Relations and quantizations in the category of probabilistic bundles.

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June 14, 2004

Abstract

In this paper we generalise the classical notion of state of a probabilistic system to include a nontrivial minimal bound on the class of observables. A product is introduced and the resulting structure is shown to be a monoidal category. Probabilistic relations are defined and compositions of relations is introduced. The resulting structure is also a category, the category of probabilistic relations. Finally we embedd the category of probabilistic relations into a category of bimodules and show that in this context it is possible to quantize some aspects of the probabilistic description.

1 Introduction

>From a deterministic point of view a physical system is described in terms of its state space. This is a set whose elements label the possible distinguishable states the system in question might assume. The state of the system at any point in time is from this point of view fully determined only when a unique label from the set of possible labels is assigned.

In many cases the state spaces are however so large or our means of observations so weak that a deterministic state description can not be found. For such situations a probabilistic approach is used. The states of a physical system are now labelled by probability spaces $\langle \Omega, \mathcal{B}, \mu \rangle$. Here we deviate somewhat from the usual idea that the state of a system is a point in a probability space. The resoning is that in order to specify the state of a system we must specify the observable events and assign a probability to each such event. The spesification of the algebra of observable events and the assignment of probabilities to them is really what constitute a state. This is a point of view that includes both classical systems and quantum systems. For the quantum case the algebra of events form a orthomodular lattice whereas in the classical case we have a boolean algebra. For the quantum case the algebra of events can be represented as the algebra of subspaces of a Hilbertspace whereas in the classical case we get a algebra of subsets of some set. These are however only convenient representations of the state and is usually not unique. In this paper we assume that for each state a particular representation in the form of a probability space is choosen and thus a state becomes a probability space. From this point of view the notion of mappings of spaces assumes a central role, they are the only means of comparing different states of a system. Properties of the system must be described in terms of maps between states. This leads us directly to the categorical point of view which is the point of view we have on probability theory. The book [14] gives and excellent elementary introduction the categorical view of mathematics, a more advanced introduction can be found in the book [7]

Let a state, that is a probability space, $\langle \Omega, \mathcal{B}, \mu \rangle$ be given. Here Ω is a set, $\mathcal{B}$ is a class of subsets of Ω that are the observables for the system in question and μ is a probability measure that assign a probability to each observable in $\mathcal{B}$. The class of observables is assumed to contain the empty set $\emptyset$,the whole set Ω and is closed under countable union and complement. The fact that $\mathcal{B}$ contains at least $\emptyset$ and Ω means that for any state at least these two events are observable. We are thus putting a lower bound on the class of observables, meaning that the class of observables should at least contain the empty set and the whole set.

In this paper we explore the consequences of assuming lower bounds on the set of observables different from the standard ones described above. More precisely we assume that for each probabilistic state $X = \langle \Omega_X, \mathcal{B}_X, \mu_X \rangle$ there is assigned a sub σ-algebra $\mathcal{G}_X \subset \mathcal{B}_X$. All the algebras $\mathcal{G}_X$ are assumed to be isomorphic to a single algebra $\mathcal{G}$. For the standard situation $\mathcal{G}$ is equal to the Boolean algebra B_2 of all subsets of a one element set. Probabilistic states with an assigned subalgebra $\mathcal{G}_X$ is called a probabilistic bundle. The probabilistic bundles are the objects in a category where arrows are absolutely continuous maps. We construct a product in this category and prove that the product defines a monoidal structure on the category of probabilistic bundles. For finite distributive lattices the monoidal structure was studied in [11].

The notion of independence between two probabilistic states is defined in terms of the product and is shown here to depend on the choice of minimal bound on the set of observables. This means that states that from a classical point of view are dependent can be independent for some choice of minimal bound $\mathcal{G} \neq B_2$.

>From a deterministic point of view a relation between two sets Ω_X and Ω_Y is a subset of the product $\Omega_X \times \Omega_Y$. A probabilistic relation is triple $\langle \Omega_X \times \Omega_Y, \mathcal{B}_{XY}, \mu_{XY} \rangle$ where $\mathcal{B}_{XY}$ is a class of deterministic relations that is assumed to form a σ-algebra and where μ_{XY} is a measure on $\mathcal{B}_{XY}$. A probabilistic relation is thus itself a probability space. We show that any morphism of probabilistic bundles is in fact a special kind probabilistic relation, and we define a composition of relations that reduce to the composition of maps when the relations corresponds to maps. The probabilistic bundles together with probabilistic relations is shown to form a (weak) category $\mathcal{R}(A)$ and we extend the product to this category.

The category $\mathcal{R}(A)$ formalizes the notion of state space from a probabilistic point of view. Objects in this category are states and arrows are relations between states. In the last part of this paper we show that the category $\mathcal{R}(A)$ can be mapped into the category of $\mathcal{A}$-$\mathcal{A}$ bimodules. Here $\mathcal{A}$ is a C^* algebra related to the minimal bound $\mathcal{G}$. This functor thus gives an algebraic description of the state space, it is a subcategory of a category of bimodules. We show that with this algebraic embedding it becomes possible to deform or quantize aspects of the probabilistic state spaces.

2 A Monoidal Category of Probability Spaces

2.1 Probability Spaces

In this section we define what a probability space will mean in this paper. Certain natural restrictions are made on the general concept of an abstract probability space in order for our constructions to make sense. We organize probability theory using the categorical framework as this leads to constructions that are natural. In fact category theory was invented in order to formalize the notion of canonical or natural constructions in mathematics. The basic idea in category theory is to organize any modeling and theory construction in terms of objects and arrows. These are the only primitives in the theory. Note that arrows are also called morphisms, both terms will be used in this paper. Each arrow points from one object to another and there is a associative composition of arrows defined. The description of a category is completed when an arrow that acts as left and right identity for composition is identified for each object. We get a concrete example of a category if we let objects be sets with some structure and arrows be structure preserving maps. For such examples arrow are maps, but in general they need not be maps. In this paper we will discuss probabilistic relations and show that they are arrows in a category. It will be evident that they are in general not maps of sets.

The goal of this section is to define a category whose objects are probability spaces and whose arrows are maps of probability spaces. Natural restrictions are made on both probability spaces and maps of spaces in order to get a well behaved category. In this way category theory will constrain what probability spaces and what maps between probability spaces should be. Recall that probability spaces are to be thought of as states of physical systems so we have placed restrictions on what kind of states to allow. The allowed states form the objects in our category. Thus the category is in effect the space of states of physical systems. Arrows are possible transformations of states. Probabilistic relations should then be thought of as the possible relations that can exist between physical states.

A *measurable space* [5] is a pair $X = \langle \Omega_X, \mathcal{B}_X \rangle$ where Ω_X is a set and $\mathcal{B}_X$ is a σ-algebra on Ω_X. A measurable map $f : X \to Y$ is a map of sets $\Omega_X \to \Omega_Y$ such that $f^{-1}(A) \in \mathcal{B}_X$ for all $A \in \mathcal{B}_Y$.

Let Ω be a set and let τ be a topology on Ω. In this paper the term topology is taken to mean a second countable, locally compact Hausdorff topology.

Note that any such space is metrizable, Polish and σ-compact.A *Borel structure* corresponding to a topology τ is the smallest σ-algebra containing the topology τ and is denoted by $\mathcal{B}(\tau)$. A *Borel space* is a measurable space where the σ-algebra is the Borel structure. Any continuous map $f : \langle \Omega_X, \tau_X \rangle \to \langle \Omega_Y, \tau_Y \rangle$ is measurable with respect to the Borel structures $\mathcal{B}(\tau_X)$ and $\mathcal{B}(\tau_Y)$.

Note that with our restriction on the topology any Borel space is a standard Borel space in the terminology of descriptive set theory.

Let $\langle \Omega, \mathcal{B}(\tau) \rangle$ be a Borel space where τ is a topology on Ω and let $Y \subset \Omega$ be a subset. Define the restriction of τ and $\mathcal{B}(\tau)$ to Y by $\tau^Y = \{A \cap Y \mid A \in \tau\}$ and $\mathcal{B}(\tau)^Y = \{B \cap Y \mid B \in \mathcal{B}(\tau)\}$. Clearly $\mathcal{B}(\tau)^Y$ is a σ-algebra on Y. We could however also define a σ-algebra on Y by $\mathcal{B}(\tau^Y)$. These two σ-algebras on Y are always the same $\mathcal{B}(\tau)^Y = \mathcal{B}(\tau^Y)$.

Note that for a general set Y, τ_Y is not a topology in the restricted sense used in this work. In order to ensure that it is locally compact the set Y must be closed or an intersection of an open and a closed set.

A *measure space* is a triple $X = \langle \Omega_X, \mathcal{B}_X, \mu_X \rangle$ where $\langle \Omega_X, \mathcal{B}_X \rangle$ is a measurable space and μ_X is measure on $\langle \Omega_X, \mathcal{B}_X \rangle$. By *probability space* we mean a measure space X where $\mathcal{B}_X = \mathcal{B}(\tau_X)$ for some topology on Ω_X and where $\mu_X(\Omega_X) = 1$. Note that since τ_X is second countable every measure is automatically regular [3].

Let $X = \langle \Omega_X, \mathcal{B}_X \rangle$ be a measurable space and let ν_X and μ_X be two measures on X. Recall that ν_X is absolute continuous with respect to μ_X if $\nu_X(U) = 0$ whenever $\mu_X(U) = 0$. When ν_X is absolute continuous with respect to μ_X we write $\nu_X \le \mu_X$. If we have both $\mu_X \le \mu_Y$ and $\mu_Y \le \mu_X$ then the measures are equivalent and we write $\mu_X \approx \mu_Y$.

Let X and Y be two probability spaces and let $f : \langle \Omega_X, \mathcal{B}_X \rangle \to \langle \Omega_Y, \mathcal{B}_Y \rangle$ be a measurable map. Define a second probability measure $f_* \mu_X$ on $\langle \Omega_Y, \mathcal{B}(\tau_Y) \rangle$ by

$$f_* \mu_X(V) = \mu_X(f^{-1}(V))$$

for $V \in \mathcal{B}(\tau_Y)$.

A map f of probability spaces is *absolute continuous* iff $f_* \mu_X \le \mu_Y$. Obviously, the identity map is absolute continuous and composition of absolute continuous maps is absolute continuous too. This completes our description of the category of probability spaces

Definition 1 *The category of probability spaces will be denoted by $\mathcal{P}$. Objects in this category are probability spaces as defined in this section and arrows are measurable, absolutely continuous maps.*

The existence and properties of all constructions in this paper are ultimately relying on properties of the category $\mathcal{P}$. In the next section we will begin exploring these properties. Here we will only describe the terminal object and isomorphisms in the category. Recall that a terminal object in a category C is a object T such that there exists a unique arrow from any object X to T. The importance of terminal objects in category theory is that they can be used to define points. Let X be a object in a category with a terminal object T. Then a point in X is a arrow from T to X. This describes the notion of points in terms of the primitives of category theory, objects and arrows. Not all categories have a terminal object, but for the one that does it is important to identify. The terminal object in the category of probability spaces is the one point probability space $T = \langle \Omega_T, \mathcal{B}(\tau_T), \mu_T \rangle$ where $\Omega_T = \{*\}$, $\mathcal{B}(\tau_T) = \{\varnothing, \{*\}\}$ and μ_T is the only possible measure on this space. Evidently all maps $f : \Omega_T \to \Omega_X$ are measurable and if $p_f = f(*) \in \Omega_X$ then it is easy to see that f is a point in X if and only if μ_X is a discontinous measure with $\mu_X(p_f) \neq 0$. Thus objects X with a continuous probability measure μ_X have no points! This is another reason for not considering the points of Ω_X to represent states of a physical system modelled by the probability space $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X \rangle$. Many states have no points. In category theory points are derived not defined. If $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X \rangle$ with $\mu_X = \delta_x$, the delta measure consentrated at a point $x \in \Omega_X$, then X is a state in the usual sense. It is trivial but satisfying that in this case the state X has exactly one point in the categorical sense. This point is the arrow $f : T \to X$ defined by $f(*) = x$.

An isomorphisms in any category is an arrow $f : X \to Y$ such that there exists an arrow $g : Y \to X$ with $f \circ g = 1_Y$ and $g \circ f = 1_X$. Since in our case an arrow, f, is a measurable map we must clearly require that f is bijectivė. Note that a bijective measurable map between measurable spaces does not in general have a measurable inverse. For our case however the Borel structures are generated by Polish spaces and for this case any injective measurable map $f : X \to Y$ will by the Kuratowski Theorem [5] satisfy $f(\mathcal{B}(\tau_X)) \subset \mathcal{B}(\tau_Y)$. Therefore any bijective measurable map has a measurable inverse.

$>$From this it follows that a morphism $f : X \to Y$ of probability spaces that is bijective at the level of sets, is an isomorphism of probability spaces iff $f_* \mu_Y \approx \mu_X$.

2.2 The Monoidal Structure

We will now start exploring the category $\mathcal{P}$ by showing that it has a product. In a category there can be many types of products, some derived from a universal limiting construction.All such products can be subsumed under the notion of a monoidal structure on a category. Anyone applying mathematics is constantly making use of monoidal structures. Two standard examples of monoidal structures are the cartesian product in the category of sets and the tensor product in the category of vector spaces. The first can be described in terms of a universal limit whereas the other can not. Note that since a category consists of both objects and arrows a product is defined only when we know how to take products of both objects and arrows. Since arrows can be composed the product of arrows should behave natural with respect to this composition. In category theory this is expressed by saying that the product is a bifunctor. The main reason for the prevalence of products in mathematics is that interactions are described in terms of products. In probability theory products are used to define dependence and independence of random variables.

A monoidal structure in a category is basically a product in the category that is associative up to natural isomorphism and has a unit object up to natural isomorphism. What this means is that if X,Y and Z are objects in the category and if the product is denoted by $\otimes$ then we require that there exists a isomorphism $\alpha_{XYZ} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$. Similarly if I is the unit object we require that there exists isomorphisms $\beta_X : I \otimes X \to X$ and $\gamma_X : X \otimes I \to X$. The isomorphisms can not be arbitrarily choosen for different objects, they must form the components of a natural transformation. In addition they must satisfie a set of equations known as the MacLane Coherence Conditions. These equations ensure that associativity and unit isomorphisms can be extended consistently to products of finitely many objects. The conditions that must be satisfied by α,γ and β are the following.

For all objects X,Y,Z and T we must have

$$\begin{aligned} \alpha_{X\otimes Y,Z,T} \circ \alpha_{X,Y,Z\otimes T} &= (\alpha_{X,Y,Z} \otimes 1_T) \circ \alpha_{X,Y\otimes Z,T} \circ (1_X \otimes \alpha_{Y,Z,T}) \\ (\gamma_X \otimes 1_Y) \circ \alpha_{X,I,Y} &= 1_X \otimes \beta_Y \\ \gamma_I &= \beta_I \end{aligned}$$

These are the MacLane coherence conditions. The naturality conditions are expressed as follows. For any arrows $f : X \to X'$,$g : Y \to Y'$ and $h : Z \to Z'$ we must have

$$\begin{aligned} ((f \otimes g) \otimes h) \circ \alpha_{X,Y,Z} &= (f \otimes (g \otimes h)) \circ \alpha_{X',Y',Z'} \\ f \circ \beta_X &= \beta_{X'} \circ (1_I \otimes f) \\ f \circ \gamma_X &= \gamma_{X'} \circ (f \otimes 1_I) \end{aligned}$$

In general such equations are difficult to solve, there is a very large number of variables and equations. However in some simple situations the naturality conditions can be used to reduce the system of equations to a much smaller set. In the last section of this paper we will solve a similar set of equations that determine what we think of as quantizers for the category. Each solution of those equations will give us a coherent way of deforming all structures defined in terms of the monoidal structure.

