Characteristic distributions on 4-dimensional almost complex manifolds

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Abstract

In this paper the Nijenhuis tensor characteristic distributions on a non-integrable four-dimensional almost complex manifold is investigated for integrability, singularities and equivalence.

Introduction

For a non-integrable four-dimensional almost complex manifold we will canonically define a distribution Π^2 by the Nijenhuis tensor N_J. In §1 we complete the description [K1] of invariants of an almost complex structure in dimension four, using this distribution. In §2-§3 we describe singularities of Π^2. We show they are standard if our field of planes is considered as a distribution, but they become quite specific if it is considered as a differential system.

In §4-5 we study moduli and hyperbolicity of the germ of a neighborhood of a pseudoholomorphic curve. §6 is devoted to a geometric meaning of the integrability of the Nijenhuis tensor characteristic distribution Π^2 and its relation to a question of V. Arnold.

In [HH] Hirzebruch and Hopf proved the following topological result: If a 4-dimensional manifolds admits a rank 2 distribution, it admits an almost complex structure as well. Moreover if the manifold admits two almost complex structures, defining opposite orientations, then it admits a rank 2 distribution.

We associate to a non-integrable almost complex structure a rank 2 distribution, realizing the above topological correspondence (to one side) canonically on the differential level. Note that any almost complex structure on a 4-dimensional manifold can be perturbed to be non-integrable outside a discrete set.

1. Local classification of almost complex structures in dimension 4

Let $(M, J \in \mathrm{Aut}(TM))$ be an almost complex manifold of dimension 4, $J^2 = -\mathbf{1}$. Its Nijenhuis tensor is the following $(2,1)$-tensor

$$N_J \in \Lambda^2 T^*M \otimes TM, \quad N_J(\xi,\eta) = [J\xi, J\eta] - J[J\xi,\eta] - J[\xi,J\eta] - [\xi,\eta]. \quad (1)$$

Integrability of J is expressed via it as $N_J = 0$ ([NW]).

This tensor satisfies the property $N_J(J\xi,\eta) = N_J(\xi, J\eta) = -JN_J(\xi,\eta)$ and so can be considered as an antilinear map $N_J : \Lambda^2\mathbb{C}^2 \to \mathbb{C}^2$, $\mathbb{C}^2 = (T_xM^4, J)$. The image is invariant under J and if $N_J \neq 0$ it is a complex line $\mathbb{C} \subset \mathbb{C}^2$.

Thus in the domain, where the structure J is non-integrable, a canonical distribution is defined:

Definition 1. *Let us call* $\Pi^2 = \operatorname{Im} N_J \subset TM$ *the* Nijenhuis tensor characteristic distribution *on a 4-dimensional almost complex manifold* (M^4, J).

This distribution Π^2 is in general situation nonintegrable. Therefore it has a nontrivial derivative $\Pi^3 = \partial\Pi^2$, which is defined as the differential system with $C^\infty(M)$-module of sections $\mathcal{P}_3 = C^\infty(\Pi^3)$ generated by the self-commutator of the submodule $\mathcal{P}_2 = C^\infty(\Pi^2) \subset \mathcal{D}(M)$: $\mathcal{P}_3 = [\mathcal{P}_2, \mathcal{P}_2]$. Π^3 is not a distribution everywhere and its singularities form a stratified submanifold Σ_1^2 of codim $= 2$.

The distribution Π^3 on $M \setminus \Sigma_1^2$ is generically nonintegrable, so that $\partial\Pi^3 = TM$ (or $[\mathcal{P}_2, \mathcal{P}_3] = \mathcal{D}(M)$) outside a stratified submanifold Σ_2^2 of codim $= 2$.

