (M^4, J)$ be a PH-curve. Existence of a PH-foliation $\mathcal{H}$ of its neighborhood with PH-transports is a condition of codimension infinity.*

Proof. The requirements of PH-transports means that projection along $\mathcal{H}$ is a PH-map. Thus by Corollary 10 the Nijenhuis tensor characteristic distribution is integrable and tangent to $\mathcal{H}$. Also the Nijenhuis tensor should be locally projectible along $\mathcal{H}$. These are two conditions of codim $= \infty$. $\square$

Here is another generalization of Arnold's theory of holomorphic curves neighborhoods:

Theorem 25. *A small neighborhood $\mathcal{O}_L$ of a PH-curve $L = \Sigma^2_g$ is Kobayashi hyperbolic iff $g \geq 2$. For $g = 0$ the punctured neighborhood $\mathcal{O}_L \setminus L$ is not hyperbolic and for $g = 1$ it is not hyperbolically imbedded into $\mathcal{O}_L$.*

We refer to [Kob] for the basics of hyperbolic spaces. In almost complex context the corresponding notions were introduced in [KO] and a non-integrable version of Brody criterion was established. Its application together with a theorem of Lang (§3.6 [Kob]) and compactness from [Gro] yield the above statement.

6. Deformations of PH-curves

In this section we continue to study PH-curves $L \simeq (\Sigma^2_g, j)$. Let $\mathcal{X} = C^\infty(\Sigma, M; A)$ be the space of all smooth maps $u : \Sigma^2_g \to M^{2n}$ representing a fixed homology class $A \in H_2(M)$ and $\varrho : \mathcal{E} \to \mathcal{X}$ be the bundle with the fiber $\varrho^{-1}(u) = \mathcal{E}_u = \Omega^{0,1}(u^*TM)$ being the space of anti-linear maps $T\Sigma \to TM$ over u. For Fredholm theory these spaces should be completed to appropriate functional spaces ([MS]), whose precise choice is not crucial due to elliptic regularity. But we will not specify them, because it is irrelevant for our geometric approach.

PH-curves $L = \mathrm{Im}[u : (\Sigma^2_g, j) \to (M, J)]$ in the class A are zeros of the section $\bar\partial_J = \frac{1}{2}(\mathbf{1} + J \circ j^*) \circ d : \mathcal{X} \to \mathcal{E}$ and their union forms the moduli space $\mathcal{M}(A, J) = \bar\partial_J^{-1}(0)$. To study regularity of a point $u \in \mathcal{M}(A, J)$ Gromov [Gro] considers the linearization $D_u = D\bar\partial_J : C^\infty(u^*TM) \to \Omega^{0,1}(u^*TM)$. This Gromov's operator can be explicitly written ([MS, IS]) as

$$D_u(v) = \bar\partial_{u,J}(v) + \tfrac{1}{4} N_J(v, \partial_J(u)). \tag{19}$$

The operator descends to the normal bundle in virtue of the following diagram:

$$\begin{array}{ccccccc} 0 \to C^\infty(T\Sigma) & \xrightarrow{du} & C^\infty(u^*TM) & \xrightarrow{\text{proj}} & C^\infty(u^*TM)/C^\infty(T\Sigma) & \to 0 \\ \bar\partial \big\downarrow & & D_u \big\downarrow & & \check D_u \big\downarrow & \\ 0 \to \Omega^{0,1}(T\Sigma) & \xrightarrow{du} & \Omega^{0,1}(u^*TM) & \xrightarrow{\text{proj}} & \Omega^{0,1}(u^*TM)/\Omega^{0,1}(T\Sigma) & \to 0. \end{array}$$

As before we consider only regular PH-curves $L = u(\Sigma)$ (singularities may enlarge the sheaf of holomorphic sections of the normal bundle, see [IS]), in which case $C^\infty(u^*TM)/C^\infty(T\Sigma) = C^\infty(N_L M)$ and similar for $\Omega^{0,1}$.