The reader not familiar with categories,natural transformations and Coherence conditions are referred to the two books mentioned in the introduction. In order to see further examples of how such system of equations are solved and how they are related to quantization we refer to previous publications by us [8],[9],[4].

A *monoidal category* is a 6 tuple $\langle C, \otimes, I, \alpha, \beta, \gamma \rangle$ where each element in this 6 tuple has the properties just described.

We will now review the usual construction of products of probability spaces and show that it defines a monoidal structure on the category $\mathcal{P}$. First we need to define a bifunctor and then show that it has the additional properties required from a monoidal structure. This is simple but instructive so we do it in detail.

Let $X = \langle \Omega_X, \mathcal{B}(\tau_X) \rangle$ and $Y = \langle \Omega_Y, \mathcal{B}(\tau_Y) \rangle$ be Borel spaces. We can define two natural σ-algebras on $\Omega_X \times \Omega_Y$. These are $\mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)$ and $\mathcal{B}(\tau_X \otimes \tau_Y)$, where $\mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)$ is the σ-algebra generated by the set of measurable boxes

$\mathcal{S} = \{A \times B \mid A \in \mathcal{B}(\tau_X),\ B \in \mathcal{B}(\tau_Y)\}$ and where $\mathcal{B}(\tau_X \otimes \tau_Y)$ is the Borel structure generated by the product topology.

Note that the product topology $\tau_X \otimes \tau_Y$ is second countable, locally compact and Hausdorf if each of τ_X and τ_Y have these properties [12]. The product topology is thus a topology in the restricted sense used in this paper.

Lemma 2 *Let τ_X and τ_Y be topologies on Ω_X and Ω_Y. Then*

$$\mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y) = \mathcal{B}(\tau_X \otimes \tau_Y).$$

Proof. Let $\pi_1 : \Omega_X \times \Omega_Y \to \Omega_X$ and $\pi_2 : \Omega_X \times \Omega_Y \to \Omega_Y$ be the projections on the first and second factor.

Clearly $\pi_1^{-1}(\tau_X) \subset \mathcal{B}(\tau_X \otimes \tau_Y)$ and $\pi_2^{-1}(\tau_Y) \subset \mathcal{B}(\tau_X \otimes \tau_Y)$ and therefore $\pi_1 : \langle \Omega_X \times \Omega_Y, \mathcal{B}(\tau_X \otimes \tau_Y)\rangle \to \langle \Omega_X, \mathcal{B}(\tau_X)\rangle$ and $\pi_2 : \langle \Omega_X \times \Omega_Y, \mathcal{B}(\tau_X \otimes \tau_Y)\rangle \to \langle \Omega_Y, \mathcal{B}(\tau_Y)\rangle$ are measurable maps.

Then $U \times \Omega_Y = \pi_1^{-1}(U)$ and $\Omega_X \times V = \pi_2^{-1}(V)$ are contained in $\mathcal{B}(\tau_X \otimes \tau_Y)$ for all $U \in \mathcal{B}(\tau_X)$ and $V \in \mathcal{B}(\tau_Y)$, and therefore $\mathcal{B}(\tau_X \otimes \tau_Y)$ contains all boxes $U \times V$ with $U \in \mathcal{B}(\tau_X)$ and $V \in \mathcal{B}(\tau_Y)$.

But $\mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)$ is the smallest σ-algebra containing all boxes and therefore $\mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y) \subset \mathcal{B}(\tau_X \otimes \tau_Y)$.

Denote by $\tau(\mathcal{S})$ the topology generated by a basis $\mathcal{S}$. Let us assume that τ_X and τ_Y have countable bases $\mathcal{S}_X$ and $\mathcal{S}_Y$. Then $\tau_X \otimes \tau_Y$ has countable basis $\mathcal{S}_X \times \mathcal{S}_Y$ and therefore elements of $\tau_X \otimes \tau_Y$ consists of countable union of open boxes $A \times B \in \mathcal{S}_X \times \mathcal{S}_Y$. All open boxes are in $\mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)$ and since it is closed with respect to countable unions we have $\tau_X \otimes \tau_Y \subset \mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)$. But $\mathcal{B}(\tau_X \otimes \tau_Y)$ is the smallest σ-algebra containing $\tau_X \otimes \tau_Y$ and therefore we must have $\mathcal{B}(\tau_X \otimes \tau_Y) \subset \mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)$. ∎

Let X and Y be any pair of probability spaces. We define their product to be the object

$$X \otimes Y = \langle \Omega_X \times \Omega_Y, \mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y), \mu_X \otimes \mu_Y \rangle$$

where $\mu_X \otimes \mu_Y$ is the product measure[1]. This define a map of objects $\otimes(X, Y) = X \otimes Y$. Note that the previous proposition and the remarks preceeding it show that $X \otimes Y$ is really a probability space and thus a object in the category $\mathcal{P}$. This makes $\otimes$ well defined on objects.The next step is to extend $\otimes$ to arrows.

Let $f : X \to X'$ and $g : Y \to Y'$ be arrows. Define a map of sets $f \otimes g : \Omega_X \times \Omega_Y \to \Omega_{X'} \times \Omega_{Y'}$ by $f \otimes g = f \times g$, where $f \times g$ is the usual cartesian product of maps. The following proposition show that $\otimes$ extends to a bifunctor on the category $\mathcal{P}$.

Proposition 3 *The product $f \otimes g$ is a arrow in $\mathcal{P}$, and $\otimes : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is a bifunctor on the category of probability spaces.*

Proof. The map $f \otimes g$ is clearly measurable with respect to $\mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)$ and $\mathcal{B}(\tau_{X'}) \otimes \mathcal{B}(\tau_{Y'})$.

Since f and g are morphisms we have $f_*\mu_X \leq \mu_{X'}$ and $g_*\mu_Y \leq \mu_{Y'}$. Furthermore,

$$\begin{aligned}&(f_*(\mu_X) \otimes g_*(\mu_Y))(C' \times D') = f_*(\mu_X)(C')g_*(\mu_Y)(D') \\ &= \mu_X(f^{-1}(C'))\mu_Y(g^{-1}(D')) = (\mu_X \otimes \mu_Y)((f \otimes g)^{-1}(C' \times D')) \\ &= (f \otimes g)_*(\mu_X \otimes \mu_Y)(C' \times D').\end{aligned}$$

Then, by the uniqueness theorem for product measures [2] and Fubini's theorem, we have

$$(f \otimes g)_*(\mu_X \otimes \mu_Y) = f_*(\mu_X) \otimes g_*(\mu_Y) \leq \mu_{X'} \otimes \mu_{Y'}.$$

The second part of the proposition is evident since it is well known that the cartesian product is a bifunctor. ■

To show that $\otimes$ is a monoidal structure is now very simple since we know that the cartesian product is a monoidal structure in the category of sets with unit object the one point set $I = \{*\}$. The maps of sets defining this monoidal structure are $\alpha_{XYZ} : \Omega_X \times (\Omega_Y \times \Omega_Z) \to (\Omega_X \times \Omega_Y) \times \Omega_Z$ defined by

$$\alpha_{XYZ}((x, (y, z))) = ((x, y), z)$$

and $\beta_X : \Omega_T \times \Omega_X \to \Omega_X$, $\gamma_X : \Omega_X \times \Omega_T \to \Omega_X$ defined by

$$\begin{aligned}\beta_X(*, y) = y \\ \gamma_X(y, *) = y\end{aligned}$$

These maps are obviously arrows in $\mathcal{P}$ and this taken together yields the following result

Theorem 4 $\langle \otimes, T, \alpha, \beta, \gamma \rangle$ *is a monoidal structure on the category of probability spaces.*

The reader might want to verify that α,β,and γ solves the MacLane Coherence conditions and the naturality requirements. For the category $\mathcal{P}$ it is furthermore easy to show that this is the only possible choise of α,β and γ that solves the coherence conditions for the given bifunctor. This is true because the naturality can be used to describe all maps $\alpha_{X,Y,Z}$, β_X and γ_X in terms of $\alpha_{T,T,T}$, β_T and γ_T. and for these maps there is only one possibility because T consists of a single point.

3 A Monoidal Structure on the Category of Probabilistic Bundles

3.1 Probabilistic Bundles

We will now construct a new category based on the category $\mathcal{P}$. The construction is an example of the usual comma construction in Category Theory. We will form a new category by considering objects over a fixed object in the category of probability spaces $\mathcal{P}$. The objects in this new category is naturally thought of as bundles over a fixed basespace.

Definition 5 *A probabilistic bundle over A is a 4-tuple $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X, \varphi_X \rangle$ where $\langle \Omega_X, \mathcal{B}(\tau_X), \mu_X \rangle$ is a probability space and $\varphi_X : \Omega_X \to \Omega_A$ is a continuous surjective map such that*

$$(\varphi_X)_* \mu_X \approx \mu_A$$

Note that according to this definition a probabilistic bundle over A is a arrow $f : X \to A$ in $\mathcal{P}$. It is however clear that not any such arrow is a probabilistic bundle. We now need to define arrows between probabilistic bundles.

Definition 6 *Let $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X, \varphi_X \rangle$ and $Y = \langle \Omega_Y, \mathcal{B}(\tau_Y), \mu_Y, \varphi_Y \rangle$ be probabilistic bundles. A arrow $f : X \to Y$ of probabilistic bundles is a arrow $f : \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X \rangle \to \langle \Omega_Y, \mathcal{B}(\tau_Y), \mu_Y \rangle$ of probability spaces such that*

$$\varphi_X = \varphi_Y \circ f$$

Finally we collect objects and arrows into a category.

Definition 7 *The category of probabilistic bundles over $A \in \mathcal{P}$ is a category whose objects are probabilistic bundles and arrows are arrows of probabilistic bundles as just defined. We denote this category by $\mathcal{P}(A)$.*

The restriction we have placed on the arrows in $\mathcal{P}$ are choosen exactly such that the resulting category of probabilistic bundles can support a monoidal structure. This is Categorical modelling at work. We will usually identify the object A in $\mathcal{P}$ with the probabilistic bundle $\langle \Omega_A, \mathcal{B}(\tau_A), \mu_A, 1_A \rangle$ and denote both by the symbol A. Using this notation it is evident that A is a terminal object in the category $\mathcal{P}(A)$. The unique morphism from any object X to A is φ_X. A point in a probabilistic bundle is a arrow $s : A \to X$ in $\mathcal{P}(A)$. Using the definitions just stated it is evident that this is equivalent to the existence of a measurable map $s : \langle \Omega_A, \mathcal{B}(\tau_A) \rangle \to \langle \Omega_X, \mathcal{B}(\tau_X) \rangle$ such that

$$\varphi_X \circ s = 1_{\Omega_A}$$
$$s_* \mu_A \leq \mu_X$$

The first identity say that s is a measurable section in the bundle $\varphi_X : \langle \Omega_X, \mathcal{B}(\tau_X) \rangle \to \langle \Omega_A, \mathcal{B}(\tau_A) \rangle$. Such sections certainly exists for the measurable spaces considered in this paper. The second condition on s require a certain singularity for the measure μ_X. For example if $X = \mathbb{R}^2$,$A = \mathbb{R}$ with the usual Lebesgue measures and φ_X the projection on the first component, then no smooth section $s : \mathbb{R} \to \mathbb{R}^2$ is a point in $\mathcal{P}(A)$ since the image $s(\mathbb{R}) \subset \mathbb{R}^2$ has measure zero with respect to the Lebesgue measure on $\mathbb{R}^2$.

The construction of the monoidal structure will rely heavily on the properties of the conditional expectation and we will now recall some of its properties and introduce some important notation that we will use in the rest of the paper.

Let X be a probability space and let $f : \Omega_X \to R$ be a real valued measurable map with respect to $\mathcal{B}(\tau_X)$ and the standard Borel structure on $\mathbb{R}$.

Let $\mathcal{G} \subset \mathcal{B}(\tau_X)$ be a sub σ-algebra of $\mathcal{B}(\tau_X)$ and assume that $\int |f| d\mu_X < \infty$. Then there exists [1],[13], a function $f_{\mathcal{G}} : \Omega_X \to R$ that is $\mathcal{G}$ measurable and

$$\int_S f_{\mathcal{G}} d\mu_X = \int_S f d\mu_X \qquad \text{for all } S \in \mathcal{G}$$

The function $f_{\mathcal{G}}$ is unique up to sets of measure zero in $\mathcal{G}$ and is the *conditional expectation* of f with respect to $\mathcal{G}$. General rules for manipulating conditional expectations can be found, for example, in [13]. We will need the following result

Lemma 8 *Let $\varphi : \Omega_X \to \Omega_Y$ be a surjective $\mathcal{B}_X - \mathcal{B}_Y$ measurable map. Then a function $f : \Omega_X \to R$ is $\varphi^{-1}(\mathcal{B}_Y)$ measurable iff there exists a unique $\mathcal{B}_Y$ measurable function $h : \Omega_Y \to R$ such that*

$$f = h \circ \varphi$$

Let $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X, \varphi_X \rangle$ be a probabilistic bundle over A and $\mathcal{G}_X = \varphi_X^{-1}(\mathcal{B}(\tau_A))$ be a sub σ-algebra of $\mathcal{B}(\tau_X)$ of "cylinder" sets. We denote the conditional expectation of a function $f : \Omega_X \to R$ with respect to the subalgebra $\mathcal{G}_X$ by $(f)_X$. The fact that $(f)_X$ is defined only up to cylinder sets of measure zero means that we really have an equivalence class of functions where two functions from the class differ only on a cylinder set of measure zero. For each representative u of $(f)_X$ there exists according to lemma 8 a unique $\mathcal{G}_X$ measurable function h_u such that $u = h_u \circ \varphi_X$. Let u and v be two representatives of $(f)_X$ and assume that h_u and h_v differ on a set U, that is not of measure zero with respect to μ_A. Then the representatives u and v of $(f)_X$ differ on the set $\varphi_X^{-1}(U)$. But then $\varphi_X^{-1}(U)$ must have measure zero and therefore $(\varphi_X)_*(\mu_X)(U) = 0$. This is however a contradiction because $\mu_A \leq (\varphi_X)_*\mu_X$. Thus h_u and h_v differ only on a set of measure zero. We will denote the class of functions on Ω_A corresponding to $(f)_X$ through lemma 8 by $e_X(f)$. Let $C \in \mathcal{B}(\tau_X)$ and let χ_C be the characteristic function corresponding to C. Then χ_C is obviously $\mathcal{B}(\tau_X)$ measurable and $\int |\chi_C| d\mu_X = \mu_X(C) < \infty$ since the measure is finite. Therefore $(\chi_C)_X$ is defined and is by definition $\mathcal{G}_X$ measurable. For this special case we will define $e_X(C) = e_X(\chi_C)$. The function $e_X(C)$ is clearly essentially bounded by the properties of conditional expectation. Note that we obviously have $(\chi_\emptyset)_X = \chi_\emptyset$ and $(\chi_{\Omega_X})_X = \chi_{\Omega_X}$ and therefore $e_X(\emptyset) = 0$ and $e_X(\Omega_X) = 1$ up to sets of measure zero.