If $x \notin \Sigma_1^2$, then $\Pi_x^2 \subset \Pi_x^3$ has a transversal measure. In fact since the J-antilinear isomorphism $N_J(\cdot, \xi_3) : \Pi_x^2 \to \Pi_x^2$ is orientation reversing, there exist vectors $\xi_1, \xi_2 \in \Pi_x^2$, $\xi_3 \in \Pi_x^3 \setminus \Pi_x^2$ such that $N_J(\xi_1, \xi_3) = \xi_1$, $N_J(\xi_2, \xi_3) = -\xi_2$. These ξ_1, ξ_2 are defined up to multiplication by a constant, while $\xi_3 (\operatorname{mod} \Pi_x^2)$ is defined up to multiplication by ± 1. Therefore Π^3/Π^2 is normed. By a similar reason T_xM/Π_x^3 is normed outside Σ_1^2 via the vector $\xi_4 = J\xi_3$.

Note that Π_x^3/Π_x^2 is oriented. Actually $[\xi_1, \xi_2] \operatorname{mod} \Pi_x^2$ depends only on the values of ξ_1, ξ_2 at the point x. It is a vector $f\xi_3 \operatorname{mod} \Pi_x^2$ for some f. So if we require $\xi_2 = J\xi_1$ then ξ_3 can be chosen so that $f > 0$. This produces a coorientation on $\Pi_x^2 \subset \Pi_x^3$ and then via J a coorientation on $\Pi_x^3 \subset T_xM$.

Moreover the requirement $f = 1$ determines canonically vector field ξ_1 (still however up to ± 1) and hence $\xi_2 = J\xi_1$. Then we set $\xi_3 = [\xi_1, \xi_2]$ and $\xi_4 = J\xi_3$. So the pair (ξ_1, ξ_2) is defined canonically up to a sign and the pair (ξ_3, ξ_4) is absolutely canonical. The following statement generalizes theorem 7 [K1]:

Theorem 1. *Let almost complex structure J be of general position. Then at a generic points $x \in M^4$ the canonical frame $(\xi_1, \xi_2, \xi_3, \xi_4)$ is defined. It restores uniquely the almost complex operator J and the tensor N_J by the tables:*

X	JX
ξ_1	ξ_2
ξ_2	$-\xi_1$
ξ_3	ξ_4
ξ_4	$-\xi_3$

$N_J(\uparrow, \leftarrow)$	ξ_1	ξ_2	ξ_3	ξ_4
ξ_1	0	0	ξ_1	$-\xi_2$
ξ_2	0	0	$-\xi_2$	$-\xi_1$
ξ_3	$-\xi_1$	ξ_2	0	0
ξ_4	ξ_2	ξ_1	0	0

Note that reducing a geometric structure to a frame ($\{e\}$-structure) solves completely the equivalence problem. The idea is as follows. Consider the moduli of the problem, i.e. functions c_{jk}^i given by the formula $[\xi_j, \xi_k] = \sum c_{jk}^i \xi_i$. Denote by $\mathbb{A} = \{c_{jk}^i\}$ the space of all invariants and by $\Phi : M \to \mathbb{A}$ the "momentum map" $x \mapsto \{c_{jk}^i(x)\}$. Then two equivalent structures have the same images and the equivalence follows. See [S] for more details.

2. Singularities of a Nijenhuis tensor characteristic distribution

A distribution $V = V_1$ is called completely non-holonomic if one of its successive derivatives $V_i = \partial V_{i-1}$ equals the whole tangent bundle TM and the minimal such $i = r$ is called the degree of non-holonomy (can vary from point to point). The growth vector of a distribution at a point $x \in M$ is the sequence of the dimensions $(\mathrm{rk}_x V_1, \dots, \mathrm{rk}_x V_{r(x)})$.

Generically a Nijenhuis tensor characteristic distribution is completely non-holonomic outside a discrete subset in M. In an open dense set the growth vector is $(2, 3, 4)$. Then it is an *Engel distribution*, which has the following local normal form ([E]):

$$\Pi^2 = \langle \xi_1 = \partial_3, \xi_2 = \partial_4 - x_3\partial_2 - x_2\partial_1 \rangle; \qquad \partial_i := \partial/\partial x_i.$$

Locally this Π^2 can be realized as a Nijenhuis tensor characteristic distribution ([K2]). In fact, consider two transversal symmetries of the distribution: $\eta_1 = \partial_1, \eta_2 = \partial_2 - x_4\partial_1$. Define the almost complex structure by the formula

$$J\xi_1 = \varphi\xi_2, J\eta_1 = \eta_2; \qquad \varphi \neq 0. \tag{2}$$

Then one easily checks that $\mathrm{Im}\, N_J = \Pi^2$ whenever $(\partial_{\eta_1}\varphi)^2 + (\partial_{\eta_2}\varphi)^2 \neq 0$.