Proposition 26. *The Gromov operator $\check D_u : C^\infty(N_L M) \to \Omega^{0,1}(N_L M)$ coincides with the Cauchy-Riemann operator $\bar\partial_{\check J}$ of the NB-II structure $\check J$.*

Of course, an indication of this result is Proposition 7.

Proof. This follows from Theorem 15 because the operator $\bar\partial_{\check J}$ with the PH-bundle structure from formula (10) coincides with the expression (19). □

Now we introduce the Dolbeault cohomology groups $H^{0,0}_{\bar\partial_{\check J}}(N^{II}_L M) = \mathrm{Ker}(\check D_u)$ and $H^{0,1}_{\bar\partial_{\check J}}(N^{II}_L M) = \mathrm{CoKer}(\check D_u)$ (of course, Sobolev spaces are needed to insure via Fredholm property that the dimensions are finite). Vanishing of the former is equivalent to non-existence of deformations for the PH-curve L, while vanishing of the latter means transversality of $\bar\partial_J$ to the zero section of ϱ at u, whence u is a regular point of the moduli space $\mathcal{M}(A, J)$.

Consider the case $\dim M = 4$, where $N^I_L M$ is a holomorphic line bundle. The following statement is essentially contained in [Gro, HLS, IS] (the two statements below are equivalent via Kodaira-Serre duality).

Theorem 27. *If $c_1(N^I_L M) < 0$, then $H^{0,0}_{\bar\partial_{\check J}}(N^{II}_L M) = 0$. If $c_1(N^I_L M) > 2g - 2$, then $H^{0,1}_{\bar\partial_{\check J}}(N^{II}_L M) = 0$.* □

For higher-dimensional M in the case $g = 0$ one proceeds as follows: By Grothendieck's theorem a holomorphic line bundle over S^2 splits into line bundles $N^I_L M = \oplus \mathfrak{L}_i$ and then one gets the vanishing theorem requiring the corresponding inequality for the Chern class of each line bundle $\mathfrak{L}_i$.

Now for the rest of the section we study a particular interesting case of an elliptic PH curve ($g = 1$) in four-dimensional manifold M^4 and its deformation. We wish to get a non-deformation criterion, which is based on the whole structure of $N^{II}_L M$, not only of $N^I_L M$.

Let self-intersection number of the curve be $L \cdot L = p$. If $p < 0$, the curve is not deformed by the positivity of intersections ([M1]). For $p > 0$ the virtual moduli space has positive dimension (by the index computation for the linearized Cauchy-Riemann operator).

Consider now topologically trivial normal bundles, $p = 0$, when the elliptic curves are generically discrete and persistent under a small perturbation of the structure J (this case was studied in [Ku] and the number of non-parametrized

PH-tori was estimated). We will formulate an explicit sufficient condition of non-deformation and persistence.

Let $\pi : (E, J) \to T^2$ be a PH line bundle. Due to corollary 16 there exist coordinates $(z = x + iy, w = s + it)$ on E, with the gluing rule (17), such that

$$\begin{cases} J\partial_x = \quad \partial_y + s \cdot \xi - t \cdot J\xi, & J\partial_s = \quad \partial_t, \\ J\partial_y = -\partial_x - s \cdot J\xi - t \cdot \xi, & J\partial_t = -\partial_s. \end{cases}$$

The vertical vector field $\xi = \frac{1}{2} JN_J(\partial_s, \partial_x)$ can be decomposed $\xi = 2\beta_1\partial_t - 2\beta_2\partial_s$ with $\beta = \beta(z)$, $\beta = \beta_1 + i\beta_2$.