3.2 The Monoidal Structure

We will first define the product of probabilistic bundles and then proceed to extend it to arrows. The product is based on the fibered product of sets.

Let X and Y be two objects in $\mathcal{P}(A)$ and let $\Omega_{X \otimes_A Y}$ be the fibered product of the underlying sets Ω_X and Ω_Y, that is

$$\Omega_{X \otimes_A Y} = \{(x, y) \mid x \in \Omega_X, y \in \Omega_Y,\ \varphi_X(x) = \varphi_Y(y)\}.$$

Define a map of sets $\varphi_{X\otimes_A Y} : \Omega_{X\otimes_A Y} \to \Omega_A$ by

$$\varphi_{X\otimes_A Y}(x, y) = \varphi_X(x) = \varphi_Y(y)$$

Note that $\varphi^{-1}_{X\otimes_A Y}(a) = \varphi^{-1}_X(a) \times \varphi^{-1}_Y(a)$ so that the fibers of the fibered product of the bundles X and Y is just the product of the fibers of the two bundles. This is perhaps a simpler way to view the bundle $\varphi_{X\otimes_A Y} : \Omega_{X\otimes_A Y} \to \Omega_A$. We are going to construct an object $X \otimes_A Y$ in $\mathcal{P}(A)$ with underlying set $\Omega_{X\otimes_A Y}$ and underlying map $\varphi_{X\otimes_A Y}$. This will be the object part of a bifunctor that will form a monoidal structure on the category of probabilistic bundles.

Note that $\Omega_{X\otimes_A Y}$ is a subset of $\Omega_X \times \Omega_Y$. Let $\triangle_A \subset \Omega_A \times \Omega_A$ be the diagonal. Then $\triangle_A$ is a closed set with respect to the product topology because the topological space $\langle \Omega_A, \tau_A\rangle$ is Hausdorff.

This means that $\Omega_{X\otimes_A Y} = (\varphi_X \otimes \varphi_Y)^{-1}(\triangle_A)$ is a closed and thus measurable subset of $\Omega_X \times \Omega_Y$ since $\varphi_{X\otimes_A Y}$ is by definition continuous.

Define

$$\tau_{X\otimes_A Y} = (\tau_X \otimes \tau_Y) \cap (\Omega_{X\otimes_A Y}).$$

Since the set $\Omega_{X\otimes_A Y}$ is closed, the topology $\tau_{X\otimes_A Y}$ is a topology in our restricted sense. This is the point where we need the continuity of φ_X and φ_Y.

>From lemma 2 it is easy to see that

$$\mathcal{B}(\tau_{X\otimes_A Y}) = (\mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)) \cap \Omega_{X\otimes_A Y}.$$

Therefore, $\mathcal{B}(\tau_{X\otimes_A Y})$ is the σ-algebra generated by fibered rectangles (or, cubes) $C \times_A D$. From the previous remarks it is clear that $\mathcal{B}(\tau_{X\otimes_A Y})$ is a subalgebra of $\mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)$.

We will now define a measure on the measurable space $\langle \Omega_{X\otimes_A Y}, \mathcal{B}(\tau_{X\otimes_A Y})\rangle$ in such a way that it reduce to the product measure when $A = T$, the trivial one-point probability space introduced in section 2. Let $C \in \mathcal{B}(\tau_X)$ and $D \in \mathcal{B}(\tau_Y)$ be arbitrary measurable subsets. Define a set function, θ_A on measurable boxes $C \times D$ by

$$\theta_A(C \times D) = \int e_X(C) e_Y(D) d\mu_A$$

This is clearly well defined . It is however not at all obvious that it can be extended to a measure on $\langle \Omega_{X\otimes_A Y}, \mathcal{B}(\tau_{X\otimes_A Y})\rangle$. We will now prove this by using the simplest elements from the theory of positive operator valued measures and integrals. Our main source for this theory is [10].

Recall that a positive operator valued measure (POV) on a measurable space $\langle \Omega, \mathcal{B}\rangle$ is a set function, $E : \mathcal{B} \to \mathcal{O}(H)$ where $\mathcal{O}(H)$ is the set of bounded operators on a Hilbert space H. For E to be a POV we must require that $E(C) \geq 0$ is a positive operator for all elements $C \in \mathcal{B}$ and that it is countable additive on disjoint unions of sets.

$$E(\cup_{i=1}^{\infty} C_i) = \sum_{i=1}^{\infty} E(C_i)$$

Limits must be taken in the strong sense, that is pointwise convergence in norm. Also recall that a spectral measure is a POV when $E(C)$ is a projector for all C.

We will show that the formula for θ_A can be related to the product of two POV's and throught this prove that θ_A extends to a measure.

Let $H_A = L_2(\Omega_A, \mathcal{B}(\tau_A), \mu_A)$. Then H_A is a Hilbertspace and we have a standard spectral measure P_A defined by

$$P_A(V)\xi = \chi_C \xi$$

Using the notion of conditional expectation we now define a operator valued set function on $\mathcal{B}(\tau_X)$ with values in $\mathcal{O}(H_A)$ by

$$E_X(C) = \int e_X(C) dP_A$$

Lemma 9 *E_X is a POV.*

Proof. We know that $e_X(C) \geq 0$ except for a set of measure zero with respect to μ_A. But any such set also has measure zero with resect to P_A. Therefore

$$E_X(C) = \int e_X(C) dP_A \geq 0$$

Let $\xi \in H_A$ and define a setfunction ν_ξ by

$$\nu_\xi(C) = \langle E_X(C)\xi, \xi \rangle_A \in [0, 1]$$

where the bracket denotes inner product in H_A. If we can prove that ν_ξ is a measure for all $\xi \in H_A$, then E_X is a POV [10].

Let $\mu_A^\xi(V) = \langle P_A(V)\xi, \xi \rangle_A$. Then μ_A^ξ is a measure that is absolutely continous with respect to μ_A with density $|\xi|^2$. Let furthermore h be the density of μ_A with respect to $(\varphi_X)_* \mu_X$ and let g be its inverse so that $gh = 1$ up to measure zero with respect to μ_A. The densities g and h exists because we are assuming that $(\varphi_X)_* \mu_X$ is equivalent to μ_A. Using these densities and standard properties of integration with respect to POV's we have

$$\begin{aligned}
\nu_\xi(C) &= \int e_X(C) d\mu_A^\xi \\
&= \int e_X(C) |\xi|^2 d\mu_A \\
&= \int e_X(C) |\xi|^2 g d(\varphi_X)_* \mu_X \\
&= \int (\chi_C)_X \; \varphi_X^*(|\xi|^2 g) d\mu_X \\
&= \int_C \varphi_X^*(|\xi|^2 g) d\mu_X
\end{aligned}$$

This proves that ν_ξ is a measure that is in fact absolutely continuous with respect to μ_X and we can conclude that E_X is a POV. ■

Let now X and Y be given objects in $\mathcal{P}(A)$ and let E_X and E_Y be the two POV's constructed as above. We have now two POV's acting on the same Hilbert space H_A. It is possible to define the product of them only if they commute. But by standard properties of integration with respect to spectral measures we have

$$\begin{aligned} E_X(C)E_Y(D) &= \int e_X(C)dP_A \int e_Y(D)dP_A \\ &= \int e_X(C)e_Y(D)dP_A \\ &= \int e_Y(D)dP_A \int e_X(C)dP_A \\ &= E_Y(D)E_X(C) \end{aligned}$$

thus our two measures commute. But then we can conclude that the formula

$$E(C \times D) = E_X(C)E_Y(D) = \int e_X(C)e_Y(D)dP_A$$

extends to a unique POV on $\mathcal{B}(\tau_{X\otimes_A Y})$.

Theorem 10 *The set function θ_A defined by*

$$\theta_A(C \times D) = \int e_X(C)e_Y(D)d\mu_A$$

extends to a unique probability measure on the measurable space $\langle \Omega_{X\otimes_A Y}, \mathcal{B}(\tau_{X\otimes_A Y})\rangle$.

Proof. In general, if E is a POV acting on a Hilbert space H then for all $\xi \in H$ the set function $\nu_\xi(C) = \langle E(C)\xi, \xi\rangle_H$ is a measure.Since μ_A is a finite measure we know that the constant function 1_A is an element in H_A. But then since

$$\begin{aligned} \langle E(C \times D)1_A, 1_A\rangle_{H_A} &= \int e_X(C)e_Y(D)d\mu_A \\ &= \theta_A(C \times D) \end{aligned}$$

we can conclude that θ_A extends to a measure on the measurable space $\langle \Omega_{X\otimes_A Y}, \mathcal{B}(\tau_{X\otimes_A Y})\rangle$. It is a probability measure because $\theta_A(\Omega_X \times \Omega_Y) = \int \mu_A = 1$. ■

Proposition 11 *Let $L_\infty(X)$ be the space of essentially bounded measurable function on $\langle \Omega_X, \mathcal{B}(\tau_X), \mu_X\rangle$. Then we have for any $f \in L_\infty(X)$*

$$\int f dE_X = \int e_X(f)dP_A$$

Proof. Let $\xi \in H_A$ and define two measures

$$\mu_X^\xi(C) = \langle E_X(C)\xi, \xi\rangle_{H_A}$$
$$\mu_A^\xi(V) = \langle P_A(V)\xi, \xi\rangle_{H_A}$$

Then by the general properties of integration with respect to operator measures we have proved the theorem if for all $\xi \in H_A$

$$\int f d\mu_X^\xi = \int e_X(f) d\mu_A^\xi$$

Let M_ξ and N_ξ be functionals on $L_\infty(X)$ defined by the lefthand side and righthand side of the previous equation. It is straight forward to prove that both M_ξ and N_ξ are bounded on $L_\infty(X)$.Since the space of simple functions is dense in $L_\infty(X)$ we have proved the theorem if we can show that $M_\xi(s) = N_\xi(s)$ for all simple functions. But for simple functions the identity follows directly from the linearity of conditional expectation. ∎

Lemma 12 *Let p, q be the projections of $\Omega_X \times \Omega_Y$ on Ω_X and Ω_Y. Then we have*

$$p_*(\theta_A) \leq \mu_X$$
$$q_*(\theta_A) \leq \mu_Y$$

Proof. Let $C_0(X)$ be the algebra of continuous functions with compact support on a probability space $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X\rangle$. Let $f \in C_0(X)$ and let $f \otimes 1$ be the function defined by $(f \otimes 1)(x, y) = f(x)$and let h be the density of μ_A with respect to $(\varphi_X)_*\mu_X$. Then we have using [10] and the previous lemma

$$\int f dp_*(\theta_A) = \int f \circ p d\theta_A = \int (f \otimes 1) d\theta_A = \langle (\int f \otimes 1 dE_X) 1_A, 1_A\rangle$$
$$= \langle (\int f dE_X) 1_A, 1_A\rangle = \langle (\int e_X(f) dP_A) 1_A, 1_A\rangle$$
$$= \int e_X(f) d\mu_A = \int (f)_X (h \circ \varphi_X) d\mu_X = \int f(h \circ \varphi_X) d\mu_X$$
$$= \int f d\nu_X$$

Where $\nu_X(U) = \int_U (h \circ \varphi_X) d\mu_X$. Since all measures are regular we can conclude that $p_*(\theta_A) = \nu_X$. But clearly $\nu_X \leq \mu_X$ so $p_*\theta_A \leq \mu_X$.

The second part of the lemma is proved in a similar way. ∎

Lemma 13 *The support of the measure θ_A is contained in the closed set $\Omega_{X \otimes_A Y}$.*

Proof. Let $S = \Omega_{X \otimes_A Y}$ and let $(x, y) \in S^c$ be a element in the complement of S. Define $a = \varphi_X(x)$ and $b = \varphi_Y(y)$. Since $(x, y) \in S^c$ we have $a \neq b$.

Since $\langle \Omega_A, \mathcal{B}(\tau_A) \rangle$ is Hausdorf there exists open sets U_x, V_y in Ω_A such that $a \in U_x, b \in V_y$ and $U_x \cap V_y = \emptyset$. Let $C_x = \varphi_X^{-1}(U_x)$ and $D_y = \varphi_Y^{-1}(V_y)$. Then $C_x \in \mathcal{B}(\tau_X), D_y \in \mathcal{B}(\tau_Y)$ and $C_x \times D_y \subset S^c$ with $(x, y) \in C_x \times D_y$. Thus $\mathcal{O} = \{C_x \times D_y \mid (x, y) \in S^c\}$ is a open cover for S^c. But $\langle \Omega_X \times \Omega_Y, \mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y) \rangle$ is second countable and thus Lindlöf and therefore there exists a countable subcover $\mathcal{G} = \{C_i \times D_i\}_{i=1}^{\infty}$ of S^c. But by the properties of conditional expectation we have $e_X(C_x) = \chi_{U_x}$ and $e_Y(D_y) = \chi_{V_y}$ up to sets of μ_A-measure zero, so $e_X(C_x)e_Y(D_y) = \chi_{U_x \cap V_y} = 0$ and therefore that $\theta_A(C_x \times D_y) = 0$.

Since θ_A is a measure we have

$$\theta_A(S^c) = \theta_A(\cup_{i=1}^{\infty} C_i \times D_i) = \sum_{i=1}^{\infty} \theta_A(C_i \times D_i) = 0.$$

Since the support of the measure θ_A is the smallest closed set F such that $\theta_A(F^c) = 0$ we can conclude that $F \subset \Omega_{X \otimes_A Y}$. ■

Recall that he Borel algebra $\mathcal{B}(\tau_{X \otimes_A Y})$ is a subalgebra of $\mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)$.

Definition 14 *The restriction of the measure θ_A to the subalgebra $\mathcal{B}(\tau_{X \otimes_A Y})$ we shall denote by $\mu_{X \otimes_A Y}$. This measure we will also denote by $\mu_X \otimes_A \mu_Y$ and called a* fibered product *of the measures μ_X and μ_Y .*

Using this definition, lemma 13 and lemma 12 we can generalize proposition 11 to the following very useful result

Proposition 15 *Let f and g be any essentially bounded functions on X and Y and let $(f \otimes g)^A$ be the restriction of $f \otimes g$ to $\Omega_{X \otimes_A Y}$. Then*

$$\int (f \otimes g)^A d\mu_{X \otimes_A Y} = \int (f \otimes g) d\theta_A = \int e_X(f) e_Y(g) d\mu_A$$

Proof. The first identity is just a direct consequence of lemma 13. We now prove the second identity. Let t be a fixed simple function on Ω_Y and define maps $M(f)$ and $N(f)$ for functions on Ω_X by

$$M(f) = \int (f \otimes t) d\theta_A$$
$$N(f) = \int e_X(f) e_Y(t) d\mu_A$$

We will now prove that $M(f)$ is well defined on $L_\infty(\mu_X)$.

