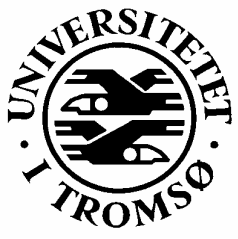

CAND. SCIENT. THESIS IN MATHEMATICS

Dynamics of the Zhang model of Self-Organized Criticality

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January 2004

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Acknowledgment

I thank my supervisor Boris Kruglikov for his inspiring lectures on dynamical systems in the fall of 2002, and for his valuable assistance through out my work on this thesis. I am also grateful to Einar Mjølhus and Kristoffer Rypdal for useful comments and discussions.

Abstract

In 1987 the concept of self-organized criticality (SOC) was proposed as an explanation of the widespread occurrence of power-law statistics and fractal structures in nature, and several models where designed to simulate self-organized critical phenomena. In [1],[2],[3] it was shown how the Zhang model of SOC can be formulated as a dynamical system of skew-product type, where the random excitations are described by a Bernoulli shift, and the relaxations are given by piecewise affine transformations. In this paper we investigate the class of dynamical systems that arise from the Zhang model. We introduce the notion of removability of singularities, and show that whenever singularities are removable the system has a well-defined attractor. Under the additional assumption of invertibility on the attractor we show that the dynamics is conjugated to a topological Markov-chain. We can then identify the physical relevant invariant measure, and show that the dynamics is ergodic and that we have exponential decay of correlations. Some simple examples of how the theory can be applied are shown. Little is known about the dynamics in the case of non-removability of singularities, but some results on the topological entropy of the dynamical systems are obtained.

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Chapter 1

Introduction

In 1987 the concept of Self-Organized Criticality (SOC) was introduced by Bak, Tang and Wiesenfeld [4]. The claim was that certain physical systems self-organize to stationary states, reminiscent of equilibrium systems at the critical point, in the sense that one has scale-invariance and long-range correlations both in space and in time. The attempt was to give an explanation of the omnipresence of fractal structures and power-law statistics in nature. This ubiquity of critical behavior indicates that it is not emerging from thermodynamic equilibrium, since such systems require fine-tuning of external parameters in order to reach the critical point. In particular SOC was proposed as a mechanism for so called $1/f$-noise, which is an indication of a particularly slow algebraic correlation decay.

SOC is proposed to be occurring in many "real" systems, but most work has been devoted to understanding idealized computer models that one believes exhibits SOC in the thermodynamic limit. The models vary slightly, but common features are external driving, relaxation through threshold interactions between nearest neighbors, and energy dissipation at the boundary. The models reach marginally stable stationary states, where a perturbation can cause a chain reaction of small events, called avalanches. To decide whether such a model is critical in the stationary state, it is natural to consider spatial correlation functions and time correlation functions. If one could show that such correlation functions become power laws in the thermodynamic limit, then this would be solid evidence that the stationary state is critical. However this approach is difficult numerically, and a different method is usually used to indicate criticality. One can define avalanche observables such as size, duration and area. In numerical studies one finds that the probability distributions of such observables are power-laws, with some cut-offs corresponding to the finite size of the system. A particular interest has been given to the avalanche size observable. If $P(s)$ is the probability of having an avalanche of size s in the stationary state, one finds that

$$P(s) \sim \frac{1}{s^{\alpha}} ,$$

where α is called the critical exponent of the size distribution. Another critical exponent D, is defined trough the relation $\tau \sim A^D$, where τ is avalanche size and A is avalanche area. It is believed that the models of SOC can be grouped in to classes of universality determined by the values of a set of such critical exponents. For some models it is possible to determine exact values of the critical exponents [9] [11], and the models are then said to be exactly solved.

The first model constructed to simulate SOC was the sandpile model of Bak, Tang and Wiesenfeld [4]. In one dimension the model is defined as follows: Take a lattice of N sites. For each site n we associate a height h_n. Let $z_n = h_n - h_{n-1}$. We go through the lattice from left to right, and if z_n is greater than some critical gradient z_c, we update the configuration by letting

$$\begin{aligned} z_n &\mapsto z_n + 1 \\ z_{n+1} &\mapsto z_{n-1} - 1 . \end{aligned}$$

This corresponds to letting a grain of sand fall from position $n-1$ to position n. The boundary at the right hand side has fixed energy equal zero, so grains of sand can be lost to the boundary. After each sequence of updates a site n is chosen at random, and $h_n \mapsto h_n + 1$. It is easy to see that the dynamics of this model will tend to an attractor where all sites are marginally stable. Hence any addition of sand will slide of to the boundary. If we fix an initial condition and let $s(n)$ be the length of "sliding" number n, then $s(n)$ is just a random number between 0 and N, independent of $s(n-1)$. This gives an uncorrelated process, so what we observe is not $1/f$-noise but $1/f^0$ noise (white noise). Bak, Tang Wiesenfeld call this a trivial critical state. It is critical only in the sense that perturbations propagate through the entire system, and it is not an example of SOC. In higher dimensions their model has much more rich and complex behavior, and power-law-statistics is observed numerically [5]. Signs of non-trivial critical behavior is also observed in models of earthquakes, forest fires and biological evolution [5].

Much effort has been put into understanding the different SOC-models. Mostly through numerical studies and the application of different mean-field methods. However, a mathematical formalism similar to that of equilibrium statistical mechanics, is not yet achieved. Even a formal definition of SOC is lacking. This is due to the fact that the mechanisms of SOC are not fully understood, even in the cases where critical exponents can be calculated exactly. It is known that the behavior the SOC models can vary greatly as one changes parameters such as dimension and critical gradients (or energies). The one dimensional sandpile of Bak, Tang and Wiesenfeld shows very trivial behavior that cannot be classified as SOC, where as the two dimensional variant show signs of SOC. Other models however, show rich behavior even in one dimension. Hence the original conjecture that SOC cannot occur in low dimensions seems not to be true. What property is it that determines if a model will show critical behavior?

This paper is devoted to the study of the SOC-model of Zhang [6]. In this model we fix a lattice of sites, and for each site i we associate an energy x_i. If all sites have energy less than some critical energy E_c, the configuration is said to be stable and we add a quantum of energy to a randomly chosen site. If the configuration is not stable, the overcritical sites will relax by letting a fixed proportion ϵ of the energy be transported, in equal parts, to its nearest neighbors. If a site on the boundary is overcritical some energy will transported out of the system.

In a series of papers by Cessac, Blanchard and Kruger [3],[2],[1] it was showed how the Zhang model could be reformulated as a dynamical system with singularities. Their study focused on the application of smooth theory, and due to the singularities in the system their work suffers from a lack of rigorous results. Rather than elaborating on their work, we will study the topological properties of the dynamical system in relation to the singularities. The notion of removability of singularities is introduced, and under this condition and some additional assumptions, we can give a complete topological and metric description of the dynamics. The results of this study surpassingly indicate chaotic but non-critical behavior for a finite size system. This could indicate that non-removability of singularities might be one of the essential criteria for SOC, but in chapter 7 we indicate how critical behavior may still emerge in the thermodynamic limit.

Although physically relevant results are obtained, this work is mainly mathematical. The Zhang model offers a class of dynamical systems that are interesting in their own right. Apart from the presence of singularities, we have non-surjectivity and non-injectivity. As a consequence there are very few applicable results from the theory of dynamical systems. If singularities are removable this problem is overcome by restricting to an attractor where the dynamics is continuous. In the last part of the paper we study topological entropy without assuming that singularities are removable. In the work of Cessac, Blanchard and Kruger, it is stated (without proof) that the topological entropy is $\log N$, where N is the number of sites in the model. We prove that their claim is correct if singularities are removable, but in the general case we can give conditions under which the topological entropy is greater than $\log N$. Whether these conditions are realized in the Zhang model is not known.

There is very little existing theory on the topological entropy of dynamical systems with the kind of singularities found in the Zhang model. The standard technique is to restrict the map to the non-compact set obtained when removing singularities from our system. It was shown in [12] how topological entropy can be defined for non-compact sets. However, this method does not

work for our dynamical system, and hence the only way to obtain a continuous map is to remove singularities and every backwards iterate of the singularity set. We do not have control over the structure of this set, so instead of applying standard approach of making our map continuous, we define topological entropy in the standard way and show that it exists despite the singularities. The particular structure of our dynamical system gives a natural measure of the exponential growth of orbits. We deduce properties of this number, and under weak conditions we show that it equals the topological entropy.

The paper is structured as follows. In chapter 2 we show how the Zhang model can be formulated as a dynamical system, and several basic properties are deduced. The notion of removability of singularities is introduced in chapter 3. We define the attractor of the dynamics, and under the assumption of invertibility on the attractor we construct a topological conjugancy to a topological Markov-chain. In chapter 4 we use the main result of chapter 3 to introduce an invariant measure on the attractor. We show that the system is mixing with respect to the invariant measure, and we show that we have exponential decay of correlations for avalanche observables. This result has important physical implications that are discussed in chapter 7. Chapter 7 also gives a short description of the concepts of SOC. In chapter 5 we show simple examples of application of the theory developed in chapters 2,3 and 4. Computer programs are written to determine if singularities are removable in the simple case of a two site system. We also give an example where this approach fails to prove removability of singularities. Chapter 6 is devoted to the study of topological entropy.

Chapter 2

Definitions and Basic Properties

In the Zhang model each site on the lattice is associated with a non-negative real number, which we call the energy of the site. The collection of energies is called an energy configuration, and can be represented as a point in N-dimensional space, where N is the number of sites in the lattice. If a configuration is unstable, the overcritical sites will lose some of their energy to their nearest neighbors, resulting in a new energy configuration. This transformation on $\mathbb{R}^N$ is denoted by F. If a configuration is stable, a site is chosen at random and an energy quantum 1 is added to this site. In [3] it was shown how the relaxation and random excitation could be formulated as a map of skew-product type on an extended phase-space. This extended phase-space has the configuration space as one factor, and the set of all possible sequences of excitations as the other factor. In this approach an initial condition consists of an ordered pair $(\mathbf{a}, x)$, where $\mathbf{a}$ is an infinite sequence of sites, and x is an initial energy configuration. If the configuration is stable, the zero element a_0 of the sequence $\mathbf{a}$ is exited, and the sequence is shifted to the left so that the site a_1 is exited the next time the configuration is stable. In [3] it was also showed how one could reformulate the dynamical system by considering the return maps to the set of stable configurations. This gives a simplification, in the sense that each avalanche is associated with an affine transformation. The set of stable configurations is partitioned in to domains, where each domain corresponds to an avalanche. The properties of these affine transformations are of importance for the proof of the main results of chapter 3 and chapter 6. Most of the statements of this chapter are stated in [3], but not formally proved before now.

Take $d \in \mathbb{N}$ and let $\Lambda \subset \mathbb{Z}^d$ be a square of cardinality $N := |\Lambda|$. Let $\phi : \Lambda \to \Lambda' := \{1, ..., N\}$ be a bijection. We define a metric d_Λ on Λ, by

$$d_\Lambda(\mathbf{k}, \mathbf{l}) = \sum_{1 \le n \le d} |k_n - l_n| \,,$$

and let $d_{\Lambda'} := \phi_* d_\Lambda$. In the following we omit primes when it is clear from the context that we are considering the metric space $(\Lambda', d_{\Lambda'})$. Elements of Λ will be called sites. We say that sites i and j are nearest neighbors if $d_\Lambda(i, j) = 1$. The boundary $\partial\Lambda$ is defined as those sites $i \in \Lambda$ that have less than $2d$ nearest neighbors.

Fix parameters $E_c > 0$ and $\epsilon \in (0, 1)$, and define $F : \mathbb{R}^N_{\ge 0} \to \mathbb{R}^N_{\ge 0}$, by

$$(Fx)_i = x_i - \theta(x_i - E_c)(1 - \epsilon)x_i + \frac{1 - \epsilon}{2d} \sum_{d_\Lambda(i,j)=1} \theta(x_j - E_c)x_j \,, \tag{2.1}$$

where

$$\theta(a) = \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{if } a \leq 0 \end{cases}.$$

Proposition 1. *For all $x \in \mathbb{R}^N_{\geq 0}$ we have*

$$\frac{1+\epsilon}{2}||x||_1 \leq ||F(x)||_1 \leq ||x||_1 \,,$$

and $||F(x)||_1 = ||x||_1$ if and only if $x_i \leq E_c$ for all $i \in \partial\Lambda$. If there is $i \in \partial\Lambda$ such that $x_i > E_c$ then

$$||F(x)||_1 < ||x||_1 - \frac{1-\epsilon}{2d}E_c \,.$$

Proof. Let $\{x_{i_k}\}_{k=1}^m$ be the entries of the vector x that are greater than E_c. Let $\{x^j_{i_k}\}_{j=1}^{n_{i_k}}$ be the set of nearest neighbors to x_{i_k}. Then

$$\begin{aligned} \sum_{i\in\Lambda}(Fx)_i &= \sum_{i\in\Lambda} x_i - (1-\epsilon)\sum_{k=1}^m x_{i_k} + \frac{1-\epsilon}{2d}\sum_{k=1}^m n_{i_k}x_{i_k} \\ &= \sum_{i\in\Lambda} x_i - (1-\epsilon)\sum_{k=1}^m (1 - \frac{n_{i_k}}{2d})x_{i_k} \,. \end{aligned}$$

We always have $d \leq n_{i_k} \leq 2d$, and $n_{i_k} = 2d$ if and only if $x_{i_k} \notin \partial\Lambda$. The inequality $n_k \geq d$ gives

$$\begin{aligned} ||Fx||_1 &\geq ||x||_1 - \frac{1-\epsilon}{2}\sum_{k=1}^m x_{i_k} \\ &\geq ||x||_1 - \frac{1-\epsilon}{2}||x||_1 \\ &= \frac{1+\epsilon}{2}||x||_1 \,, \end{aligned}$$

and the inequality $n_k \leq 2d$ gives $||Fx||_1 \leq ||x||_1$. If $x_{i_k} \in \partial\Lambda$, then $n_{i_k} \leq 2d-1$, so

$$\sum_{i\in\Lambda}(Fx)_i < \sum_{i\in\Lambda} x_i - \frac{1-\epsilon}{2d}E_c \,.$$

□

A point $x \in \mathbb{R}^N_{\geq 0}$ is called a stable configuration if $x_i \leq E_c$ for all $i \in \Lambda$. The set of stable configurations is $M := [0, E_c]^N$. We say that a site $i \in \Lambda$ of the configuration x is relaxed if $x_i \leq E_c$, and exited if $x_i > E_c$.

For $x \in \mathbb{R}^N_{\geq 0}$ and $n \in \mathbb{N}$ define

$$\alpha_i(n, x) := |\{t \leq n \,|\, (F^t x)_i > E_c\}| \,.$$

Lemma 1. *If there is a positive integer β such that*

$$\alpha_i(n, x) \geq \beta \cdot \left(\left[\frac{2d}{1-\epsilon}\right] + 1\right),$$

then $\alpha_j(n, x) \geq \beta$ whenever $d_\Lambda(i, j) = 1$.

Proof. Let $\gamma = [2d/(1-\epsilon)] + 1$. There is a finite increasing sequence $\{t_k\}$, such that $(F^{t_k}x)_i > E_c$ for $k = 1, ..., \gamma\beta$. Let j be a nearest neighbor of i and suppose that there is a positive integer $l < \beta$ such that $(F^t x)_j \leq E_c$ for all $t \in [t_{l\gamma}, t_{(l+1)\gamma}]$. It follows from (2.1) that

$$(F^{t_{(l+1)\beta}-1}x)_j \geq \gamma \cdot \frac{1-\epsilon}{2d} E_c > E_c \,.$$

Contradiction. Hence, in each interval $[t_{l\gamma}, t_{(l+1)\gamma})$ there is τ such that $(F^\tau x)_j > E_c$. There are β such intervals, so $\alpha_j(n, x) \geq \beta$. □

Proposition 2. *For each $x \in \mathbb{R}^N_{\geq 0}$ there exists $m(x) \in \mathbb{Z}_{\geq 0}$ such that $F^{m(x)}(x) \in M$. Moreover if we let $m(x)$ be the minimal such number, then the function $m : \mathbb{R}^N_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is bounded on any bounded subset of $\mathbb{R}^N_{\geq 0}$.*

Proof. We say that $i \in \Lambda$ is infinitely often overcritical in the relaxation of $x \in \mathbb{R}^N_{\geq 0}$ if there exists an infinite increasing sequence $\{t^{(i)}_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$, such that $(F^{t^{(i)}_k} x)_i > E_c$ for all $k \in \mathbb{N}$. Suppose i is infinitely often overcritical in the relaxation of x, and that j is a nearest neighbor of i that does not have this property. Then there exists T such that

$$\forall t \geq T \, : (F^t x)_j \leq E_c \,.$$

On the other hand

$$(F^t x)_j \geq (F^T x)_j + \sum_{T < t^{(i)}_k < t} \frac{1-\epsilon}{2d} (F^{t^{(i)}_k} x)_i \,.$$

Since $(F^{t^{(i)}_k} x)_i > E_c$, this gives a contradiction. Hence, if there is one site that is infinitely often overcritical in the relaxation of x, then all sites have this property, including sites in $\partial\Lambda$. Suppose that $i \in \partial\Lambda$, and let $m(t) = |\{k \in \mathbb{N} \,|\, t^{(i)}_k < t\}|$. Then it follows from proposition 1 that

$$\sum_{i\in\Lambda} (F^t x)_i < \sum_{i\in\Lambda} x_i - m(t) \cdot \frac{1-\epsilon}{2d} E_c \,.$$

If $m(t) \to \infty$ as $t \to \infty$, this is a contradiction. Then no point can be infinitely often overcritical in the relaxation of x. This proves the first statement.

Take a site $i \in \Lambda$ and construct a path $i, j_1, j_2, ..., j_{\xi_i}$ of nearest neighbors, such that $j_{\xi_i} \in \partial\Lambda$. Suppose there is $n \in \mathbb{N}$ such that

$$\alpha_i(n, x) > \Big[\frac{2d}{1-\epsilon} \frac{||x||_1}{E_c} + 1\Big] \Big(\Big[\frac{2d}{1-\epsilon}\Big] + 1\Big)^{\xi_i}$$

By lemma 1 we have

$$\alpha_{j_{\xi_i}}(n, x) > \Big[\frac{2d}{1-\epsilon} \frac{||x||_1}{E_c} + 1\Big] ,$$

and then it follows from proposition 1 that

$$||F^n x||_1 \leq ||x||_1 - \Big[\frac{2d}{1-\epsilon} \frac{||x||_1}{E_c} + 1\Big] \cdot \frac{1-\epsilon}{2d} E_c < 0 \,.$$

Contradiction. Let

$$\xi = \Big[\frac{\mathrm{diam}(\Lambda) + 1}{2}\Big] .$$

Then

$$\alpha_i(n, x) \leq \alpha(x) := \Big[\frac{2d}{1-\epsilon} \frac{||x||_1}{E_c} + 1\Big] \Big(\Big[\frac{2d}{1-\epsilon}\Big] + 1\Big)^{\xi} ,$$

for all $i \in \Lambda$ and all $n \in \mathbb{N}$. Suppose $F^t(x) \notin M$ for all $t \leq T$. If $T > N\alpha(x)$, then there must be a site $i \in \Lambda$ that is greater than E_c for more than $\alpha(x)$ different times. This is impossible so

$$m(x) \leq N\alpha(x) \leq N\Big(\frac{2d}{1-\epsilon}\frac{||x||_1}{E_c} + 1\Big)\Big(\frac{2d}{1-\epsilon} + 1\Big)^{\xi}.$$

It follows that $m(x)$ is bounded on any bounded subset of $\mathbb{R}^N_{\geq 0}$. □

Define $\Sigma_\Lambda = \Lambda^{\mathbb{Z}}$ and let $\sigma_\Lambda : \Sigma_\Lambda \to \Sigma_\Lambda$ be the left shift on Σ_Λ. We define a map

$$\hat{F} : \Sigma_\Lambda \times \mathbb{R}^N_{\geq 0} \to \Sigma_\Lambda \times \mathbb{R}^N_{\geq 0}\,,$$

by

$$\hat{F}(\mathbf{a}, x) = \begin{cases} (\sigma_\Lambda \mathbf{a}, x + e_{a_0}) & \text{if } x \in M \\ (\mathbf{a}, Fx) & \text{if } x \notin M \end{cases},$$

where $e_1, ..., e_N$ is the standard basis in $\mathbb{R}^N$. We denote points in $\Sigma_\Lambda \times \mathbb{R}^N_{\geq 0}$ by $\hat{x} = (\mathbf{a}, x)$, and we define projections

$$\begin{aligned} \pi^u : \Sigma_\Lambda \times \mathbb{R}^N_{\geq 0} &\to \Sigma_\Lambda \\ (\mathbf{a}, x) &\mapsto \mathbf{a}\,, \end{aligned}$$

and

$$\begin{aligned} \pi^s : \Sigma_\Lambda \times \mathbb{R}^N_{\geq 0} &\to \mathbb{R}^N_{\geq 0} \\ (\mathbf{a}, x) &\mapsto x\,. \end{aligned}$$

Proposition 3. *For all $\hat{x} \in \Sigma_\Lambda \times \mathbb{R}^N_{\geq 0}$ there exists $n(\hat{x}) \in \mathbb{N}$ such that*

$$\sum_{i\in\Lambda}(Fy)_i < \sum_{i\in\Lambda} y_i - \frac{1-\epsilon}{2d}E_c\,,$$

for $y = (\pi^s \circ \hat{F}^{n(\hat{x})})(\hat{x})$. If we let $n(\hat{x})$ be the minimal such number, then $n(\hat{x})$ is bounded on $\Sigma_\Lambda \times M$.