Every PH-curve in E, homologous to the zero section T^2, is of the form $L = f(T^2)$ for some section $f \in C^\infty(\pi)$. This follows from the properties of PH-maps $\pi : L \to T^2$ of degree 1 (also from positivity of intersections for the spheres-compactification of the fibers). Let us deduce the equation for $f = f_1 + if_2$. The tangent bundle to the curve $w = f(z)$ is spanned by

$$\eta_1 = \partial_x + \partial_x(f_1)\partial_s + \partial_x(f_2)\partial_t, \quad \eta_2 = \partial_y + \partial_y(f_1)\partial_s + \partial_y(f_2)\partial_t. \tag{20}$$

The pseudoholomorphicity condition $J\eta_1 = \eta_2$ along $f(T^2)$ is equivalent to the equation

$$f_{\bar z} + \beta \bar f = 0. \tag{21}$$

Below we use the normalization of §5 for the pair (ω, λ) from (17), characterizing the holomorphic line bundle NB-I over an elliptic curve $L = T^2$.

Proposition 28. *Let J be a PH line bundle structure and the corresponding complex structure J_0 from* (10) *have the multiplier λ (normalized as in §5). Determine the function $\Lambda \in C^\infty(T^2)$ from the equation $\frac{1}{2} JN_J(\partial_w, \partial_z) = \Lambda\partial_{\bar w}$. Let $\lambda \neq 1$ if $\Lambda \equiv 0$ and if $\Lambda \not\equiv 0$ assume the inequality:*

$$|\partial_{\bar z}\Lambda| \le (1 - \varepsilon)|\Lambda|^2 - |\tau\Lambda|, \quad \tau = \ln|\lambda| / \operatorname{Im}\omega, \tag{22}$$

for some $\varepsilon > 0$. Then the zero section T^2 is the only PH-torus in E.

Notice that if $\Lambda \neq 0$, the inequality can be achieved via a simple rescaling. Its meaning is then that the structure J is far from being integrable (J_0).

Proof. We have $\Lambda = -2i\bar\beta$ because

$$\begin{aligned} &\frac{1}{2} JN_J(\partial_w, \partial_z) = -2(\beta_2 + i\beta_1)\partial_{\bar w}, && \frac{1}{2} JN_J(\partial_{\bar w}, \partial_z) = 0, \\ &\frac{1}{2} JN_J(\partial_w, \partial_{\bar z}) = 0, && \frac{1}{2} JN_J(\partial_{\bar w}, \partial_{\bar z}) = -2(\beta_2 - i\beta_1)\partial_w, \end{aligned} \tag{23}$$

Let us show that equation (21) has no nonzero solutions.

Our torus neighborhood is the product of the cylinder $\mathcal{C}^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z \in [0, \operatorname{Im}\omega)\}/2\pi\mathbb{Z}$ and $\mathbb{C}(w)$ glued by the rule $(z, w) \mapsto (z + \omega, \lambda w)$. The boundary $\partial\mathcal{C}^2$ consists of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and its ω-shift.

Introduce the real-valued linear function $\sigma = i\tau\frac{1}{2}(z-\bar{z})$ and note that the function $h = e^{2\sigma}$ satisfies: $h(z+2\pi) = h(z)$, $h(z+\omega) = h(z)/|\lambda|^2$. So using formula (21) and its consequence $f_{\bar{z}z} = -\beta_z\bar{f} + |\beta|^2 f$ we get:

$$0 = \Big(\frac{\lambda\bar{\lambda}}{|\lambda|^2} - 1\Big)\oint_{S^1}\frac{i}{2}\,e^{2\sigma}f_{\bar{z}}\bar{f}d\bar{z} = \iint_{\mathcal{C}^2}\frac{i}{2}d(e^{2\sigma}f_{\bar{z}}\bar{f}d\bar{z}) =$$
$$\iint_{\mathcal{C}^2} e^{2\sigma}\Big[\,2|\beta|^2|f|^2 - (i\tau\beta + \beta_z)\bar{f}^2\,\Big]\frac{i}{2}dz\wedge d\bar{z}.$$