Let $f \in L_\infty(\mu_X)$ and $g \in L_\infty(\mu_Y)$ be arbitrary essentially bounded functions on $\langle \Omega_X, \mathcal{B}(\tau_X) \rangle$ and $\langle \Omega_Y, \mathcal{B}(\tau_Y) \rangle$. Thus there exists sets $V_f \subset \Omega_X$ and $V_g \subset \Omega_Y$ with $\mu_X(V_f) = \mu_Y(V_g) = 0$ and f bounded on the complement V_f^c of V_f and g bounded on the complement V_g^c of V_g. But then $f \otimes g$ is obviously bounded on the set $V_f^c \times V_g^c \subset \Omega_X \times \Omega_Y$. Furthermore we have

$$(V_f^c \times V_g^c)^c \subset (V_f \times \Omega_Y) \cup (\Omega_X \times V_g)$$

But then

$$\theta_A((V_f^c \times V_g^c)^c) \leq \theta_A(V_f \times \Omega_Y) + \theta_A(\Omega_X \times V_g)$$
$$= p_*\theta_A(V_f) + q_*\theta_A(V_g) = 0$$

where the last identity follows from lemma 12. Therefore $f \otimes g$ is in $L_\infty(\theta_A)$. This argument proves that $M(f)$ is well defined on $L_\infty(\mu_X)$. It is obvious that $N(f)$ is well defined on the same space by the properties of conditional expectation. In fact simple estimates show that M and N are both continous linear functionals. But the definition of θ_A and linearity of conditional expectation show that $M(s) = N(s)$ for all simple functions. The density of simple functions in $L_\infty(\mu_X)$ and the continuity now show that

$$\int (f \otimes t) d\theta_A = \int e_X(f) e_Y(t) d\mu_A$$

We now fix f and consider the two sides of the above equation to be functions on $L_\infty(\mu_Y)$. Repeating the argument above for $L_\infty(\mu_Y)$ we finally prove the proposition. ■

We can now use the previus result to prove the following extension of lemma 12 that we will need later.

Proposition 16 *Let $p : \Omega_X \times \Omega_Y \to \Omega_X$ and $q : \Omega_X \times \Omega_Y \to \Omega_Y$ be the projections. Then we have*

$$p_*(\theta_A) \approx \mu_X$$
$$q_*(\theta_A) \approx \mu_Y$$

Proof. By lemma 12 we only need to prove that $p_*\theta_A \geq \mu_X$ and $q_*\theta_A \geq \mu_Y$. Let h be the density of $(\varphi_X)_*\mu_X$ with respect to μ_A and let $f \in C_0(X)$ be any continuous function with compact support on X. Then we have

$$\int f d\mu_X = \int e_X(f) h d\mu_A = \int e_X(f) e_Y(h \circ \varphi_Y) d\mu_A =$$
$$\int (f \otimes (h \circ \varphi_Y)) d\theta_A = \int (f \otimes 1)(1 \otimes (h \circ \varphi_Y)) d\theta_A =$$
$$\int (f \circ p) d\rho = \int f dp_*\rho$$

where $\rho \leq \theta_A$ is the measure defined by $\rho(V) = \int_V (1 \otimes (h \circ \varphi_Y)) d\theta_A$. By uniqueness of regular measures we can conclude that $\mu_X = p_*\rho$. But $\rho \leq \theta_A$ implies that $p_*\rho \leq p_*\theta_A$ and therefore $\mu_X \leq p_*\theta_A$. The second part of the proposition is proved in an entirely similar way. ■

Proposition 17 $(\varphi_{X \otimes_A Y})_* \mu_{X \otimes_A Y} = \mu_A$

Proof. We have for any $U \in \mathcal{B}(\tau_A)$ that

$$
\begin{aligned}
&[(\varphi_{X \otimes_A Y})_*(\mu_{X \otimes_A Y})](U) = (\mu_{X \otimes_A Y})((\varphi_{X \otimes_A Y})^{-1}(U)) \\
&= (\mu_{X \otimes_A Y})((\varphi_X^{-1}(U) \times \Omega_Y) \cap \Omega_{X \otimes_A Y}) = \theta_A(\varphi_X^{-1}(U) \times \Omega_Y) \\
&= \int e_X(\varphi_X^{-1}(U)) e_Y(\Omega_Y) d\mu_A = \int e_X(\varphi_X^{-1}(U)) d\mu_A \\
&= \int \chi_U d\mu_A = \mu_A(U)
\end{aligned}
$$

■

We can conclude that

Proposition 18 $X \otimes_A Y = \langle \Omega_{X \otimes_A Y}, \mathcal{B}(\tau_{X \otimes_A Y}), \mu_{X \otimes_A Y}, \varphi_{X \otimes_A Y} \rangle$ *is a probabilistic bundle.*

We have now defined a product of objects $\otimes_A(X, Y) = X \otimes_A Y$ and we will next proceed to extend it to a product of arrows and show that it is a bifunctor.

Let $f : X \to X'$ and $g : Y \to Y'$ be morphisms of probabilistic bundles. Define a map of sets $f \otimes_A \dot{g} : \Omega_{X \otimes_A Y} \to \Omega_{X' \otimes_A Y'}$ by

$$f \otimes_A g = f \times_A g$$

Recall that the map $f \times_A g$ is the restriction of $f \times g$ to the subset $\Omega_{X \otimes_A Y}$. The map $f \otimes_A g$ is clearly measurable with respect to the Borel algebras $\mathcal{B}(\tau_{X \otimes_A Y})$ and $\mathcal{B}(\tau_{X' \otimes_A Y'})$. In order to prove that it is a arrow in the category of probabilistic bundles we must now show that it is absolute continuous.

Let the probabilistic bundle X be given by $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X, \varphi_X \rangle$. Define a new probabilistic bundle $f_* X$ by $f_* X = \langle \Omega_{f_* X}, \mathcal{B}(\tau_{f_* X}), \mu_{f_* X}, \varphi_{f_* X} \rangle$ where $\Omega_{f_* X} = \Omega_{X'}$,$\mathcal{B}(\tau_{f_* X}) = \mathcal{B}(\tau_{X'})$,$\varphi_{f_* X} = \varphi_{X'}$ and $\mu_{f_* X} = f_* \mu_X$.

We need the following lemma

Lemma 19 *Let* $C' \in \mathcal{B}(\tau_{X'})$. *Then* $e_{f_* X}(C') = e_X(f^{-1}(C'))$

Proof. Let $U \in \mathcal{B}(\tau_A)$ be any measurable set and let $V = \varphi_{f_* X}^{-1}(U) = \varphi_{X'}^{-1}(U)$.

Then we have

$$
\begin{aligned}
&\int_V (\chi_{C'})_{f_* X} d\mu_{f_* X} = \int_V \chi_{C'} d(f_* \mu_X) = f_* \mu_X(C' \cap V) \\
&= \mu_X(f^{-1}(C') \cap f^{-1}(\varphi_{X'}^{-1}(U))) = \mu_X(f^{-1}(C') \cap \varphi_X^{-1}(U)) \\
&= \int_{\varphi_X^{-1}(U)} \chi_{f^{-1}(C')} d\mu_X
\end{aligned}
$$

On the other hand using the change of variable formula we have

$$\int_V [e_X(f^{-1}(C')) \circ \varphi_{f_*X}] d\mu_{f_*X} = \int \chi_{\varphi_{X'}^{-1}(U)}(e_X(f^{-1}(C')) \circ \varphi_{X'}) d(f_*\mu_X)$$
$$= \int (\chi_{\varphi_{X'}^{-1}(U)} \circ f)(e_X(f^{-1}(C')) \circ \varphi_{X'} \circ f) d\mu_X = \int \chi_{\varphi_X^{-1}(U)}[e_X(f^{-1}(C')) \circ \varphi_X] d\mu_X$$
$$= \int_{\varphi_X^{-1}(U)} (\chi_{f^{-1}(C')})_X d\mu_X = \int_{\varphi_X^{-1}(U)} \chi_{f^{-1}(C')} d\mu_X$$

Therefore by uniqueness $(\chi_{C'})_{f_*X} = e_X(f^{-1}(C')) \circ \varphi_{f_*X}$ and $e_{f_*X}(C') = e_X(f^{-1}(C'))$. ■

Using this lemma we can prove

Proposition 20 $(f \otimes_A g)_*(\mu_{X \otimes_A Y}) = \mu_{f_*X \otimes_A g_*Y}$

Proof. We have

$$[(f \otimes_A g)_*(\mu_{X \otimes_A Y})](C' \times_A D') = (\mu_{X \otimes_A Y})(f^{-1}(C') \times_A g^{-1}(D'))$$
$$= \int e_X(f^{-1}(C')) e_Y(g^{-1}(D')) d\mu_A = \int e_{f_*X}(C') e_{g_*Y}(D') d\mu_A$$
$$= \mu_{f_*X \otimes_A g_*Y}(C' \times_A D')$$

The proposition now follows from the uniqueness of measure theorem. ■

For probabilistic bundles X and Y define a relation $X \leq Y$ iff $\Omega_X = \Omega_Y$,$\mathcal{B}(\tau_X) = \mathcal{B}(\tau_Y)$,$\varphi_X = \varphi_Y$ and $\mu_X \leq \mu_Y$. If h is the density of μ_X with respect to μ_Y we say that $X \leq Y$ with density h. If $X \leq Y$ and $Y \leq X$ we write $X \approx Y$.

Lemma 21 *Let $X \leq Y$ with density h and let the densities of μ_A with respect to $(\varphi_X)_*\mu_X$ and of $(\varphi_Y)_*\mu_Y$ with respect to μ_A be g_X and f_Y. Then we have for $C \in \mathcal{B}(\tau_X)$*

$$e_X(C) = e_Y(F_Y G_X h \chi_C)$$

where $F_Y = f_Y \circ \varphi_Y$ and $G_X = g_X \circ \varphi_Y$.

Proof. Let f_X be the density of $(\varphi_X)_*\mu_X$ with respect to μ_A and g_X the density of μ_A with respect to $(\varphi_X)_*\mu_X$. Note that $f_X g_X = 1$ up to sets of measure zero.Let $V_A \in \mathcal{B}(\tau_A)$ be any measurable set. Then we have

$$\int_V e_X(C) f_X d\mu_A = \int_V e_X(C) d((\varphi_X)_*\mu_X) =$$
$$\int_{\varphi_X^{-1}(V)} (\chi_C)_X d\mu_X = \int_{\varphi_X^{-1}(V)} \chi_C d\mu_X =$$
$$\int_{\varphi_X^{-1}(V)} h\chi_C d\mu_Y = \int_V e_Y(h\chi_C) d((\varphi_Y)_*\mu_Y) =$$
$$\int_V e_Y(h\chi_C) f_Y d\mu_A$$

>>From this we can conclude that $e_X(C)f_X = e_Y(h\chi_C)f_Y$ up to sets of measure zero. But then since $f_X g_X = 1$ up to sets of measure zero we get $e_X(C) = e_Y(h\chi_C)g_X f_Y$. For any $\mathcal{B}(\tau_X)$ measurable function k and $\varphi_X^{-1}(\mathcal{B}(\tau_A))$ measurable function l we have $l(k)_X = (lk)_X$. Using this result we have $e_X(C) = e_Y(h\chi_C)g_X f_Y = e_Y(h\chi_C(g_X \circ \varphi_Y)(f_Y \circ \varphi_Y)) = e_Y(F_Y G_X h\chi_C)$. ∎

We can now prove a generalization of a well known result from the theory of product measures.

Proposition 22 *Let X_i and Y_i be probabilistic bundles with $X_i \leq Y_i$ for $i = 1, 2$. Then we have*

$$X_1 \otimes_A X_2 \leq Y_1 \otimes_A Y_2$$

Proof. Let the density of X_i with respect to Y_i be h_i for $i = 1, 2$. Then we have

$$\mu_{X_1 \otimes_A X_2}(C \times_A D) = \int e_{X_1}(C)e_{X_2}(D)d\mu_A$$

$$= \int e_{Y_1}(F_{Y_1}G_{X_1}h_1\chi_C)e_{Y_2}(F_{Y_2}G_{X_2}h_2\chi_D)d\mu_A$$

$$= \int (F_{Y_1}G_{X_1}h_1\chi_C \otimes F_{Y_2}G_{X_2}h_2\chi_D)^A d\mu_{Y_1 \otimes_A Y_2}$$

$$= \int_{C \times_A D} (F_{Y_1}h_1 G_{X_1} \otimes F_{Y_2}G_{X_2}h_2)^A d\mu_{Y_1 \otimes_A Y_2}$$

But then by uniqueness we can conclude that $\mu_{X_1 \otimes_A X_2} \leq \mu_{Y_1 \otimes_A Y_2}$ with density $(F_{Y_1}h_1 G_{X_1} \otimes F_{Y_2}G_{X_2}h_2)^A$. ∎

We now use the previous results to prove that

Proposition 23 *$f \otimes_A g$ is a morphism of probability spaces*

Proof. We have using propositions 20 and 22 that

$$(f \otimes_A g)_*(\mu_{X \otimes_A Y}) = \mu_{f_* X \otimes_A g_* Y} \leq \mu_{X' \otimes_A Y'}$$

∎

This proposition show that $\otimes_A$ is a bifunctor on the category of conditional probability spaces. We will now show that it is in fact an monoidal structure.