Moreover the following statement holds:

Proposition 2. *Let Π be an analytic distribution of rank 2 in R^4. Then it can be locally realized as a Nijenhuis tensor characteristic distribution.*

Proof. Let Π^2 be generated by $\xi_1 = \partial_3$ and $\xi_2 = \partial_4 + h_1\partial_1 + h_2\partial_2$. A pair of generators can always be chosen in such a form. Consider ξ_2 as a vector field in $\mathbb{R}^3(x_1, x_2, x_4)$ depending on a parameter x_3. It has two independent symmetries $\eta_1, \eta_2 \in \mathcal{D}(\mathbb{R}^3)$: $[\eta_i, \xi_2] = 0$. Let's differentiate these fields by the parameter: $\partial_3\eta_i = [\partial_3, \eta_i] = a_i^j\eta_j + b_i\xi_2$.

Define the almost complex structure by the formula

$$J\xi_1 = \varphi\xi_2, J\eta_1 = \alpha\eta_1 + \beta\eta_2; \quad \beta, \varphi \neq 0.$$

The condition $\mathrm{Im}\, N_J = \Pi^2$ is equivalent to the following system

$$\begin{cases} \varphi\partial_{\xi_2}\alpha &= \alpha\partial_{\xi_1}\alpha - \frac{1+\alpha^2}{\beta}\partial_{\xi_1}\beta + [a_1^1(1+\alpha^2) - a_1^2\alpha\frac{1+\alpha^2}{\beta} + a_2^1\alpha\beta - a_2^2(1+\alpha^2)] \\ \varphi\partial_{\xi_2}\beta &= \beta\partial_{\xi_1}\alpha - \alpha\partial_{\xi_1}\beta + [a_1^1\alpha\beta + a_1^2(1-\alpha^2) + a_2^1\beta^2 - a_2^2\alpha\beta] \end{cases}$$

and the inequality $(\partial_{\eta_1}\varphi - b_1\alpha - b_2\beta)^2 + (\partial_{\eta_2}\varphi - b_1\frac{1-\alpha^2}{\beta} + b_2\alpha)^2 > 0$. The system is in the Cauchy-Kovalevskaya form and so possesses a local solution. After this the inequality is arranged to hold. □

Theorem 3. *Nijenhuis tensors characteristic distributions in the domain of non-integrability for J have the same singularities as the usual two-dimensional distributions in $\mathbb{R}^4$.*

Proof. Let us at first define the degeneration locus of a distribution. Introduce the partial order on the growth vectors: $(m_1, \dots, m_s) \le (n_1, \dots, n_r)$ iff $s \ge r$ and $m_i \le n_i$ for $i = 1, \dots, r$. Fix one growth vector I. Then the degeneration locus $\Sigma_I \subset M$ is the set of points with the growth vector less or equal to I. Proposition 2 (it holds formally as well – on the jets of the structure) and the Thom transversality theorem imply that for a typical J the sets Σ_I are nice subvarieties, stratifying the manifold M. The statement follows. $\square$

The generic degenerations of two-plane fields in $\mathbb{R}^4$, up to codimension 3, were classified by Zhitomirskii [Z]. Let us show how generic codimension 2 singularities are realized as a Nijenhuis tensor characteristic distribution.