Proof. We say that $i \in \Lambda$ is infinitely often overcritical in the orbit of $\hat{x}$ if there exists an increasing sequence $\{t_k^{(i)}\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $(\pi^s \circ \hat{F}^{t_k^{(i)}}\hat{x})_i > E_c$ for all $k \in \mathbb{N}$. We now make the same argument as in the proof of proposition 2. If i is infinitely often overcritical, and j a nearest neighbor to i, then j must be infinitely often overcritical. If not, then there exists T such that

$$\forall t \geq T\ : (\pi^s \circ \hat{F}^t\hat{x})_j \leq E_c.$$

However

$$(\pi^s \circ \hat{F}^t\hat{x})_j \geq (\pi^s \circ \hat{F}^T\hat{x})_j + \sum_{T<t_k^{(i)}<t} \frac{1-\epsilon}{2d}(\pi^s \circ \hat{F}^{t_k^{(i)}}\hat{x})_i\,,$$

so we have a contradiction. Hence either all sites are infinitely often overcritical, or no site is infinitely often overcritical. Suppose no site is infinitely often overcritical. Then there exists $T \in \mathbb{N}$ such that

$$\forall t \geq T\ \ \forall i \in \Lambda\ : (\pi^s \circ \hat{F}^t\hat{x})_i \leq E_c\,.$$

Whenever $\pi^s \circ \hat{F}^t(\hat{x}) \in M$ we have

$$\sum_{i\in\Lambda}(\pi^s \circ \hat{F}^{t+1}(\hat{x}))_i = \sum_{i\in\Lambda}(\pi^s \circ \hat{F}^t(\hat{x}))_i + 1\,,$$

so

$$\sum_{i\in\Lambda}(\pi^s\circ\hat{F}^{T+N[E_c+1]}(\hat{x}))_i > NE_c\,.$$

This is a contradiction, so all sites must be infinitely often overcritical. Let $n(\hat{x})$ be the smallest integer such that a site in $\partial\Lambda$ is overcritical. Then it follows from proposition 1 that

$$\sum_{i\in\Lambda}\left(F(\pi^s\circ\hat{F}^{n(\hat{x})}\hat{x})\right)_i < \sum_{i\in\Lambda}(\pi^s\circ\hat{F}^{n(x)}\hat{x})_i - \frac{1-\epsilon}{2d}E_c\,.$$

Let $E_t = \max\{||\pi^s\circ\hat{F}^t(\hat{x})||_1\,|\,\pi^s(\hat{x})\in M\}$. Then

$$E_1 = E_c + 1\,,$$

and

$$E_{t+1}\leq \max\left\{\max\{\epsilon E_t, E_c\} + (1-\epsilon)E_t, E_c+1\right\},$$

for $t\geq 1$. It is easy to see that if $\epsilon E_1 \leq$E$_c$, then $\epsilon E_t \leq E_c$ for all $t\geq 0$. If

$$E_c < \frac{\epsilon}{1-\epsilon}\,,$$

then $E_t \leq E_c + 1$ for all $t\geq 0$. Hence

$$E_t \leq \max\left\{\frac{E_c}{\epsilon}, E_c+1\right\},$$

for all $t\geq 0$. Form proposition 2 it follows that there is an integer $h(M)>0$, such that for any $\hat{x}\in\Sigma_\Lambda\times M$ we have

$$|\{t\leq h(M)\,|\,\pi^s\circ\hat{F}^t(\hat{x})\in M\}| > N\max\left\{\frac{E_c}{\epsilon}, E_c+1\right\}.$$

Then we must have $n(\hat{x})\leq h(M)$, because otherwise we would have $||\pi^s\circ F^{h(M)}(\hat{x})||_1 > NE_{h(M)}$, which gives a contradiction. □

For $x\in\mathbb{R}^N_{\geq 0}$ and $i\in\Lambda$ we define

$$\tau(x,i) := \min\{t\in\mathbb{N}\,|\,F^t(x+e_i)\in M\}\,.$$

Proposition 2 assures us that this minimum exists, and that there is τ_m such that $\tau(x,i)\leq\tau_m$ for all $i\in\Lambda$ and all $x\in M$. Let $\hat{x}=(\mathbf{a},x)$. For $t=1,...,\tau(x,a_0)$ we define

$$C_t(\hat{x}) = \{i\in\Lambda\,|\,(\pi^s\circ\hat{F}^t\hat{x})_i > E_c\}\,,$$

and

$$A(\hat{x}) = \left(C_1(\hat{x}),...,C_{\tau(x,a_0)}(\hat{x})\right).$$

We call $A(\hat{x})$ the avalanche of the point $\hat{x}$. Let $\hat{M} := \Sigma_\Lambda\times M$ and define an equivalence relation $\sim$ on $\hat{M}$ by

$$\hat{x}\sim\hat{y}\Leftrightarrow A(\hat{x}) = A(\hat{y})\,.$$

We denote the corresponding partition of $\hat{M}$ by $\mathcal{P}_0$. From the definition it is clear that $A(\hat{x})$ depends on a_0 and x only. Hence partition elements are of the form $\hat{M}_{ij} := [i]\times M_{ij}$, where $[i] := \{\mathbf{a}\in\Sigma_\Lambda\,|\,a_0 = i\}$, and

$$\forall i\in\Lambda: \bigcup_j M_{ij} = M\,.$$

We see that for each $i \in \Lambda$, the domains $M_{i1}, M_{i2}, ...$ are separated by segments of hyperplanes of type $F^t(x + e_i) - E_c = 0$. Since the number of such hyperplane segments that intersect M is bounded by τ_m, we have finite number of domains $M_{i1}, ..., M_{iq_i}$ for each $i \in \Lambda$. Let

$$q := \sum_{i \in \Lambda} q_i \,.$$

Then $|\mathcal{P}_0| = q$.

By definition there is a unique avalanche for each partition element $[i] \times M_{ij}$. We denote this avalanche by A_{ij}. Its duration is $\tau_{ij} := \tau(i, x)$, for $x \in M_{ij}$, and define its size to be

$$s_{ij} = \sum_{t=1}^{\tau_{ij}} |C_t| \,.$$

Let

$$\rho(x) = |\{i \in \Lambda \,|\, x_i > E_c\}| \,.$$

Then

$$s_{ij} = \sum_{t=0}^{\tau_{ij}-1} \rho(F^t(x + e_i)) \,,$$

for any $x \in M_{ij}$.

Next we define a map

$$\begin{aligned} \mathcal{T} : \hat{M} &\to \hat{M} \\ (\mathbf{a}, x) &\mapsto (\sigma_\Lambda \mathbf{a}, T_{a_0} x) \,, \end{aligned}$$

where

$$T_i(x) := F^{\tau(x,i)}(x + e_i) \,.$$

We define $T_{ij} := T_i|_{M_{ij}}$.

Lemma 2. *Let A and B be square matrices with non-negative entries. If the sum over each column of A and B is less than or equal to 1, then the sum over each column of AB is less than or equal to 1. If A and B are as above and the sum over some column of AB is less than 1, then at least one of A and B must have a column, such that the sum over this column is less than 1.*

Proof. If $\sum_k A_{kl} \leq 1$ and $\sum_k B_{kl} \leq 1$, for all l, then

$$\sum_k (AB)_{kl} = \sum_k (\sum_i A_{ki} B_{il}) = \sum_i (\sum_k A_{ki}) B_{il} \leq \sum_i B_{il} \leq 1 \,.$$

Suppose that $\sum_k (AB)_{kl} < 1$. If the sum over each column of A and the sum over each column of B is 1. Then

$$\sum_k (AB)_{kl} = \sum_k (\sum_i A_{ki} B_{il}) = \sum_i (\sum_k A_{ki}) B_{il} = \sum_i B_{il} = 1 \,.$$

Contradiction. □

Lemma 3. *If A is a square matrix with non-negative entries, and the sum over each column of A is less than or equal 1, then the spectral radius of A is less than or equal to 1.*

Proof. Suppose $\lambda \neq 0$ is an eigenvalue of A. Let $v \neq 0$ be a corresponding eigenvector. Then

$$\begin{aligned} |\lambda| &= \frac{||Av||_1}{||v||_1} \\ &= \frac{|(Av)_1| + ... + |(Av)_N|}{|v_1| + ... + |v_N|} \\ &= \frac{|\sum_l A_{1l}v_l| + ... + |\sum_l A_{Nl}v_l|}{|v_1| + ... + |v_N|} \\ &\leq \frac{\sum_l A_{1l}|v_l| + ... + \sum_l A_{Nl}|v_l|}{|v_1| + ... + |v_N|} \\ &= \frac{\sum_k A_{k1}|v_1| + ... + \sum_k A_{kN}|v_N|}{|v_1| + ... + |v_N|} \\ &\leq 1\,. \end{aligned}$$

□

Theorem 1. *For all i, j we have*

1. *There exists a linear map L_{ij} such that $T_{ij}(x) = L_{ij}(x + e_i)$.*
2. *$r(L_{ij}) \leq 1$, where $r(L_{ij})$ is the spectral radius of L_{ij}.*
3. *There is $c \in (0, 1)$ and number $T \in \mathbb{N}$ such that for every $\theta > T$ and a sequence*

$$\{(i_t, j_t) \,|\, 1 \leq t \leq \theta\}$$

with

$$\bigcap_{t=0}^{\theta} (T_{i_t j_t} \circ ... \circ T_{i_1 j_1})^{-1}(M_{i_{t+1} j_{t+1}}) \neq \emptyset\,,$$

it holds:

$$r(L_{i_\theta j_\theta} \cdot ... \cdot L_{i_1 j_1}) < c\,.$$

Proof.

1. For each $x \in \mathbb{R}^N_{\geq 0}$ we define a matrix $Q(x)$ by

$$Q_{kl}(x) = \begin{cases} \frac{1}{2d}\theta(x_l - E_c) & \text{if } d_\Lambda(k, l) = 1 \\ 0 & \text{otherwise} \end{cases},$$

and the diagonal matrix $J(x)$ by

$$J_{kl}(x) = (1 - (1 - \epsilon)\theta(x_l - E_c))\delta_{kl}$$

Let

$$S(x) = J(x) + (1 - \epsilon)Q(x)\,.$$

Then we see from (2.1) that $F(x) = S(x)x$. Let $x^{[1]} = x + e_i$ and $x^{[t]} = F(x^{[t-1]})$ for $t \in \{2, ..., \tau(i, x)\}$. Then

$$T_i(x) = S(x^{[\tau(x,i)]}) \cdot ... \cdot S(x^{[1]})(x + e_i)$$

So let

$$L_i(x) = S(x^{[\tau(x,i)]}) \cdot ... \cdot S(x^{[1]})\,.$$

If $x, y \in M_{ij}$ then $\tau(x, i) = \tau(y, i)$ and the same components of $x^{[t]}$ and $y^{[t]}$ are grater than E_c for each $t = 1, ..., \tau(x, i)$. Thus $S(x^{[t]}) = S(y^{[t]})$ for these values of t. Then $L_i(x) = L_i(y)$. We can define $L_{ij} := L_i(x)$ for $x \in M_{ij}$. We get $T_i|_{M_{ij}}(x) = L_{ij}(x + e_i)$.

2. We fix $l \in \{1, ..., N\}$. If $x_l > E_c$ we have

$$\sum_{k=1}^{N} S_{kl}(x) = \epsilon + \frac{1-\epsilon}{2d} |\{k \in \Lambda \,|\, d(k,l) = 1\}| \leq 1 \,,$$

and if $x_l \leq E_c$ the sum over column l of $Q(x)$ is zero, so

$$\sum_{k=1}^{N} S_{kl}(x) = 1 \,.$$

Then it follows from lemma 2 that

$$\sum_{k=1}^{N} (L_{ij})_{kl} \leq 1$$

for any k, l. From lemma 3 we deduce that the spectral radius of L_{ij} is less than or equal to 1.

3. By proposition 3 there is x-independent $T \in \mathbb{N}$ such that

$$||L_{i_T j_T} \cdot ... \cdot L_{i_1 j_1} x||_1 < ||x||_1 \,.$$

whenever

$$x \in \bigcap_{t=0}^{T} (T_{i_t j_t} \circ ... \circ T_{i_1 j_1})^{-1} (M_{i_{t+1} j_{t+1}}) \,,$$

This implies that

$$\frac{\sum_k (L_{i_T j_T} \cdot ... \cdot L_{i_1 j_1})_{k1} x_1 + ... + \sum_k (L_{i_T j_T} \cdot ... \cdot L_{i_1 j_1})_{kN} x_N}{x_1 + ... + x_N} < 1 \,,$$

so there is $l' \in \{1, ..., N\}$ such that sum over column l' of $L_{i_T j_T} \cdot ... \cdot L_{i_1 j_1}$ is less than 1. Let $x_t = T_{i_t j_t} \circ ... \circ T_{i_1 j_1}(x)$. The diagonal of $S(x_t)$ is always positive, so

$$(S(x_{t+1}) S(x_t))_{kl} = \sum_m S_{km}(x_{t+1}) S_{ml}(x_t) \geq \epsilon S_{kl}(x_{t+1}) \,.$$

Moreover we have that $S_{kl}(x_{t+1}) > 0$ if $(x_t)_l > E_c$ and $d(k,l) = 1$. From the proof of proposition 2 it is clear that there is an integer $n > 0$ such that for any initial condition $\hat{x} \in \hat{M}$, all sites have been overcritical after n iterations. We can assume that $T \geq n$. Then $L_{i_T j_T} \cdot ... \cdot L_{i_1 j_1}$ is a positive matrix. Let λ be the greatest eigenvalue of $L_{i_T j_T} \cdot ... \cdot L_{i_1 j_1}$ and $v \neq 0$ a corresponding eigenvector. It follows from the Perron-Frobenius theorem that λ is real, and that v is positive. We have

$$\lambda \leq \frac{\sum_k v_1 + ... + \sum_k (L_{i_T j_T} \cdot ... \cdot L_{i_1 j_1})_{l'} v_{l'} + ... + \sum_k v_N}{v_1 + ... + v_N} < 1 \,.$$

Let c be the maximum spectral radius of all the products $L_{i_T j_T} \cdot ... \cdot L_{i_1 j_1}$ of length T, where

$$\bigcap_{t=0}^{T} (T_{i_t j_t} \circ ... \circ T_{i_1 j_1})^{-1} (M_{i_{t+1} j_{t+1}}) \neq \emptyset \,. \tag{2.2}$$

Then $c < 1$, and

$$r(L_{i_T j_T} \cdot ... \cdot L_{i_1 j_1}) < 1 \,,$$

whenever equation (2.2) holds. □

An immediate consequence of the last statement of theorem 1 is:

Corollary 1. *For all $c \in (0,1)$ and there exists $T_c \in \mathbb{N}$ such that for every $\theta > T_c$ and a sequence*

$$\{(i_t, j_t) \,|\, 1 \leq t \leq \theta\}$$

with

$$\bigcap_{t=0}^{\theta} (T_{i_t j_t} \circ ... \circ T_{i_1 j_1})^{-1}(M_{i_{t+1} j_{t+1}}) \neq \emptyset \,,$$

it holds:

$$r(L_{i_\theta j_\theta} \cdot ... \cdot L_{i_1 j_1}) < c \,.$$

Lemma 4. *For $E_c \geq \epsilon/(1-\epsilon)$ we have $d(i,j) \neq 1$ whenever $(\pi^s \circ \mathcal{T}(\hat{x}))_i, (\pi^s \circ \mathcal{T}(\hat{x}))_j > E_c$.*

Proof. Take $\hat{x} \in M$ and consider the avalanche $A(\hat{x}) = \{C_t(\hat{x})\}$ of length τ. For each $t \in \{1, ..., \tau\}$ we let $E_t = \max\{x_i \,|\, i \in C_t(\hat{x})\}$. It is cleat that $E_1 = E_c + 1$, and from the proof of proposition 3 we know that

$$E_t \leq \max\left\{\frac{E_c}{\epsilon}, E_c + 1\right\}$$

for all $t \leq 0$. So whenever $E_c \geq \epsilon/(1-\epsilon)$ we have $\epsilon E_t \leq E_c$. This implies that for all $i \in \Lambda$ and all $t \in \mathbb{N}$ we have $(F^{t+1}x)_i \leq E_c$ whenever $(F^t x)_i > E_c$ and $(F^t x)_j \leq E_c$ for all nearest neighbors j of i. The statement follows by induction. In fact, all avalanches start with a single site. Let $C_1(\hat{x}) = \{i\}$. Then $d(i,j) = 1$ for all $j \in C_2(\hat{x})$. This implies that any two elements of $C_2(\hat{x})$ can be connected with a path of length 2, so no two sites of $C_2(\hat{x})$ are nearest neighbors. And for any $k \in C_3(\hat{x})$ there is $j \in C_2(\hat{x})$ such that $d(k,j) = 1$. This implies that any two elements of $C_3(\hat{x})$ can be connected by a path of length 4. But if there exists a path of even length between two points in Λ, then all path connecting these points are of even length, so the distance between any two points of $C_3(\hat{x})$ is even. This implies that there are no nearest neighbors in $C_3(\hat{x})$. We can repeat the argument, and the result follows. □

Proposition 4. *The linear maps L_{ij} are all invertible whenever $\epsilon > 1/2$ or $E_c \geq \epsilon/(1-\epsilon)$. If we have $E_c \geq \epsilon/(1-\epsilon)$ then*

$$\det L_{ij} = \epsilon^{s_{ij}} \,.$$

Proof. Take arbitrary $x \in M$. First we observe that since the sum over each column of $Q(x)$ is less than or equal to 1, we have $||Q(x)v||_1 \leq ||v||_1$ for each $v \in \mathbb{R}^N$. This follows from the proof of lemma 3. This implies that

$$\begin{aligned} ||S(x)v||_1 &= ||(J(x) + (1-\epsilon)Q(x))v||_1 \\ &\geq ||J(x)v||_1 - (1-\epsilon)||Q(x)v||_1 \\ &\geq (\epsilon - (1-\epsilon))||v||_1 \\ &= (2\epsilon - 1)||v||_1 \,. \end{aligned}$$

If $\epsilon > 1/2$, then $S(x)v \neq 0$ for all $v \neq 0$, so we have invertibility.

The second statement is proved by reducing the matrix $S(x)$. If $x_k \leq E_c$, then row k of $S(x)$ is of the form $(0, ..., 0, 1, 0, ...0)$, where the 1 is in the diagonal position. We can start the decomposition of $\det S(x)$ with row k, and hence we see that row k and column k can be removed from $S(x)$ without changing the determinant. We remove all rows and columns that correspond to relaxed sites. If $\rho(x)$ is the number of overcritical sites of x, we get a $\rho(x) \times \rho(x)$ matrix $S_{red}(x)$. If site k is overcritical then $J_{kk}(x) = \epsilon$. If $E_c \geq \epsilon/(1-\epsilon)$, then it follows from lemma 4 that all nearest neighbors of k are relaxed. Hence row k of $Q(x)$ has only zero entries. This shows that $S_{red}(x) = \mathrm{diag}(\epsilon, ..., \epsilon)$. Then

$$\det S(x) = \det S_{red}(x) = \epsilon^{\rho(x)} \,,$$

so $\det L_{ij} = \epsilon^{s_{ij}}$. □

Remark: In the original model of Zhang one has $\epsilon = 0$. In this case the maps L_{ij} are not invertible. In [3] it is stated that the existence of non-trivial kernels can occur for $\epsilon = 0$ only. This is not true. A simple counter-example is the case $N = 2$, $E_c = 1/3$ and $\epsilon = 1/3$. For $x_1 > 0$ and $2x_1 + 3x_2 \leq 1$ we have

$$T_1\Big(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Big) = \frac{1}{9}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 + 1 \\ x_2 \end{bmatrix} .$$

Hence the corresponding linear map is degenerate.

Corollary 2. *If $\epsilon > 1/2$ or $E_c \geq \epsilon/(1-\epsilon)$ then vol$(\pi^s \circ \mathcal{T}^t(\hat{M})) > 0$ for each $t \in \mathbb{N}$.*

Proof. For each $i \in \Lambda$ there is a finite number of domains M_{ij}. Each of them is a polygon, and since

$$\sum_{j=1}^{q_i} \mathrm{vol}(M_{ij}) = \mathrm{vol}(M) = E_c^N > 0\,,$$

there must be j, such that $\mathrm{vol}(M_{ij}) > 0$. Then

$$\mathrm{vol}(\pi^s \circ \mathcal{T}(\hat{M})) \geq |\det(L_{ij})| \cdot \mathrm{vol}(M_{ij}) > 0\,.$$

Now $\pi^s \circ \mathcal{T}(\hat{M})$ has positive volume, and it is a finite disjoined union of domains $M_{ij} \cap (\pi^s \circ \mathcal{T}(\hat{M}))$. Hence we can repeat the argument above. The result follows by induction. □

We define topology on $\hat{M}$ by introducing a metric

$$d((\mathbf{a}, x), (\mathbf{b}, y)) = \max\{d^u(\mathbf{a}, \mathbf{b}), d^s(x, y)\}\,,$$

where

$$d^u(\mathbf{a}, \mathbf{b}) = \begin{cases} 2^{-k} & \text{where } k = \min\{|i| \,|\, a_i \neq b_i\} \\ 0 & \text{if } \mathbf{a} = \mathbf{b} \end{cases} ,$$

and

$$d^s(x, y) = ||x - y||_1\,.$$

This defines the Tychonoff product topology on $\Lambda^{\mathbb{Z}} \times M$. The map σ_Λ is a homeomorphism in the in the topology induced on Σ_Λ by d^u.

Proposition 5. *$\mathcal{T}$ is continuous on $\hat{M}_{ij}$.*

Proof. Since $\mathcal{T}|_{[i]\times M_{ij}} = \sigma_\Lambda \times T_{ij}$, and both σ_Λ and T_{ij} are continuos, the statement follows from a theorem of Tychonoff. □

Chapter 3

Symbolic Coding

The boundaries between the partition elements defined in the previous chapter are singularities of our dynamical system. We say that singularities are removable if there is an integer m such that the m-th iterate of any initial condition is bounded away from this set of singularities. The notion of removability of singularities is seemingly essential for the understanding of the dynamics of the Zhang model. In this chapter we show that if we have removability of singularities, then we have a well defined attractor of the dynamics, and that the dynamics on the attractor is continuos. Under the additional condition of invertibility on the attractor we show that our dynamical system is topologically conjugated to a topological Markov-chain. In other words the dynamics is essentially equivalent to a symbolic dynamical system consisting of the left-shift map on the set of infinite sequences over a finite alphabet. The set of legal sequences is determined by a transition matrix A, and the structure of A determines the topological properties of the dynamics. As a consequence of the symbolic coding we can calculate the topological entropy and the exponential growth of periodic points. If we assume transitivity of A, we can also show that periodic points are dense on the attractor and that we have topological mixing. Hence, if our assumptions hold, the dynamics on the attractor has all the characteristics of a chaotic dynamical system. The idea of performing symbolic coding on the Zhang model was introduced in [3], but no statements where proved and no examples of symbolic coding where given. It was even stated that one could only expect to obtain a semi-conjugancy to a topological Markov-chain. The new ideas presented in this paper is that symbolic coding must be seen in relation to the property of removability of singularities, and that the partitions used to preform symbolic coding should be constructed from the connected components of some iterate of the phase-space, and not on the domains of continuity. The corollaries of theorem 2 are based on a simple observation of the structure of the transition matrix A, and general theory of topological Markov-chains [7].