At the level of sets $\times_A$ is a monoidal structure with unit object A and associativity and unit constraints given by α^A, β^A and γ^A where α^A is the restriction of the associativity constraint for $\times$ on the category of sets and where

$$\beta_X^A(a, x) = x, \quad \text{and} \quad \gamma_X^A(x, a) = x.$$

In order to ensure that $\otimes_A$ is a monoidal structure we only need to verify that the maps α^A, β^A and γ^A are isomorphisms in the category of probabilistic bundles. They are all continuous and thus measurable maps. Furthermore we have

Lemma 24

$$(\beta_X^A)_*(\mu_{A\otimes_A X}) \approx \mu_X, \quad (\gamma_X^A)_*(\mu_{X\otimes_A A}) \approx \mu_X.$$

Proof. Let h_X be the density of μ_A with respect to $(\varphi_X)_*\mu_X$ and f_X the density of $(\varphi_X)_*\mu_X$ with respect to μ_A. Note that $h_X f_X = 1$ up to sets of μ_A measure zero. Let ν_X be the measure defined by $\nu_X = \int_U (h_X \circ \varphi_X) d\mu_X$. Then clearly $\nu_X \leq \mu_X$. But we also have $\mu_X(U) = \int_U d\mu_X = \int_U (f_X \circ \varphi_X)(h_X \circ \varphi_X) d\mu_X = \int_U (f_X \circ \varphi_X) d\nu_X$ and therefore we also have $\nu_X \leq \mu_X$. The two measures μ_X and ν_X are thus equivalent. For β^A we have

$$[(\beta_X^A)_*\mu_{A\otimes_A X}](U) = \mu_{A\otimes_A X}(\Omega_A \times_A U) = \int e_A(\Omega_A) e_X(U) d\mu_A = \int e_X(U) d\mu_A$$

$$= \int (\chi_U)_X (h_X \circ \varphi_X) d\mu_X = \int_U (h_X \circ \varphi_X) d\mu_X = \nu_X(U)$$

and therefore $(\beta_X^A)_*\mu_{A\otimes_A X} = \nu_X \approx \mu_X$ ∎

>From this lemma we have

Corollary 25 β_X^A *and* γ_X^A *are isomorphisms in the category of probabilistic bundles.*

For α_{XYZ}^A we need the following result

Lemma 26 $e_{X\otimes_A Y}(C \times_A D) = e_X(C) e_Y(D)$

Proof. We have seen in proposition 17 that $(\varphi_{X\otimes_A Y})_*(\mu_{X\otimes_A Y}) = \mu_A$. Using this and the change of variable formula we have for all $U \in \mathcal{B}(\tau_A)$

$$\int_{(\varphi_{X\otimes_A Y})^{-1}(U)} (e_X(C) e_Y(D) \circ \varphi_{X\otimes_A Y}) d(\mu_{X\otimes_A Y}) =$$
$$\int (\chi_U e_X(C) e_Y(D)) \circ (\varphi_{X\otimes_A Y}) d(\mu_{X\otimes_A Y}) = \int_U e_X(C) e_Y(D) d\mu_A$$

On the other hand we have

$$\int_{(\varphi_{X\otimes_A Y})^{-1}(U)} (\chi_{C\times_A D})_{X\otimes_A Y} d(\mu_{X\otimes_A Y})$$
$$= \int \chi_{(\varphi_{X\otimes_A Y})^{-1}(U)} (\chi_{C\times_A D})_{X\otimes_A Y} d(\mu_{X\otimes_A Y})$$
$$= \int (\chi_{\varphi_X^{-1}(U)\cap C\times_A D})_{X\otimes_A Y} d(\mu_{X\otimes_A Y}) = \int \chi_{\varphi_X^{-1}(U)\cap C\times_A D} d(\mu_{X\otimes_A Y})$$
$$= (\mu_{X\otimes_A Y})(\varphi_X^{-1}(U) \cap C \times_A D) = \int e_X(\varphi_X^{-1}(U) \cap C) e_Y(D) d\mu_A$$
$$= \int_U e_X(C) e_Y(D) d\mu_A$$

Therefore by uniqueness we can conclude that

$$e_{X\otimes_A Y}(C\times_A D) = e_X(C)e_Y(D)$$

■

Using this lemma we have

Proposition 27 α^A_{XYZ} *is a isomorphism in the category of probability bundles*

Proof.

$$\begin{aligned}
&(\alpha^A_{XYZ})_*\mu_{X\otimes_A(Y\otimes_A Z)}((C\times_A D)\times_A E)\\
&= \mu_{X\times\otimes_A(Y\otimes_A Z)}(C\times_A (D\times_A E))\\
&= \int e_X(C)e_{Y\otimes_A Z}(D\times_A E)d\mu_A = \int e_X(C)e_Y(D)e_Z(E)d\mu_A\\
&= \int e_{X\otimes_A Y}(C\times_A D)e_Z(E)d\mu_A = \mu_{(X\otimes_A Y)\otimes_A Z}((C\times_A D)\times_A E)
\end{aligned}$$

But this means that $(\alpha^A_{XYZ})_*\mu_{X\otimes_A(Y\otimes_A Z)} = \mu_{(X\otimes_A Y)\otimes_A Z}$ ■

In summary we have proved the following fundamental property of the category of probabilistic bundles.

Theorem 28 $\langle\otimes_A, A, \alpha^A, \beta^A, \gamma^A\rangle$ *is a monoidal structure on the category of probabilistic bundles.*

Note that when A is equal to the one point probability space T we obviously have $\mu_{X\otimes_T Y} = \mu_X\otimes\mu_Y$.

Example 29 *Let $\Omega_A = \{a_1, ..., a_n\}$ be a finite set. The topology on A is the power set $\tau_A = P(\Omega_A)$, so all set are clopen and $\mathcal{B}(\tau_A) = P(\Omega_A)$. A probability measure on the measurable space $\langle\Omega_A, P(\Omega_A)\rangle$ is a sequence $\mu_A = \{q_i\}_{i=1}^n$ with $q_i \geq 0$ and $\sum_{i=1}^n q_i = 1$. The probability space $A = \langle\Omega_A, P(\Omega_A), \mu_A\rangle$ is clearly a object in $\mathcal{P}$. Let X and Y be any pair object in $\mathcal{P}(A)$. The functions φ_X and φ_Y induce partitions of Ω_X and Ω_Y into clopen sets $\{C_i\}_{i=1}^n$ and $\{D_i\}_{i=1}^n$and furthermore $\varphi_X^{-1}(\mathcal{B}(\tau_A)) = \{C_i\}_{i=1}^n, \varphi_Y^{-1}(\mathcal{B}(\tau_A)) = \{D_i\}_{i=1}^n$.We have*

$$\begin{aligned}
\Omega_X\times_A\Omega_Y &= \cup_{i=1}^n C_i\times_A D_i\\
(\chi_C)_X &= \sum_{\mu_X(C_i)>0} P_X(C|C_i)\chi_{C_i}\\
(\chi_D)_Y &= \sum_{\mu_Y(D_i)>0} P_Y(D|D_i)\chi_{D_i}
\end{aligned}$$

where

$$\begin{aligned}
P_X(C|C_i) &= \frac{\mu_X(C\cap C_i)}{\mu_X(C_i)}\\
P_Y(D|D_i) &= \frac{\mu_Y(D\cap D_i)}{\mu_Y(D_i)}
\end{aligned}$$

Therefore

$$\theta_A(C \times D) = \sum_{I=1}^{n} P_X(C|C_i)P_Y(D|D_i)q_i$$

For the case of $n = 1$ *we get* $\Omega_X \times_A \Omega_Y = \Omega_X \times \Omega_Y$ *and* $\theta_A(C \times D) = \mu_X(C)\mu_Y(D)$. *This is the standard product measure.*

Example 30 *Let* $\Omega_X = \Omega_Y = I = [-1, 1]$ *and let* $\mathcal{B}(\tau_X) = \mathcal{B}(\tau_Y)$ *be the standard Borel structure on* I *and* $\mu_X = \mu_Y = \frac{1}{2}\lambda$ *where* λ *is the lebesques measure on* R *restricted to the interval* I. *Let* A *be the interval* $J = [0, 1]$, *standard Borel structure and* $\mu_A = \lambda$. *Let* $\varphi_X = \varphi_Y$ *be the absolute value function. The maps* φ_X *and* φ_Y *are continuous and* $(\varphi_X)_*\mu_X = (\varphi_Y)_*\mu_Y = \mu_A$ *so* X *and* Y *are objects in* $\mathcal{P}(A)$. *For any set* C *in* R *let*

$$C_- = \{x \in R \mid -x \in C\}$$

We observe that

$$\varphi_X^{-1}(\mathcal{B}(\tau_A)) = \{V \cup V_- \mid V \in \mathcal{B}(\tau_A)\}$$

An easy calculation gives us the following characterization

Proposition 31 $f : \Omega_X \to R$ *is measurable with respect to* $\varphi_X^{-1}(\mathcal{B}(\tau_A))$ *if and only if* f *is an even function. For absolute integrable* $\mathcal{B}(\tau_X)$*-measurable function* f *we have*

$$e_X(f) = \frac{1}{2}(f(x) + f(-x))$$

Therefore the measure θ_A *will in this case be*

$$\theta_A(C \times D) = \int_0^1 \frac{1}{4}(\chi_C + \chi_{C_-})(\chi_D + \chi_{D_-})d\lambda$$

This measure is supported on the set

$$\Omega_X \times_A \Omega_Y = \{(x, y) \in R^2 \mid \quad |x| = |y|\}$$

4 Markov Relations

If Ω_X and Ω_Y are sets then a relation between Ω_X and Ω_Y is a subset of $\Omega_X \times \Omega_Y$ and a relation between measurable spaces $\langle\Omega_X, \mathcal{B}(\tau_X)\rangle$ and $\langle\Omega_Y, \mathcal{B}(\tau_Y)\rangle$ is a measurable subset in $\langle\Omega_X \times \Omega_Y, \mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)\rangle$. The collection of all measurable sets in the product $\langle\Omega_X \times \Omega_Y, \mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)\rangle$ form the algebra of observable relations. We get a probabilistic relation by assigning a probability to all observable relations. This is the obvious and natural notion of a probabilistic relation. In the following we introduce relations of probabilistic bundles and show that they form the arrows in a larger category that we call the category of Markov relations. We explore some of the properties of this category and its relation to the category of probabilistic bundles.

Let $p_{XY} : \Omega_X \otimes \Omega_Y \to \Omega_X$ and $q_{XY} : \Omega_X \times \Omega_Y \to \Omega_Y$ be the projections and let p^A_{XY} and q^A_{XY} be their restriction to the set $\Omega_X \times_A \Omega_Y \subset \Omega_X \times \Omega_Y$.

Definition 32 *A* Markov relation *between the probabilistic bundles X and Y is a probability measure μ on the measurable space $\langle \Omega_X \times_A \Omega_Y, \mathcal{B}(\tau_X) \otimes_A \mathcal{B}(\tau_Y) \rangle$ such that*

$$(p^A_{XY})_* \mu \geq \mu_X, \quad \text{and} \quad (q^A_{XY})_* \mu \leq \mu_Y.$$

We will denote a relation μ between X and Y by $\mu : X \rightsquigarrow Y$.

We will first show that any arrow $f : X \to Y$ between probabilistic bundles will give rise to a Markov relation between X and Y.

Let $f : X \to Y$ be a morphism of probabilistic bundles and let $\Gamma_f : \Omega_X \to \Omega_X \times \Omega_Y$ be the graph of f, $\quad \Gamma_f(x) = (x, f(x))$.

The graph of f is clearly a $\mathcal{B}(\tau_X) - \mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y)$ measurable map and we define a probability measure μ_f on the measurable space $\langle \Omega_X \times \Omega_Y, \mathcal{B}(\tau_X) \otimes \mathcal{B}(\tau_Y) \rangle$ by

$$\mu_f = (\Gamma_f)_* \mu_X.$$

Proposition 33 *The measure μ_f has its support in the closed set $\Omega_X \times_A \Omega_Y$.*

Proof. Indeed, let $C \in \mathcal{B}(\tau_X)$ and $D \in \mathcal{B}(\tau_Y)$ and assume that $C \times D$ is in the complement of $\Omega_X \times_A \Omega_Y$. Note that for any box we have $\mu_f(C \times D) = \mu_X(C \cap f^{-1}(D))$. Take $x \in C \cap f^{-1}(D)$, then $x \in C$, and there exists $y \in D$ with $y = f(x)$. But then $\varphi_Y(y) = \varphi_Y(f(x)) = \varphi_X(x)$ because f is a morphism of probabilistic bundles. Therefore we have $(x, y) \in C \times_A D$. But this last set is empty. We can thus conclude that $C \cap f^{-1}(D) = \varnothing$ and so $\mu_f(C \times D) = 0$. The proof of the proposition in completed by an argument like in lemma 13. ∎

The measure μ_f therefore restricts to a measure on the measurable space $\langle \Omega_X \times_A \Omega_Y, \mathcal{B}(\tau_X) \otimes_A \mathcal{B}(\tau_Y) \rangle$. Moreover, we have $(p^A_{XY})_* \mu_f = \mu_X$ and $(q^A_{XY})_* \mu_f = f_* \mu_X \leq \mu_Y$ and this proves that $\mu_f : X \rightsquigarrow Y$ is a Markov relation. Markov relations thus exists. The following proposition gives is another example of a Markov relation.

Proposition 34 *Let X and Y be probability bundles. Then*

$$\mu_{X \otimes_A Y} : X \rightsquigarrow Y$$

is a Markov relation

Proof. We know that $\mu_{X \otimes_A Y}$ is a measure on $\langle \Omega_X \times_A \Omega_Y, \mathcal{B}(\tau_X) \otimes_A \mathcal{B}(\tau_Y) \rangle$ and from lemma 16 we have $(p^A_{XY})_* \mu \approx \mu_X$, and $(q^A_{XY})_* \mu \approx \mu_Y$. ∎

4.1 Composition of Markov Relations

We will now show that Markov relations, like morphisms, can be composed and that they form the arrows in a category that includes the category of probability bundles.

First we must introduce some notation that will be used in the rest of this section. For any pair of probabilistic bundles X and Y let the subalgebras G^p_{XY} and G^q_{XY} of $\mathcal{B}(\tau_X) \otimes_A \mathcal{B}(\tau_Y)$ be given by $G^p_{XY} = (p^A_{XY})^{-1}(\mathcal{B}(\tau_X))$ and

$G^q_{XY} = (q^A_{XY})^{-1}(\mathcal{B}(\tau_Y))$. For any relation $\mu : X \rightsquigarrow Y$ and measurable function F on $\langle \Omega_X \times \Omega_Y, \mathcal{B}(\tau_X) \otimes_A \mathcal{B}(\tau_Y), \mu\rangle$ with $\int |F|\, d\mu < \infty$ let $(F)^p_{XY}$ and $(F)^q_{XY}$ be the conditional expectation of F with respect to the subalgebras G^p_{XY} and G^q_{XY}. The function $e^p_{XY}(F)$ is well defined on Ω_X up to sets of $(p^A_{XY})_*\mu$ -measure zero by the identity $(F)^p_{XY} = e^p_{XY}(F) \circ p^A_{XY}$. Similarly the function $e^q_{XY}(F)$ is well defined on Ω_Y up to sets of $(q^A_{XY})_*\mu$-measure zero by the identity $(F)^q_{XY} = e^q_{XY}(F) \circ q^A_{XY}$.

Let now $\mu : X \rightsquigarrow Y$ and $\nu : Y \rightsquigarrow Z$ be relations and let $\theta = (q^A_{XY})_*\mu$ be the measure induced on Y by μ.

Let $C \in \mathcal{B}(\tau_X)$ and $D \in \mathcal{B}(\tau_Z)$ be measurable sets. Since $\theta = (q^A_{XY})_*\mu \leq \mu_Y \leq (p^A_{YZ})_*\nu$ the functions $e^q_{XY}(C \times \Omega_Y)$ and $e^p_{YZ}(\Omega_Y \times D)$ are well defined on $\langle \Omega_Y, \mathcal{B}(\tau_Y)\rangle$ up to sets of θ-measure zero. The following set function is well defined by Hölder

$$\alpha(C \times D) = \int e^q_{XY}(C \times \Omega_Y) e^p_{YZ}(\Omega_Y \times D) d\theta$$

Using arguments entirely similar to the ones we used for the fibered product we can show that this set function extends to a unique measure defined on $\langle \Omega_{X\otimes Z}, \mathcal{B}(\tau_{X\otimes Z})\rangle$. We denote this unique measure by the notation $\nu \circ \mu$. The following result is proved in a way that is similar to the corresponding results for the fibered product and is ommited.

Theorem 35 *$\nu \circ \mu$ is a probability measure on the product supported on the set $\Omega_X \times_A \Omega_Y$ and we have*

$$(p_{XZ})_*(\nu \circ \mu) \geq \mu_X$$
$$(q_{XZ})_*(\nu \circ \mu) \leq \mu_Z$$

Furthermore if $\mu : X \rightsquigarrow Y$ and $\nu : Y \rightsquigarrow Z$ are relations with $(q_{XY})_*\mu \approx \mu_Y$,$(p_{XY})_*\mu \approx \mu_X$ and similar for ν. Then

$$(p_{XZ})_*(\nu \circ \mu) \approx \mu_X$$
$$(q_{XZ})_*(\nu \circ \mu) \approx \mu_Z$$

if equivalence of measures is replaced with identity, then the same holds for the composition $\nu \circ \mu$.