There are 2 different types of such singularities, defined by the growth vectors $I_1 = (2, 2, 4)$ and $I_2 = (2, 3, 3, 4)$. All other growth vectors are subordinated to these two and hence the singular set is

$$\Sigma = \Sigma_1^2 \cup \Sigma_2^2, \qquad \Sigma_i^2 = \Sigma_{I_i}.$$

Generically the loci Σ_i^2 are smooth 2-dimensional submanifolds ([Z]), which intersect non-transversally along a curve Σ_1^1. There is also a curve $\Sigma_2^1 \subset \Sigma_2^2$ separating the locus into the elliptic/hyperbolic parts $\Sigma_{2\pm}^2$.

The codimension 2 loci of $\Pi^2 = \langle \xi_1, \xi_2 \rangle$ have the following normal forms:

$$\begin{aligned} \Sigma_1^2 \setminus \Sigma_1^1 : &\quad \xi_1 = \partial_3, \xi_2 = \partial_4 - x_3 x_4 \partial_2 - x_3^2 \partial_1 \\ \Sigma_{2+}^2 : &\quad \xi_1 = \partial_3, \xi_2 = \partial_4 - \big(\frac{1}{3} x_3^3 + x_3 x_4^2\big)\partial_2 - x_3 \partial_1 \\ \Sigma_{2-}^2 : &\quad \xi_1 = \partial_3, \xi_2 = \partial_4 - x_3^2 x_4 \partial_2 - x_3 \partial_1 \end{aligned}$$

In each of these cases the choice $\eta_1 = \partial_1, \eta_2 = \partial_2$ and formula (2) will lead to realization $\Pi^2 = \operatorname{Im} N_J$. The cases of higher degenerations are studied similarly.

3. Singularities of $\Pi = \operatorname{Im} N_J$ as of a differential system

As differential systems Nijenhuis tensors characteristic distributions have singularities different from those of the usual differential systems in $\mathbb{R}^4$: The rank of a Nijenhuis tensor characteristic distribution is even and so is 2 or 0.

Proposition 4. *For a generic structure J the set, where $N_J = 0$ (the rank of Π falls to zero), is a discrete set $\Sigma^0 \subset M^4$. For each point of Σ^0 there is a centered coordinate neighborhood (x_1, y_1, x_2, y_2) around it such that the almost complex structure is given by the formula*

$$J\partial_{x_i} = \alpha_i \partial_{x_i} + (1 + \beta_i)\partial_{y_i}, \qquad J\partial_{y_i} = -\frac{1 + \alpha_i^2}{1 + \beta_i}\partial_{x_i} - \alpha_i \partial_{y_i}, \qquad i = 1, 2,$$

where the functions α_i, β_i are of the second order of smallness at the origin.

Proof. Singularities of the differential system $\Pi = \operatorname{Im} N_J$ are given by the vector equation $N_J(\xi,\eta)=0$ for some J-independent vector fields ξ,η, and so are generically isolated points given by the integrability condition $N_J=0$.

To get the other claim recall ([K1]) that an almost complex structure can be approximated by a complex structure to the second order of smallness at the integrability points. Let (w_1,w_2) be the corresponding complex coordinates. By a theorem of Nijenhuis and Woolf [NW] (see also proposition 9 below) there are two J-holomorphic foliations by disks C^1-close to the foliations $\{w_i = \text{const}\}$ at the origin, $i=1,2$. Let z_1 be a complex coordinate on the disk of the first family passing the origin and z_2 — on the second. They define the complex coordinate system (z_1,z_2) in a neighborhood of the origin with the required properties. $\square$

Remark 1. *For* $\dim M > 4$ *the set, where* $N_J=0$*, is generically empty.*

Let $\alpha_i^\diamond,\beta_i^\diamond$ be the quadratic parts of α_i,β_i. Using the coordinate system from proposition 4 we calculate: $\Pi^2 = \operatorname{Im} N_J = \langle \xi_1, \xi_2 = J\xi_1\rangle$, where linearizations of the generators at the origin are