Let

$$\mathcal{S} = \bigcup_{i,j} (\partial M_{ij} - \partial M)\,.$$

Definition 1. *We say that singularities are removable if there exists $m \in \mathbb{N}$ such that*

$$\overline{\pi^s \circ \mathcal{T}^m(\hat{M})} \cap \mathcal{S} = \emptyset\,.$$

If singularities are removable we let

$$X := \overline{\pi^s \circ \mathcal{T}^m(\hat{M})}\,.$$

$\mathcal{T}$ is continuous on $\Sigma_\Lambda \times X$, which gives

$$\begin{aligned}\mathcal{T}(\Sigma_\Lambda \times X) &= \mathcal{T}(\Sigma_\Lambda \times \overline{\pi^s \circ \mathcal{T}^m(\hat{M})}) \\ &= \overline{\mathcal{T}(\Sigma_\Lambda \times \pi^s \circ \mathcal{T}^m(\hat{M}))} \\ &\subset \overline{\Sigma_\Lambda \times \pi^s \circ \mathcal{T}^m(\hat{M})} \\ &= \Sigma_\Lambda \times X\,.\end{aligned}$$

$\mathcal{T}|_{\Sigma_\Lambda \times X}$ is a continuous self-mapping of a compact space, and it follows that

$$\mathcal{A} := \bigcap_{t=0}^{\infty} \mathcal{T}^t(\Sigma_\Lambda \times X)$$

is non-empty, closed and invariant. We call $\mathcal{A}$ the attractor of the dynamics. Let $Y_0 = X$ and recursively define

$$Y_t := \bigcup_{ij} T_{ij}(Y_{t-1} \cap M_{ij})\,.$$

It then follows that

$$\begin{aligned}\mathcal{T}^t(\Sigma_\Lambda \times X) &= \mathcal{T}^{t-1}\mathcal{T}(\bigcup_{i,j}[i] \times (X \cap M_{ij})) \\ &= \mathcal{T}^{t-1}(\Sigma_\Lambda \times \bigcup_{i,j} T_{ij}(X \cap M_{ij})) \\ &= \mathcal{T}^{t-1}(\Sigma_\Lambda \times Y_1) \\ &= ... \\ &= \Sigma_\Lambda \times Y_t\,.\end{aligned}$$

We now see that

$$\mathcal{A} = \bigcap_{t=0}^{\infty} \Sigma_\Lambda \times Y_t = \Sigma_\Lambda \times Y\,,$$

where

$$Y := \bigcap_{t=0}^{\infty} Y_t\,.$$

The set X is a finite disjoined union of connected components. Let $X_1, ..., X_s$ be the connected components that have non-empty intersection with Y, and let $Y^\nu = X_\nu \cap Y$. This gives a partition of the attractor:

$$\mathcal{A} = \bigcup_{i \in \Lambda} \bigcup_{\nu=1}^{s} [i] \times Y^\nu\,.$$

Denote the partition by $\mathcal{P}_\mathcal{A}$, and renumerate such that

$$\mathcal{P}_\mathcal{A} = \{P_1, ..., P_r\}\,,$$

where $r = Ns$. We define the $r \times r$ matrix A by

$$A_{kl} = \begin{cases} 1 & \text{if } \mathcal{T}^{-1}(P_l) \cap P_k \neq \emptyset \\ 0 & \text{otherwise} \end{cases}\,,$$

and we let

$$\Omega_A = \{(...\omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2...)\,|\,\forall t:\ A_{\omega_t\omega_{t+1}} = 1\} \subset \{1,...,r\}^{\mathbb{Z}}\,.$$

We give $\{1,..,r\}^{\mathbb{Z}}$ the Tychonoff product topology, and Ω_A the subspace topology induced by the product topology on $\{1,..,r\}^{\mathbb{Z}}$. This topology can be defined by the metric

$$d_A(\omega,\alpha) = \begin{cases} 2^{-k} & \text{where } k = \min\{|i|\,|\,\omega_i \neq \alpha_i\} \\ 0 & \text{if } \omega = \alpha \end{cases}\,.$$

Let $\sigma_A : \Omega_A \to \Omega_A$ be the left-shift homeomorphism.

To prove topological conjugancy we need an assumption.

Assumption 1. *$T_i|_Y$ is invertible for each $i \in \Lambda$.*

Theorem 2. *If singularities are removable and assumption 1 holds, then $\mathcal{T}|_{\mathcal{A}}$ is topologically conjugated to σ_A. That is; there exists a homeomorphism $h : \mathcal{A} \to \Omega_A$ such that $h \circ \sigma_A = \mathcal{T}|_{\mathcal{A}} \circ h$.*

Proof. If $T_i|_Y$ is invertible for each $i \in \Lambda$, then $\mathcal{T}$ is invertible on $\mathcal{A}$. We can then define a map $h : \mathcal{A} \to \Omega_A$, by

$$h(\hat{x}) = \omega = (...\omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2...)\,,$$

where $\omega_t \in \{1,...,r\}$ is the unique number such that

$$\mathcal{T}^t(\hat{x}) \in P_{\omega_t}\,.$$

It is clear that if $\mathcal{T}^t(\hat{x}) \in P_{\omega_t}$ for all $t \in \mathbb{Z}$, then $\mathcal{T}^t(\mathcal{T}\hat{x}) \in P_{\omega_{t+1}}$ for all $t \in \mathbb{Z}$. So $h(\mathcal{T}(\hat{x})) = \sigma_A(h(\hat{x}))$. What remains is to prove that h is a homeomorphism.

First we prove that h is surjective. Take a legal sequence

$$\omega = (...\omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2...) \in \Omega_A\,.$$

Each ω_t corresponds to a pair (i_t,ν_t), and since the sequence is legal we have

$$T_{i_t}(X_{\nu_t}) \cap X_{\nu_{t+1}} \neq \emptyset\,.$$

The set $T_{i_t}(X_{\nu_t})$ is the continuous image of the connected component X_{ν_t}, and hence it is connected. Moreover

$$T_{i_t}(X_{\nu_t}) \subset \bigcup_{\nu=1}^{s} X_\nu\,,$$

so we must have

$$T_{i_t}(X_{\nu_t}) \subset X_{\nu_{t+1}}\,.$$

Then we have a nested sequence

$$... \subset T_{i_{-1}}(X_{\nu_{-1}}) \subset X_{\nu_0} \subset T_{i_1}^{-1}(X_{\nu_1}) \subset (T_{i_1} \circ T_{i_2})^{-1}(X_{\nu_2}) \subset ...\,,$$

of closed sets. This has a non-empty intersection. Take x from the intersection. Clearly $x \in Y$. Let $\mathbf{a} = ...i_{-1}i_0i_1...$, and $\hat{x} = (\mathbf{a}, x)$. Then $\hat{x} \in \mathcal{A}$ and $h(\hat{x}) = \omega$.

Next we prove that h is injective. For each $\omega \in \Omega_A$ there is a unique sequence (i_t, j_t), such that

$$P_{\omega_t} \subset [i_t] \times M_{i_t,j_t}\,.$$

Suppose $h(\hat{x}) = h(\hat{y}) = \omega$. Then

$$\pi^s(\hat{x}), \pi^s(\hat{y}) \in \bigcap_{t=0}^{\infty} (T_{i_t j_t} \circ ... \circ T_{i_1 j_1})^{-1}(M_{i_{t+1} j_{t+1}}) \,.$$

Lemma 1 states that there exists $T \in \mathbb{N}$ such that

$$r(L_{i_{tT} j_{tT}} \cdot ... \cdot L_{i_{(t-1)T} j_{(t-1)T}}) < c < 1 \,,$$

for all $t \in \mathbb{N}$. This implies that

$$\begin{aligned} d(\mathcal{T}^{-tT}\hat{x}, \mathcal{T}^{-tT}\hat{y}) &\geq d^s(\pi^s \circ \mathcal{T}^{-tT}(\hat{x}), \pi^s \circ \mathcal{T}^{-tT}(\hat{y})) \\ &\geq (1/c)^t d^s(x, y) \,. \end{aligned}$$

We let $t \to \infty$ and see we get a contradiction if not $x = y$. It follows directly from the definition of the partition $\mathcal{P}_{\mathcal{A}}$, that if $h(\hat{x}) = h(\hat{y})$, then $\pi^u(\hat{x}) = \pi^u(\hat{y})$. Hence we have injectivity of h.

Define

$$\gamma := \min\{d^s(x, y) \,|\, x \in X_\nu, y \in X_\mu, \nu \neq \mu\} \,.$$

This minimum exists and it is positive. We also define

$$C := \min_{i,j}\{||L_{ij}||_1\} \,.$$

Let $\hat{x} = (\mathbf{a}, x)$ and $\hat{y} = (\mathbf{b}, y)$. Suppose $d(\hat{x}, \hat{y}) < \min\{C^{-l}\gamma, 2^{-l}\}$. Then $a_t = b_t$ for $t = -l, ...l$, and since

$$d^s(\pi^s \circ \mathcal{T}^{-t}(\hat{x}), \pi^s \circ \mathcal{T}^{-t}(\hat{y})) \leq C^{-t} d^s(x, y) \,,$$

we have $\nu_{-l}, ..., \nu_l$ such that $\mathcal{T}^{-t}(x), \mathcal{T}^{-t}(y) \in X_{\nu_t}$ for $t = -l, ..., l$. This implies that if $\omega^{(1)} = h(\hat{x})$ and $\omega^{(2)} = h(\hat{y})$, then $\omega_t^{(1)} = \omega_t^{(2)}$ for $t = -l, ..., l$. Hence $d_A(h(\hat{x}), h(\hat{y})) \leq 2^{-l}$. This shows that h is continuous. Since h is a bijection, $\mathcal{A}$ is compact and Ω_A is Hasudorff, continuity of h^{-1} follows from continuity of h. □

Remark: It follows from the proof of theorem 1 that we have a semi-conjugancy (surjective) $h : \mathcal{A} \to \Omega_A^+$, even if assumption 1 does not hold. However, the estimates obtained by this semi-conjugancy are not better then what we obtain from the trivial semi-conjugancy $\mathcal{A} \to \Sigma_\Lambda$.

Proposition 6. *N is an eigenvalue of A and $r(A) = N$.*

Proof. Take $k \in \{1, ..., r\}$. Then there is i, ν such that $P_k = [i] \times Y^\nu$. There is a unique $\alpha \in \{1, ..., s\}$ such that

$$T_i(X_\nu) \subset Y^{\nu_\alpha} \,.$$

Let $P_{k_l} = [l] \times Y^{\nu_\alpha}$. Then $\mathcal{T}(P_k) \cap P_{k_l} \neq \emptyset$ for $l = 1, ..., N$. The domains P_{k_l} are the only partition elements that intersect $\mathcal{T}(P_k)$. Hence the matrix A has exactly N 1-s in each row. Let $v := (1, 1, ..., 1)^T$. It is easy to see that $Av = Nv$. This proves that N is an eigenvalue. If A is transitive we can use the Perron-Frobenius theorem stating that there is a unique eigenvalue with a positive eigenvector, that this eigenvalue is simple and that all other eigenvalues have less absolute value. Since v is positive, it follows that N is this Perron-Frobenius eigenvalue.

If A is not transitive we make the following perturbation to get a positive matrix:

$$\tilde{A} = A + \begin{bmatrix} \delta/N & \cdots & \delta/N \\ \vdots & & \vdots \\ \delta/N & \cdots & \delta/N \end{bmatrix} \,.$$

It has an eigenvalue $\tilde{\lambda} = N + \delta$, with the same eigenvector v. Hence $N + \delta$ is the Perron-Frobenius eigenvalue of $\tilde{A}$. Then all other eigenvalues have absolute value less than $N+\delta$. Since the spectrum changes continuously, this implies that all eigenvalues of A have absolute value less than or equal to N. □

A consequence of proposition 6 is:

Corollary 3. *If singularities are removable and assumption 1 holds, then* $h_{top}(\mathcal{T}|_{\mathcal{A}}) = \log N$.

This result will be proved later without assuming invertibility on the attractor. We therefore omit the proof in this chapter.

Corollary 4. *Let* $P_n(\mathcal{T}) = |\{\hat{x} \in \mathcal{A} \,|\, \mathcal{T}^n(\hat{x}) = \hat{x}\}|$. *If singularities are removable and assumption 1 holds, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log |P_n(\mathcal{T}|_{\mathcal{A}})| = \log N$$

Proof. A finite sequence $k = \omega_1, ..., \omega_m = l$, of elements in $\{1, ..., r\}$, is called a legal path connecting k and l, if $A_{\omega_t \omega_{t+1}} = 1$ for $t = 1, ..., m-1$. We claim that the number of different legal paths of length m that connect k and l, is $(A^m)_{kl}$. This is proved by induction. The number of legal paths of length 1 that connect k and l, is obviously A_{kl}. Suppose that the number of legal paths of length $m-1$ connecting k and l', is $(A^{m-1})_{kl'}$. Then the number of legal paths of length m, that connect k to l' in $m-1$ steps and then l' to l in one step, is $(A^{m-1})_{kl'} A_{l'l}$. Then the number of paths of length m connecting k and l is obtained by summing over all l'. That is

$$\sum_{l'=1}^{r} (A^{m-1})_{l'l} A_{l'k} = (A^m)_{kl} \,.$$

Observe that $P_n(\mathcal{T}|_{\mathcal{A}}) = P_n(\sigma_A)$. A periodic point of period n corresponds to a path of length n connecting a number $k \in \{1, ..., r\}$ with itself. Thus $P_n(\sigma_A) = \mathrm{Tr}(A^n)$. Let $\lambda_1, ..., \lambda_r$ be the spectrum of A. Then

$$\mathrm{Tr}(A^n) = \lambda_1^n + ... + \lambda_r^n \,,$$

and since $r(A) = N$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log(\lambda_1^n + ... + \lambda_r^n) = \limsup_{n\to\infty} \frac{1}{n} \log(N^n) = \log N \,.$$

□

In order to prove additional results we make another assumption:

Assumption 2. *The matrix A is transitive, i.e. there exists $n \in \mathbb{N}$, such that all entries of A^n are positive.*

First we prove a result about the connected components X_ν. If all domains M_{ij} have positive volume, and the maps L_{ij} are invertible, then the volume of each connected component X_ν is obviously positive. It can however happen that some domains M_{ij} have dimension less than N. This occurs, for instance, if $E_c = 1$, when the set $\{x \in M \,|\, x_i = 0\}$ is a domain of continuity for the map T_i. Another example is for $N = 2$, $E_c = 1/3$ and $\epsilon = 1/3$. In this case the one point set $\{(0,0)\}$ is a domain of continuity for both T_1 and T_2. In the cases where some domains M_{ij} have zero volume, one could imagine that some of the connected components X_ν have zero volume, even if the linear maps are invertible. We show that this is not possible if assumption 2 holds:

Proposition 7. *If singularities are removable, $E_c \geq \epsilon/(1-\epsilon)$ or $\epsilon > 1/2$, and assumption 2 holds, then each connected component of X has positive volume.*

Proof. Suppose $A_{kl} = 1$. Then there exists a point $\hat{x} \in P_k$, such that $\mathcal{T}(\hat{x}) \in P_l$. Let $P_k = [i_1] \times Y^{\nu_1}$ and $P_l = [i_2] \times Y^{\nu_2}$, $X_{\nu_1} \subset M_{i_1 j_1}$. Then

$$T_{i_1 j_1}(X_{\nu_1}) \cap X_{\nu_2} \neq \emptyset \,.$$

The continuous image of X_{ν_1} under $T_{i_1 j_1}$ is connected and contained in the disjoined union of connected components $X_{\nu_1}, ..., X_{\nu_s}$. Hence

$$T_{i_1 j_1}(X_{\nu_1}) \subset X_{\nu_2} \,.$$

This implies that $\mathrm{vol}(X_{\nu_2}) \geq \mathrm{vol}(T_{i_1 j_1}(X_{\nu_1}))$. So if $\pi^s(P_k)$ has positive volume and $A_{kl} = 1$, then $\pi^s(P_l)$ has positive volume. From corollary 1 we know that X has positive volume, so there exists connected component of X with positive volume. Assume, with no loss of generality, this to be X_{ν_1}. Since A is transitive, we have $(A^n)_{kl} > 0$ for all $l = 1, ..., r$. Then

$$\sum_{\gamma_1, ... \gamma_{n-1} = 1}^{r} A_{k\gamma_1} A_{\gamma_1 \gamma_2} \cdot ... \cdot A_{\gamma_{n-1} l} > 0 \,,$$

so there exists sequence $\gamma_1, ... \gamma_{n-1}$ such that

$$A_{k\gamma_1} = A_{\gamma_1 \gamma_2} = ... = A_{\gamma_{n-1} l} = 1 \,.$$

Since $\mathrm{vol}(\pi^s(P_k)) > 0$ we have $\mathrm{vol}(\pi^s(P_{\gamma_1})) > 0$. Then $\mathrm{vol}(\pi^s(P_{\gamma_2})) > 0$ and so on. Finally we have $\mathrm{vol}(\pi^s(P_l)) > 0$. Since l was arbitrary, we have that each connected component of X has positive volume. □

Corollary 5. *If singularities are removable and assumptions 1 and 2 hold, then*

1. $\mathcal{T}|_{\mathcal{A}}$ *is topologically mixing.*
2. *Periodic points are dense in* $\mathcal{A}$.

Proof. If A is transitive, it implies that there is $n \in \mathbb{N}$ such that A^n is positive. This means that any two numbers $k, l \in \{1, ..., r\}$ can be connected by a legal path of length m, for $m \geq n$. We can now prove the statements of corollary:

1. Topological mixing means that for any two sets U and V, that are open in $\mathcal{A}$, there exists integer T such that $\mathcal{T}^t(U) \cap V \neq \emptyset$ for all $t \geq T$. The property is obviously an invariant of topological conjugancy, so it is sufficient to prove topological mixing for (Ω_A, σ_A). Since symmetric cylinders

$$[\omega_{-k} ... \omega_k] := \{\gamma \in \Omega_A \,|\, \gamma_{-k} = \omega_{-k}, ..., \gamma_k = \omega_k\} \,,$$

form a basis for the topology on Ω_A, we only need to prove that for cylinders $[\omega_{-k} ... \omega_k]$ and $[\alpha_{-k} ... \alpha_k]$, there is T such that for each $t > T$ there exists sequence $\gamma^{(t)} \in \Omega_A$, with

$$\gamma^{(t)} \in \sigma_A^t([\omega_{-k} ... \omega_k]) \cap [\alpha_{-k} ... \alpha_k] \,.$$

Since any k, l can be connected by a path of length m, for $m \geq n$, we can construct a legal path β of length $t > n$, connecting α_k and ω_{-k}. Let

$$\gamma^{(t)} = (... \alpha_{-k} ... \alpha_k \beta_1 ... \beta_t \omega_{-k} ... \omega_k ...) \;,\; \gamma_0^{(t)} = \alpha_0 \,.$$

Clearly $\gamma^{(t)} \in \sigma_A^t([\omega_{-k} ... \omega_k]) \cap [\alpha_{-k} ... \alpha_k]$ for $t > n$.

2. If ω is a periodic point for σ_A, then $h^{-1}(\omega)$ is a periodic point for $\mathcal{T}$, and the image of a dense set under a homeomorphism, is dense. Hence if periodic points are dense in Ω_A, then they are dense in $\mathcal{A}$. We prove that periodic points are dense in Ω_A by observing that every symmetric cylinder $[\omega_{-k} ... \omega_k]$ contains a periodic point. We can take a legal sequence β of length $t > n$, that connects ω_k and ω_{-k}. Then we can construct a point

$$p = (... \beta_t \omega_{-k} ... \omega_k \beta_1 ... \beta_t \omega_{-k} ... \omega_k \beta_1 ...) \;,\; p_0 = \omega_0 \,.$$

Clearly p is periodic and $p \in [\omega_{-k} ... \omega_k]$.

□

Chapter 4

Metric Properties

If singularities are removable and assumptions 1 and 2 hold, the results of chapter 3 provide a complete topological description of the dynamics on the attractor. In this chapter these properties are assumed. We can define an invariant measure on the Markov-chain, hence inducing a measure on the attractor of the Zhang model. It is implicit in the Zhang model that sites are exited at random with a fixed probability $1/N$. This corresponds to a uniform Bernoulli measure on the set Σ_Λ. We show that there is a unique invariant probability measure on the attractor that preserves this property, and this is hence the physical relevant measure. As a corollary of this result we obtain that the physical relevant measure is the unique measure of maximal entropy. The invariant measure is of Markov type, and it follows from transitivity that we have mixing, and hence ergodicity. We prove this for our specific measure in proposition 9, and it also follows from a general result for Markov measures [7]. By applying the standard approach of transfer operators [10] we also show that the correlation functions of avalanche observables have exponential decay of correlations. Physical implications of this result are discussed in section 7. In [3],[2],[1] one attempts to define an invariant measure by iteration of the Lebesgue measure on the phase-space. The attempt is to construct a measure of SRB-type. However they fail to even prove existence of this measure, and no statistical properties can be deduced from this approach.