These properties identify $\nu \circ \mu$ as a Markov relation.

Corollary 36 *Let $\mu : X \rightsquigarrow Y$ and $\nu : Y \rightsquigarrow Z$ be Markov relations. Then*

$$\nu \circ \mu : X \rightsquigarrow Z$$

is a Markov relation.

We have seen that any arrows in $\mathcal{P}(A)$ give rise to Markov relations. Let $f : X \to Y$ and $g : Y \to Z$ be arrows in $\mathcal{P}(A)$ and let $\mu_f : X \rightsquigarrow Y$ and

$\mu_g : Y \rightsquigarrow Z$ be the corresponding relations, $\mu_f = (\Gamma_f)_* \mu_X$ and $\mu_g = (\Gamma_g)_* \mu_Y$. Since we already know how to compose arrows in $\mathcal{P}(A)$ the following result is very satisfying and testify to the naturalness of our construction of composition of Markov relations.

Proposition 37

$$\mu_g \circ \mu_f = \mu_{g \circ f}$$

Proof. Let $C \in \mathcal{B}(\tau_X)$ and $D \in \mathcal{B}(\tau_Z)$.Then we have

$$(\mu_g \circ \mu_f)(C \times_A D) = \int e^q_{XY}(C \times_A \Omega_Y) e^p_{YZ}(\Omega_Y \times_A D) d\theta$$
$$= \int e^q_{XY}(C \times_A \Omega_Y) e^p_{YZ}(\Omega_Y \times_A D) d(q^A_{XY})_* \mu_f$$
$$= \int \chi_{C \times_A \Omega_Y} (e^p_{YZ}(\Omega_Y \times_A D) \circ q^A_{XY}) d\mu_f$$
$$= \int (\chi_{C \times_A \Omega_Y} \circ \Gamma_f)(e^p_{YZ}(\Omega_Y \times_A D) \circ q^A_{XY} \circ \Gamma_f) d\mu_X$$
$$= \int \chi_C (e^p_{YZ}(\Omega_Y \times_A D) \circ f) d\mu_X$$

and

$$\int_{(p^A_{YZ})^{-1}(V)} (\chi_D \circ g \circ p^A_{YZ}) d\mu_g = \int_{V \times_A \Omega_Z} (\chi_D \circ g \circ p^A_{YZ}) d(\Gamma_g)_* \mu_Y$$
$$= \int (\chi_{V \times_A \Omega_Z} \circ \Gamma_g)(\chi_D \circ g \circ p^A_{YZ} \circ \Gamma_g) d\mu_Y = \int \chi_V (\chi_D \circ g) d\mu_Y$$
$$= \mu_Y(V \cap g^{-1}(D)) = \mu_g(V \times_A D) = \int \chi_{V \times_A D} d\mu_g$$
$$= \int \chi_{V \times_A \Omega_Z} \chi_{\Omega_Y \times_A D} d\mu_g = \int_{(p^A_{YZ})^{-1}(V)} \chi_{\Omega_Y \times_A D} d\mu_g$$

By uniqueness of conditional expectation we can conclude that

$$e^p_{YZ}(\Omega_Y \times_A D) = \chi_D \circ g$$

But then the previous calculation give

$$(\mu_g \circ \mu_f)(C \times_A D)$$
$$= \int \chi_C (\chi_D \circ g \circ f) d\mu_X$$
$$= \mu_X(C \cap (g \circ f)^{-1}(D))$$
$$= \mu_{g \circ f}(C \times_A D)$$

The proposition now follows from the uniqueness of measure theorem. ■

For any pair of probabilistic bundles X and Y there is a special relation $\mu_{X\otimes_A Y} : X \rightsquigarrow Y$. For the special case when $A = T$ the fibred product $\mu_{X\otimes_T Y} = \mu_X \otimes \mu_Y$ is just the usual product of two measures. For this case the measure signify independence between the states X and Y. In the general case we will also think of $\mu_{X\otimes_A Y}$ as signifying independence between the probabilistic bundles X and Y. The following proposition support this interpretation by showing that the composition preserve this special class of relations.

Proposition 38

$$(\mu_X \otimes \mu_Y) \circ (\mu_Y \otimes_A \mu_Z) = \mu_X \otimes_A \mu_Z$$

Proof. Let f be the density of $(\varphi_Y)_*\mu_Y$ with respect to μ_A and g the density of $(\varphi_Y)_*\mu_Y$ with respect to μ_A.

Then the density of μ_Y with respect to $(q^A_{XY})_*(\mu_X \otimes_A \mu_Y)$ is $\varphi^*_Y(f)$.

Note that $fg = 1$ up to sets of measure zero with respect to μ_A.

Using this we have for a given $C \in \mathcal{B}(\tau_X)$ and any $V \in \mathcal{B}(\tau_Y)$

$$\int_{(q^A_{XY})^{-1}(V)} (\chi_{C\times_A \Omega_Y})^q_{XY} d(\mu_X \otimes_A \mu_Y) = \int \chi_{C\times_A V} d(\mu_X \otimes_A \mu_Y)$$
$$= (\mu_X \otimes_A \mu_Y)(C \times_A V) = \int e_X(C)e_Y(V)d\mu_A = \int e_X(C)e_Y(V)gd(\varphi_Y)_*\mu_Y$$
$$= \int \chi_V \varphi^*_Y(e_X(C)g)d\mu_Y = \int \chi_V \varphi^*_Y(e_X(C))\varphi^*_Y(g)\varphi^*_Y(f)d(q^A_{XY})_*(\mu_X \otimes_A \mu_Y)$$
$$= \int_{(q^A_{XY})^{-1}(V)} (q^A_{XY})^*(\varphi^*_Y(e_X(C)))d(\mu_X \otimes_A \mu_Y)$$

By the uniqueness of conditional expectation we have the identity $e^q_{XY}(C\times_A \Omega_Y) = \varphi^*_Y(e_X(C))$. In a similar way we get for any $D \in \mathcal{B}(\tau_Z)$ the identity $e^p_{YZ}(\Omega_Y \times_A D) = \varphi^*_Y(e_Z(D))$. Using these two identities and proposition 17 we have

$$\begin{aligned}
&((\mu_X \otimes \mu_Y) \circ (\mu_Y \otimes_A \mu_Z))(C \times_A D) \\
&= \int e^q_{XY}(C \times_A \Omega_Y)e^p_{YZ}(\Omega_Y \times_A D)d(q^A_{XY})_*(\mu_X \otimes_A \mu_Y) \\
&= \int \varphi^*_Y(e_X(C)e_Z(D))d(q^A_{XY})_*(\mu_X \otimes_A \mu_Y) \\
&= \int e_X(C)e_Z(D)d(\varphi_{X\otimes_A Y})_*(\mu_X \otimes_A \mu_Y) \\
&= \int e_X(C)e_Z(D)d\mu_A = (\mu_X \otimes_A \mu_Z)(C \times_A D)
\end{aligned}$$

The proposition now follows by the uniqueness of measure theorem. ■

4.2 The category of markov Relations

In this section we show that the composition is associative and that probabilistic bundles in general form a weak category.

Let $\mu : X \rightsquigarrow Y$, $\nu : Y \rightsquigarrow Z$ and $\omega : Z \rightsquigarrow W$ be relations. These relations can be composed in two different ways, $\omega \circ (\nu \circ \mu)$ and $(\omega \circ \nu) \circ \mu$. These two compositions are in general equal. In order to prove this we will need the following two lemmas.

Lemma 39 *We have*

$$(p^A_{YW})_*(\omega \circ \nu) = (p^A_{YZ})_*\nu$$

and if g is the density of $(q^A_{XY})_*\mu$ with respect to $(p^A_{YZ})_*\nu$, then the density of $(q^A_{XZ})_*(\nu \circ \mu)$ with respect to $(q^A_{YZ})_*\nu$ is $e^q_{YZ}((p^A_{YZ})^*(g))$

We will also need the following result

Lemma 40 *The following identities*

$$e^p_{YZ}((q^A_{YZ})^*(e^p_{ZW}(\Omega_Z \times_A D))) = e^p_{YW}(\Omega_Y \times_A D)$$
$$e^q_{XZ}(C \times_A \Omega_Z)e^q_{YZ}((p^A_{YZ})^*(g)) = e^q_{YZ}((p^A_{YZ})^*(e^q_{XY}(C \times_A \Omega_Y)g))$$

hold.

Proof. Let $V \in \mathcal{B}(\tau_Y)$, then we have

$$\int_{(p^A_{YZ})^{-1}(V)} (p^A_{YZ})^*(e^p_{YW}(\Omega_Y \times_A D))d\nu = \int_V e^p_{YW}(\Omega_Y \times_A D)d(p^A_{YZ})_*\nu$$
$$= \int_V e^p_{YW}(\Omega_Y \times_A D)d(p^A_{YW})_*(\omega \circ \nu)$$
$$= \int \chi_{V \times_A D} d(\omega \circ \nu) = (\omega \circ \nu)(V \times_A D)$$
$$= \int e^q_{YZ}(V \times_A \Omega_Z)e^p_{ZW}(\Omega_Z \times_A D)d(q^A_{YZ})_*\nu$$
$$= \int_{(p^A_{YZ})^{-1}(V)} (q^A_{YZ})^*(e^p_{ZW}(\Omega_Z \times_A D))d\nu$$

The first identity now follow from the uniqueness of the conditional expectation. For the second identity let $V \in \mathcal{B}(\tau_Z)$. Then we have

$$\begin{aligned}
&\int_{(q^A_{YZ})^{-1}(V)} (q^A_{YZ})^*(e^q_{XZ}(C \times_A \Omega_Z)e^q_{YZ}((p^A_{YZ})^*(g)))d\nu \\
&= \int_V e^q_{XZ}(C \times_A \Omega_Z)e^q_{YZ}((p^A_{YZ})^*(g))d(q^A_{YZ})_*\nu \\
&= \int_V e^q_{XZ}(C \times_A \Omega_Z)d(q^A_{XZ})_*(\nu \circ \mu) \\
&= \int_{(q^A_{XZ})^{-1}(V)} \chi_{C\times_A \Omega_Z} d(\nu \circ \mu) \\
&= \int \chi_{C\times_A V} d(\nu \circ \mu) = (\nu \circ \mu)(C \times_A V) \\
&= \int e^q_{XY}(C \times_A \Omega_Y)e^p_{YZ}(\Omega_Y \times_A V)d(q^A_{XY})_*\mu \\
&= \int e^q_{XY}(C \times_A \Omega_Y)e^p_{YZ}(\Omega_Y \times_A V)gd(p^A_{YZ})_*\nu \\
&= \int (p^A_{YZ})^*(e^q_{XY}(C \times_A \Omega_Y)g)\chi_{\Omega_Y \times_A V} d\nu \\
&= \int_{(q^A_{YZ})^{-1}(V)} (p^A_{YZ})^*(e^q_{XY}(C \times_A \Omega_Y)g)d\nu
\end{aligned}$$

The uniqueness of conditional expectation now gives the second identity. ■

Proposition 41

$$\omega \circ (\nu \circ \mu) = (\omega \circ \nu) \circ \mu$$

Proof. let $C \in \mathcal{B}(\tau_X)$ and $D \in \mathcal{B}(\tau_W)$. Then we have

$$\begin{aligned}
&(\omega \circ (\nu \circ \mu))(C \times_A D) \\
&= \int e^q_{XZ}(C \times_A \Omega_Z)e^p_{ZW}(\Omega_Z \times_A D)d(q^A_{XZ})_*(\nu \circ \mu) \\
&= \int e^q_{XZ}(C \times_A \Omega_Z)e^q_{YZ}((p^A_{YZ})^*(g))e^p_{ZW}(\Omega_Z \times_A D)d(q^A_{YZ})_*\nu \\
&= \int e^q_{YZ}((p^A_{YZ})^*(e^q_{XY}(C \times_A \Omega_Y)g))e^p_{ZW}(\Omega_Z \times_A D)d(q^A_{YZ})_*\nu \\
&= \int (p^A_{YZ})^*(e^q_{XY}(C \times_A \Omega_Y)g)(q^A_{YZ})^*(e^p_{ZW}(\Omega_Z \times_A D))d\nu \\
&= \int e^q_{XY}(C \times_A \Omega_Y)e^p_{YZ}((q^A_{YZ})^*(e^p_{ZW}(\Omega_Z \times_A D)))gd(p^A_{YZ})\nu \\
&= \int e^q_{XY}(C \times_A \Omega_Y)e^p_{YZ}((q^A_{YZ})^*(e^p_{ZW}(\Omega_Z \times_A D)))d(q^A_{XY})_*\mu \\
&= \int e^q_{XY}(C \times_A \Omega_Y)e^p_{YW}(\Omega_Y \times_A D)d(q^A_{XY})_*\mu = ((\omega \circ \nu) \circ \mu)(C \times_A D)
\end{aligned}$$

The proposition now follows from the uniqueness of measure theorem. ■

Proposition 42 *Let $\mu : X \rightsquigarrow Y$ be a relation. Then we have*

$$\mu \circ \mu_{1_X} \leq \mu$$
$$\mu_{1_Y} \circ \mu = \mu$$

Proof. Let $C \in \mathcal{B}(\tau_X)$ and $D \in \mathcal{B}(\tau_Z)$ and let $V \in \mathcal{B}(\tau_X)$. Then we have

$$\int_{(q^A_{XX})^{-1}(V)} (\chi_C \circ q^A_{XX}) d\mu_{1_X} = \int (\chi_{\Omega_{\Omega_X \times V}} \circ \Gamma_{1_X})(\chi_C \circ q^A_{XX} \circ \Gamma_{1_X}) d\mu_X$$
$$= \int \chi_V \chi_C d\mu_X = \mu_X(C \cap V) = \mu_{1_X}(C \times_A V) = \int_{(q^A_{XX})^{-1}(V)} \chi_{C \times_A \Omega_X} d\mu_{1_X}$$

By uniqueness of the conditional expectation we can conclude that $e^q_{XX}(C \times_A \Omega_X) = \chi_C$. Let g be the density of μ_X with respect to $(p^A_{XY})_* \mu$.

$$(\mu \circ \mu_{1_X})(C \times_A D) = \int e^q_{XX}(C \times_A \Omega_X) e^p_{XY}(\Omega_X \times_A D) d\theta$$
$$= \int \chi_C e^p_{XY}(\Omega_X \times_A D) d(q^A_{XX})_* \mu_{1_X} = \int \chi_C e^p_{XY}(\Omega_X \times_A D) d\mu_X$$
$$= \int \chi_C e^p_{XY}(\Omega_X \times_A D) g d(p^A_{XY})_* \mu = \int (\chi_C \circ p^A_{XY}) \chi_{\Omega_X \times_A D} (p^A_{XY})^*(g) d\mu$$
$$= \int \chi_{C \times_A \Omega_X} \chi_{\Omega_X \times_A D} (p^A_{XY})^*(g) d\mu = \int_{C \times_A D} (p^A_{XY})^*(g) d\mu$$

By uniqueness of measure and Radon-Nikodym we have $\mu \circ \mu_{1_X} \leq \mu$. The second part of the proposition is proved in a entirely similar way. ■

The relation μ_{1_X} will thus not in general act as a left and right identity under composition.