$$\xi_1^0 = \bigl(-\tfrac{\partial\beta_1^\diamond}{\partial x_2} - \tfrac{\partial\alpha_1^\diamond}{\partial y_2}\bigr)\partial_{x_1} + \bigl(\tfrac{\partial\alpha_1^\diamond}{\partial x_2} - \tfrac{\partial\beta_1^\diamond}{\partial y_2}\bigr)\partial_{y_1} + \bigl(\tfrac{\partial\beta_2^\diamond}{\partial x_1} + \tfrac{\partial\alpha_2^\diamond}{\partial y_1}\bigr)\partial_{x_2} + \bigl(\tfrac{\partial\beta_2^\diamond}{\partial y_1} - \tfrac{\partial\alpha_2^\diamond}{\partial x_1}\bigr)\partial_{y_2}$$

and $\xi_2^0 = J_0\xi_1^0$ (J_0 is the constant coordinate extension of J from the origin).

Thus we see that the linearization of the considered differential system is special, not as for the usual differential systems. If we consider linear vector fields ξ_i^0 as linear operators, we represent the 1st order approximation of Π by a 2-dimensional subspace $V^2 \subset \mathrm{gl}(4)$. The condition $V^2 = \langle X_1, X_2 = JX_1\rangle$ for some $J^2 = -\mathbf{1}$ characterizes admissible 2-planes.

The higher order terms in ξ_1,ξ_2 are special as well.

4. Moduli of a PH-curve neighborhood

Let $\mathcal{C}^2$ be a pseudoholomorphic (PH-)curve, i.e. a surface with J-invariant tangent bundle. At every point $x\in\mathcal{C}$ we have two J-invariant planes $T_x\mathcal{C}^2$ and Π^2_x, which generically intersects by zero, except at a finite number of points $\Sigma_0'\subset\mathcal{C}$. The sets $\Sigma_1' = \Sigma_1^2\cap\mathcal{C}$ and $\Sigma_2'=\Sigma_2^2\cap\mathcal{C}$ are generically finite as well. The arrangement of all these points

$$\Sigma' = \Sigma_0'\cup\Sigma_1'\cup\Sigma_2'\subset\mathcal{C}$$

gives a (finite-dimensional) invariant of $\mathcal{C}$.

For points $x\in\mathcal{C}\setminus\Sigma_1'$ we define field of directions $L^1 = T\mathcal{C}\cap\Pi^3$. The integral curves of this 1-distribution foliate the set $\mathcal{C}\setminus\Sigma_1'$ and in general $\mathcal{C}$ foliates with only nondegenerate singular points. Denote the number of elliptic points by $e(L^1)$ and the number of hyperbolic points by $h(L^1)$. One can prove:

Proposition 5. *Under* C^1*-small perturbation of the structure* J *the foliation* L^1 *has minimal number of singularities:* $\min\{e(L),h(L)\}=0$, $\max\{e(L),h(L)\} = |\chi(\mathcal{C})|$. *For instance if* $\mathcal{C}=T^2$ *we get a foliation without singularities.* $\square$

Due to §1 the foliation L^1 is oriented, cooriented and has parallel and transverse measures outside Σ'. Thus there exist canonical vector fields v_1 along L^1 and $v_2 = Jv_1$ transverse to it. Consequently the curve $\mathcal{C}$ has a lot of dynamical invariants like winding classes of v_1 and v_2. Moreover decomposing

$$[v_1, v_2] = \gamma_1 v_1 + \gamma_2 v_2.$$

we obtain two invariant (under pseudoholomorphic isomorphisms) functions γ_1, γ_2. These together with the germs of the functions c^i_{jk} from §1 form *moduli* of the $\mathcal{C}$-neighborhoods germ. They solve the equivalence problem for PH-embeddings $\mathcal{C}^2 \to M^4$ (of general position).

Example. Let $M = T^2(\varphi, \psi) \times \mathbb{R}^2(x, y)$ be equipped with the structure

$$\begin{array}{ll} J\partial_x = \partial_y; & J\partial_\varphi = \frac{2-\rho y^2}{2}\partial_\psi + \frac{y^2}{2}\partial_\varphi + x\partial_x; \\ J\partial_y = -\partial_x; & J\partial_\psi = \frac{4+y^4}{2\rho y^2-4}\partial_\varphi - \frac{y^2}{2}\partial_\psi + \frac{xy^2}{\rho y^2-2}\partial_x + \frac{2x}{\rho y^2-2}\partial_y, \end{array}$$

Then $\mathcal{C} = \{x = y = 0\}$ is a PH-torus and the winding number of v_1 is ρ. Similarly one shows the other considered invariants are non-trivial.