Assume that A is transitive and define the matrix

$$\Pi := \frac{1}{N} A = ||\pi_{ij}||_{r\times r} \,.$$

It has an eigenvalue $\lambda_1 = 1$, and by the Perron-Frobenius theorem the rest of the spectrum $\mathrm{Sp}(\Pi) = \{\lambda_1, ..., \lambda_m\}$ is inside the unit circle. That is; $1 > |\lambda_2| \geq ... \geq |\lambda_m|$. The eigenspace of λ_1 is $\mathbb{R}v$, where $v = (1, 1, ..., 1)^T$. Let $p = (p_1, ..., p_r)^T$ be the Perron-Frobenius eigenvector of A^T, normalized so that

$$\sum_{k=1}^{r} p_k = 1 \,,$$

and define the matrix J by

$$J = \begin{bmatrix} p_1 & p_2 & \cdots & p_r \\ \vdots & \vdots & & \vdots \\ p_1 & p_2 & \cdots & p_r \end{bmatrix} . \tag{4.1}$$

Lemma 5. *There is a matrix norm $||\cdot||$, such that for each $\delta > 0$ there exists $c_\delta > 0$ with*

$$||\Pi^n - J|| \leq c_\delta (|\lambda_2| + \delta)^n \,.$$

Proof. We make a change of basis in $\mathbb{R}^r$ to bring Π to the Jordan normal form:

$$M\Pi M^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_m \end{bmatrix},$$

where $B_1, ..., B_k$ are blocks of the form

$$B_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ 0 & 0 & \lambda_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_k \end{bmatrix}.$$

Take any vector $u \in \mathbb{R}^r$, and let $w = Mu$. Let n_k be the size of B_k. Define

$$n^{(i)} = \sum_{k=1}^{i} n_k \,,$$

and define blocks

$$w^{(i)} = (w_{n^{(i)}+1}, ..., w_{n^{(i+1)}})^T \,.$$

Then

$$\begin{aligned} M\Pi^n M^{-1} w - w_1 e_1 &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_m \end{bmatrix} \begin{bmatrix} w_1 \\ w^{(2)} \\ w^{(3)} \\ \vdots \\ w^{(m)} \end{bmatrix} - \begin{bmatrix} w_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ B_2^n w^{(2)} \\ B_3^n w^{(3)} \\ \vdots \\ B_m^n w^{(m)} \end{bmatrix}. \end{aligned}$$

Hence

$$||M\Pi^n M^{-1} w - w_1 e_1||_1 = ||B_1^n w^{(1)}||_{1,1} + ... + ||B_m^n w^{(m)}||_{1,m} \,,$$

where $||\cdot||_{1,k}$ is the L_1-norm on $\mathbb{R}^{n_k}$. For all $\delta > 0$ there exists a norm $||\cdot||$ on $\mathbb{R}^{n_k}$ such that $||B_k|| \leq |\lambda_k| + \delta$. Since all norms are equivalent in finite dimension we have $c_{k,\delta} > 0$ such that $||\cdot||_{1,k} \leq c_{k,\delta}||\cdot||$. Let

$$c_\delta = m \cdot \max_k \{c_{k,\delta}\} \,.$$

Then

$$||M\Pi^n M^{-1} w - w_1 e_1||_1 \leq c_\delta (|\lambda_2| + \delta)^n ||w||_1 \,.$$

Let $P_1 : \mathbb{R}^r \to \mathbb{R}^r$, $w \mapsto (w \cdot e_1)e_1$, be the projection to the first coordinate axis. Define an operator J by

$$J = M^{-1} P_1 M \,.$$

We define a norm by $||u|| = ||Mu||_1$. In the corresponding matrix norm we have

$$||\Pi^n - J|| \leq c_\delta(|\lambda_2| + \delta)^n \,.$$

It is clear that the image of J is $\mathbb{R}v$, and that the image of J^T is $\mathbb{R}p$. In fact, v is an eigenvector of Π with eigenvalue 1, so it is also an eigenvector, with the same eigenvalue, for $J = \lim_{n\to\infty} \Pi^n$. Similary p is an eigenvector of Π^T with eigenvalue 1, so it is also an eigenvector, with the same eigenvalue, for $J^T = \lim_{n\to\infty}(\Pi^T)^n$. The claim follows now from the fact that $\mathrm{rank}(J) = 1$. The latter ($\mathrm{rank}(J) = 1$) implies that the rows are proportional with coefficients $(1, ..., 1)$ due to $Jv = v = (1, ..., 1)^T$, and also that the columns are proportional with coefficients $(p_1, ..., p_r)$ due to $J^T p = p = (p_1, ..., p_r)^T$. Thus J has the form (4.1). □

Define μ_A by

$$\mu_A([\omega_{-n}...\omega_n]) = p_{\omega_{-n}} \prod_{t=-n}^{n-1} \pi_{\omega_t\omega_{t+1}} \,,$$

and observe that we can write

$$\mu_A([\omega_{-n}...\omega_n]) = p_{\omega_{-n}} N^{-2n} \,,$$

for an admissible path $(\omega_{-n}...\omega_n)$. It can be uniquely extended to the Borel σ-algebra, where it defines a measure. Each $\omega_t \in \{1, ..., r\}$ corresponds to a partition element $P_{\omega_t} = [i_t] \times Y^{\nu_t}$. So for each ω_t there is a pair (i_t, ν_t). For each $\nu_t \in \{1, ..., s\}$ there is unique $\nu_{t+1} \in \{1, ..., s\}$ such that $T_{i_t}(Y^{\nu_t}) \subset Y^{\nu_{t+1}}$. Hence the cylinder $[\omega_{-n}...\omega_n]$ is completely determined by $i_{-n}...i_n$ and ν_{-n}, and there are $sN^{2n+1} = rN^{2n}$ cylinders of length n. For each $k \in \{1, ..., r\}$ there are N^{2n} cylinders with measure $p_k N^{-2n}$. The cyllinders are disjoined and cover Ω_A, so

$$\mu_A(\Omega_A) = N^{2n} \sum_{k=1}^{r} p_k N^{-2n} = \sum_{k=1}^{r} p_k = 1 \,.$$

In other words we have a probability measure. It is obviously σ_A-invariant.

Definition 2. *A function $\phi : \Omega_A \to \mathbb{R}$ is called an avalanche observable if there exists $\phi' : \{1, ..., r\} \to \mathbb{R}$, such that $\phi(\omega) = \phi'(\omega_0)$.*

We define the correlation function of an observable ϕ by

$$C_\phi(n) := \int \phi(\phi \circ \sigma_A^n)\, d\mu_A - \Big(\int \phi \, d\mu_A\Big)^2 \,.$$

Proposition 8. *The correlation function C_ϕ of each avalanche observable ϕ decays exponentially. More precisely, for each $\delta > 0$ and each norm $||\cdot||$ on the space of avalanche observables, there exists a constant $c_\delta > 0$ such that*

$$|C_\phi(n)| \leq c_\delta(|\lambda_2| + \delta)^n ||\phi||^2 \,.$$

Proof. Let V be the r-dimensional real vectorspace of avalanche observables. We have a basis $\{e_k\}$, where $e_k(\omega) = e'_k(\omega_0) = \delta_{k\omega_0}$. We define a linear operator $L : V \to V$ by the formula

$$(L\phi)(\omega) = \sum_{\bar\omega_1=1}^{r} \pi_{\omega_0\bar\omega_1} \phi'(\bar\omega_1) \,.$$

With respect to the basis $\{e_k\}$ we have

$$L_{lk} = (Le_k)'(l) = \sum_{\alpha=1}^{r} \pi_{l\alpha}\delta_{k\alpha} = \pi_{lk} \,.$$

So the matrix of L is Π. Let

$$\Omega_A^{(n)} = \{\omega_0...\omega_n \,|\, A_{\omega_t\omega_{t+1}} = 1, t = 0, ..., n-1\}\,,$$

be the set of finite legal paths of length n. Then

$$(L^n\phi)'(\alpha) = \frac{1}{N^n} \sum_{\omega\in\Omega_A^{(n)},\omega_0=\alpha} \phi'(\omega_n)\,.$$

Hence

$$\begin{aligned}
\sum_{\alpha=1}^{r} p_\alpha (L^n\phi)'(\alpha) &= \frac{1}{N^n} \sum_{\omega\in\Omega_A^{(n)}} p_{\omega_0}\phi'(\omega_n) \\
&= \sum_{\omega\in\Omega_A^{(n)}} \phi'(\omega_n)\mu_A([\omega_0...\omega_n]) \\
&= \int \phi\circ\sigma_A^n \, d\mu_A \\
&= \int \phi \, d\mu_A \\
&= \sum_{\alpha=1}^{r} p_\alpha\phi'(\alpha)\,.
\end{aligned}$$

Since the matrix of J has equal rows, we have

$$\lim_{n\to\infty}(L^n\phi)'(\alpha) = (J\phi)'(\alpha) = (J\phi)'(\beta) = \lim_{n\to\infty}(L^n\phi)'(\beta)\,,$$

for all $\alpha, \beta \in \{1, ..., r\}$. This implies that

$$\lim_{n\to\infty}(L^n\phi)'(\alpha) = \sum_{\beta=1}^{r} p_\beta\phi'(\beta)$$

for each $\alpha \in \{1, ..., r\}$. Moreover, by lemma 5 we have a norm $||\cdot||$ on V, such that for each $\delta > 0$ there exists C_δ with

$$\Big|(L^n\phi)'(\alpha) - \sum_{\beta=1}^{r} p_\beta\phi'(\beta)\Big| = |(L^n\phi)'(\alpha) - (J\phi)'(\alpha)| \le C_\delta(|\lambda_2| + \delta)^n||\phi||\,.$$

Then

$$\begin{aligned}
|C_\phi(n)| &= \Big|\int \phi(\phi\circ\sigma_A^n)\, d\mu_A - \Big(\int \phi\, d\mu_A\Big)^2\Big| \\
&= \Big|\frac{1}{N^n}\sum_{\omega\in\Omega_A^{(n)}} p_{\omega_0}\phi'(\omega_0)\phi'(\omega_n) - \Big(\sum_{\alpha=1}^{r} p_\alpha\phi'(\alpha)\Big)^2\Big| \\
&= \Big|\frac{1}{N^n}\sum_{\alpha=1}^{r} p_\alpha\phi'(\alpha) \sum_{\omega\in\Omega_A^{(n)},\omega_0=\alpha} \phi'(\omega_n) - \Big(\sum_{\beta=1}^{r} p_\beta\phi'(\beta)\Big)\Big(\sum_{\alpha=1}^{r} p_\alpha\phi'(\alpha)\Big)\Big| \\
&= \Big|\sum_{\alpha=1}^{r} p_\alpha\phi'(\alpha)\Big((L^n\phi)(\alpha) - \sum_{\beta=1}^{r} p_\beta\phi'(\beta)\Big)\Big| \\
&\le \max_\alpha |\phi'(\alpha)| \cdot C_\delta(|\lambda_2| + \delta)^n||\phi||\,,
\end{aligned}$$

Since all norms on V are equivalent, the statement follows. □

Proposition 9. *σ_A is mixing with respect to μ_A*

Proof. Take cylinders $A = [\omega_{-n}...\omega_n]$ and $B = [\alpha_{-n}...\alpha_n]$. For $t > 2n$ we have

$$\begin{aligned}
\mu_A(\sigma^{-t}(A) \cap B) &= \mu_A(\bigcup_{\gamma_1,...,\gamma_t} [\gamma_1...\gamma_t\omega_{-n}...\omega_n] \cap [\alpha_{-n}...\alpha_n]) \\
&= \mu_A(\bigcup_{\gamma_{2n+2},...,\gamma_t} [\alpha_{-n}...\alpha_n\gamma_{2n+2}...\gamma_t\omega_{-n}...\omega_n]) \\
&= \sum_{\gamma_{2n+2},...,\gamma_t} \mu_A([\alpha_{-n}...\alpha_n\gamma_{2n+2}...\gamma_t\omega_{-n}...\omega_n]) \\
&= p_{\alpha_{-n}} \sum_{\gamma_{2n+2},...,\gamma_t} \pi_{\alpha_{-n}\alpha_{-n+1}}...\pi_{\alpha_n\gamma_{2n+2}}...\pi_{\gamma_t\omega_{-n}}...\pi_{\omega_{n-1}\omega_n} \\
&= \mu_A(A)\mu_A(B) \cdot p_{\omega_{-n}}^{-1} \sum_{\gamma_{2n+2},...,\gamma_t} \pi_{\alpha_n\gamma_{2n+2}}...\pi_{\gamma_t\omega_{-n}} \\
&= \mu_A(A)\mu_A(B) \cdot p_{\omega_{-n}}^{-1} (\Pi^{t-2n})_{\alpha_n\omega_{-n}} .
\end{aligned}$$

As $t \to \infty$ we know that $\Pi^{t-2n} \to J$. Hence $(\Pi^{t-n})_{\alpha_n\omega_{-n}} \to p_{\omega_{-n}}$, so

$$\mu_A(\sigma^{-t}(A) \cap B) \to \mu_A(A)\mu_A(B) .$$

Since any measurable set can be approximated arbitrary well by a finite disjoined union of symmetrical cylinders, this proves that σ_A is mixing. □

Mixing always implies ergodicity. Hence proposition 7 proves that σ_A is ergodic with respect to μ_A. We now define a measure μ on $\hat{M}$ by

$$\mu := h_*^{-1}\mu_A .$$

We immediately get that μ is a $\mathcal{T}$-invariant, Borel, probability measure. Moreover μ is supported on the attractor $\mathcal{A}$. Proposition 7 shows that μ is mixing, and thus ergodic.

From a physical perspective we must ask if μ is the right measure to consider. The Zhang model is originally constructed so that a site is exited with probability $1/N$. This corresponds to a uniform Bernoulli-measure on Σ_Λ. This measure is given by

$$\mu_{Ber}([a_{-n}...a_n]) = N^{-2n-1} .$$

A physical relevant measure should somehow preserve this structure on Σ_Λ. To prove that the measure μ is the only natural measure on $\hat{M}$, we must first prove that any invariant measure is supported on $\mathcal{A}$.

The ω-limit set, $\omega(\hat{x})$, of a point $\hat{x}$ consists of the points $\hat{y} \in \hat{M}$ for which there exists sequence $n_k \to \infty$, such that

$$\lim_{k\to\infty} \mathcal{T}^{n_k}(\hat{x}) = \hat{y} .$$

Lemma 6. *For all $\hat{x} \in \hat{M}$ we have $\omega(\hat{x}) \subset \mathcal{A}$.*

Proof. Suppose $\hat{y} \in \omega(\hat{x})$. Then there is a sequence n_k such that $\mathcal{T}^{n_k}(\hat{x}) \to \hat{y}$. If $\hat{y} \notin \mathcal{A}$, there is $T \in \mathbb{N}$ such that

$$\hat{y} \notin \mathcal{T}^T(\Sigma_\Lambda \times X) .$$

Let us choose k so that $n_k > T + m$, where m is the integer of definition 1. Then we have:

$$... \subset \mathcal{T}^{n_{k+1}}(\Sigma_\Lambda \times M) \subset \mathcal{T}^{n_k}(\Sigma_\Lambda \times M) \subset \mathcal{T}^T(\Sigma_\Lambda \times X) .$$

Hence the sequence $\mathcal{T}^{n_k}(\hat{x}), \mathcal{T}^{n_{k+1}}(\hat{x}), ...$ is contained in the closed set $\mathcal{T}^T(\Sigma_\Lambda \times X)$, and so $\hat{y} \in \mathcal{T}^T(\Sigma_\Lambda \times X)$. This gives a contradiction. □

The support of the measure μ is

$$\mathrm{supp}(\mu) = \{\hat{x} \in \hat{M} \,|\, \mu(U) > 0 \text{ for any open } U \ni x\}\,.$$

Proposition 10. *Any invariant Borel probability-measure on $\hat{M}$ has support in $\mathcal{A}$.*

Proof. A point $\hat{x} \in \hat{M}$ is said to positively recurrent if $\hat{x} \in \omega(\hat{x})$. The set of positively recurrent points is denoted by R. It follows from lemma 6 that $R \subset \mathcal{A}$. The Poincaré Recurrence theorem states that for any invariant measure μ, and any measurable set U, the subset of those points in U that return to U only a finite number of times, has measure zero. Let $U_1, U_2, ...$ be a countable basis for the topology on $\hat{M}$. If the measure is Borel, these sets are measurable, so the set of points with the property that if $\hat{x} \in U_k$ then infinitely many positive iterates of $\hat{x}$ are in U_k, has full measure. Since the sets U_k form a basis for the topology, these points are positively recurrent. Hence $\mu(R) = 1$. The set $\hat{M} - \mathrm{supp}(\mu)$ can be covered by a countable union of open sets of measure zero. Hence $\mu(\hat{M} - \mathrm{supp}(\mu)) = 0$. If $\mathrm{supp}(\mu) \not\subset \bar{R}$, then we have a set $\mathrm{supp}(\mu) - \bar{R}$, which is open in the support of μ. Hence $\mu(\mathrm{supp}(\mu) - R) > 0$. This is impossible, so $\mathrm{supp}(\mu) \subset \bar{R} \subset \mathcal{A}$. □

Theorem 3. *μ is the unique $\mathcal{T}$-invariant, Borel probability measure on $\hat{M}$, such that*

$$\pi^u_* \mu = \mu_{Ber}\,.$$

Proof. Take a measure ρ on $\hat{M}$. Since the support of any invariant Borel probability measure is in $\mathcal{A}$, we have a measure $\rho_A := h_* \rho$ on Ω_A. Define $\tilde{\pi}^u = \pi^u \circ h^{-1}$. Then is is clear that $\pi^u_* \rho = \mu_{Ber}$ is equivalent to $\tilde{\pi}^u_* \rho_A = \mu_{Ber}$. Hence we must show that μ_A is the unique σ_A-invariant measure such that $\tilde{\pi}^u_* \mu = \mu_{Ber}$.

First we must show that the property is actually satisfied by μ_A. We use the fact that each $\omega \in \{1, ..., r\}$ can be uniquely represented as a pair (i, ν), where $i \in \Lambda$ and $\nu \in \{1, ..., s\}$. Thus we can write

$$\Omega_A = \{...(i_{-1}, \nu_{-1})(i_0, \nu_0)(i_1, \nu_1)... \,|\, T_{i_t}(X_{\nu_n}) \subset X_{\nu_{t+1}}\}\,,$$

and

$$\tilde{\pi}^u(...(i_{-1}, \nu_{-1})(i_0, \nu_0)(i_1, \nu_1)...) = (...i_{-1} i_0 i_1 ...) \in \Sigma_\Lambda\,.$$

Since each set X_ν is a connected component, that can only be mapped to one connected component for each $i \in \Lambda$, then ν_{-n} and $i_{-n}, ..., i_n$ uniquely determine $\nu_{-n+1}, \nu_{-n+2}, ..., \nu_n$. Hence

$$\begin{aligned}
\mu_A((\tilde{\pi}^u)^{-1}([i_{-n}...i_n])) &= \mu_A(\bigcup_{\nu_{-n}=1}^{s} [(i_{-n}, \nu_{-n})...(i_n, \nu_n)]) \\
&= \sum_{\nu_{-n}=1}^{s} \mu_A([(i_{-n}, \nu_{-n})...(i_n, \nu_n)]) \\
&= \sum_{\nu_{-n}=1}^{s} p_{(i_{-n}, \nu_{-n})} \frac{1}{N^{2n}}\,.
\end{aligned}$$

If $\mathcal{T}([i'] \times Y^{\nu'}) \cap ([i] \times Y^\nu) \neq \emptyset$, then

$$\mathcal{T}([i'] \times Y^{\nu'}) \cap ([k] \times Y^\nu) \neq \emptyset$$

for any $k \in \Lambda$. Then columns (i, ν) and (k, ν) of A are identical. This means that

$$p_{(i,\nu)} = (A^T p)_{(i,\nu)} = \sum_{j=1}^{r} A_{j,(i,\nu)} p_j = \sum_{j=1}^{r} A_{j,(k,\nu)} p_j = p_{(k,\nu)}\,.$$

Hence

$$1 = \sum_{i\in\Lambda}\sum_{\nu=1}^{s} p_{(i,\nu)} = N\sum_{\nu=1}^{s} p_{(i,\nu)}\,,$$

so

$$\sum_{\nu=1}^{s} p_{(i,\nu)} = \frac{1}{N}\,.$$

Then

$$\mu_A((\tilde{\pi}^u)^{-1}([i_{-n}...i_n])) = \frac{1}{N^{2n+1}} = \mu_{Ber}([i_{-n}...i_n])\,.$$

Next we prove uniqueness. Suppose there is another invariant probability measure ρ_A, with $\tilde{\pi}^u_*\rho_A = \mu_{Ber}$. Then the Bernoulli shift (σ_N, μ_{Ber}) is a metric factor of (σ_A, ρ_A). This implies that the metric entropy $h_{\mu_{Ber}}(\sigma_N)$ of the Bernoulli shift is less than or equal to the metric entropy, $h_{\rho_A}(\sigma_A)$, of σ_A with respect to ρ_A. The entropy of the Bernoulli shift is $\log N$. By the variational principle the maximal entropy of σ_A is the topological entropy, which also is $\log N$. Thus ρ_A is a measure of maximal entropy. We can apply the same arguments to μ_A. Then both ρ_A and μ_A are measures of maximal entropy. It is known that a topologically transitive Markov chain has a unique measure of maximal entropy [7]. Hence $\rho_A = \mu_A$. □

As an immediate consequence of the proof we get that

$$h_\mu(\mathcal{T}) = h_\mu(\mathcal{T}|_{supp(\mu)}) \le h_\mu(\mathcal{T}|_{\mathcal{A}}) = h_{\mu_A}(\sigma_A) = \log N\,.$$

On the other hand the Bernoulli shift is a factor of $(\mathcal{T}, \mu)$, so

$$\log N \le h_{\mu_{Ber}}(\sigma_N) \le h_\mu(\mathcal{T})\,,$$

and thus:

Corollary 6. $h_\mu(\mathcal{T}) = \log N$.

Chapter 5

Examples

Application of the theory developed in chapters 3 and 4 requires that singularities are removable and that assumptions 1 and 2 hold. In this chapter we give two examples where these conditions are satisfied, and one example where we cannot determine if singularities are removable. The examples we present describe the simple situation of a system with only two sites. However example C shows that the behavior of the Zhang model can be complex even in this case. In examples A and B, we use a computer program to prove removability of singularities. In these two examples both assumptions 1 and 2 are satisfied. Hence the results of the two previous chapters can be applied. Moreover we can compute the transition matrices, and by using the method presented in the proofs of lemma 5 and proposition 8, we show how the correlation function of an avalanche observable can be explicitly computed.