It is however clear from the proof of the proposition that if we restrict to the class of relations that satisfies $(p^A_{XY})_* \mu = \mu_X$ then the relation μ_{1_X} is really a right and left identity under composition.

We can now form a structure $\mathcal{R}(A)$ where objects are probabilistic bundles and where arrows are relations. Because of proposition 42 the structure $\mathcal{R}(A)$ will in general not be a category but we will abuse the term and define $\mathcal{R}(A)$ to be the category of relations of probabilistic bundles.

For the restricted class of relations with $(p^A_{XY})_* \mu = \mu_X$ it will be a true category.

For any pair of objects we have at least one relation, namely $\mu_X \otimes_A \mu_Y$. The statement that this is a relation follows from lemma 13 and proposition 16. This is the relation signifying independency between X and Y in the context of probabilistic bundles.

The following result supports this interpretation

Proposition 43 *Let $f : X \to Y$ be a morphism of probabilistic relations*

$$\mu_f \circ (\mu_X \otimes_A \mu_X) = (1 \otimes_A f)_*(\mu_X \otimes_A \mu_X)$$

Proof. Let g and f be densities as defined in the proof of the previous proposition for the case $X = Y$.

Then we have

$$\begin{aligned}
&\mu_f \circ (\mu_X \otimes_A \mu_X)(C \times_A D) \\
&= \int e^q_{XX}(C \times_A \Omega_X) e^p_{XY}(\Omega_X \times_A D) d(q^A_{XX})_*(\mu_X \otimes_A \mu_X) \\
&= \int \varphi^*_X(e_X(C))\chi_{f^{-1}(D)} d(q^A_{XX})_*(\mu_X \otimes_A \mu_X) \\
&= \int \varphi^*_X(e_X(C))\chi_{f^{-1}(D)} \varphi^*_X(g) d\mu_X \\
&= \int e_X(C) e_X(f^{-1}(D)) g d(\varphi_X)_*\mu_X = \int e_X(C) e_X(f^{-1}(D)) d\mu_A \\
&= (\mu_X \otimes_A \mu_X)(C \times_A f^{-1}(D)) = (1 \otimes_A f)_*(\mu_X \otimes_A \mu_X)(C \times_A D)
\end{aligned}$$

■

We will now extend the monoidal structure $\otimes_A$ to the category of relations $\mathcal{R}(A)$.

Let $\mu : X \rightsquigarrow X'$ and $\nu : Y \rightsquigarrow Y'$ be relations and let $\sigma_{X'Y} : \Omega_{X'} \times \Omega_Y \to \Omega_Y \times \Omega_{X'}$ be the map, $\sigma_{X'Y}(x', y) = (y, x')$.

Using this we can define a measurable map $(1_X \times \sigma_{X'Y} \times 1_{Y'})$ that restricts to a well defined continuous map

$$(1_X \otimes_A \sigma_{X'Y} \otimes_A 1_{Y'}) : (\Omega_X \times_A \Omega_{X'}) \times_A (\Omega_Y \times_A \Omega_{Y'}) \to (\Omega_X \times_A \Omega_Y) \times_A (\Omega_{X'} \times_A \Omega_{Y'}).$$

Let the expression $\mu \boxtimes_A \nu$ be defined by

$$\mu \boxtimes_A \nu = (1_X \otimes_A \sigma_{X'Y} \otimes_A 1_{Y'})_*(\mu \otimes_A \nu)$$

Proposition 44 *$\mu \boxtimes_A \nu : X \otimes_A Y \rightsquigarrow X' \otimes_A Y'$ is a relation*

Proof. $\mu : X \rightsquigarrow X'$ is a relation and therefore we have $(p^A_{XX'})_*\mu \geq \mu_X$ and $(q^A_{XX'})_*\mu \leq \mu_{X'}$. But then

$$\begin{aligned}
(\varphi_{X \otimes_A X'})_*\mu &= (\varphi_X \circ p^A_{XX'})_*\mu \geq (\varphi_X)_*\mu_X \geq \mu_A \\
(\varphi_{X \otimes_A X'})_*\mu &= (\varphi_{X'} \circ q^A_{XX'})_*\mu \leq (\varphi_{X'})_*\mu_{X'} \leq \mu_A
\end{aligned}$$

so $(\varphi_{X \otimes_A X'})_*(\mu) \approx \mu_A$. In a similar way we get $(\varphi_{Y \otimes_A Y'})_*\nu \approx \mu_A$. Therefore the product $\mu \otimes_A \nu$ is defined and is supported on the set $(\Omega_X \times_A \Omega_{X'}) \times_A (\Omega_Y \times_A \Omega_{Y'})$. But then $\mu \boxtimes_A \nu$ is a well defined probability measure supported on the set $(\Omega_X \times_A \Omega_Y) \times_A (\Omega_{X'} \times_A \Omega_{Y'})$ since the map $(1_X \otimes_A \sigma_{X'Y} \otimes_A 1_{Y'})$ is continuous. It is easy to see that

$$\begin{aligned}
(q^A_{X \otimes_A Y, X' \otimes_A Y})_*(\mu \boxtimes_A \nu) &= (q^A_{XX'})_*\mu \otimes_A (q^A_{YY'})_*\nu \leq \mu_{X'} \otimes_A \mu_{Y'} \\
(p^A_{X \otimes_A Y, X' \otimes_A Y})_*(\mu \boxtimes_A \nu) &= (p^A_{XX'})_*\mu \otimes_A (p^A_{YY'})_*\nu \geq \mu_X \otimes_A \mu_Y
\end{aligned}$$

and this completes the proof. ■

The definition of the product of relations in this particular way is justified by the following result

Proposition 45 *Let $f : X \to X'$ and $g : Y \to Y'$ be morphisms of probabilistic bundles. Then we have*

$$\mu_f \boxtimes_A \mu_g = \mu_{f \otimes_A g}$$

Proof. Let $C \in \mathcal{B}(\tau_X)$ and $E \in \mathcal{B}(\tau_{X'})$ and let $V \in \mathcal{B}(\tau_A)$. Then we have

$$\int_{(\varphi_{X\otimes_A X'})^{-1}(V)} (\chi_{C\times_A E})_{X\otimes_A X'} d\mu_f = \int \chi_{(C\cap\varphi_X^{-1}(V))\times_A E} d\mu_f$$
$$= \mu_f((C \cap \varphi_X^{-1}(V)) \times_A E) = \mu_X(C \cap \varphi_X^{-1}(V) \cap f^{-1}(E))$$
$$= \int \varphi_X^*(\chi_V)\chi_{C\cap f^{-1}(E)} d\mu_X$$
$$= \int \varphi_X^*(\chi_V e_X(C \cap f^{-1}(E))) d\mu_X$$
$$= \int f^*(\varphi_{X'}^*(\chi_V e_X(C \cap f^{-1}(E)))) d\mu_X$$
$$= \int \varphi_{X'}^*(\chi_V e_X(C \cap f^{-1}(E))) df_*\mu_X$$
$$= \int \varphi_{X'}^*(\chi_V e_X(C \cap f^{-1}(E))) d(q^A_{XX'})_*\mu_f$$
$$= \int_{\varphi_{X\otimes_A X'})^{-1}(V)} (\varphi_{X\otimes_A X'})^*(e_X(C \cap f^{-1}(E))) d\mu_f$$

By the uniqueness of conditional expectation we have the identity $e_{X\otimes_A X'}(C\times_A E) = e_X(C\cap f^{-1}(E))$. In a similar way we get the identity $e_{Y\otimes_A Y'}(D \times_A F) = e_Y(D \cap g^{-1}(F))$ for all $D \in \mathcal{B}(\tau_Y)$ and $F \in \mathcal{B}(\tau_{Y'})$. Using these identities we have

$$(\mu_f \boxtimes_A \mu_g)(C \times_A D \times_A E \times_A F) = (\mu_f \otimes_A \mu_g)(C \times_A E \times_A D \times_A F)$$
$$= \int e_{X\otimes_A X'}(C \times_A E) e_{Y\otimes_A Y'}(D \times_A F) d\mu_A$$
$$= \int e_X(C \cap f^{-1}(E)) e_Y(D \cap g^{-1}(F)) d\mu_A$$
$$= (\mu_X \otimes_A \mu_Y)(C \cap f^{-1}(E) \times_A D \cap g^{-1}(F))$$
$$= (\mu_X \otimes_A \mu_Y)((C \times_A D) \cap (f^{-1}(E) \cap g^{-1}(F))) = \mu_{f\otimes_A g}(C \times_A D \times_A E \times_A F)$$

and the proposition follows from the uniqueness of measure theorem. ■

This proposition show that the product $\boxtimes_A$ is an extension of the product $\otimes_A$.

5 Quantizations

5.1 Algebra of probabilistic bundles and relations

Let X be any probabilistic bundle and let $L_\infty(X)$ be the space of essentially bounded complex valued functions on X. Elements of X are classes of essentially bounded functions differing on sets of measure zero with respect to μ_X. The space $L_\infty(X)$ is not only a Banach space but also a C^*- algebra with respect to the natural operations.

Let $f : X \to Y$ be a morphism of probabilistic bundles and let $f^* : L_\infty(Y) \to L_\infty(X)$ be the map defined by $f^*([h]) = [h \circ f]$.

It is easy to see that f^* is well defined, continuous and commutes with the product and $*$-operation. It is thus a morphism of C^* algebras.

For any probabilistic bundle we have a map $\varphi_X : X \to A$. This gives a morphism of C^* algebras $(\varphi_X)^* : L_\infty(A) \to L_\infty(X)$. This morphism is moreover a monomorphism and give each C^* algebra $L_\infty(X)$ the structure of a $L_\infty(A) - L_\infty(A)$ bimodule with the left and right action of $L_\infty(A)$ on $L_\infty(X)$ induced by the map $(\varphi_X)^*$.

$$\mu^l_X : L_\infty(A) \otimes_\pi L_\infty(X) \to L_\infty(X),$$
$$\mu^r_X : L_\infty(X) \otimes_\pi L_\infty(A) \to L_\infty(X),$$

where $\mu^l_X([h] \otimes [h']) = (\varphi_X)^*([h])[h']$ and similarly for the right action.

Here $E \otimes_\pi F$ is the projective tensor product of two Banach spaces E and F.

Using this product there is a bijective correspondence between the set of continuous bilinear maps from $E \times F$ to G and the set of continuous linear maps from $E \otimes F$ to G for any Banach space G. It defines furthermore a monoidal structure on the category of Banach spaces with $\mathbb{C}$ as the unit object. In the following we denote the algebra $L_\infty(A)$ by the symbol $\mathcal{A}$.

Let $\mathcal{C}_\mathcal{A}$ be the category whose objects are banach spaces and $\mathcal{A} \to \mathcal{A}$ bimodules and morphisms are morphisms of Banach spaces that commutes with the left and right action of $\mathcal{A}$. These morphisms are thus $\mathcal{A} - \mathcal{A}$ bimodule morphisms.

The observations made above can be formulated as the existence of a contravariant functor $F : \mathcal{P}(A) \to \mathcal{C}_\mathcal{A}$ defined on objects and morphisms by

$$F(X) = L_\infty(X),$$
$$F(f) = f^*.$$

We will now show that the functor F can be extended to Markov relations.

Let $\mu : X \rightsquigarrow Y$ be a Markov relation and let $[h] \in L_\infty(Y)$. The condition $(p^A_{XY})_*\mu \geq \mu_X$ ensure that the expression $[e^p_{XY}((q^A_{XY})^*([h]))]$ is a well defined element in $L_\infty(X)$.

Define

$$\mu^*([h]) = [e^p_{XY}((q^A_{XY})^*([h]))].$$

Then the properties of conditional expectation ensure that μ^* is a $\mathcal{A}-\mathcal{A}$ bimodule morphism.

Definition 46 *Let $\mu : X \rightsquigarrow Y$ be a Markov relation. Then $\mu^* : L_\infty(Y) \to L_\infty(X)$ is the pullback by μ.*

The pullback by a relation μ has the following two fundamental properties

Proposition 47 *Let $f : X \to Y$ be a morphism of probabilistic bundles and the $\mu_f : X \rightsquigarrow Y$ be the corresponding relation. Then we have*

$$(\mu_f)^*([h]) = f^*([h])$$

Proof. Let $C \in \mathcal{B}(\tau_Y)$ and let $[\chi_C] \in L_\infty(Y)$ be the corresponding element in $L_\infty(Y)$. Then we have

$$\int_{(p^A_{XY})^{-1}(V)} (p^A_{XY})^*(\chi_{f^{-1}(C)})d\mu_f = \int_{(p^A_{XY})^{-1}(V)} (p^A_{XY})^*(\chi_{f^{-1}(C)})d(\Gamma_f)_*\mu_X = \int_V \chi_{f^{-1}(C)}d\mu_X$$

$$= \mu_X(V \cap f^{-1}(C)) = \mu_f(V \times_A C) = \int \chi_{V\times_A C}d\mu_f = \int_{(p^A_{XY})^{-1}(V)} \chi_{\Omega_X\times_A C}d\mu_f$$

By the uniqueness of conditional expectation we have $e^p_{XY}(\Omega_X \times_A C) = [\chi_{f^{-1}(C)}]$.

But then we have

$$\mu^*([\chi_C]) = e^p_{XY}(\Omega_X \times_A C) = [\chi_{f^{-1}(C)}] = f^*([\chi_C]).$$

The proposition is therefore true for characteristic functions and by linearity for simple functions. The proposition now follows by using the dominated convergence theorem ■

Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of probabilistic bundles. From this proposition and proposition 37 we observe that we have the natural identity $(\mu_g \circ \mu_f)^* = {\mu_f}^* \circ {\mu_g}^*$. This property in fact holds for any pair of composable relations.

Proposition 48 *Let $\mu : X \rightsquigarrow Y$ and $\nu : Y \rightsquigarrow Z$ be relations of probabilistic relations. Then we have*

$$(\nu \circ \mu)^* = \mu^* \circ \nu^*$$

Proof. The proposition follows for characteristic functions immediately from lemma 40.

For $C \in \mathcal{B}(\tau_Z)$ we have

$$(\nu\circ\mu)^*([\chi_C]) = e^p_{XZ}(\Omega_X\times_A C) = e^p_{XY}((q^A_{XY})^*(e^p_{YZ}(\Omega_Y\times_A C))) = \mu^*(\nu^*([\chi_C])).$$

By linearity the proposition is true for simple functions and by dominated convergence and properties of conditional expectation it is true for all functions in $L_\infty(Z)$. ■

Abusing the language slightly as explained previously we can summarize these investigations in the following theorem

Theorem 49 *F is a contravariant functor from the category of relations of probabilistic bundles $\mathcal{R}(A)$ to the category of $\mathcal{A}-\mathcal{A}$ bimodules $\mathcal{C}_A$.*

There are other natural functors one could define on the category of probabilistic bundles. One could associate the Hilbert space $L_2(X)$ of square integrable functions to each probability space X. Markov relations $\mu : X \rightsquigarrow Y$ would now be mapped to bounded linear maps $\mu^* : L_2(Y) \longrightarrow L_2(X)$. Let this functor be H. The functor H makes possible an interesting interpretation of standard notions like transitivity and symmetry for classical relations. Let us consider the special case of Markov relations $\mu : X \longrightarrow X$. If $t : \Omega_X \times_A \Omega_X \longrightarrow \Omega_X \times_A \Omega_X$ is the transposition we clearly have a action of t on μ by $t \cdot \mu = t_* \mu$. The following definitions are natural generalizations of the corresponding classical notions

Definition 50 *A Markov relation is symmetric iff $t \cdot \mu = \mu$ and transitive iff $\mu \circ \mu = \mu$.*

For a given Markov relation let *Supp*(μ) be the support of the measure μ. The support of a Markov relation is clearly a classical relation on Ω_X. We can prove under quite general conditions that *Supp*(μ) is transitive and symmetric if μ is transitive and symmetric. The defined notions are thus the natural ones for Markov relations.Using the results from above it is evident that a symmetric transitive Markov relation $\mu : X \rightsquigarrow Y$ is mapped to a orthogonal projector μ^* on the Hilbert space $L_2(X)$. Symmetric and transitive classical relations are "almost" equivalence relations so orthogonal projectors are in this way closely linked to the notion of equivalence relations. We will not pursue these matters further here but will defer them to a future publication.