5. Hyperbolicity of a PH-curve neighborhood

In this section we consider the case of PH-tori $\mathcal{C} = T^2$. We assume for simplicity that the normal bundle is topologically trivial, though in general case the result is the same.

Recall that the Kobayashi pseudometric d_M measures the distance between points via pseudoholomorphic disks ([Ko, KO]). An almost complex manifold is called Kobayashi hyperbolic if d_M is a metric. Let $\|\cdot\|$ be a norm on TM.

Proposition 6. *Let $\mathcal{O}$ be a small neighborhood of a pseudoholomorphic torus $T^2 \subset (M^4, J)$. Then the domain $\mathcal{O} \setminus T^2$ is not Kobayashi-hyperbolic.*

Moreover for some constant $C > 0$ and any $R > 0$ there exists a smooth family of PH-disks $f^R_\alpha : D_R \to \mathcal{O}$, with uniformly bounded norms $\|(f^R_\alpha)_*(z)\| \le C$ and $\|(f^R_\alpha)_*(0)\| = 1$, that fills some smaller neighborhood $\mathcal{O}' \subset \mathcal{O}$ of T^2:

$$\mathcal{O}' \subset \cup_\alpha f^R_\alpha(D_R).$$

Proof. Let us take the universal covering $\hat{\mathcal{O}} \simeq \mathbb{C} \times D^2$ of $\mathcal{O}$. The torus is covered by the entire line $\mathbb{C} \to T^2$. Changing the structure J at infinity in $\hat{\mathcal{O}}$ and near the boundary to the integrable one we glue the manifold to the product $S^2 \times S^2$ with the line $\mathbb{C}$ being glued to the first factor S^2_1. Then the introduction of the taming symplectic product-structure $\omega = \omega_1 \oplus \omega_2$ yields a foliation of $S^2_1 \times S^2_2$ by PH-spheres S^2 in the homology class of the first factor if we additionally demand that the homology class $[S^2_1]$ of the first sphere-factor is symplectically indecomposable (for example, if $\omega_1(S^2_1) = k\omega_2(S^2_2)$, $k \in \mathbb{N}$). Here we use the fact that the dimension is 4: due to positivity of intersections [M1] we actually have a foliation ([M2]).

This foliation of $S^2 \times S^2$ gives a family of big PH-disks on $\hat{\mathcal{O}}$ parametrized by the radius R of disk in $\mathbb{C}$ out of which the almost complex structure is changed. The estimates follow from the Brody reparametrization lemma as in [KO]. Pulling-back we get the required family. □

We now consider filling by pseudoholomorphic cylinders $\mathcal{C}_R = [-R; R] \times S^1 \subset \mathbb{C} \setminus \{0\}$, which is topologically different from the disk-filling (Fig.1).

Proposition 7. *In the statement of proposition 6 we can change disks D_R to the cylinders $\mathcal{C}_R$ and get for every $R > 0$ a filling family of PH-cylinders $f_\alpha^R : \mathcal{C}_R \to \mathcal{O}$ with uniformly bounded norms and normalization $\|(f_\alpha^R)_*(0)\| = 1$:*