Taking the real part we deduce:

$$\iint_{\mathcal{C}^2} e^{2\sigma}(2|\beta|^2 - |\tau\beta| - |\beta_z|)|f|^2\,dx\wedge dy \le 0.$$

Since by the assumption $2(1-\varepsilon)|\beta|^2 - |\tau\beta| \ge |\beta_z|$ for some positive ε, we should have $f = 0$ or $\beta = 0$. If $\beta \equiv 0$, then the Fourier decomposition of the 2π-periodic holomorphic function f and the condition $f(z+\omega) = \lambda f(z)$ for $e^{-\operatorname{Im}\omega} < |\lambda| \le 1$, $\lambda \ne 1$, imply $f \equiv 0$. If β vanishes only on a domain $D \subset T^2$, then f is holomorphic in D and vanishes in $T^2 \setminus D$, whence $f \equiv 0$.

So there are no PH-tori $\tilde{T}^2$, homologous to the zero section, with $f \not\equiv 0$. If the homology class of $\tilde{T}^2$ is a multiple of the zero section $[\tilde{T}^2] = k[T^2]$ a k-finite covering finishes the proof. □

Proposition 29. *The linearized equation for close PH-tori can be written as*

$$f_{\bar{z}} + \alpha f + \beta\bar{f} = 0. \tag{24}$$

The function $\alpha = 0$ for the normal coordinate w on $N_{T^2}M$, equipped with NB-I complex structure and with the gluing rule (17). Alternatively $\alpha = \mathrm{const}$ for a global well-defined coordinate w.

Proof. Since we are interested in the linearized equation, which is determined by 1-jet of J, we can use the normal form given by Theorem 20 (we simplify it for dimension 4): $J \doteq J_0 + \frac{1}{2}JN_J(r,\cdot)$. We write the complex structure J_0 in coordinates (z,w) of $\mathcal{O}(T^2)$ with the gluing rule (17) ($p=0$): $J_0\partial_z = i\partial_z$, $J_0\partial_w = i\partial_w$. Note that in these coordinates $r = w\partial_w + \bar{w}\partial_{\bar{w}}$.

The most general form of the Nijenhuis tensor along T^2 is the following: $-\frac{1}{2}JN_J(\partial_z,\partial_w) \doteq a\partial_{\bar{z}} + b\partial_{\bar{w}}$, where $a = a(z,\bar{z})$, $b = b(z,\bar{z})$ are smooth functions on T^2. Then we obtain ($\{z = \mathrm{const}\}$ is assumed a PH-foliation, as in Proposition 6):

$$J\partial_z \doteq i\partial_z + aw\partial_{\bar{z}} + bw\partial_{\bar{w}}, \quad J\partial_w = i\partial_w.$$

If $w = f(z,\bar{z})$ is a surface, then $\eta = \partial_z + f_z\partial_w + \bar{f}_z\partial_{\bar{w}}$ and $\bar{\eta} = \partial_{\bar{z}} + f_{\bar{z}}\partial_w + \bar{f}_{\bar{z}}\partial_{\bar{w}}$ span a complexified tangent plane to its graph.

Thus $w = f(z,\bar{z})$ is a PH-curve iff $J\eta - i\eta - aw\bar{\eta} \doteq 0$ ($w = f$ is a function of the first order of smallness on T^2, so we disregard $wf_{\bar{z}}$), which is equivalent to the equation $f_{\bar{z}} = \beta\bar{f}$ with $\beta = \frac{i}{2}\bar{b}$. The first statement is proved.

We obtain the second statement, introduce a global coordinate by the change $w \mapsto w\cdot\exp\Big(\dfrac{z-\bar{z}}{\bar{\omega}-\omega}\ln\lambda\Big)$, which yields equation (24) with $\alpha = \dfrac{i}{2}\dfrac{\ln\lambda}{\operatorname{Im}\omega}$. □

Remark 8. *Equation (24) with $\alpha = 0$, $\beta = const$ was considered by Moser [Mo]. The proposition proves a remark on p. 430 that* "the linearized equation can be brought into form (24) with $\alpha = \text{const}$".

