Transitive and transversal actions of pseudogroups on submanifolds

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Abstract

In this paper we consider the equivalence problem of submanifolds with respect to a transitive pseudogroup action. The corresponding differential invariants are determined formally as well (l-variants and l-covariants). We investigate their properties and calculate for several basic examples. The main application concerning the action of the Lie transformations pseudogroup in the jet spaces corresponds to the equivalence problem of differential equations.

Dedicated to A. P. Norden
on the occasion of his centennial anniversary

Introduction

Transformation groups or more generally pseudogroups, originated in the works of S. Lie and E. Cartan [Lie, C], play a central role in geometry and analysis.

A pseudogroup $G \subset \mathrm{Diff}_{\mathrm{loc}}(M)$ acting on a manifold M consists of a collection of local diffeomorphisms φ, each bearing own domain of definition $\mathrm{dom}(\varphi)$, that satisfies the following properties:

1. If $\varphi, \psi \in G$, then $\varphi \circ \psi \in G$ whenever defined,
2. If $\varphi \in G$, then $\varphi^{-1} \in G$,
3. $\mathrm{id}_M \in G$,
4. $\varphi \in G$ iff for every open subset $U \in \mathrm{dom}(\varphi)$ the restriction $\varphi|_U \in G$,
5. The pseudogroup is of order l if this is the minimal number such that $\varphi \in G$ whenever for each point $a \in \mathrm{dom}(\varphi)$ the l-jet is admissible: $[\varphi]^l_a \in G^l$.

The last property means that a pseudogroup is defined by differential equations of maximal order l and will be explained below. It uses the infinitesimal language. In fact, a transformation $\varphi \in G$ defines a map (l-th prolongation) of the jet space $\varphi_{(l)} : J^l_r(M) \to J^l_r(M)$, which obeys the following properties:

- $(\varphi \circ \psi)_{(l)} = \varphi_{(l)} \circ \psi_{(l)}$,
- $\mathrm{id}_{(l)} = \mathrm{id}^l_M$,
- $\left(\varphi^{-1}\right)_{(l)} = {\varphi_{(l)}}^{-1}$.

They are basic in the formal approach to pseudogroups, cf. [E, Lib, SS].

In this paper we develop a more general notion of infinitesimal pseudogroup. Namely, an l-pseudogroup is a transformation group in finite jets, not necessarily integrable. Even such finite order pseudogroups are important in producing invariants for differential equations and curvatures for geometric structures.

Formal integrability criterion for infinitesimal pseudogroups is based on the well-developed algebraic machinery, described in the paper. The passage from formal integrability to the local one is not automatic and is generically wrong. However the former implies the latter in the following cases:

- Finite type pseudogroups (the symbol $\mathfrak{g}^k \equiv 0$ for big k). Actually this condition implies that the integrated pseudogroup is finite-dimensional.
- Analytic pseudogroups. It is a consequence of Cartan-Kähler theorem, which holds for general differential equations [M, KLV].
- Elliptic pseudogroups of analytic type, see [S, M].
- Transitive flat pseudogroups, see [BM, P].

Only in special cases the global integrability (or equivalence) problem can be handled, see [S, GS, T].

The goal of this paper is to develop the invariants theory of pseudogroups actions on submanifolds. For Lie groups this is the well-known differential invariants theory. On the level of finite jets we are lead to what we will call l-covariants. Their calculus is governed by certain cohomologies similar to formal Spencer cohomologies and we establish their relation to the equivalence problem of submanifolds under the action.

Lie pseudogroups consist of pseudo-automorphisms of geometric structures. We provide a series of calculations for them. The most important application concerns the pseudogroup of Lie transformations in the space of jets, i.e. the transformations preserving the Cartan distribution. They are equivalences of differential equations and thus we establish a way to produce the invariants.

1. Formal pseudogroups

Let M be a smooth manifold and $J^l_r(M)$ be the corresponding jet space. Its points a_l are the l-jets $[N]^l_a$ of codimension r submanifolds $N \subset M$, $a \in N$.

Denote the natural projections by $\rho_{i,j}: J^i_r(M) \to J^j_r(M)$. The fibers bear a canonical affine structure ([KLV, Ly]), associated with the vector structure, described below. It is sufficient to specify it for $\mathfrak{F}(a_{l-1}) = \rho^{-1}_{l,l-1}(a_{l-1})$.