5.2 Quantizations

Functors between monoidal categories should preserve monoidal structures up to natural isomorphism. This is the natural notion of morphism between monoidal categories[7]. We have interpreted these natural isomorphisms in terms of the physical notion of quantization and defined the notion of quantizers in the categorical context. This idea has been explored in several papers [8],[9] and preprints.

The coherence conditions for the monoidal structures leads to conditions for the quantizers. These conditions take the form of commutative diagrams within the categories in question; these are the quantizer equations. Each solution of the quantizer equations leads to a uniform way of quantizing all structures defined in terms of the monoidal structures.

Thus in order to quantize we need a monoidal structure. In the category $\mathcal{C}_\mathcal{A}$ the tensor product of bimodules over $\mathcal{A}$ is an obvious candidate for such a structure. We now briefly recall its construction.

For any two objects E and F in $\mathcal{C}_\mathcal{A}$ we define a continuous map $\varphi : E \otimes_\pi \mathcal{A} \otimes_\pi F \rightarrow E \otimes_\pi F$ by

$$\varphi = (1_E \otimes \mu^l_F) - (\mu^r_X \otimes 1_F)$$

Let $M = \overline{\varphi(E \otimes \mathcal{A} \otimes F)}$ be the closure of the image of φ in $E \otimes F$ and define

$$E \otimes_{\mathcal{A}} F = (E \otimes_{\pi} F)/M$$

This is a Banach space and the canonical projection $\pi : E \otimes_{\pi} F \rightarrow E \otimes_{\mathcal{A}} F$ is a continuous open map with $\ker(\pi) = M$.

The product $\otimes_{\mathcal{A}}$ sets up a bijective correspondence between the set of continuous linear maps g from $E \otimes F$ to G that satisfies $M \subset \ker g$ and the set of continuous linear maps from $E \otimes_{\mathcal{A}} F$ to G.

Note that $\ker g$ is always closed so that $M \subset \ker g$ if and only if the image of φ is included in the kernel of g. Let $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ be morphisms of $\mathcal{A} - \mathcal{A}$ bimodules.

Define a continuous map $F : X \otimes Y \rightarrow X' \otimes_{\mathcal{A}} Y'$by

$$F(x \otimes y) = f(x) \otimes_{A} g(y)$$

For this map we evidently have $M \subset \ker F$ so we get induced a unique map $f \otimes_{\mathcal{A}} g : X \otimes_{\mathcal{A}} Y \rightarrow X' \otimes_{\mathcal{A}} Y$. It is now easy to see that with this value on maps the product $\otimes_{\mathcal{A}}$ defines a monoidal structure on the category $\mathcal{C}_{\mathcal{A}}$ with associativity constraint $\alpha^{\mathcal{A}}$ given by $\alpha^{\mathcal{A}}(x \otimes_{\mathcal{A}} (y \otimes_{\mathcal{A}} z)) = (x \otimes_{\mathcal{A}} y) \otimes_{\mathcal{A}} z$, unit object $\mathcal{A}$ and unit constraints

$$\beta_X^{\mathcal{A}} : \mathcal{A} \otimes_{\mathcal{A}} L_{\infty}(X) \rightarrow L_{\infty}(X)$$
$$\gamma_X^{\mathcal{A}} : L_{\infty}(X) \otimes_{\mathcal{A}} \mathcal{A} \rightarrow L_{\infty}(X)$$

defined by $\beta_X^{\mathcal{A}}([h] \otimes_{\mathcal{A}} x) = \mu_X^l([h] \otimes x)$ and similar for $\gamma_X^{\mathcal{A}}$.

These maps are well defined because μ_X^l and μ_X^r define the $\mathcal{A} - \mathcal{A}$ bimodule structure on X and then by definition their kernels contains the image of the map φ. They are clearly invertible.

Collecting together these observations we have

Theorem 51 *$\langle \otimes_{\mathcal{A}}, \mathcal{A}, \alpha^{\mathcal{A}}, \mu^{\mathcal{A}}, \gamma^{\mathcal{A}} \rangle$ is a monoidal structure for the category $\mathcal{C}_{\mathcal{A}}$.*

We will apply this general formalism to the case of quantizations of the identity functor on the category $\mathcal{C}_{\mathcal{A}}$.

For this case a quantization is a isomorphism

$$q_{XY} : X \otimes_{\mathcal{A}} Y \rightarrow X \otimes_{\mathcal{A}} Y,$$

that is natural in X and Y and that is a solution of the following set of coherence conditions

$$q_{X \otimes_{\mathcal{A}} Y, Z} \circ (q_{X,Y} \otimes_{\mathcal{A}} 1_Z) = q_{X, Y \otimes_{\mathcal{A}} Z} \circ (1_X \otimes_{\mathcal{A}} q_{Y,Z}),$$
$$\beta_X^{\mathcal{A}} \circ q_{\mathcal{A},X} = \beta_X^{\mathcal{A}},$$
$$\gamma_X^{\mathcal{A}} \circ q_{X,\mathcal{A}} = \gamma_X^{\mathcal{A}}.$$

Let $\mathcal{A}^2 = \mathcal{A} \otimes_{\pi} \mathcal{A}$,$\mathcal{A}^3 = \mathcal{A} \otimes_{\pi} \mathcal{A} \otimes_{\pi} \mathcal{A}$ etc. Note that $\mathcal{A}^n$ is an object in the category $\mathcal{C}_{\mathcal{A}}$.

Define maps $\widetilde{\gamma} : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}^2 \otimes_{\mathcal{A}} \mathcal{A}^2$ and $\widetilde{\triangle} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}^2 \otimes_{\mathcal{A}} \mathcal{A}^2$ by

$$\begin{aligned}\widetilde{\gamma}(a, b, c) &= (a \otimes_\pi b) \otimes_{\mathcal{A}} (1 \otimes_\pi c),\\ \widetilde{\triangle}(a, b) &= (a \otimes_\pi 1) \otimes_{\mathcal{A}} (1 \otimes_\pi b).\end{aligned}$$

The maps $\widetilde{\gamma}$ and $\widetilde{\triangle}$ are multilinear and continuous and thus define unique continuous linear maps $\gamma : \mathcal{A}^3 \to \mathcal{A}^2 \otimes_{\mathcal{A}} \mathcal{A}^2$ and $\triangle : \mathcal{A}^2 \to \mathcal{A}^2 \otimes_{\mathcal{A}} \mathcal{A}^2$. They clearly respect the $\mathcal{A} - \mathcal{A}$ bimodule structure and are thus morphisms in $\mathcal{C}_{\mathcal{A}}$.

For a given pair of objects X and Y in $\mathcal{C}_{\mathcal{A}}$ and points $x \in X$ and $y \in Y$ define morphism $\varphi_x : \mathcal{A}^2 \to X$ and $\varphi_y : \mathcal{A}^2 \to Y$ by

$$\varphi_x(a \otimes b) = axb\,, \varphi_y(a \otimes b) = ayb.$$

Here for simplicity we write ax for $\mu^l_X(a \otimes x)$ etc.

Using these morphisms and naturallity it is easy to show that any quantizer q_{XY} must be of the form

$$q_{XY}(x \otimes_{\mathcal{A}} y) = \gamma(q) \cdot (x \otimes_{\mathcal{A}} y)$$

where for any element $s \in \mathcal{A}^2 \otimes_{\mathcal{A}} \mathcal{A}^2$ of the form $s = \sum_i (a_i \otimes b_i) \otimes_{\mathcal{A}} (c_i \otimes d_i)$ we define

$$s \cdot (x \otimes_{\mathcal{A}} y) = \sum_i (a_i x b_i) \otimes_{\mathcal{A}} (c_i y d_i)$$

Using this representation of q_{XY} we can reduce the equations for the quantizer to equations in the algebra $\mathcal{A}^3$.

The resulting equations are

$$\begin{aligned}(1_{\mathcal{A}^2} \otimes_{\mathcal{A}} \triangle)(\gamma(q))(1 \otimes_{\mathcal{A}} \gamma(q)) &= (\triangle \otimes_{\mathcal{A}} 1_{\mathcal{A}^2})(\gamma(q))(\gamma(q) \otimes_{\mathcal{A}} 1_{\mathcal{A}^2}),\\ (\mu \otimes 1_{\mathcal{A}})(q) &= 1,\\ (1_{\mathcal{A}} \otimes \mu)(q) &= 1,\end{aligned}$$

where $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the product on $\mathcal{A}$

$$\mu([f] \otimes [g]) = [fg]$$

In addition to this set the condition that q_{XY} is a morphism in $\mathcal{C}_{\mathcal{A}}$ leads to the equations

$$\triangle(h)\gamma(q) = \gamma(q)\triangle(h) \qquad \forall\ h \in \mathcal{A}^2.$$

These last equations are however trivially satisfied for our case since the algebra $\mathcal{A}$ is a commutative algebra.

As an example of what we can get by solving these equations let us assume that the probability space $A = \langle \Omega_A, \mathcal{B}(\tau_A), \mu_A \rangle$ is finite with $\Omega_A = \{1, 2, 3, ..., n\}$. Then since we are only considering Hausdorf topological spaces we have $\mathcal{B}(\tau_A) = \mathcal{P}(\Omega_A)$ where $\mathcal{P}(\Omega_A)$ is the power set of Ω_A and thus all functions are measurable. The measure μ_A is then defined by a set of numbers $\{p_i\}_{i=1}^n$. Let us for simplicity assume that $p_i \neq 0$ for all i. Then there are no sets of measure zero and so for any function we have $[f] = \{f\}$ and the algebra $\mathcal{A}$ is equal to the algebra of all complex valued functions on the finite set Ω_A. Let $\theta_i : \Omega_A \to \mathbb{C}$ be defined by $\theta_i(j) = 1$ if and only if $i = j$ otherwise it is equal to zero. The set of functions $\{\theta_i\}_{i=1}^n$ is evidently a basis for $\mathcal{A}$ and the algebra structure is given by

$$\theta_i \theta_j = \begin{cases} \theta_i & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

An element in $\mathcal{A}^3$ can be expanded in the basis $\{\theta_i \otimes \theta_j \otimes \theta_k\}$ by

$$q = \sum_{i,j,k} q_{ijk} \theta_i \otimes \theta_j \otimes \theta_k$$

The quantizer equations for this case reduce to a large set of quadratic equations of a very simple form.

$$\begin{aligned} q_{ijk} q_{jmk} &= q_{imk} q_{ijm} && \forall\, i, j, k, m \\ q_{iik} &= 1 && \forall i, k \\ q_{ikk} &= 1 && \forall i, k \end{aligned}$$

These equations have nontrivial solutions even for the case of $n = 2$. For this case there is one family of solutions given by

$$q = 1 + t(\theta_2 \otimes \theta_1 \otimes \theta_2 + \theta_1 \otimes \theta_2 \otimes \theta_1) \qquad t \in \mathbb{C}$$

The action of q_{XY} is for this solution given by

$$q_{XY}(x \otimes_A y) = x \otimes_A y + t(\theta_2 x \theta_1 \otimes_A y\theta_2 + \theta_1 x \theta_2 \otimes_A y \theta_1)$$

Note that the quantizer acts trivially if the $\not{\mathcal{A}} - \mathcal{A}$ bimodule structure on X or Y are symmetric, meaning $ax = xa$ etc. This is certainly true for the objects $L_\infty(X)$ and $L_\infty(Y)$ where X and Y denote probabilistic bundles in $\mathcal{P}(A)$. Thus we can not use these quantizers to deform the algebra structure on $L_\infty(X)$. This is true not only for this particular case but also true for any n.

Let X and Y be objects in $\mathcal{C}_\mathcal{A}$ and let $Hom(X, Y)$ be the set of continuous linear maps from X to Y. The set $Hom(X, Y)$ is clearly a Banach space since X and Y are Banach spaces. It is also a $\mathcal{A} - \mathcal{A}$ bimodule if we define for $h \in Hom(X, Y)$ and $a \in \mathcal{A}$

$$\begin{aligned} (ah)(x) &= a(h(x)) \\ (ha)(x) &= h(ax) \end{aligned}$$

Therefore $Hom(X, Y)$ is an object in the category $\mathcal{C}_A$.Define a map $\widetilde{c}$: $Hom(Y, Z) \times Hom(X, Y) \to Hom(X, Z)$ by

$$\widetilde{c}(h', h) = h' \circ h$$

It is easy to see that this map is continuous and $\mathcal{A}-\mathcal{A}$ bilinear. It therefore define a morphism

$$c : Hom(Y,Z) \otimes_A Hom(X,Y) \to Hom(X,Z)$$

This map c is clearly just the composition of morphisms in the category $\mathcal{C}_A$. We have thus internalized the operation of composition of morphisms. The quantizers will in general act nontrivially on $Hom(Y,Z) \otimes_A Hom(X,Y)$ and will thus lead to nontrivial quantizations of composition through

$$c_q(g \otimes_A f) = c(\gamma(q) \cdot (g \otimes_A f))$$

As an example of this let us consider the case with $n = 2$. Let $\Omega_X = \Omega_A$, φ_X be the transposition on Ω_A,$\varphi_X(1) = 2$,$\varphi_X(2) = 1$ and $\mu_X = (\varphi_X)_*\mu_A$. Then $\langle \Omega_X, \mathcal{P}(\Omega_X), \mu_X, \varphi_X \rangle$ is a object in $\mathcal{P}(A)$ with $L_\infty(X) = \mathbb{C}^2$. The set $Hom(L_\infty(X), L_\infty(X))$ is a algebra in $\mathcal{C}_A$ and through the basis $\{\theta_i\}$ isomorphic to the set of 2×2 complex matrices. Composition of linear maps turns into matrix product and the quantized composition c_q will turn into a deformed way of computing matrix products. For the family of quantizers computed above we find the new product to be

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} *_q \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + t \begin{bmatrix} bc' & 0 \\ 0 & cb' \end{bmatrix}$$

It is easy to verify directly that this product is associative,distributive and has the identity matrix as a unit. This algebra is isomorphic to the original 2×2 matrix algebra for all $t \neq -1$. For $t = -1$ it is not semisimple with a 2 dimensional nilpotent Jacobson radical.

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