Theorem 30. *Let the normal bundle of a PH-curve $T^2 \subset (M^4, J)$ be topologically trivial and its NB-II structure be described by the function Λ, as in Proposition 28, satisfying inequality (22). Then the curve is isolated and persistent under small perturbations of J.*

Proof. This follows from Propositions 7 and 28. Alternatively, since index of the linearized Cauchy-Riemann operator $P(f) = f_{\bar z} + af + b\bar f$, $f \in C^\infty(T^2, \mathbb{C})$, is zero, the required properties follow from non-existence of non-zero solutions of the equation $P(f) = 0$. □

Certainly a big perturbation of J can destroy the properties. Another criterion of deformations non-existence with an additional requirement of complex transports is given by Proposition 24.

A. Minimal almost complex connections

In this appendix we prove a theorem, which is basically due to Lichnerowicz. Our proof, however, differs from the original one ([L]).

Recall that a linear connection on an almost complex manifold (M, J) can always be taken J-linear. In fact, for any connection ∇ we can define

$$\hat\nabla_X = \frac{1}{2}(\nabla_X - J\nabla_X J).$$

One easily checks that $\hat\nabla$ is a linear connection satisfying $\hat\nabla JY = J\hat\nabla Y$.

Also let us recall that every tensor uniquely decomposes into its J-linear and anti-linear parts. For instance if T is a $(2,1)$-tensor, it has the decomposition

$$T = T^{++} + T^{+-} + T^{-+} + T^{--}, \qquad \text{where}$$

$$T^{\varepsilon_1\varepsilon_2}(JX, Y) = \varepsilon_1 JT^{\varepsilon_1\varepsilon_2}(X, Y), \quad T^{\varepsilon_1\varepsilon_2}(X, JY) = \varepsilon_2 JT^{\varepsilon_1\varepsilon_2}(X, Y);$$

$$T^{\varepsilon_1\varepsilon_2}(X, Y) = \frac{1}{4}\big[T(X, Y) - \varepsilon_1 JT(JX, Y) - \varepsilon_2 JT(X, JY) - \varepsilon_1\varepsilon_2 T(JX, JY)\big].$$

Theorem 31. *For any almost complex connection ∇ the totally antilinear part of its torsion is $T_\nabla^{--} = \frac{1}{4}N_J$. There are connections, called minimal, for which $T_\nabla = \frac{1}{4}N_J$. These connections are sections of an affine bundle $\mathcal{M}_{(M,J)}$ associated with the vector bundle $S^2T^*M \otimes_{\mathbb{C}} TM$ over M.*

Proof. The first formula follows directly from the definitions. There are also other formulae expressing the Nijenhuis tensor via a covariant differentiation (see [K1] for flat connections).

Consider now an almost complex connection ∇. We can make a gauge transformation $\nabla \mapsto \tilde\nabla = \nabla + A$, $A \in C^\infty(T^*M \otimes (T^*M \otimes_{\mathbb{C}} TM))$, with the

J-linearity condition imposed to keep $\tilde{\nabla}$ almost complex. Then the torsion is changed by the rule:

$$T_{\tilde{\nabla}} = T_{\nabla} + \Box(A),$$

where $\Box = \mathrm{alt} : \otimes^2 T^*M \otimes TM \to \Lambda^2 T^*M \otimes TM$ is the alternation operator. Introducing the decomposition

$$A = A^+ + A^-, \qquad A^\varepsilon(X,Y) = \frac{1}{2}\big[A(X,Y) - \varepsilon JA(JX,Y)\big],$$

we compute the components of $\Box(A) = \sum_{\varepsilon_i = \pm} \Box^{\varepsilon_1\varepsilon_2}(A)$:

$$\Box^{++}(A) = A^+ - A^+\tau,\ \Box^{+-}(A) = -A^-\tau,\ \Box^{-+}(A) = A^-,\ \Box^{--}(A) = 0,$$

where $\tau(X,Y) = (Y,X)$. We can make all the components of the torsion vanishing, save for T_{∇}^{--}, using the graded commutation relations $T_{\nabla}^{\varepsilon_1\varepsilon_2} \circ \tau = -T_{\nabla}^{\varepsilon_2\varepsilon_1}$. Actually we get a minimal connection $\tilde{\nabla}$ with the gauge

$$A = -\frac{1}{2}T_{\nabla}^{++} - T_{\nabla}^{-+}.$$

This proves the second part of the statement.

The last one follows from the above formulae for $\Box^{\varepsilon_1\varepsilon_2}(A)$: The gauge transformation $\nabla \mapsto \nabla + A$ does not change the minimality iff A^+ is symmetric and $A^- = 0$, i.e. $A \in C^\infty(S^2T^*M \otimes_{\mathbb{C}} TM)$. $\Box$

Proposition 32. *Let $\nabla \in \mathcal{M}_{(M,J)}$ be a minimal connection. Then*

$$4\mathfrak{S}\{R_{\nabla}(X,Y)Z\} = \mathfrak{S}\{N_J(N_J(X,Y),Z)\} + \mathfrak{S}\{(\nabla_X N_J)(Y,Z)\},$$

where $\mathfrak{S}$ denotes the cyclic sum.

Proof. This is a direct corollary of the first Bianchi's identity. $\Box$

Remark 9. *Thus the field of the Nijenhuis tensors $N_J \in C^\infty(\Lambda^2T^*M \otimes_{\bar{\mathbb{C}}} TM)$ on a manifold M is not arbitrary. For a general position tensor N_J this follows also from a result of [K1]: Such a tensor field N restores the structure $\pm J$, which in turn determines N_J and we obtain the constraint $N = N_J$.*

The formula of the proposition involves the curvature R_{∇}, but neither it, nor even its anti-linear part R_{∇}^{--} is independent of $\nabla \in \mathcal{M}_M$. However we have:

Proposition 33. *The operator $\Box_{X\wedge Y}Z = \nabla_{N_J(X,Y)}Z - 4R_{\nabla}^{--}(X,Y)Z$ is independent of $\nabla \in \mathcal{M}_M$, tensorial in X,Y and is an N_J-twisted differentiation in Z: $\Box_{X\wedge Y}(fZ) = (N_J(X,Y)f)Z + f\Box_{X\wedge Y}Z$.*

Proof. In fact, if $\tilde{\nabla} = \nabla + A \in \mathcal{M}_M$ is another minimal connection, then $A \in C^\infty(S^2T^*M \otimes_{\mathbb{C}} TM)$ and we calculate: $R_{\tilde{\nabla}}^{--} = R_{\nabla}^{--} + \frac{1}{4}A_{N_J}$. $\Box$

Using $\Box$ we can multiply invariants of an almost complex structure. For instance, $\Box N_J \in C^\infty(\Lambda^2_{\mathbb{C}}T^*M \otimes (\Lambda^2T^*M \otimes_{\bar{\mathbb{C}}} TM))$. There are other ways to get invariants – by prolongation-projection method and via the Frölicher-Nijenhuis bracket ([K1]), but they are different from this differentiation.