Denote $\mathfrak{t}_a = T_aN = [N]^1_a$ and $\mathfrak{v}_a = T_aM/T_aN$. Let $a_l \in J^l_r(M)$, $a_{l-1} = \rho_{l,l-1}(a_l)$. Then $T_{a_l}\mathfrak{F}(a_{l-1}) \simeq S^l\mathfrak{t}^*_a \otimes \mathfrak{v}_a$ and we get the exact sequence:

$$0 \to S^l\mathfrak{t}^*_a \otimes \mathfrak{v}_a \to T_{a_l}J^l_r(M) \stackrel{(\rho_{l,l-1})_*}{\longrightarrow} T_{a_{l-1}}J^{l-1}_r(M) \to 0.$$

For a vector bundle $\rho: E \to B$ of rank r, the corresponding space of jets of sections $J^l\rho$ is an open subset in $J^l_r(E)$. In particular, we realize the jet space for maps $J^l(N,M)$. Denote by $D^l(M) \subset J^l(M,M)$ the open dense subset, consisting of the l-jets of local diffeomorphisms. Being equipped with the partially defined composition operation, it is an example of finite order pseudogroup.

To define this notion in general, recall some basic facts from the geometric theory of differential equations, see [KLV, Gu, Ly] for details. The prolongation of differential equation $\mathcal{E} \subset J^l_r(M)$ is defined as

$$\mathcal{E}^{(1)} = \{a_{l+1} = [s]^{l+1}_a \mid \text{ the jet-section } j_l(s) \text{ is tangent to } \mathcal{E} \text{ at } a_l\} \subset J^{l+1}_r(M).$$

This can be equivalently written as $\mathcal{E}^{(1)} = \{a_{l+1} \,|\, L(a_{l+1}) \subset T_{a_l}\mathcal{E}\}$, where for $a_{l+1} = [s]^{l+1}_a$ we set: $L(a_{l+1}) = T_{a_l}j_l(s)$, $a_l = \rho_{l+1,l}(a_{l+1})$.

Definition 1. *An l-pseudogroup is a collection of subbundles $G^j \subset D^j(M)$, $0 < j \le l$, such that the following properties are satisfied:*

1. *If $\varphi_j, \psi_j \in G^j$, then $\varphi_j \circ \psi_j \in G^j$ whenever defined,*
2. $\mathrm{id}^j_M \in G^j$,
3. *If $\varphi_j \in G^j$, then $\varphi_j^{-1} \in G^j$,*
4. *The map $\rho_{j,j-1}: G^j \to G^{j-1}$ is a bundle.*

We also assume, as usual in the differential equations theory, that $G^0 = J^0(M,M)$ $= M \times M$, which is equivalent to transitivity *of the pseudogroup action.*

An l-pseudogroup is called l-integrable if $G^j \subset (G^{j-1})^{(1)}$ for all $0 < j \le l$.

Pseudogroups $G = \{G^j\}^l_{j=1}$ defined by this approach can be studied for integrability by the standard prolongation-projection method ([GS, KLV, S]).

Denote $G^j_{a,b} = \{\varphi_j \in G^j \,|\, \varphi_0(a) = b\}$, $G^j_a = G^j_{a,a}$ – the subgroup of G^j and $\mathfrak{G}^j_a = \mathrm{Ker}[\rho_{j,j-1}: G^j_a \to G^{j-1}_a]$ – its (normal) subgroup, which is abelian for $j > 1$ and for $j = 1$: $\mathfrak{G}^1_a = G^1_a \subset \mathrm{Gl}(T_aM)$.

Definition 2. *Let $\varphi_j \in G^j$ be a point and $\rho_{j,0}(\varphi_j) = (a,b) \in M \times M$. The symbol of the pseudogroup G is given by:*

$$\mathfrak{g}^j(\varphi_j) = \mathrm{Ker}\big[(\rho_{j,j-1})_*: T_{\varphi_j}G^j \to T_{\varphi_{j-1}}G^{j-1}\big]$$

(and the same with the change $G^i \mapsto G^i_{a,b}$). It can be represented as a subspace $\mathfrak{g}^j(\varphi_j) \subset S^j(T^*_aM) \otimes T_bM \stackrel{\varphi_1^{-1}}{\simeq} S^j(T^*_aM) \otimes T_aM$, and in the last form is identified with the Lie algebra $\mathfrak{g}^j_a$ of the Lie group $\mathfrak{G}^j_a$.