$$\mathcal{O}' \subset \cup_\alpha f_\alpha^R(\mathcal{C}_R).$$

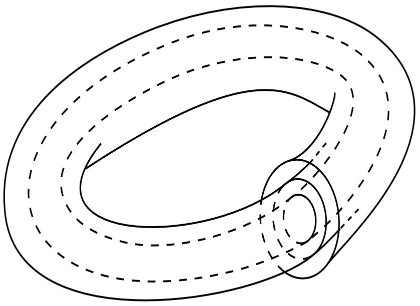

Figure 1: Filling by PH-cylinders

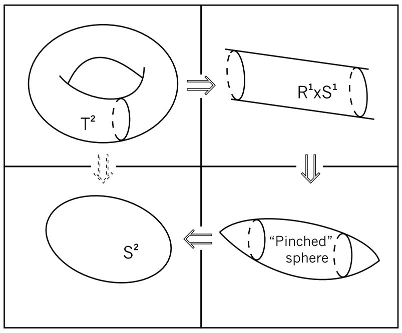

Figure 2: Cutting and Gluing

Proof. Actually take a covering of the neighborhood $\mathcal{O}$ which corresponds to one cycle of the torus. The torus is covered by the entire cylinder $\mathcal{C}_\infty \to T^2$. We can change the almost complex structure J at infinity so that it makes possible to "pinch" each end of the cylinder. This means we perturb the structure J so that it is standard integrable outside some $\mathcal{C}_{R_2} \subset \mathcal{C}_\infty$ and the support is also a big cylinder $\mathcal{C}_{R_1}$. Then we glue the ends to the disks. This operation gives us a sphere S^2 instead of the cylinder $\mathcal{C}_\infty = \mathbb{R} \times S^1$. We can also assume that neighborhoods of two cylinder ends are pinched (Fig.2).

Thus we have a neighborhood U of the sphere S_0^2. It is foliated by PH-spheres close to S_0^2. Actually, we can change the structure J near the boundary of this neighborhood, glue and get the manifold-product $\hat{M} = S^2 \times S^2$. As before it is foliated by PH-spheres. Thus U is foliated by PH-spheres and in the preimage they give a PH-foliation by cylinders. □

Remark 2. *Neighborhoods of PH-spheres $\mathcal{C} = S^2$ are also non-hyperbolic and if the normal bundle is topologically trivial can be foliated by close PH-spheres.*

For PH-curves of higher genus $\mathcal{C} = S_g^2$, $g > 1$, one expects a small neighborhood $\mathcal{O}$ to be Kobayashi hyperbolic. If an almost complex structure J is C^∞-close to an integrable one near $\mathcal{C}$ this is proved in [KO].

6. Arnold's question

In [A2](1993-25) Arnold asks about almost complex version for his Floquet-type theory of elliptic curves neighborhoods ([A1]) in the spirit of the Moser's KAM-type theorem ([Mo]). Namely he asks if a germ of neighborhood $\mathcal{O}$ of a PH-torus $\mathcal{C} = T^2 \subset (M^4, J)$ is determined by its normal bundle $N_{\mathcal{C}}M$.

The following result is a direct consequence of the definition:

Proposition 8. *If $F : M^4 \to \mathcal{C}^2$ is a (local) PH-surjection and the structure J is non-integrable, then the Nijenhuis tensor characteristic distribution Π^2 is integrable and is tangent to the fibers of F.* □

Thus there is a functional obstruction to the equivalence of the $\mathcal{C}$-germ in M^4 and of the $\mathcal{C}$-germ in the normal bundle (we do not discuss here the normal bundle: If $\dim M = 4$, the almost complex structure on $N_{\mathcal{C}}M$ can be obtained via linearization along a family of transversal PH-disks; For the general case see [K3]). Integrability and transversality of Π^2 to the torus $\mathcal{C}$ is a necessary, but by no means sufficient condition for the existence of an equivalence: There are other functional moduli.

In search of a proper generalization of the Arnold's result we notice that a neighborhood of an elliptic curve in a complex surface is foliated by half-infinite cylinders: They are given as $|z| = const$ in the representation of the neighborhood as $\mathbb{C}^2(z, w)/(z, w) \sim (z + 2\pi, w) \sim (z + \nu, \lambda w)$, where $\nu \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ (see [A1] for the representation). The hypothesis is then that for a non-integrable perturbation J of the complex structure J_0, most of the cylinders persist (as in the Moser's theory).