For fixed values of parameters it is possible to prove that singularities are removable by iterating the phase-space. One should determine the partition elements M_{ij} and the maps T_{ij}. Then one can compute the polygons $T_{ij}(M_{ij})$. It the next step one must compute the intersections $T_{ij}(M_{ij}) \cap M_{kl}$, and $T_{kl}(T_{ij}(M_{ij}) \cap M_{kl})$. As the number of polygon domains grows exponentially, it is in reality impossible to make more than a few such iterations by hand, even in the simplest cases. We have however written a computer program that carry out this task for the case $N = 2$. This program lets us compute $\pi^s \circ \mathcal{T}^t(\hat{M})$ for t as great as about 20. As a result we can determine if there exists $m \leq 20$ such that

$$\overline{\pi^s \circ \mathcal{T}^m(\hat{M})} \cap \mathcal{S} = \emptyset \,.$$

These calculations are carried out with rational numbers, and are thus exact. Hence we can give "proof by computer", of removability of singularities in simple examples.

Example A: We study the case $N = 2$, $E_c = 7/2$ and $\epsilon = 1/2$. It is easy to see that we have six domains of continuity, and six corresponding affine transformations. These transformations are

$$T_{1j}\Big(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Big) = L_{1j} \begin{bmatrix} x_1 + 1 \\ x_2 \end{bmatrix} ,$$

and

$$T_{2j}\Big(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Big) = L_{2j} \begin{bmatrix} x_1 \\ x_2 + 1 \end{bmatrix} ,$$

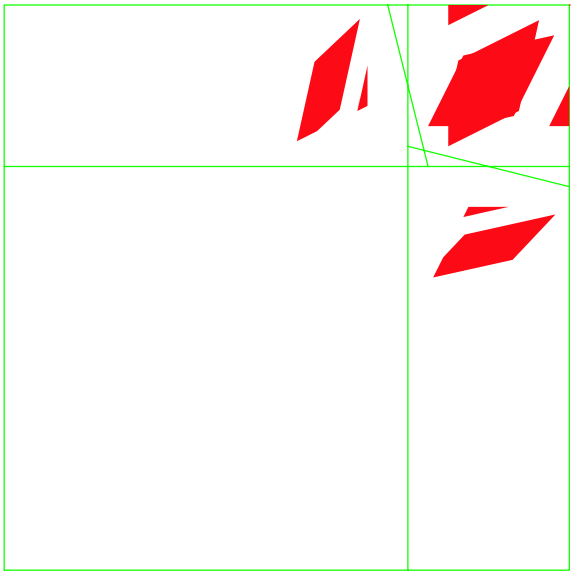

Figure 5.1: Shows the set $\pi^s \circ \mathcal{T}^5(\hat{M})$ and lines of singularity for $N = 2$, $E_c = 7/2$ and $\epsilon = 1/2$.

for $j = 1, 2, 3$. The linear maps L_{ij} are:

$$L_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad L_{12} = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1 \end{bmatrix} \quad L_{13} = \begin{bmatrix} 9/16 & 1/4 \\ 1/8 & 1/2 \end{bmatrix}$$

$$L_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad L_{22} = \begin{bmatrix} 1 & 1/4 \\ 0 & 1/2 \end{bmatrix} \quad L_{23} = \begin{bmatrix} 1/2 & 1/8 \\ 1/4 & 9/16 \end{bmatrix}.$$

The domains of continuity are $[i] \times M_{ij}$ for $i = 1, 2$ and $j = 1, 2, 3$, where the sets M_{ij} are:

$$\begin{aligned}
M_{11} &= \{(x_1, x_2) \in M \,|\, x_1 \leq 5/2\} \\
M_{12} &= \{(x_1, x_2) \in M \,|\, x_1 > 5/2, (x_1 + 1)/4 + x_2 \leq 7/2\} \\
M_{13} &= \{(x_1, x_2) \in M \,|\, x_1 > 5/2, (x_1 + 1)/4 + x_2 > 7/2\} \\
M_{21} &= \{(x_1, x_2) \in M \,|\, x_2 \leq 5/2\} \\
M_{22} &= \{(x_1, x_2) \in M \,|\, x_2 > 5/2, (x_2 + 1)/4 + x_1 \leq 7/2\} \\
M_{23} &= \{(x_1, x_2) \in M \,|\, x_2 > 5/2, (x_2 + 1)/4 + x_1 > 7/2\}.
\end{aligned}$$

Figure 5.1 shows the set $X := \pi^s \circ \mathcal{T}^5(\hat{M})$. Form the picture we see that singularities are removable. There is more than one connected component in each domain M_{ij}, but in this case we do not have to use connected components to generate a partition for symbolic coding. Let $X_1 = X \cap (M_{13} \cap M_{23})$, $X_2 = X \cap (M_{12} \cap M_{21})$ and $X_3 = X \cap (M_{11} \cap M_{22})$. Then X_1 consists of the points in the upper right corner of figure 5.1, X_2 of the points in the lower right corner and X_3 of the points in the upper left corner. It is easy to verify that

$$\begin{aligned}
T_1(X_1) &\subset X_2 \qquad T_2(X_1) \subset X_3 \\
T_1(X_2) &\subset X_3 \qquad T_2(X_2) \subset X_1 \\
T_1(X_3) &\subset X_1 \qquad T_2(X_3) \subset X_2.
\end{aligned}$$

From this we see that we have invertibility of T_1 and T_2 on X, and hence on Y. We can then construct a symbolic coding of the dynamics on the attractor. Let $R_1 = [1] \times X_1, R_2 = [1] \times$

$X_2, ..., R_6 = [2] \times X_3$. We draw an arrow from R_k to R_l if $\mathcal{T}(R_k) \cap R_l \neq \emptyset$. This gives

$$\begin{aligned} R_1 &\to R_2, R_5 \qquad & R_4 &\to R_3, R_6 \\ R_2 &\to R_3, R_6 \qquad & R_5 &\to R_1, R_4 \\ R_3 &\to R_1, R_4 \qquad & R_6 &\to R_2, R_5 \,. \end{aligned}$$

From this we can construct the 6×6 transition matrix A:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

The matrix A^3 is positive, so A is transitive. Hence we know that periodic points are dense on the attractor $\mathcal{A}$, that we have topological mixing on the attractor and that $\mathcal{T}$ is mixing with respect to the invariant measure μ. The spectrum of A is $\mathrm{Sp}(A) = \{-1, 0, 2\}$. This implies that the spectrum of the matrix Π is $\mathrm{Sp}(\Pi) = \{-1/2, 0, 1\}$. Hence we have exponential correlation decay, $C_\phi(n) \leq C(1/2)^n$.

The proof of lemma 5 and proposition 8 also gives us a method for computing the correlation functions of avalanche observables. In this example the matrix A can be diagonalized, and then a simple computation gives

$$\Pi^n - J = \frac{1}{3}(-\frac{1}{2})^n \begin{bmatrix} 1 & -2 & 1 & 1 & -2 & 1 \\ 1 & 1 & -2 & 1 & 1 & -2 \\ -2 & 1 & 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & -2 \\ -2 & 1 & 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & 1 & -2 & 1 \end{bmatrix} .$$

Let ϕ be the observable of avalanche size. Points of R_1 and R_4 have avalanches of size 3, the points of R_2 and R_6 have avalanches of size 2, and R_3 and R_5 correspond to avalanche size 0. Hence the observable ϕ corresponds to the vector $w = (3, 2, 0, 3, 0, 2)^T$. Since there are 2 1-s in each column of A we have $p = (1/6, ..., 1/6)^T$, so

$$C_\phi(n) = \frac{1}{6} \sum_{k=1}^{6} w_k ((\Pi^n - J)w)_k = \frac{20}{9}(-1/2)^n \,.$$

We note that the attractor of this example is particularly simple. The linear maps L_{13} and L_{23} are contracting. Moreover products $L_{12}L_{22}$ and $L_{22}L_{12}$, as well as any admissible product $L_{ij}L_{kl}$, are contracting. This implies that each partition element X_ν is contracted to a point. Hence the attractor consists of three points. It is easy to see there is only one set of three points such that $\mathcal{T}(\mathcal{A}) = \mathcal{A}$. These points are $(3, 3) \in X_1$, $(3, 2) \in X_2$ and $(3, 2) \in X_3$. Hence $\mathcal{A} = \Sigma_\Lambda \times \{(3, 3), (3, 2), (2, 3)\}$.

Example B: Our second example is $N = 2$, $E_c = 1/4$ and $\epsilon = 1/2$. There are 14 domains of continuity $M_{1,1}, ..., M_{1,14}$ for the map T_1, and 14 corresponding affine transformations. By symmetry there are 14 domains of continuity for the map T_2, and 14 corresponding affine transformations. The maps and the domains of continuity are given in the appendix. We apply our computer program, and iterate the phase-space. The result is that $\pi^s \circ \mathcal{T}^5(\hat{M})$ does not intersect singularities. The set $\pi^s \circ \mathcal{T}^5(\hat{M})$ is made up of 752 convex polygons, that generate 13 connected components. We denote these components by $X_1, ..., X_{13}$. In figure 5.2 the size of the polygons is increased to make them visible in the figure. As a result it may look as if they intersect singularities. Figure 5.3 is a schematic illustration of how these components are situated with respect to the lines of singularity. We construct a partition of $\pi^s \circ \mathcal{T}^5(\hat{M})$ by taking sets $\{[i] \times X_\nu\}$, where $i = 1, 2$. This gives 26 partition elements:

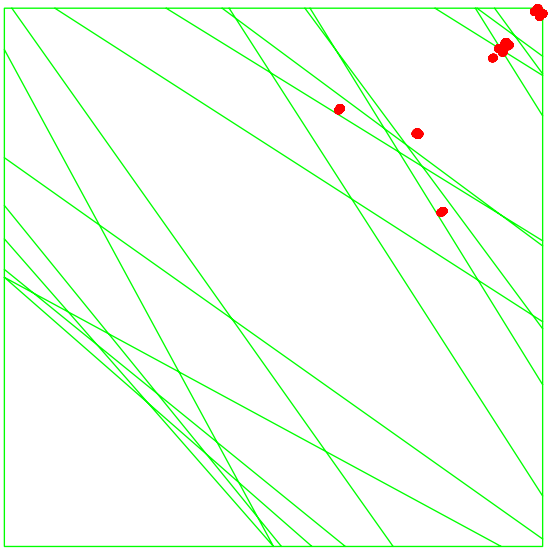

Figure 5.2: Shows the set $\pi^s \circ \mathcal{T}^5(\hat{M})$ and lines of singularity for $N = 2$, $E_c = 1/4$ and $\epsilon = 1/2$. The sizes of points are increased to make them visible in the figure.

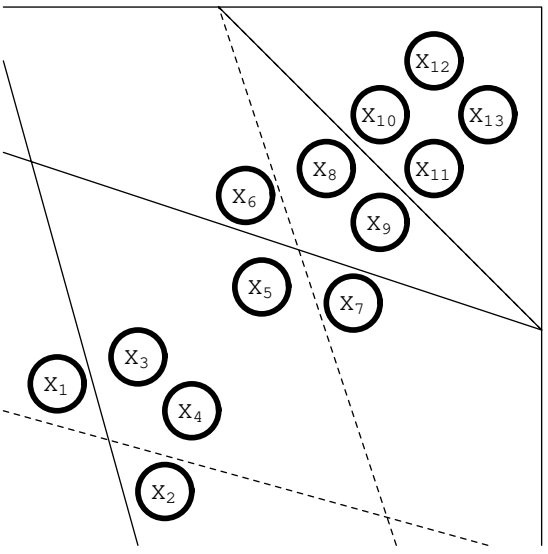

Figure 5.3: Schematic illustration of the 13 connected components of $\pi^s \circ \mathcal{T}^5(\hat{M})$ for the example where $N = 2$, $E_c = 1/4$ and $\epsilon = 1/2$. The connected components are represented as circles, and the lines represent lines of singularity. The continuous lines are the singularities of the map T_1 and the dashed lines are the singularities of the map T_2.

$$
\begin{aligned}
R_1 &= [1] \times X_1 \\
R_2 &= [1] \times X_2 \\
&\vdots \\
R_{14} &= [2] \times X_1 \\
&\vdots \\
R_{26} &= [2] \times X_{13}
\end{aligned}
$$

For each of the maps T_1 and T_2 a domain X_ν can only be mapped to one of the connected components. By computer it is easy to verify that we have:

$$
\begin{array}{ll}
T_1(X_1) \subset X_6 & T_2(X_1) \subset X_5 \\
T_1(X_2) \subset X_5 & T_2(X_2) \subset X_7 \\
T_1(X_3) \subset X_8 & T_2(X_3) \subset X_9 \\
T_1(X_4) \subset X_8 & T_2(X_4) \subset X_9 \\
T_1(X_5) \subset X_{10} & T_2(X_5) \subset X_{11} \\
T_1(X_6) \subset X_1 & T_2(X_6) \subset X_{13} \\
T_1(X_7) \subset X_{12} & T_2(X_7) \subset X_2 \\
T_1(X_8) \subset X_1 & T_2(X_8) \subset X_2 \\
T_1(X_9) \subset X_1 & T_2(X_9) \subset X_2 \\
T_1(X_{10}) \subset X_3 & T_2(X_{10}) \subset X_4 \\
T_1(X_{11}) \subset X_3 & T_2(X_{11}) \subset X_4 \\
T_1(X_{12}) \subset X_3 & T_2(X_{12}) \subset X_4 \\
T_1(X_{13}) \subset X_3 & T_2(X_{13}) \subset X_4
\end{array}
$$

This determines which transitions that are legal between partition elements. We draw an arrow from R_k to R_l if $\mathcal{T}(R_k) \cap R_l \neq \emptyset$. This gives:

$$
\begin{array}{ll}
R_1 \to R_6, R_{19} & R_{14} \to R_5, R_{18} \\
R_2 \to R_5, R_{18} & R_{15} \to R_7, R_{20} \\
R_3 \to R_8, R_{21} & R_{16} \to R_9, R_{22} \\
R_4 \to R_8, R_{21} & R_{17} \to R_9, R_{22} \\
R_5 \to R_{10}, R_{23} & R_{18} \to R_{11}, R_{24} \\
R_6 \to R_1, R_{14} & R_{19} \to R_{13}, R_{26} \\
R_7 \to R_{12}, R_{25} & R_{20} \to R_2, R_{15} \\
R_8 \to R_1, R_{14} & R_{21} \to R_2, R_{15} \\
R_9 \to R_1, R_{14} & R_{22} \to R_2, R_{15} \\
R_{10} \to R_3, R_{16} & R_{23} \to R_4, R_{17} \\
R_{11} \to R_3, R_{16} & R_{24} \to R_4, R_{17} \\
R_{12} \to R_3, R_{16} & R_{25} \to R_4, R_{17} \\
R_{13} \to R_3, R_{16} & R_{26} \to R_4, R_{17}
\end{array}
$$

We can now construct the 26×26 transition matrix A. Indeed $A_{1,6}$ and $A_{1,19}$ are the non-zero elements of row 1, and $A_{2,5}$ and $A_{2,18}$ are the non-zero elements of row 2, and so on. Figure 5.4 shows the complete matrix. Zero elements are white and 1-s are colored black. We observe that there are exactly 2 1-s in each row. There can be more than 2 1-s in each column. We see from the matrix that different domains can be mapped to the same domain under the map T_1. However, we see from figure 5.2 that X_3, X_4, X_5, X_7 are all contained in the same domain of continuity for T_1. The same is true for X_6, X_8, X_9, and for X_{10}, X_{11}, X_{12}, X_{13}. It is then easy to verify that whenever $T_1(X_{\nu_1})$ and $T_1(X_{\nu_2})$ intersect, then X_{ν_1} and X_{ν_2} are contained in same domain of continuity M_{1j}. Then $T_1(X_{\nu_1}) = T_{1j}(X_{\nu_1})$ and $T_1(X_{\nu_2}) = T_{1j}(X_{\nu_2})$. The affine map T_{1j} is invertible, so T_1 is invertible on $\pi^s \circ \mathcal{T}^5(\hat{M})$. By symmetry the map T_2 is also invertible on $\pi^s \circ \mathcal{T}^5(\hat{M})$. Then all the conditions for theorem 2 are satisfied. We have a topological conjugancy $h : \mathcal{A} \to \Omega_A$. Moreover the matrix A is transitive. In fact A^{13} is a positive matrix. Hence all corollaries of theorem 2 and all results from chapter 4 are applicable to this example. Periodic points are dense on the attractor, we have topological mixing on the attractor, and the topological

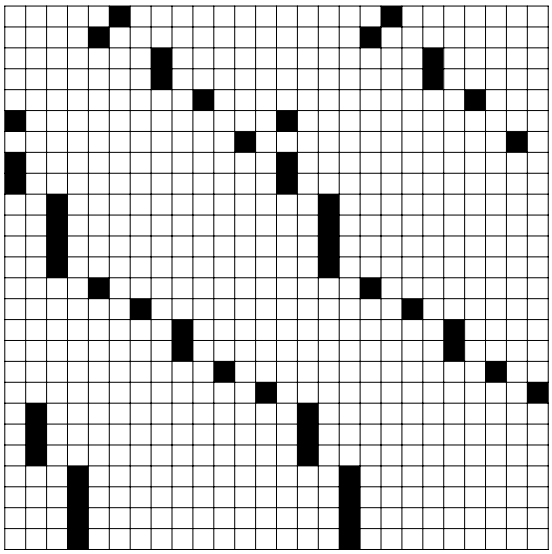

Figure 5.4: Presentation of the 26×26 transition matrix A for the symbolic coding in the example with $N = 2$, $E_c = 1/4$ and $\epsilon = 1/2$. Zero elements are represented as white squares, and 1-s are represented as black squares.

entropy is $\log 2$. From chapter 2 we know that there is a unique Borel, probability measure μ that projects to the Bernoulli measure on Σ_Λ, and that $\mathcal{T}$ is mixing with respect to this measure. Moreover we know that for any avalanche observable ϕ, the correlation function $C_\phi(n)$ decays exponentially, with rate given by the second largest eigenvalue of the matrix $\Pi = 1/N \cdot A$. The absolute value of the second greatest eigenvalue of A is approximately 1.919. Hence

$$|C_\phi(n)| \leq C|\lambda_2|^n \,,$$

where $|\lambda_2| \approx 0.960$. The matrix Π has five non-zero eigenvalues less that 1:

$$\begin{aligned}
\lambda_2 &\approx -0.81 + 0.52i \\
\lambda_3 &\approx -0.81 - 0.52i \\
\lambda_4 &\approx 0.31 + 0.85i \\
\lambda_5 &\approx 0.31 - 0.85i \\
\lambda_6 &= -1/2
\end{aligned}$$

Each of them has multiplicity 1. If we bring Π to Jordan normal form by a transformation M, then it is easy to see that $M^{-1}\Pi^n M$ will be diagonal for n sufficiently great. Denote the entries of M by m_{ij} and the entries of M^{-1} by r_{ij}. Then

$$(\Pi^n - J)_{ij} = \sum_{k=2}^{6} m_{ik} r_{kj} \lambda_k^n \,.$$

Take an avalanche observable ϕ and let w be the corresponding vector. Then

$$C_\phi(n) = \sum_{i=1}^{26} p_i w_i ((\Pi^n - J)w)_i = \sum_{k=2}^{6} c_k(w) \lambda_k^n \,,$$

where

$$c_k(w) = \sum_{i,j=1}^{26} p_i w_i w_j m_{ik} r_{kj} \,.$$

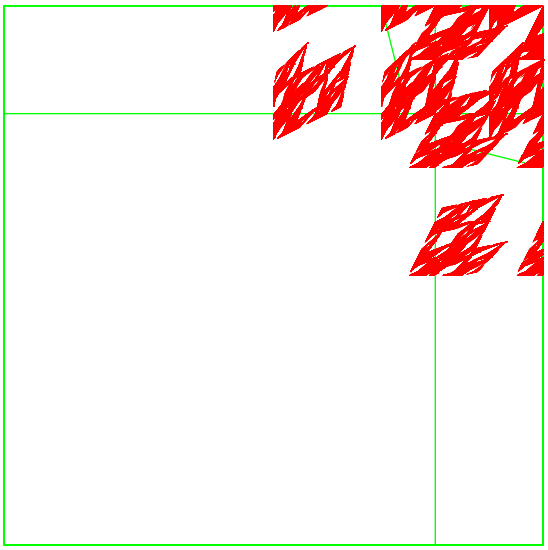

Figure 5.5: Shows the set $\pi^s \circ \mathcal{T}^{20}(\hat{M})$ and singularities for $N = 2$, $E_c = 5$ and $\epsilon = 1/2$.

The vector p can be computed by finding the positive eigenvector of A^T:

$$p = \frac{1}{68}(4, 4, 3, 3, 4, 2, 2, 3, 3, 2, 2, 1, 1, 4, 4, 3, 3, 4, 2, 2, 3, 3, 2, 2, 1, 1)^T \,.$$

In principle we can preform the Jordan decomposition and find the elements m_{ij} and r_{ij}, and then we can compute the correlation function of any avalanche observable. Since the correlation function is real we must have

$$c_2(w) = c_3(w)\,,$$

and

$$c_4(w) = c_5(w)\,.$$

Hence the correlation function is on the form

$$C_\phi(n) = a(w)(\lambda_2^n + \bar{\lambda}_2^n) + b(w)(\lambda_4^n + \bar{\lambda}_4^n) + c(w)\lambda_6^n\,.$$

Moreover we see that the expectation value of the observable ϕ is $p \cdot w$. If we let ϕ be the avalanche size observable, then a calculation shows that

$$w = (5, 6, 7, 7, 7, 8, 7, 8, 8, 9, 9, 9, 9, 6, 5, 7, 7, 7, 7, 8, 8, 8, 9, 9, 9, 9)^T\,.$$

So the average avalanche size is

$$\int \phi \, d\mu = p \cdot w = \frac{123}{17}\,.$$

Example C: The last example is $N = 2$, $E_c = 5$ and $\epsilon = 1/2$. In this case there are 3 domains of continuity for each of the maps T_1 and T_2. The maps and the domains of continuity are given in the appendix. We iterate the phase-space 20 times using the our computer program. The result is shown in figure 5.5. We see that still after 20 iterations, the image of M intersects singularities. If singularities are removable in this example is not known.