B. Normal bundle in other geometries.

The proposed construction of the tangent and normal bundle structures is more general and can be carried out for other geometric structures (the submanifold $L \subset M$ should allow restriction of the structure). One chooses a Cartan connection ∇ (i.e. preserving the structure) on M with a kind of minimality: Its torsion should be equal to the corresponding structural (Weyl) tensor ([St, KL]), which is realized (we consider, for simplicity, the case of the first order structures) via a splitting σ of the exact sequence

$$0 \to \mathfrak{g}^{(1)} \xrightarrow{i} \mathfrak{g} \otimes T_x^* M \xrightarrow{\delta} \Lambda^2 T_x^* M \otimes T_x M \overset{\sigma}{\dashleftarrow} H^{0,2}(\mathfrak{g}) \to 0.$$

Here the last term is the Spencer δ-cohomology group (space of structural functions), $\mathfrak{g} \subset \mathrm{aut}(T_x M)$ is the symbol of the geometric structure and $\mathfrak{g}^{(1)} = \mathrm{Ker}\,\delta$ is its prolongation ([KL]). The freedom in a choice of ∇ is thus reduced to $\mathfrak{g}^{(1)}$.

For an almost complex structure J: $\mathfrak{g} = \mathrm{gl}_{\mathbb{C}}(T_x M, J)$ and the prolongation is $\mathfrak{g}^{(1)} = S^2 T_x^* M \otimes_{\mathbb{C}} T_x M$, cf. Theorem 31.

For a symplectic structure Ω: $\mathfrak{g} = \mathrm{sp}(T_x M, \Omega)$, $\mathfrak{g}^{(1)} = S^3 T_x^* M$ and a canonical normal bundle structure $\hat{\Omega}$ appears. By the symplectic neighborhood theorem ([W]) it is completely determined by the restriction Ω_L and the isomorphism class of the normal bundle with fiber-wise symplectic structure, usually called "symplectic normal bundle" (ν_L, Ω).

For a Riemannian structure g: $\mathfrak{g} = \mathrm{so}(T_x M, g)$, $\mathfrak{g}^{(1)} = 0$. ∇ is the Levi-Civita connection. It splits the normal bundle $N_L M$ and leads to the normal bundle structure $\hat{g}$. Another approach to $\hat{g}$ is similar to (3): One constructs a normal foliation $\mathcal{W}$ around L via geodesics $\gamma \subset \mathcal{W}$ in all normal directions $T_x^{\perp} L$, $x \in L$, and applies the dilatations R_t along geodesics.

Defining in this way the normal structure on $N_L M$ we obtain two structural tensors on L: One original on $L \subset M$ and the other from the normal bundle on the zero section $L \subset N_L M$. There are relations between these tensors. For an almost complex structure J we described them in §3.

Consider now a Riemannian metric g. Our structural tensors are: Riemannian curvature R_g along L and the normal bundle curvature $R_{\hat{g}}$ at zero section.

To describe the relations consider the curvature of the normal bundle $R^{\perp}$. It is the curvature tensor of the normal connection $\nabla^{\perp}$, given by the orthogonal decomposition in $TM|_L = TL \oplus N_L M$, $R = R^{\|} + R^{\perp}$. Note that $R^{\perp}(X, Y) = R_{\hat{g}}(X, Y)$ for $X, Y \in TL$ and the left-hand side is not defined for others X, Y.

Let $\mathrm{II} : TL \otimes TL \to N_L M$ be the second quadratic form of L and $A : TL \otimes N_L M \to TL$ be the shape (Peterson) operator given by $g(A(X, V), Y) = g(\mathrm{II}(X, Y), V)$, $X, Y \in TL$, $V \in N_L M$. The Ricci equation reads:

$$[R_g(X, Y)V]_{\perp} = R^{\perp}(X, Y)V + \mathrm{II}(X, A(Y, V)) - \mathrm{II}(Y, A(X, V)),$$

where $X, Y, Z \in TL$, $V \in N_L M$.

In particular when L is totally geodesic $\mathrm{II} = 0$ and $A = 0$, so that the equation mean $R_g(X, Y) = R_{\hat{g}}(X, Y)$ for $X, Y \in TL$ at the points of L.

Similar calculations occurs in other geometries, like projective or conformal, they can be deduced from the basic structure equations [N] of these geometries.

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