Let us sketch how to prove existence of one such a half-cylinder. In proposition 7 we have constructed a pre-compact family of finite cylinders f_α^R for different R. If it winds up to the curve $\mathcal{C}$ (as in the holomorphic normal form with $|\lambda| \neq 1$), then one can extract a subsequence $f_{\alpha_k}^{R_k}$ with $R_k \to \infty$ converging to a pseudoholomorphic curve due to the standard technique ([G, MS]). This is the required half-cylinder.

There are no tools however to complete this construction to a PH-foliation (also a filling is problematic – a remark of V. Bangert). Note though that even if we construct a foliation, it is not necessary so nice as its holomorphic original. To explain this let us notice the following fact, which is a corollary of a theorem by Nijenhuis and Woolf [NW]:

Proposition 9. *Small neighborhood $\mathcal{O}$ of a PH-curve $\mathcal{C} \subset M^4$ can be foliated by transversal PH-disks D^2.* □

Now consider a neighborhood of a PH-curve $\mathcal{C}$ with topologically trivial normal bundle and suppose we have a foliating family $f_\alpha : \mathcal{B} \to \mathcal{O}$ with unbounded or compact leaves in it. Let D_φ, $\varphi \in \mathcal{C}$, be the family of normal disks from proposition 9. Then every path $\gamma(t)$ on $\mathcal{C}$ with $\gamma(0) = \varphi_0$, $\gamma(1) = \varphi_1$ gives a mapping $\Phi_\gamma : D_{\varphi_0} \to D_{\varphi_1}$ of shift along the leaves of f_α. For a loop γ we have an automorphism of D_φ. Since f_α is a foliation there is no local holonomy: $\Phi_\gamma = \mathrm{id}$ for contractible loops γ. Thus we can consider the map $\pi_1(\mathcal{C}) \to \mathrm{Aut}(D_\varphi)$.

Definition 2. *We call* $\Phi_\gamma \in \mathrm{Aut}(D_\varphi)$ *the monodromy map along* $\gamma \in \pi_1(\mathcal{C})$ *and* $\Phi_\gamma : D_{\varphi_0} \to D_{\varphi_1}$ *the transport map.*

For example there is no monodromy for the sphere $\mathcal{C} = S^2$ and each choice of local coordinates in a normal disk D_{φ_0} gives coordinates for the others D_φ.

Let now $\mathcal{C} = T^2(2\pi, \nu)$ and we have a foliating family f_α of half-infinite cylinders. Since every leaf $\mathcal{B}$ is a cylinder, there is no monodromy along one generating cycle. Normalize it to be the cycle $\varphi \mapsto \varphi + 2\pi$. Denote by Φ_ν the monodromy along the other cycle $\varphi \mapsto \varphi + \nu$.

Unlike the complex case, the almost complex monodromy can be non-holomorphic mapping of the fibers: It is possible to construct examples of PH-foliations with any prescribed monodromy Φ_ν.

Moreover even if the monodromy is complex, the transport maps $\Phi_\gamma : (D_{\varphi_0}, J) \to (D_{\varphi_1}, J)$ can be non-complex. In fact there are functional obstructions for the transports to be complex:

Theorem 10. *Let* $\mathcal{C}$ *be a PH-curve in a 4-dimensional manifold* (M, J) *and let* $f_\alpha : \mathcal{B} \to \mathcal{O}$ *be a local PH-foliating family in some neighborhood* $\mathcal{O}$ *of* $\mathcal{C}$*. Then if all transport maps* Φ_γ *are holomorphic, then the Nijenhuis tensor characteristic distribution* Π^2 *is integrable and is tangent to the leaves of* f_α*.*

Actually this is because the foliation provides a local bundle $\pi : \mathcal{O} \to D_\varphi$ and so proposition 8 apply. Again the integrability is not a sufficient condition: There are other moduli.

So we see that existence of foliating PH-family with complex transports (as in the original holomorphic case) is generically obstructed, and the obstructions are of the same nature as for the existence of equivalence between a germ of a neighborhood of a PH-curve $\mathcal{C}$ and its normal bundle (though in the first case the Nijenhuis tensor characteristic distribution is tangent to the curve $\mathcal{C}$, while in the second one it is transversal).

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