Chapter 6

Growth of Orbits and Topological Entropy

In this section we study the complexity of the dynamics by considering the exponential growth of the number of distinguishable orbit segments. First we identify orbit segments if they are in the same domain of continuity for each iterate, and define p as the exponential growth of equivalence classes. We show that $p \geq \log N$, and that we have equality if singularities are removable. We also give sufficient conditions for $p > \log N$. Next we define topological entropy, and show that it exists even if singularities are non-removable. We prove that $h_{top}(\mathcal{T}) = \log N$ if singularities are removable, and under weak conditions we show that $h_{top}(\mathcal{T}) = p$. In [3] it is argued that the topological entropy of the Zhang model always is $\log N$, since the affine maps T_{ij} are contracting. The results of this section show that the contracting property of the maps T_{ij} is not sufficient for the equality $h_{top}(\mathcal{T}) = \log N$. If the statement is true in general is not known.

In chapter 1 we defined an equivalence relation $\sim$ on $\hat{M}$. This gave the partition

$$\mathcal{P}_0 = \{[i] \times M_{ij}\}\,.$$

Now we define a sequence of new equivalence relations $\sim_n$ by

$$\hat{x} \sim_n \hat{y} \Leftrightarrow \forall t \in \{0, ..., n\} : \mathcal{T}^t \hat{x} \sim \mathcal{T}^t \hat{y}\,.$$

Let the partition corresponding to $\sim_n$ be $\mathcal{P}_n$. This gives a sequence

$$\mathcal{P}_0 \leq \mathcal{P}_1 \leq \mathcal{P}_2 \leq ...$$

of refinements of $\mathcal{P}_0$. Let $p_n := |\mathcal{P}_n|$. We define

$$p = \limsup_{n\to\infty} \frac{1}{n} \log p_n\,.$$

We denote the equivalence class of $\hat{x}$, with respect to $\sim_n$, by $(\hat{x})_n$. We say that a set $E \subset \hat{M}$ is n-spanning if

$$\bigcup_{\hat{x}\in E} (\hat{x})_n = \hat{M}\,.$$

A set $F \subset \hat{M}$ is said to be n-separating if

$$\hat{x}, \hat{y} \in F,\, \hat{x} \neq \hat{y} \Rightarrow (\hat{x})_n \neq (\hat{y})_n\,.$$

If E is n-spanning, then E must contain at least one point from each partition element, so $|E| \geq p_n$. If F is n-separating, then F cannot contain two points from the same partition element. Hence $|F| \leq p_n$. This shows that p_n is the cardinality of a minimal n-spanning set, which is equal to the cardinality of a maximal n-separating set.

Proposition 11. $p \geq \log N$.

Proof. Take $\hat{x} = (\mathbf{a}, x) \in \hat{M}$. For each finite sequence $i_0...i_n$, $i_t \in \Lambda$, we define

$$\hat{x}_{i_0...i_{n-1}} = (...a_{-2}a_{-1}i_0...i_n a_{n+1}..., x)\,.$$

Let F be the set of these points. Then $|F| = N^{n+1}$. Take $\hat{y}, \hat{z} \in F$. If $\hat{y} \neq \hat{z}$, then by construction there is $t \in \{0, ..., n\}$ such that $\sigma_\Lambda^t(\pi^u \hat{y}) \neq \sigma_\Lambda^t(\pi^u \hat{z})$. Let $i_{\hat{y}} = \sigma_\Lambda^t(\pi^u \hat{y})$ and $i_{\hat{z}} = \sigma_\Lambda^t(\pi^u \hat{z})$. Then there are $j_{\hat{y}}$ and $j_{\hat{z}}$ such that $\mathcal{T}^t(\hat{y}) \in [i_{\hat{y}}] \times M_{i_{\hat{y}} j_{\hat{y}}}$, and $\mathcal{T}^t(\hat{z}) \in [i_{\hat{z}}] \times M_{i_{\hat{z}} j_{\hat{z}}}$. Since $i_{\hat{y}} \neq i_{\hat{z}}$, we have

$$([i_{\hat{y}}] \times M_{i_{\hat{y}} j_{\hat{y}}}) \cap ([i_{\hat{z}}] \times M_{i_{\hat{z}} j_{\hat{z}}}) = \emptyset\,.$$

Thus $\mathcal{T}^t \hat{y} \not\sim \mathcal{T}^t \hat{z}$, which implies $\hat{y} \not\sim_n \hat{z}$. We conclude that F is a n-separating set, and $p_n \geq |F| = N^{n+1}$. Then

$$p = \limsup_{n\to\infty} \frac{1}{n} \log p_n \geq \limsup_{n\to\infty} \frac{1}{n} \log N^{n+1} = \log N\,.$$

□

For $\mathbf{a} \in \Sigma_\Lambda$ we define

$$[\mathbf{a}]_n = [a_0...a_{n-1}] = \{\mathbf{i} \in \Sigma_\Lambda \,|\, i_0 = a_0, ..., i_{n-1} = a_{n-1}\}\,.$$

From the proof of proposition 11 we see that $(\mathbf{a}, x) \sim_n (\mathbf{b}, y)$ implies $[\mathbf{a}]_{n+1} = [\mathbf{b}]_{n+1}$.

Lemma 7. *If $\hat{x} \sim_n \hat{y}$, then $\pi^s \circ \mathcal{T}^n(\hat{x})$ and $\pi^s \circ \mathcal{T}^n(\hat{y})$ are in the same connected component of $\pi^s \circ \mathcal{T}^n(\hat{M})$.*

Proof. Suppose $\hat{x} \sim_n \hat{y}$. Let $\hat{x} = (\mathbf{a}, x)$ and $\hat{y} = (\mathbf{b}, y)$. We know that $[\mathbf{a}]_n = [\mathbf{b}]_n$, and that there is j such that $\pi^s \circ \mathcal{T}^{n-1}(\hat{x}), \pi^s \circ \mathcal{T}^{n-1}(\hat{y}) \in M_{a_{n-1} j}$. We have

$$\pi^s \circ \mathcal{T}^n(\hat{M}) = \bigcup_{ij} T_{ij}(M_{ij} \cap \pi^s \circ \mathcal{T}^{n-1}(\hat{M}))\,,$$

where T_{ij} is continuos on the connected domains M_{ij}. So if $\pi^s \circ \mathcal{T}^n(\hat{x})$ and $\pi^s \circ \mathcal{T}^n(\hat{y})$ are not in the same connected component of $\pi^s \circ \mathcal{T}^n(\hat{M})$, then the only possibility is that $\pi^s \circ \mathcal{T}^{n-1}(\hat{x})$ and $\pi^s \circ \mathcal{T}^{n-1}(\hat{y})$ are not in the same connected component of $\pi^s \circ \mathcal{T}^{n-1}(\hat{M})$. We repeat the argument. Since $\mathcal{T}^0(\hat{x})$ and $\mathcal{T}^0(\hat{y})$ are obviously in the same connected component of M, we get that $\pi^s \circ \mathcal{T}^n(\hat{x})$ and $\pi^s \circ \mathcal{T}^n(\hat{y})$ are in the same connected component of $\pi^s \circ \mathcal{T}^n(\hat{M})$, after repeating the argument n times. □

Proposition 12. $p = \log N$ *if singularities are removable.*

Proof. If singularities are removable there exists $m \in \mathbb{N}$, so that $\pi^s \circ \mathcal{T}^m(\hat{M}) \cap \mathcal{S} = \emptyset$. Fix $n > m$ and let

$$E_n = \{\hat{x}_1, ..., \hat{x}_{p_n}\}$$

be a minimal n-spanning set. For each $\hat{x}_k = (\mathbf{a}^{(k)}, x_k)$, and each $l \in \Lambda$ we define

$$\hat{x}_{kl} = (...a_{-1}^{(k)} a_0^{(k)} a_1^{(k)} ... a_n^{(k)} l a_{n+2}^{(k)} ..., x_k)\,.$$

Take $\hat{y} = (\mathbf{b}, y) \in \hat{M}$, and suppose that $\hat{y} \sim_n \hat{x}_k$. Then $b_n = a_n^{(k)}$, and by lemma 7 we know that there is a connected component X_ν such that $\pi^s \circ \mathcal{T}^n(\hat{y}), \pi^s \circ \mathcal{T}^n(\hat{x}_k) \in X_\nu$. There exists j such that $T_{b_n}(X_\nu) \subset M_{b_{n+1}j}$, so $\mathcal{T}^{n+1}(\hat{y}), \mathcal{T}^{n+1}(\hat{x}_{kb_n}) \in [b_{n+1}] \times M_{b_{n+1}j}$. Then $y \sim_{n+1} x_{kb_n}$. Hence $E_{n+1} := \{\hat{x}_{kl}\}$ is a $n+1$-spanning set. Then $p_{n+1} \leq N p_n$. This gives

$$p \leq \limsup_{n\to\infty} \frac{1}{n} \log(N^{n-m} p_m) = \log N \,.$$

From proposition 10 we have $p \geq \log N$, so this proves $p = \log N$. □

If singularities are not removable we can still get upper bounds on p. Let

$$\mathcal{P}_0 = \{[i] \times M_{ij}\} = \{R_1, ..., R_q\} \,.$$

Then a partition element of $\mathcal{P}_n$ corresponds uniquely to a path of length n between edges $R_1, ..., R_q$. Such a path can be represented as a sequence of length $n+1$ in the numbers $1, ..., q$. Define $\Omega_q := \{1, ..., q\}^{\mathbb{Z}_{\geq 0}}$, and define $H : \hat{M} \to \Omega_q$, by

$$H(\hat{x}) = (\omega_0 \omega_1 ...) \,,$$

where ω_t is the unique number so that

$$\mathcal{T}^t(\hat{x}) \in R_{\omega_t} \,.$$

We call $H(\hat{x})$ the sequence of the point $\hat{x}$. Which transitions that are possible may change as the dynamics is contracted on to the attractor. Therefore we define a sequence of $q \times q$ transition matrices by

$$(B_t)_{kl} = \begin{cases} 1 & \text{if } \mathcal{T}(\mathcal{T}^t(\hat{M}) \cap R_k) \cap R_l \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \,,$$

We say that a sequence $\omega \in \Omega_q$ is legal if $(B_t)_{\omega_t \omega_{t+1}} = 1$ for all $t \geq 0$, and we say that the sequence ω is realized if there exists $\hat{x} \in \hat{M}$ such that $\omega = H(\hat{x})$. Let

$$\Omega_q^{(n)} = \{\omega_0 ... \omega_n \,|\, \omega_t \in \{1, ..., q\}\} \,,$$

be the set of all paths of length n. We define

$$L^{(n)} = \{\omega_0 ... \omega_n \,|\, (B_t)_{\omega_t \omega_{t+1}} = 1, t = 0, ..., n\} \subset \Omega_q^{(n)} \,,$$

and

$$R^{(n)} = \{(H\hat{x})_0 ... (H\hat{x})_n \,|\, \hat{x} \in \hat{M}\} \subset L^{(n)} \,.$$

Then $L^{(n)}$ is the set of legal paths of length n, and $R^{(n)}$ is the set of realized paths of length n. The number of realized paths corresponds to the number of different partition elements of $\mathcal{P}_n$, so $p_n = |R^{(n)}|$. Define $\tilde{p}_n := |L^{(n)}|$ to be the number of legal paths of length n. Then $p_n \leq \tilde{p}_n$.

Lemma 8. $\tilde{p}_n = \sum_{k,l=1}^q (B_0 ... B_{n-1})_{kl}$.

Proof. We prove by induction that the number of legal paths of length n from k to l is $(B_0 ... B_{n-1})_{kl}$. For $n = 1$ it is obvious. Suppose that the number of paths of length t from k to l' is $(B_0 ... B_{t-1})_{kl'}$. Then the number of paths that go from k to l' in t steps and then from l' to l in one step, is $(B_0 ... B_{t-1})_{kl'} (B_t)_{l'l}$. Summation over all l' gives the number of paths from k to l in $t+1$ steps:

$$\sum_{l'=1}^{q} (B_0 ... B_{t-1})_{kl'} (B_t)_{l'l} = (B_0 ... B_t)_{kl} \,.$$

This proves our claim, and the statement of the proposition follows immediately. □

Since

$$... \subset \mathcal{T}^2(\hat{M}) \subset \mathcal{T}(\hat{M}) \subset \hat{M}\,,$$

it is clear that if a transition $R_k \to R_l$ is not possible for any points in $\mathcal{T}^t(\hat{M})$, then it is not possible for any points in $\mathcal{T}^{t+1}(\hat{M})$ either. Thus no new 1-s are added to the matrices B_t as t increases. Hence the sequence $B_1, B_2, ...$ changes only a finite number of times. This implies that there exists $m \in \mathbb{N}$, such that $B_{m+t} = B_m$ for all $t \geq 0$. Define $B := B_m$.

Proposition 13.

$$p \leq \tilde{p} := \limsup_{n\to\infty} \frac{1}{n} \log \tilde{p}_n = \log r(B)\,.$$

Proof. The first inequality follows from the fact that $p_n \leq \tilde{p}_n$. Let $||\cdot||$ be the matrix norm given by

$$||A|| = \sum_{kl} |a_{kl}|\,.$$

Then there is $C > 0$ such that $\tilde{p}_n = ||B_0...B_{n-1}|| \geq Cr(B)^{n-m}$, so

$$\limsup_{n\to\infty} \frac{1}{n} \log \tilde{p}_n \geq \limsup_{n\to\infty} \frac{1}{n} \log \left(Cr(B)^{n-m+1}\right) = \log r(B)\,.$$

For each $k \in \{0, ..., m\}$ and each $\delta > 0$ there is norm $||\cdot||_{k,\delta}$ such that

$$||B_k||_{k,\delta} \leq r(B_k) + \delta\,.$$

All norms are equivalent in finite dimensions, so there is $c_{k,\delta}$ such that $||\cdot|| \leq c_{k,\delta}||\cdot||_{k,\delta}$. This implies that

$$\tilde{p}_n = ||B_0...B_{n-1}|| \leq c_{0,\delta}(r(B_0) + \delta) \cdot ... \cdot c_{n-1,\delta}(r(B_{n-1}) + \delta)\,,$$

so

$$\begin{aligned}
\tilde{p} &= \limsup_{n\to\infty} \frac{1}{n} \log \tilde{p}_n \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \log \left(c_{0,\delta}(r(B_0) + \delta) \cdot ... \cdot c_{m-1,\delta}(r(B_{m-1}) + \delta) c_{m,\delta}(r(B) + \delta)^{n-m+1}\right) \\
&= \log(r(B) + \delta)\,.
\end{aligned}$$

Since $\delta > 0$ is arbitrary we have $\tilde{p} \leq \log r(B)$. □

If there is a partition element $R_k \in \mathcal{P}_0$ such that $\mathcal{T}^t(\hat{M}) \cap R_k = \emptyset$ for a sufficiently great t, then row k and column k of B have only zero entries. These entries of B do not contribute to the growth of $\tilde{p}_n$ and can be removed from the matrix. We define B' to be the square matrix obtained by removing rows and columns with only zero entries. We have $r(B') = r(B)$, and we can replace B with B' in proposition 13.

We say that singularities are strictly non-removable if $\pi^s \circ \mathcal{T}^n(\hat{M}) \cap \mathcal{S} \neq \emptyset$ for all $n \in \mathbb{N}$.

Proposition 14. *If singularities are strictly non-removable and B' is transitive, then $\tilde{p} > \log N$.*

Proof. Take a partition element $R_k = [i] \times M_{ij} \in \mathcal{P}_0$, such that $\mathcal{T}^t(\hat{M}) \cap R_k \neq \emptyset$ for all $t \in \mathbb{N}$. Recall that

$$\bigcup_j M_{aj} = M$$

for all $a \in \Lambda$, so for each $a \in \Lambda$ we have j_a such that

$$T_{ij}(M_{ij} \cap \pi^s \circ \mathcal{T}^t(\hat{M})) \cap M_{aj_a} \neq \emptyset$$

Let $R'_a = [a] \times M_{aj_a}$. Then

$$\mathcal{T}(\mathcal{T}^t(\hat{M}) \cap R_k) \cap R'_a \neq \emptyset$$

for all $a \in \{1, ..., N\}$ and all $t \in \mathbb{N}$, so row k of B' has at least N 1-s.

Let q' be the number of rows of B that are non-zero. Then B' is a $q' \times q'$ matrix. Suppose $r(B') \leq N$. If B' is transitive, then B'^T is transitive, so there is a Perron-Frobenius eigenvalue $\lambda \leq N$ and $w \in \mathbb{R}_+^{q'}$, such that $B'^T w = \lambda w$. Let $N + k_i$, $k_i \geq 0$, be the number of 1-s in row i of B'. Then

$$\lambda ||w||_1 = ||B'^T w||_1 = \sum_{i=1}^{q'} (N + k_i) w_i = N||w||_1 + \sum_{i=1}^{q'} k_i w_i \,,$$

so we have

$$\sum_{i=1}^{q'} k_i w_i = 0 \,.$$

Since $w_1, ..., w_{q'} > 0$, this implies that $k_1 = ... = k_{q'} = 0$, so the rows of B' have exactly N 1-s.

If singularities are strictly non-removable then there is a domain R_k such that

$$\pi^s \circ \mathcal{T}(\mathcal{T}^t(\hat{M}) \cap R_k) \cap \mathcal{S} \neq \emptyset$$

for all $t \in \mathbb{N}$. Then there is $a \in \Lambda$ and $j'_a \neq j_a$, such that

$$T_{ij}(M_{ij} \cap \pi^s \circ \mathcal{T}^t(\hat{M})) \cap M_{aj'_a} \neq \emptyset \,.$$

for all $t \in \mathbb{N}$. Hence $\mathcal{T}(\mathcal{T}^t(\hat{M}) \cap R_k)$ intersects $R'_1, ...R'_N$ and $[a] \times M_{aj'_a}$ for all $t \in \mathbb{N}$, so the number of 1-s in row k of B' is least $N + 1$. Contradiction. □

Define $M^{(t)} := \pi^s \circ \mathcal{T}^t(\hat{M})$.

Corollary 7. *Suppose that we for all $t \in \mathbb{N}$ have*

$$T_{i_1 j_1}(M^{(t-1)} \cap M_{i_1 j_1}) \cap T^{-1}_{i_3 j_3}(M^{(t+1)} \cap M_{i_3 j_3}) \cap (M^{(t)} \cap M_{i_2 j_2}) \neq \emptyset$$

whenever

$$T_{i_1 j_1}(M^{(t-1)} \cap M_{i_1 j_1}) \cap M_{i_2 j_2} \neq \emptyset$$

and

$$T^{-1}_{i_3 j_3}(M^{(t+1)} \cap M_{i_3 j_3}) \cap M_{i_2 j_2} \neq \emptyset \,.$$

Then $p > \log N$ if B' is transitive and singularities strictly are non-removable.

Proof. Let $X^t_{ij} = M^{(t)} \cap M_{ij}$. Let $R_{k_1} = [i_1] \times M_{i_1 j_1}$, $R_{k_2} = [i_2] \times M_{i_2 j_2}$ and $R_{k_3} = [i_3] \times M_{i_3 j_3}$. If $(B_{t-1})_{k_1 k_2} = (B_t)_{k_2 k_3} = 1$, then

$$T_{i_1 j_1}(X^{t-1}_{i_1 j_1}) \cap X^t_{i_2 j_2} \neq \emptyset \,,$$

and

$$T^{-1}_{i_2 j_2}(X^{t+1}_{i_3 j_3}) \cap X^t_{i_2 j_2} \neq \emptyset \,.$$

By assumption there is $x \in M_{i_1 j_1}$ such that $T_{i_1 j_1}(x) \in M_{i_2 j_2}$ and $T_{i_3 j_3}(T_{i_1 j_1}(x)) \in M_{i_3 j_3}$. Hence the finite legal path $k_1 k_2 k_3$ is realized. By induction all finite legal paths are realized, so $\tilde{p}_n = p_n$ for all $n \in \mathbb{N}$. The statement then follows from proposition 13. □

Next we study topological entropy. Given a metric d on $\hat{M}$ we define

$$d_n(\hat{x},\hat{y}) = \max_{0\le t<n}\{d(\mathcal{T}^t\hat{x},\mathcal{T}^t\hat{y})\}\,.$$

Obviously $d_n(\hat{x},\hat{y}) = d_n(\hat{y},\hat{x})$ for all $\hat{x},\hat{y}\in\hat{M}$, and $d_n(\hat{x},\hat{y}) = 0$ if and only if $\hat{x}=\hat{y}$. Moreover

$$\begin{aligned} d_n(\hat{x},\hat{y}) + d_n(\hat{y},\hat{z}) &= \max_{0\le t<n}\{d(\mathcal{T}^t\hat{x},\mathcal{T}^t\hat{y})\} + \max_{0\le t<n}\{d(\mathcal{T}^t\hat{y},\mathcal{T}^t\hat{z})\} \\ &\ge \max_{0\le t<n}\{d(\mathcal{T}^t\hat{x},\mathcal{T}^t\hat{y}) + d(\mathcal{T}^t\hat{y},\mathcal{T}^t\hat{z})\} \\ &\ge \max_{0\le t<n}\{d(\mathcal{T}^t\hat{x},\mathcal{T}^t\hat{z})\} \\ &= d_n(\hat{x},\hat{z})\,, \end{aligned}$$

so d_n is a metric on $\hat{M}$. Define

$$B_{d_n,\delta}(\hat{x}) = \{\hat{y}\in\hat{M}\,|\,d_n(\hat{x},\hat{y})<\delta\}\,.$$

We say that a set $E\subset\hat{M}$ is (d_n,δ)-spanning if

$$\bigcup_{\hat{x}\in E} B_{d_n,\delta}(\hat{x}) = \hat{M}\,.$$

If there for all $n\in\mathbb{N}$ exists finite (d_n,δ)-spanning set we define $S(d_n,\delta)$ to be the minimal number of points in a (d_n,δ)-spanning set, and we let

$$h_d(\mathcal{T}) = \lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log S(d_n,\delta)\,.$$