An l-pseudogroup G is called *formally integrable* if it is l-integrable, for all $j > l$ the prolongations $G^j = (G^l)^{(j-l)}$ exist, are pseudogroups and the projections $\rho_{j,j-1} : G^j \to G^{j-1}$ are vector bundles. Similar to the differential equations theory ([Go, Gu, S]), a criterion of formal integrability can be formulated in terms of the Spencer δ-complex:

$$0 \to \mathfrak{g}_a^l \xrightarrow{\delta} \mathfrak{g}_a^{l-1} \otimes T_a^*M \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathfrak{g}_a^{l-j} \otimes \Lambda^j T_a^*M \xrightarrow{\delta} \dots \qquad (1)$$

Its bi-graded cohomology groups are denoted by $H^{l-j,j}(G)$ or $H^{l-j,j}(\mathfrak{g})$. The obstructions to formal integrability of the l-pseudogroup G, considered as a differential equation, are some elements $W_j(G) \in H^{j-1,2}(G)$, called Weyl tensors (or curvatures), defined via the jet-spaces geometry ([Ly]).

Theorem 1. *Let G be an l-pseudogroup. Suppose that it is l-integrable, the symbols $\mathfrak{g}^j$ over G^l form a vector bundle and all the Weyl tensors W_j vanish identically for all $j \geq l$. Then the pseudogroup is formally integrable.*

Proof. The hypotheses imply integrability of G as a differential equation, see [Ly]. We need to check the obtained system $\{G^j\}_{j=0}^{\infty}$ is a pseudogroup, i.e. to check all the requirements of definition 1. We study only the first one, the others being considered similarly.

Let $G^{j+1} = (G^j)^{(1)}$, $\varphi_{j+1} \in G_{a,b}^{j+1}$, $\psi_{j+1} \in G_{b,c}^{j+1}$ and $\chi_{j+1} = \psi_{j+1} \circ \varphi_{j+1}$. We need to show that $\chi_{j+1} \in G^{j+1}$. This is equivalent to $L(\chi_{j+1}) \subset T_{\chi_j} G^j$.

To prove the inclusion consider the multiplication operator $m_j : G^j \times G^j \to G^j$. It has the differential:

$$T_{\psi_j} G^j \oplus T_{\varphi_j} G^j \xrightarrow{dm_j} T_{\chi_j} G^j.$$

The two summands on the left contain the subspaces $L(\psi_{j+1})$ and $L(\varphi_{j+1})$ respectively. But

$$L(\psi_{j+1}) \oplus L(\varphi_{j+1}) \xrightarrow{dm_j} L(\psi_{j+1}\varphi_{j+1})$$

for arbitrary $\varphi_{j+1}, \psi_{j+1} \in D^{j+1}(M)$ such that the composition is defined. The claim follows. $\square$

An l-pseudogroup G is called *q-acyclic* if $H^{i,j}(G) = 0$ for $i \geq l$, $0 \leq j \leq q$. An ∞-acyclic pseudogroup is called *involutive* For such pseudogroups G investigation of formal integrability involves only one obstruction $W_l(G)$.

If a pseudogroup G is formally integrable we obtain its infinite prolongation $G^\infty = \lim_{\text{proj}}(G^l, \rho_{l,l-1})$, which is called infinitesimal or *formal pseudogroup*. If there is local integrability (smooth or analytic), as described in the introduction, we refer to the pseudogroup as to *integrable*.

Example 1. The group of complex fractional-linear transformations of $S^2 = \mathbb{C}P^1$ (or real transformations of $S^1 = \mathbb{R}P^1$) is an integrable pseudogroup of finite type and order 3. In fact, its Lie algebra is represented as the algebra of quadratic-polynomial vector fields on the line: $\mathfrak{g} = \{\xi = (c_0 + c_1 z + c_2 z^2)\partial_z\}$.

Example 2. The pseudogroup of Lie transformations on the jet-space $M = J^k\pi$ for some bundle $\pi : E_\pi \to M$ ([KLV]) is of order 1 and infinite type. This example is central in the paper and we will provide more details in §4.

Example 3. Let $\mathcal{E}$ be a geometric structure ([Gu, Ly]) and G be its Lie pseudogroup of (jet of) automorphisms. If the structure $\mathcal{E}$ is integrable (flat), the pseudogroup is integrable as well. It can be of finite or infinite type depending on the geometric structure ([Ko]). It has the same order as the structure $\mathcal{E}$. When the geometric structure is non-integrable, the order of the pseudogroup G can increase and it can readily be non-integrable (formally or locally).

2. Equivalence of submanifolds via pseudogroups

The natural question is to distinguish when two (close, local) submanifolds $N_1, N_2 \subset M$ are equivalent under the transitive action of a pseudogroup G. In this section we study the infinitesimal problem for the l-jets and l-pseudogroups:

Definition 3. *We say that l-jets of two submanifolds N_1 and N_2 at the points $a, b \in M$ are G-equivalent if $\varphi_l[N_1]^l_a = [N_2]^l_b$ for some $\varphi_l \in G^l_{a,b}$.*

For transitive pseudogroups the equivalence problem reduces to the case $a = b$. We assume this and begin subsequently equalizing the jets of submanifolds.