If d' is another metric that defines the topology on $\hat{M}$, then d' is continuous on $\hat{M}\times\hat{M}$, so it attains a minimum $\gamma(\delta)$ on the compact set of points $(\hat{x},\hat{y})\in\hat{M}\times\hat{M}$ with $d(\hat{x},\hat{y})\ge\delta$. Hence $d'(\hat{x},\hat{y})<\gamma(\delta)$ whenever $d(\hat{x},\hat{y})<\delta$. If $d_n(\hat{x},\hat{y})<\delta$, then $d(\mathcal{T}^t\hat{x},\mathcal{T}^t\hat{y})<\delta$ for all $t=0,1,...,n-1$, so $d'(\mathcal{T}^t\hat{x},\mathcal{T}^t\hat{y})<\gamma(\delta)$ for all $t=0,1,...,n-1$. Then $d'_n(\hat{x},\hat{y})<\gamma(\delta)$, and

$$B_{d_n,\delta}(\hat{x})\subset B_{d'_n,\gamma(\delta)}(\hat{x})\,.$$

If there is finite (d_n,δ)-spanning set for all $n\in\mathbb{N}$ and $\delta>0$, then there is finite (d'_n,δ)-spanning set for all $n\in\mathbb{N}$ and $\delta>0$, and

$$S(d_n,\delta)\ge S(d'_n,\gamma(\delta))\,,$$

so

$$h_d(\mathcal{T})\ge h_{d'}(\mathcal{T})\,.$$

We can change d and d' to get the opposite inequality. This shows that the existence and value of the number $h_d(\mathcal{T})$ is independent of the choice of metric. We can then define topological entropy as

$$h_{top}(\mathcal{T}) := h_d(\mathcal{T})\,.$$

Proposition 15. *The topological entropy of $\mathcal{T}$ exists, and*

$$h_{top}(\mathcal{T})\le p\,.$$

Proof. Take $P^{(n)}_\alpha\in\mathcal{P}_n$. There is sequence $(i_0,j_0)...(i_n,j_n)$ such that

$$P^{(n)}_\alpha = [i_0...i_n]\times(M_{i_0j_0}\cap T^{-1}_{i_0j_0}(M_{i_1j_1}\cap T^{-1}_{i_1j_1}(...\cap T^{-1}_{i_{n-1}j_{n-1}}(M_{i_{n-1}j_{n-1}}))...)\,.$$

The set

$$\pi^s(P_\alpha^{(n)}) = (M_{i_0j_0} \cap T_{i_0j_0}^{-1}(M_{i_1j_1} \cap T_{i_1j_1}^{-1}(... \cap T_{i_{n-1}j_{n-1}}^{-1}(M_{i_{n-1}j_{n-1}}))...)$$

is the intersection of a finite number of polygons in M, so for all $\delta > 0$ there are points $x_1, ..., x_{m(\alpha,n,\delta)} \in \pi^s(P_\alpha^{(n)})$ such that

$$\bigcup_{l=1}^{m(\alpha,n,\delta)} \{y \in \pi^s(P_\alpha^{(n)}) \,|\, d^s(x_l, y) < \delta\} = \pi^s(P_\alpha^{(n)})\,.$$

Let $\delta = 2^{-k}$, take $\mathbf{a} \in \Sigma_\Lambda$ and define

$$\hat{x}_{b_0,...,b_{k-2},l} = (...a_{-2}a_{-1}i_0...i_nb_0...b_{k-2}a_{n+k-1}..., x_l)\,.$$

We claim that $E_{n,\alpha} := \{\hat{x}_{b_0,...,b_{k-2},l}\}$ is a $(d_n, 2^{-k})$-spanning set for $P_n^{(\alpha)}$. Take $\hat{z} = (\mathbf{c}, z) \in P_n^{(\alpha)}$. Then $c_0 = i_0, ..., c_n = i_n$, and there is l such that $d^s(z, x_l) < 2^{-k}$. Let $b_0 = c_{n+1}, b_1 = c_{n+2}, ..., b_{k-2} = c_{n+k-1}$. Then

$$\begin{aligned}
& d^s(\pi^s \circ \mathcal{T}^{n-1}(\hat{x}_{b_0,...,b_{k-2},l}), \pi^s \circ \mathcal{T}^{n-1}(\hat{z})) \\
= \;& d^s(T_{i_{n-1}j_{n-1}} \circ ... \circ T_{i_0j_0}(x_l), T_{i_{n-1}j_{n-1}} \circ ... \circ T_{i_0j_0}(z)) \\
\leq \;& d^s(T_{i_{n-2}j_{n-1}} \circ ... \circ T_{i_0j_0}(x_l), T_{i_{n-2}j_{n-1}} \circ ... \circ T_{i_0j_0}(z)) \\
= \;& d^s(\pi^s \circ \mathcal{T}^{n-2}(\hat{x}_{b_0,...,b_{k-2},l}), \pi^s \circ \mathcal{T}^{n-2}(\hat{z})) \\
\leq \;& ... \\
= \;& d^s(x_l, z) \\
< \;& 2^{-k}\,,
\end{aligned}$$

and

$$\begin{aligned}
2^{-k} > d^u(\sigma_\Lambda^{n-1}\pi^u(\hat{x}_{b_0,...,b_{k-2},l}), \sigma_\Lambda^{n-1}\mathbf{c}) \;&\geq\; d^u(\sigma_\Lambda^{n-2}\pi^u(\hat{x}_{b_0,...,b_{k-2},l}), \sigma_\Lambda^{n-2}\mathbf{c}) \\
&\geq\; ... \\
&\geq\; d^u(\pi^u(\hat{x}_{b_0,...,b_{k-2},l}), \mathbf{c})\,.
\end{aligned}$$

Then $d_n(\hat{x}_{b_0,...,b_{k-2},l}, \hat{z}) < 2^{-k}$. We now have a $(d_n, 2^{-k})$-spanning set

$$E_n = \bigcup_{\alpha=1}^{p_n} E_{n,\alpha}\,,$$

so

$$S(d_n, 2^{-k}) \leq N^{k-1} \sum_{\alpha=1}^{p_n} m(\alpha, m, 2^{-k})\,.$$

Let $m(\alpha, n, \delta)$ be the minimal number of δ-balls that is needed to cover $\pi^s(P_\alpha^{(n)})$. Define $m(n, \delta) = \max_\alpha\{m(\alpha, n, \delta)\}$. Then we see that

$$S(d_n, 2^{-k}) \leq p_n N^{k-1} \cdot m(n, 2^{-k})\,,$$

so

$$h_{top}(\mathcal{T}) \leq p + \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log m(n, \delta)\,.$$

Let $P \subset M$ be a polygon. For each $\delta > 0$ there is constant $C_\delta > 0$ such that M can be covered with $C_\delta\delta^{-N}$ d^s-balls of radius $\delta > 0$. If one of these balls intersects P, but does not have center in P, we can cover the intersection with P with a ball of radius 2δ, and center in P. Hence

$$m(n, 2\delta) \leq C_\delta\delta^{-N}\,.$$

This implies that

$$\limsup_{n\to\infty} \frac{1}{n} \log m(n,\delta) = 0\,,$$

and thus $h_{top}(\mathcal{T}) \leq p$. □

Proposition 16. *$h_{top}(\mathcal{T}) \geq \log N$, and $h_{top}(\mathcal{T}) = \log N$ if singularities are removable.*

Proof. It is known that the map $\sigma_\Lambda : \Sigma_\Lambda \to \Sigma_\Lambda$ has topological entropy equal to $\log N$. We have a surjective continuous map $\pi^u : \hat{M} \to \Sigma_\Lambda$ such that $\pi^u \circ \mathcal{T} = \sigma_\Lambda \circ \pi^u$. Hence σ_Λ is a topological factor of $\mathcal{T}$. Then it follows that

$$\log N = h_{top}(\sigma_\Lambda) \leq h_{top}(\mathcal{T})\,.$$

Suppose that singularities are removable. Then there is $n \in \mathbb{N}$ such that $\pi^s \circ \mathcal{T}^n(\hat{M})$ is a disjoined union of connected components $X_1, ..., X_s$, where for each $\nu \in \{1, ..., s\}$ and each $i \in \Lambda$ there is $j \in \{1, ..., q_i\}$ such that $X_\nu \subset M_{ij}$. Take $\hat{x} \in \hat{M}$ and $\hat{y} \in B_{d_n, 2^{-k}}(\hat{x})$. Let $\gamma > 0$ be the minimal distance between two different connected components. For $2^{-k} < \gamma$ we have $\pi^s \circ \mathcal{T}^{n-1}(\hat{x})$ and $\pi^s \circ \mathcal{T}^{n-1}(\hat{y})$ in the same connected component X_ν. Let $\hat{x} = (\mathbf{a}, x)$ and $\hat{y} = (\mathbf{b}, y)$. We have $a_t = b_t$ for $t = 0, ..., n+k-1$. There is unique j such that $X_\nu \subset M_{a_n j}$. Then

$$\begin{aligned} d^s(\pi^s \circ \mathcal{T}^n(\hat{x}), \pi^s \circ \mathcal{T}^n(\hat{y})) &= d^s(T_{a_n j}(\pi^s \circ \mathcal{T}^{n-1}(\hat{x})), T_{a_n j}(\pi^s \circ \mathcal{T}^{n-1}(\hat{y}))) \\ &\leq d^s(\pi^s \circ \mathcal{T}^{n-1}(\hat{x}), \pi^s \circ \mathcal{T}^{n-1}(\hat{y})) \\ &< 2^{-k}\,. \end{aligned}$$

Define $\hat{x}_j = (...a_{-1}a_0a_1...a_{n+k-1}ja_{n+k+1}..., x)$. For any $\hat{y} \in B_{d_n, 2^{-k}}(\hat{x})$ there is $\hat{x}_j$ such that

$$d^u(\pi^u \circ \mathcal{T}^n(\hat{x}_j), \pi^u \circ \mathcal{T}^n(\hat{y})) < 2^{-k}\,.$$

Hence $\{\hat{x}_j\}$ is a $(d_{n+1}, 2^{-k})$-spanning set for the ball $B_{d_n, 2^{-k}}(\hat{x})$. By induction we see that

$$S(d_t, 2^{-k}) \leq N S(d_{t-1}, 2^{-k})$$

for $2^{-k} < \gamma$ and $t > n$. Then $S(d_t, 2^{-k}) \leq N^{t-n} S(d_n, 2^{-k})$, so

$$h_{top}(\mathcal{T}) \leq \lim_{k\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log(N^{t-n} S(d_n, 2^{-k})) = \log N\,.$$

□

Lemma 9. *Take domains M_{ik} and M_{il}, $k \neq l$. If $T_{ik}(z) \neq T_{il}(z)$ for all $z \in \partial M_{ik} \cap \partial M_{il}$, then there exists γ such that for all $\delta \in (0, \gamma]$ we have*

$$d^s(x,y) < \delta \Rightarrow d^s(T_{ik}(x), T_{il}(y)) > \delta\,,$$

whenever $x \in M_{ik}$ and $y \in M_{il}$.

Proof. Suppose that for all $\delta > 0$ there exist $x \in M_{ik}$ and $y \in M_{il}$ such that $d^s(x,y) < \delta$ and $d^s(T_{ik}(x), T_{il}(y)) \leq \delta$. There exists a sequence $\{x_m\} \subset M_{ik}$ such that for each m there is $y_m \in M_{il}$ with $d^s(x_m, y_m) < 2^{-m}$ and $d^s(T_{ik}(x_m), T_{il}(y_m)) \leq 2^{-m}$. The sequence has a convergent subsequence $\{x_{m_n}\}$. Let the limit of this subsequence be $z \in \bar{M}_{ik}$. The point z lies on the boundary between M_{ik} and M_{il}. We have $y_{m_n} \to z$, and by continuity of the maps T_{ik} and T_{il} we have $T_{ik}(x_{m_n}) \to T_{ik}(z)$ and $T_{il}(y_{m_n}) \to T_{il}(z)$. Since the metric d^s is continuous on $M \times M$ we have

$$d^s(T_{ik}(z), T_{il}(z)) = \lim_{n\to\infty} d^s(T_{ik}(x_{m_n}), T_{ik}(y_{m_n})) = 0\,.$$

So $T_{ik}(z) = T_{il}(z)$. Contradiction. □

Theorem 4. *Let $T_{ik}(z) \neq T_{il}(z)$ for all $i \in \Lambda$, $k \neq l$ and $z \in \partial M_{ik} \cap \partial M_{il}$. Then $h_{top}(\mathcal{T}) = p$.*

Proof. Let $A, B \in \mathcal{P}_n$. Take $\hat{x} = (\mathbf{a}, x) \in A$ and $\hat{y} = (\mathbf{b}, y) \in B$. If $[\mathbf{a}]_n \neq [\mathbf{b}]_n$ then $d_{n+1}(\hat{x}, \hat{y}) \geq 2^{-1}$. If $A \neq B$, and $[\mathbf{a}]_n = [\mathbf{b}]_n$, there is $t \leq n$ such that $\pi^s \circ \mathcal{T}^t(\hat{x}) \in M_{a_t k}$ and $\pi^s \circ \mathcal{T}^t(\hat{y}) \in M_{a_t l}$, where $k \neq l$. By lemma 9 there is γ such that for all $\xi < \gamma$ we have

$$\max\{d^s(\pi^s \circ \mathcal{T}^t(\hat{x}), \pi^s \circ \mathcal{T}^t(\hat{y})), d^s(\pi^s \circ \mathcal{T}^{t+1}(\hat{x}), \pi^s \circ \mathcal{T}^{t+1}(\hat{y}))\} \geq \xi\,.$$

For $\delta < \min\{\gamma/2, 1/4\}$ no d_{n+1} ball of radius δ can contain points of both A and B. This implies that $S(d_{n+1}, \delta) \geq p_n$. Then $h_{top}(\mathcal{T}) \geq p$, and the statement follows from proposition 15. □

Corollary 8. *$h_{top}(\mathcal{T}) > \log N$ if the conditions of corollary 7 and theorem 4 are satisfied.*

Chapter 7

Physical Implications and Concluding Remarks

The notion of SOC consists of two separate concepts; *self organization* and *criticality*. Criticality here is used in the same sense as in equilibrium statistical physics, where criticality is characterized by the fact that perturbations in the configuration of the system will generate effects that propagate through the entire system. The behavior of an equilibrium system at a critical point is quite different from the behavior at a non-critical point. Here a perturbation will generate only a local effect. Let $n(\mathbf{r})$ be a field describing the system. Then the characteristic correlation length is given by the rate of convergence of the correlation function

$$G(\mathbf{r}) = \langle n(\mathbf{r}_0 + \mathbf{r}) n(\mathbf{r}_0) \rangle_{\mathbf{r}_0} - \langle n(\mathbf{r}_0) \rangle^2_{\mathbf{r}_0} \,,$$

to zero. At non-critical points the decay is exponential, i.e. $G(\mathbf{r}) \sim \exp(-r/\xi)$. The number ξ is the characteristic correlation length. However if the temperature approaches the critical temperature T_c, the correlation length diverges like $\xi \sim |T - T_c|^\nu$. At the critical point the correlation decay is no longer exponential but algebraic. That is

$$G(\mathbf{r}) \sim \frac{1}{r^\alpha} \,.$$

The divergence of the characteristic correlation length is taken as an indication that a perturbation of the system effects the system not only locally. For non-equilibrium systems it is also natural to consider time-correlation functions. For a time signal $n(t)$ this is defined as

$$C(t) = \langle n(t_0 + t) n(t_0) \rangle_{t_0} - \langle n(t_0) \rangle^2_{t_0} \,.$$

In systems that exhibit self-organized criticality one expects algebraic decay of both space-correlations, and of the time-correlation functions. In fact SOC was originally proposed as a mechanism of $1/f$-noise. Given the correlation function one can define the power spectrum of the time signal $n(t)$ by

$$S(f) = \lim_{T \to \infty} \frac{1}{T} \Big| \int_{-T}^{T} n(t) \exp(2\pi i f t) \, dt \Big|^2$$

and show that it is related to the time-correlation function of $n(t)$ by [5]:

$$S(f) = 2 \int_0^\infty C(t) \cos(2\pi f t) \, dt \,.$$

Hence we see that if we have $C(t) \sim t^{-\alpha}$, then $S(f) \sim f^{-\alpha - 1}$. By $1/f$-noise one means that $S(f) \sim f^{-\beta}$, where $\beta \simeq 1$. We see that this implies a very slow algebraic correlation decay. In the limit $\beta = 1$ the correlation function becomes a non-zero constant.

It is stated in proposition 8 that if singularities are removable and assumptions 1 and 2 hold, then we have exponential decay of correlation, and hence we have non-critical states in these situations. However, this result holds only for finite values of N. Even if there are values of E_c and ϵ such that proposition 8 holds for each N, the rate of decay may tend to 1 as $N \to \infty$. We must then ask if such a limit of non-critical states can result in a SOC-state, with phenomena such as $1/f$-noise, or is non-removability of singularities a necessary condition for SOC? We will show that the answer to this question is given by the structures of the transition matrices A defined in chapter 3.

In example B the correlation function of any avalanche observable ϕ is of the from

$$C_\phi(t) = a(\phi)(\lambda_2^t + \bar{\lambda}_2^t) + b(\phi)(\lambda_4^t + \bar{\lambda}_4^t) + c(\phi)(\lambda_6^t + \bar{\lambda}_6^t)\,.$$

Let $r_k = |\lambda_k|$, and $\theta_k = \arg(\lambda_k)$. Then the contribution to the power spectrum from the mode $\lambda_k^t + \bar{\lambda}_k^t$ becomes

$$\begin{aligned} S_k(f) &= 4\int_0^\infty r_k^t \cos(\theta_k t)\cos(2\pi f t)\,dt \\ &= \frac{-2\Big(4\pi^2 f^2 + (\log r_k)^2 + \theta_k^2\Big)\log r_k}{\Big((\log r_k)^2 + (\theta_k + 2\pi f)^2\Big)\Big((\log r_k)^2 + (\theta_k - 2\pi f)^2\Big)}\,. \end{aligned}$$

Hence, for a given observable we can compute the constants $a(\phi)$, $b(\phi)$ and $c(\phi)$, and determine the three terms of the power spectrum. We see that $S_k(f)$ is approximately constant for $f << |\log r_k|$, and that

$$S_k(f) \sim \frac{1}{f^2}\,,$$

for $f >> |\log r_k|$. In numerical simulations of various SOC-models it has been observed that power spectra become constant bellow a critical frequency f_c. In example B we have such a critical frequency $f_c = -\log|\lambda_2|$.

As $N \to \infty$, the number of different eigenvalues may go to infinity. We assume transitivity, so there is one eigenvalue $\lambda_1 = 1$ on the unit circle. Let $\mathrm{Sp}(\Pi)^* := \mathrm{Sp}(\Pi) - \{0, 1\}$. Assume for simplicity that all contributing eigenvalues of $\mathrm{Sp}(\Pi)^*$ are simple. We then have the same situation as in example B, and the correlation functions are of the form

$$C_\phi(t) = \sum_k a_k(\phi)\frac{\lambda_k^t + \bar{\lambda}_k^t}{2}\,.$$

For each N and each avalanche observable ϕ we define

$$\rho_\phi(r, \theta; \Delta r, \Delta\theta, N) = \frac{1}{\Delta r \Delta\theta} \sum_{\lambda_k \in \mathrm{Sp}(\Pi)^* \cap \Delta(r,\theta)} a_k(\phi)\,,$$

where

$$\Delta(r, \theta) = \{\lambda \in \mathbb{C} \,|\, r \le |\lambda| \le r + \Delta r\,,\ \theta \le \arg(\lambda) \le \theta + \Delta\theta\}\,.$$

Suppose there exists a density ρ_ϕ such that we have uniform convergence

$$\lim_{\substack{\Delta r, \Delta\theta \to 0 \\ N \to \infty}} \rho_\phi(r, \theta; \Delta r, \Delta\theta, N) = \rho_\phi(r, \theta)\,,$$

i.e.

$$\lim_{n\to\infty} \sup_{0\le r\le 1, 0\le\theta\le 2\pi} |\rho_\phi(r, \theta; \Delta r_n, \Delta\theta_n, N_n) - \rho_\phi(r, \theta)| = 0\,,$$

for any decreasing sequences $\Delta\theta_n, \Delta r_n \to 0$ and any increasing sequence $N_n \to \infty$. Then the power spectrum of of the observable ϕ in the thermodynamic limit can be written as

$$S(f) = \int_0^\pi \int_0^1 \frac{-\Big(4\pi^2 f^2 + (\log r)^2 + \theta^2\Big)\log r}{\Big((\log r)^2 + (\theta + 2\pi f)^2\Big)\Big((\log r)^2 + (\theta - 2\pi f)^2\Big)} \rho_\phi(r,\theta) dr d\theta \,.$$

We see that the density ρ_ϕ determines if $1/f$-noise can be found in the thermodynamic limit. This density is completely given by the matrix A and the observable ϕ, but not enough is known about the structure of A for large values of N to give any estimation of ρ_ϕ.

Construction of power spectra that scale like $1/f$ through superposition of exponentially decaying modes is an old technique introduced by van der Ziel in 1950 [5]. It was attempted for the sandpile model of Bak, Tang and Wiesenfeld in [4]. However, the modes in [4] are thought of as corresponding to avalanches, and one can then by heuristic arguments construct a density that give $S(f) \sim 1/f$. One enumerates the possible avalanches $A_1, A_2, ...$, and define $n_k(t)$ to be the number of overcritical sites in time t of avalanche k. One then assumes that the correlation function of each signal $n_k(t)$ is exponential;

$$C_{n_k}(t) = \frac{A_k}{T_k} \exp(-t/T_k) \,,$$

where

$$A_k = \int C_{n_k}(t)\, dt$$

is claimed to be proportional to the avalanche size s_k, and the characteristic correlation length T_k is claimed to be proportional to the avalanche duration τ_k. One then assumes that the avalanches occur independently of each other, so that the correlation function of the observable

$$n(t) = \sum_k n_k(t)$$

can be written as

$$C_n(t) = \sum_k P(A_k) \frac{A_k}{T_k} \exp(-t/T_k) \,,$$

where $P(A_k)$ is the probability of avalanche A_k. This gives a power spectrum

$$S(f) = \sum_k \frac{A_k P(A_k)}{1 + 4\pi^2 f^2 T_k^2}$$

By replacing A_k with s_k and T_k with τ_k, and by assuming that an avalanche is completely determined by its size and duration, Bak, Tang and Wiesenfeld obtained the approximation

$$S(f) = \int \frac{sP(s,\tau)}{1 + 4\pi^2 f^2 \tau^2}\, ds d\tau = \int \frac{\tau D(\tau)}{1 + 4\pi^2 f^2 \tau^2}\, d\tau \,,$$

where

$$D(\tau) = \int \frac{s}{\tau} P(s,\tau)\, ds \,.$$

The density $D(\tau)$ is called the distribution of life-times in the SOC-litterature. Numerically one finds that $D(\tau) \sim \tau^{-\alpha}$, in which case the power spectrum becomes

$$S(f) \sim \int \frac{\tau^{-\alpha+1}}{1 + 4\pi^2 f^2 \tau^2}\, d\tau = \frac{(2\pi)^{\alpha-1}}{4\sin(\alpha\pi/2)} \frac{1}{f^{2-\alpha}} \,.$$

Hence we have $1/f$-noise if $\alpha \simeq 1$.

Many objections have been raised concerning these arguments of Bak, Tang and Wiesenfeld. In [5] it is shown by a different method, that what is claimed to be $1/f$-noise in [4], actually is $1/f^2$-noise. The main objection is that avalanches should not be represented as exponentially decaying modes in the correlation function, but rather one should consider each avalanche as a "boxed" signal with finite support, and make the superposition of these signals prior to computing the correlation function.

However, if singularities are removable and assumptions 1 and 2 holds for physically interesting situations, then the work presented in this paper shows that the correlation functions of the Zhang model consists of exponentially decaying modes, and the original approach of Bak, Tang and Wiesenfeld hence becomes of interest. Indeed we see from their work that we can obtain a power law structure $S(f) \sim f^{-\beta}$ if the weighted density of eigenvalues is supported on the real axis, where it has the form

$$\rho(r,\theta) \sim \frac{|\log r|^{1-\beta}}{r}\delta(\theta)\,.$$

If $\beta \simeq 1$ we have $1/f$-noise.

In both [4] and [5] the method of constructing the power spectrum involved the assumption that avalanches can be considered as independent. The results of this paper show that this is not true. The invariant measure introduced on Ω_A, is not of Bernoulli type, but of Markov type, where the correlations between avalanches are given by the matrix A. By taking correlations between avalanches into account, we can no longer interpret the exponentially decaying modes of the correlation function in terms of avalanches. However, we can in principle calculate the density ρ_ϕ if we can preform symbolic coding.

To construct symbolic coding for great values of N seems to be an enormous computational task. Hence it is difficult to use the method described above to determine if we can get $1/f$ power spectra in the cases where symbolic coding is possible. Another approach is to look to the existing numerical work on the Zhang model, and to try to determine if singularities are removable for those values of parameters where critical behavior is found. Unfortunately the existing work is for a slightly different version of the model. In the original model critical energy is $E_c = 1$, $\epsilon = 0$ and the excitation energy δ (this is set to 1 in our model) is chosen from the interval $(0, 1/2)$. This corresponds to $E_c > 2$ in our model. In [8] δ is chosen from $(0, 1/4)$, which corresponds to $E_c > 4$ in our model. However, in numerical simulations [8] δ is not fixed, but chosen randomly from the interval $(0, 1/4)$. Hence the results from this study are not directly comparable with the findings of this paper. We note that the average activation energy is $\overline{\delta} = 1/8$, which corresponds to $E_c = 8$, in our study. This is a greater value of critical energy than in examples A and B, where we can preform symbolic coding. The result of the study is that time correlation of the total energy in the system decays like $\sim t^{-0.28}$ for small t, and like $t^{-0.26}$ for large t. Hence the power spectrum scales like $f^{-1.28}$ for high frequencies and like $f^{-1.26}$ for low frequencies.

The physical reasons for choosing large values of E_c, is that one achieves a slow driving of the system, i.e. the average number of excitations needed to bring stable configurations above the threshold, is much greater than the average avalanche duration. It is clear that if $E_c \geq 1$ and the physical attractor Y intersects the set $[0, E_c - 1)^N$, then the driving becomes infinitely slow as $N \to \infty$. In example A we see that no configuration on the physical attractor can be stable for more than two time-steps, and in example B we see that no configuration can be stable for more than 1 time step. In example C however, the attractor seems to intersect the set $[0, E_c - 1)^2$. If this property holds as one increases N, it means that configurations can be stable for $N + 1$ time-steps, giving an infinitely slow rate of driving in the thermodynamic limit. It has been argued by [13] and [14] that the models of SOC actually require a fine-tuning to reach the critical state, namely a fine-tuning of the driving rate to zero. In order to determine if this should be considered as fine-tuning, one must identify the parameters that control the rate of driving. Hence one must study physical attractor in relation to the parameter E_c. If there exists an open set of critical

energies for which $Y \cap [0, E_c - 1)^N \neq \emptyset$ for all N, then a slow rate of driving is obtained without fine-tuning of parameters.

The power-law statistics of SOC-models is believed to be related to fractal structures of the attractors of the models. In our work we have shown that if the singularities are removable and assumption 1 holds, the attractor $\mathcal{A}$ is homeomorphic to the set Ω_A, which is homeomorphic to a Cantor set. However the physical attractor Y does not have fractal properties for all values of E_c and ϵ. Example A even shows a situation where this set is finite. In example B the transition matrix has more than 2 1-s in each column, and so the physical attractor may have a Cantor set structure. In example C it is seen clearly that the physical attractor has fractal structures. The study of the fractal properties of SOC-models has previously focused mainly on the phenomena of so called clustering of overcritical sites on the lattice Λ, and little is known about the properties of the phase-space attractor. In [3] one attempted to introduce local Hausdorff dimensions and relate these to dynamical properties through the Ledrappier-Young formula. However it is not clear how these local dimensions are defined since the measure used is not described. Hence the relation of the fractal properties of the physical attractor to the dynamics is not yet understood.

It is observed both in [3] and in this work, that the most complex and physically interesting dynamics is found for large values of critical energy. It is also in these cases that singularities seem to be non-removable. Further work on the Zhang model should be concerned with these cases, and the challenge is to develop a theory that holds when singularities are not-removable. Moreover the theory developed in this paper should be completed by giving conditions for when assumptions 1 and 2 hold. A proof of assumption 2 will require a better understanding of the transition matrices. These matrices are closely linked to the topological properties of the dynamics, and some structure of the matrices should be found from what is already known about the dynamical properties. Moreover it is of both physical and mathematical interest to search for bifurcation values of the parameters E_c and ϵ, and to determine how fractal properties and topological properties, such as removability of singularities, are related to these parameters.

Chapter 8

Appendix

The affine maps T_{1j} of example B are

$$T_{1j}\Big(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Big) = L_{1j}\begin{bmatrix} x_1 + 1 \\ x_2 \end{bmatrix},$$

where

$$L_{1,1} = \begin{bmatrix} 7/32 & 1/4 \\ 13/64 & 3/8 \end{bmatrix} \qquad L_{1,2} = \begin{bmatrix} 89/512 & 15/64 \\ 33/256 & 7/32 \end{bmatrix}$$

$$L_{1,3} = \begin{bmatrix} 89/1024 & 15/128 \\ 221/1024 & 17/64 \end{bmatrix} \qquad L_{1,4} = \begin{bmatrix} 577/4096 & 47/256 \\ 221/2048 & 17/128 \end{bmatrix}$$

$$L_{1,5} = \begin{bmatrix} 69/256 & 11/32 \\ 13/128 & 3/16 \end{bmatrix} \qquad L_{1,6} = \begin{bmatrix} 69/512 & 11/64 \\ 173/1024 & 35/128 \end{bmatrix}$$

$$L_{1,7} = \begin{bmatrix} 725/4096 & 141/512 \\ 173/2048 & 35/256 \end{bmatrix} \quad L_{1,8} = \begin{bmatrix} 725/8192 & 141/1024 \\ 2109/16384 & 281/1024 \end{bmatrix}$$

$$L_{1,9} = \begin{bmatrix} 41/256 & 7/32 \\ 5/32 & 1/4 \end{bmatrix} \qquad L_{1,10} = \begin{bmatrix} 41/512 & 7/64 \\ 201/1024 & 39/128 \end{bmatrix}$$

$$L_{1,11} = \begin{bmatrix} 529/4096 & 95/512 \\ 201/2048 & 39/128 \end{bmatrix} \qquad L_{1,12} = \begin{bmatrix} 51/256 & 9/32 \\ 5/64 & 1/8 \end{bmatrix}$$

$$L_{1,13} = \begin{bmatrix} 51/512 & 9/64 \\ 131/1024 & 25/128 \end{bmatrix} \qquad L_{1,14} = \begin{bmatrix} 61/512 & 11/64 \\ 121/1024 & 23/128 \end{bmatrix}$$

Let

$$B^{-}_{1,j}(k) = \{x \in M \,|\, (T_{1j}x)_k \leq 1/4\},$$

and

$$B^{+}_{1,j}(k) = \{x \in M \,|\, (T_{1j}x)_k > 1/4\},$$

for $k = 1, 2$. Then the domains of continuity for the map T_1 are

$$\begin{aligned}
M_{1,1} &= B^-_{1,1}(1) \cap B^-_{1,1}(2) \\
M_{1,3} &= B^+_{1,1}(1) \cap B^-_{1,1}(2) \cap B^-_{1,2}(1) \\
M_{1,4} &= B^+_{1,1}(1) \cap B^-_{1,1}(2) \cap B^+_{1,2}(1) \cap B^-_{1,2}(2) \\
M_{1,5} &= B^+_{1,1}(1) \cap B^-_{1,1}(2) \cap B^+_{1,2}(1) \cap B^+_{1,2}(2) \\
M_{1,6} &= B^-_{1,1}(1) \cap B^+_{1,1}(2) \cap B^-_{1,5}(1) \\
M_{1,7} &= B^-_{1,1}(1) \cap B^+_{1,1}(2) \cap B^+_{1,5}(1) \cap B^-_{1,5}(2) \\
M_{1,8} &= B^-_{1,1}(1) \cap B^+_{1,1}(2) \cap B^+_{1,5}(1) \cap B^+_{1,5}(2) \cap B^-_{1,7}(1) \\
M_{1,9} &= B^-_{1,1}(1) \cap B^+_{1,1}(2) \cap B^+_{1,5}(1) \cap B^+_{1,5}(2) \cap B^+_{1,7}(1) \\
M_{1,10} &= B^+_{1,1}(1) \cap B^+_{1,1}(2) \cap B^-_{1,9}(1) \cap B^-_{1,8}(2) \\
M_{1,11} &= B^+_{1,1}(1) \cap B^+_{1,1}(2) \cap B^+_{1,9}(1) \cap B^-_{1,8}(2) \cap B^-_{1,10}(2) \\
M_{1,12} &= B^+_{1,1}(1) \cap B^+_{1,1}(2) \cap B^+_{1,9}(1) \cap B^-_{1,8}(2) \cap B^+_{1,10}(2) \\
M_{1,13} &= B^+_{1,1}(1) \cap B^+_{1,1}(2) \cap B^-_{1,9}(1) \cap B^+_{1,8}(2) \cap B^-_{1,12}(1) \\
M_{1,14} &= B^+_{1,1}(1) \cap B^+_{1,1}(2) \cap B^-_{1,9}(1) \cap B^+_{1,8}(2) \cap B^+_{1,12}(1)
\end{aligned}$$

The maps T_{2j} and the domains M_{2j} can can be computed from symmetry.

The affine maps T_{1j} of example C are

$$T_{1j}\Big(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Big) = L_{1j} \begin{bmatrix} x_1 + 1 \\ x_2 \end{bmatrix} ,$$

where

$$L_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad L_{12} = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1 \end{bmatrix} \quad L_{13} = \begin{bmatrix} 9/16 & 1/4 \\ 1/8 & 1/2 \end{bmatrix} .$$

The corresponding domains of continuity are:

$$\begin{aligned}
M_{11} &= \{x \in M \,|\, x_1 \leq 4\} \\
M_{12} &= \{x \in M \,|\, x_1 > 4, x_2 + (x_1 + 1)/4 \leq 5\} \\
M_{13} &= \{x \in M \,|\, x_1 > 4, x_2 + (x_1 + 1)/4 > 5\} .
\end{aligned}$$

The maps T_{2j} and the domains M_{2j} can be found by symmetry.

The following shows the computer program (Mathematica worksheet) used to iterate the phase-space in our examples. The program given here is for example C, but it can easily be modified for examples A and B.

```
skjæringspunkt[L1_,L2_]:=(
  A={{L1[[2]][[1]]-L1[[1]][[1]],L2[[1]][[1]]-L2[[2]][[1]]},
    {L1[[2]][[2]]-L1[[1]][[2]],L2[[1]][[2]]-L2[[2]][[2]]}};
  b={L2[[1]][[1]]-L1[[1]][[1]],L2[[1]][[2]]-L1[[1]][[2]]};
  If[Det[A]==0, Return[{-1,-1}],
  løs=Inverse[A].b;
 ];
  If[løs[[1]]<0 || løs[[1]]>1  || løs[[2]]<0 || løs[[2]]>1, Return[{-1,-1}],
  Return[(1-løs[[1]])*L1[[1]]+løs[[1]]*L1[[2]]]
 ];
 )
  polyskjærninger1[pol_]:=(
  punliste={};
  For[t=2, t≤ Length[pol],
  punkt=skjæringspunkt[{pol[[t-1]],pol[[t]]},{{4,0},{4,5}}];
  If[(!punkt=={-1,-1}) && FreeQ[pol,punkt],punliste=Append[punliste,{punkt,t-1}];
 ];
  ;t++
 ];
  punkt=skjæringspunkt[{pol[[Length[pol]]],pol[[1]]},{{4,0},{4,5}}];
  If[(!punkt=={-1,-1}),punliste=Append[punliste,{punkt,Length[pol]}];
 ];
  If[Length[punliste]==1,
  For[t=1, t≤ Length[pol],
  If[pol[[t]][[1]]==4,
  punliste=Append[punliste,{pol[[t]],t-1}]
 ];
  ;t++
 ];
  If[punliste[[1]][[2]]>punliste[[2]][[2]],
  punliste={punliste[[2]],punliste[[1]]};
 ];
 ];
  Clear[punkt,t];
  Return[punliste]
 )
  polyskjærninger2[pol_]:=(
  punliste={};
  For[t=2, t≤ Length[pol],
  punkt=skjæringspunkt[{pol[[t-1]],pol[[t]]},{{4,15/4},{5,7/2}}];
  If[(!punkt=={-1,-1}) && FreeQ[pol,punkt],punliste=Append[punliste,{punkt,t-1}];
 ];
  ;t++
 ];
  punkt=skjæringspunkt[{pol[[Length[pol]]],pol[[1]]},{{4,15/4},{5,7/2}}];
  If[(!punkt=={-1,-1}),punliste=Append[punliste,{punkt,Length[pol]}];
 ];

  If[Length[punliste]==1,
  For[t=1, t≤ Length[pol],
  If[addresse[pol[[t]]][[2]]+addresse[pol[[t]]][[3]]==2,
  punliste=Append[punliste,{pol[[t]],t-1}];
 ];
  ;t++
 ];
```

```
If[punliste[[1]][[2]]>punliste[[2]][[2]],
  punliste={punliste[[2]],punliste[[1]]};
 ];
 ];

  Clear[punkt,t];
  Return[punliste]
 )
  delpolyen[varpol_]:=(
  If[Length[polyskjærninger1[varpol]]==0,
  Return[{varpol}]
 ];
  varpolen={};
  For[r=1, r≤  polyskjærninger1[varpol][[1]][[2]],
  varpolen=Append[varpolen, varpol[[r]]];
  ;r++
 ];
  varpolen=Append[varpolen,polyskjærninger1[varpol][[1]][[1]]];
  varpolen=Append[varpolen,polyskjærninger1[varpol][[2]][[1]]];
  For[r=polyskjærninger1[varpol][[2]][[2]]+1, r≤ Length[varpol],
  varpolen=Append[varpolen, varpol[[r]]];
  ;r++
 ];
  varpolto={polyskjærninger1[varpol][[1]][[1]]};
  For[r=polyskjærninger1[varpol][[1]][[2]]+1 , r≤ polyskjærninger1[varpol][[2]][[2]],
  varpolto=Append[varpolto,varpol[[r]]];
  ;r++
 ];
  varpolto=Append[varpolto,polyskjærninger1[varpol][[2]][[1]]];
  Return[{varpolen, varpolto}]
 )
  delpolyto[varpol_]:=(
  If[Length[polyskjærninger2[varpol]]==0, Return[{varpol}]];
  varpolen={};
  For[r=1, r≤  polyskjærninger2[varpol][[1]][[2]],
  varpolen=Append[varpolen, varpol[[r]]];
  ;r++
 ];
  varpolen=Append[varpolen,polyskjærninger2[varpol][[1]][[1]]];
  varpolen=Append[varpolen,polyskjærninger2[varpol][[2]][[1]]];
  For[r=polyskjærninger2[varpol][[2]][[2]]+1, r≤ Length[varpol],
  varpolen=Append[varpolen, varpol[[r]]];
  ;r++
 ];
  varpolto={polyskjærninger2[varpol][[1]][[1]]};
  For[r=polyskjærninger2[varpol][[1]][[2]]+1 , r≤ polyskjærninger2[varpol][[2]][[2]],
  varpolto=Append[varpolto,varpol[[r]]];
  ;r++
 ];
  varpolto=Append[varpolto,polyskjærninger2[varpol][[2]][[1]]];
  Return[{varpolen, varpolto}]
 )
  addresse[p_]:=(
  a1=0; b1=0; a2=0; b2=0; a3=0; b3=0;
  If[p[[1]]≤ 4, a1=1];
  If[p[[1]]≥4 && p[[2]]≤ -(p[[1]]-5)/4+7/2, a2=1];
  If[p[[1]]≥4 && p[[2]]≥-(p[[1]]-5)/4+7/2, a3=1];
  Return[{a1,a2,a3}])
  polyaddresse[pol_]:=(
  If[Sum[addresse[pol[[q]]][[1]], {q,1,Length[pol]}]==Length[pol],
  Return[1]
 ];
  If[Sum[addresse[pol[[q]]][[2]], {q,1,Length[pol]}]==Length[pol],
  Return[2]
 ];
```

```
If[Sum[addresse[pol[[q]]][[3]], {q,1,Length[pol]}]==Length[pol],
  Return[3]
 ];
  r1=Sum[addresse[pol[[q]]][[1]], {q,1,Length[pol]}];
  r2=Sum[addresse[pol[[q]]][[2]], {q,1,Length[pol]}];
  r3=Sum[addresse[pol[[q]]][[3]], {q,1,Length[pol]}];
  If[Max[{r1,r2,r3}]==r1, Return[1]];
  If[Max[{r1,r2,r3}]==r2, Return[2]];
  If[Max[{r1,r2,r3}]==r3, Return[3]];
 )
  F={Function[X,{X[[1]]+1,X[[2]]}],
  Function[X,{(X[[1]]+1)/2,X[[2]]+(X[[1]]+1)/4}],
  Function[X,{1/2*(X[[2]]+1/4*(X[[1]]+1)),(X[[1]]+1)/2+1/4*(X[[2]]+(X[[1]]+1)/4)}]};
  punktbytt[X_]:=(
  Return[{X[[2]],X[[1]]}]
 ) punktlistebytt[punktliste_]:=(
  m=Length[punktliste];
  M={};
  For[i=1, i≤ m,
  M=Append[M,punktbytt[punktliste[[i]]]];
  ;i++
 ];
  Clear[i];
  Return[M]
 ) linjelistebytt[var_]:=(
  P={};
  For[j=1, j≤ Length[var],
  P=Append[P,punktlistebytt[var[[j]]]];
  ;j++
 ];
  Return[P]
 )
  polyiterer[PLNULL_]:=(
  PLTO={};
  For[rr=1, rr≤ Length[PLNULL],
  For[ss=1, ss≤ Length[delpolyen[PLNULL[[rr]]]],
  PLTO=Append[PLTO,delpolyen[PLNULL[[rr]]][[ss]]];
  ;ss++
 ];
  ;rr++
 ];
  PLTRE={};
  For[rr=1, rr≤ Length[PLTO],
  For[ss=1, ss≤ Length[delpolyto[PLTO[[rr]]]],
  PLTRE=Append[PLTRE,delpolyto[PLTO[[rr]]][[ss]]];
  ;ss++
 ];
  ;rr++
 ];
  QL={};
  For[qqq=1, qqq≤ Length[PLTRE],
  QL=Append[QL,polyavbild[PLTRE[[qqq]]]];
  ;qqq++
 ];
  QL=Join[QL, linjelistebytt[QL]];
  Return[QL]
 )
  startpolygon={{4,0},{5,0},{5,5},{4,5}}; start={
  {{0,0},{0,5}},
  {{0,5},{5,5}},
  {{5,5},{5,0}},
  {{5,0},{0,0}},
  {{4,0},{4,5}},
  {{4,15/4},{5,7/2}}
 };
```

```
utskrift[var2_]:=(
  plotliste={
  Graphics[{RGBColor[0,1,0],Line[start[[1]]]}],
  Graphics[{RGBColor[0,1,0],Line[start[[2]]]}],
  Graphics[{RGBColor[0,1,0],Line[start[[3]]]}],
  Graphics[{RGBColor[0,1,0],Line[start[[4]]]}],
  Graphics[{RGBColor[0,1,0],Line[start[[5]]]}],
  Graphics[{RGBColor[0,1,0],Line[start[[6]]]}],
  Graphics[{RGBColor[0,1,0],Line[punktlistebytt[start[[1]]]]}],
  Graphics[{RGBColor[0,1,0],Line[punktlistebytt[start[[2]]]]}],
  Graphics[{RGBColor[0,1,0],Line[punktlistebytt[start[[3]]]]}],
  Graphics[{RGBColor[0,1,0],Line[punktlistebytt[start[[4]]]]}],
  Graphics[{RGBColor[0,1,0],Line[punktlistebytt[start[[5]]]]}],
  Graphics[{RGBColor[0,1,0],Line[punktlistebytt[start[[6]]]]}]
 };
  For[t=1, t≤ Length[var2],
  plotliste=Append[plotliste,Graphics[{RGBColor[1,0,0],Polygon[var2[[t]]]}]];
  ;t++
 ];
  Return[Show[plotliste,AspectRatio→1]]
 ) utskrift[polyiterer[{startpolygon}]]
```

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