2.1. Action on jets of submanifolds

The pseudogroup $D^l(M)$ and hence G^l act on the space $J^l_r(M)$ by the formula $\varphi_{(l)} : [N]^l_a \mapsto [\varphi(N)]^l_{\varphi(a)}$. These actions obey the relation: $\rho_{l,s} \circ \varphi_{(l)} = \varphi_{(s)} \circ \rho_{l,s}$.

Consequently, the group $\mathfrak{G}^l_a$ acts on $\mathfrak{F}(a_{l-1})$. This action is affine for $l > 1$. For $l = 1$ it is generated by the linear collineations in the Grassmannians.

The induced (linear) action of the Lie algebra $\mathfrak{g}^l_a \ni \theta$ is described as follows: $f \mapsto \lambda(\theta) + f$, $f \in T_{a_k}\mathfrak{F}(a_{l-1})$. Here $\lambda : S^l T^*_a M \otimes T_a M \to T_{a_k}\mathfrak{F}(a_{l-1})$ maps an element θ to its image $\bar\theta \in S^l \mathfrak{t}^*_a \otimes \mathfrak{v}_a$ under the restriction-factorization map.

Thus the stabilizer of an element $a_l \in \mathfrak{F}(a_{l-1})$ equals $\mathfrak{H}^l_a = \mathfrak{G}^l_a \cap \mathrm{St}^l_a$ in the case of the Lie group, or $\mathfrak{h}^l_a = \mathfrak{g}^l_a \cap \mathrm{St}^l_a$ for the Lie algebra, where

$$\mathrm{St}^l_a = (\mathrm{Ann}\,\mathfrak{t}_a) \circ_{\mathrm{sym}} S^{l-1}T^*_a M \otimes T_a M + S^l T^*_a M \otimes \mathfrak{t}_a.$$

In particular, since the symbol of $D^l(M)$ acts transitively, we get:

$$S^l T^*_a M \otimes T_a M / \mathrm{St}^l_a \simeq S^l \mathfrak{t}^*_a \otimes \mathfrak{v}_a. \tag{2}$$

Remark 1. *Since $\mathfrak{G}^l_a$ is abelian for $l > 1$, there is no difference between the actions of Lie groups and Lie algebras. In the case of 1-jets the actions differ.*

Consider now some distinguished submanifolds of codimension r in M. Specification of this subset among all submanifolds is related to a choice of some common interior structure w.r.t. the pseudogroup action and will be discussed below. It is given by certain differential relations and thus we may consider the set of specified jets of submanifolds given by a differential equation $\mathfrak{N}$.

Thus let $\mathfrak{N} \subset J^l_r(M)$ be a G-invariant differential equation. Its symbol $h^l_a \subset S^l\mathfrak{t}^*_a \otimes \mathfrak{v}_a$ is as usual the $\rho_{l,l-1}$-vertical subspace of $T_{a_l}\mathfrak{N}$. Since the pseudogroup G acts on $\mathfrak{N}$ and so obtain the following exact sequence:

$$0 \to \mathfrak{h}^l_a \hookrightarrow \mathfrak{g}^l_a \xrightarrow{\lambda} h^l_a \xrightarrow{\varpi} \mathfrak{O}^l_a \to 0. \tag{3}$$

Definition 4. *The quotient $\mathfrak{O}^l_a = h^l_a/\lambda(\mathfrak{g}^l_a)$ is called the space of* l-covariants *of the pseudogroup G action. The dual $(\mathfrak{O}^l_a)^*$ is named the space of* l-variants*.*

Our study of formal equivalence of submanifolds under the G-action is inductive and based on the following obvious statement:

Proposition 2. *Let $[N_1]^{l-1}_a = [N_2]^{l-1}_a \in \rho_{l,l-1}(\mathfrak{N})$ and $l > 1$. The l-jets of submanifolds N_1 and N_2 from $\mathfrak{N}$ at a point $a \in M$ are G-equivalent if and only if they belongs to the same $\mathfrak{g}^l_a$-orbit on h^l_a, which are are affine subspaces of codimension equal* $\dim \mathfrak{O}^l_a = \dim h^l_a - \dim\bigl(\mathfrak{g}^l_a/\mathfrak{h}^l_a\bigr)$. *In other words, this happens iff they have the same l-variants: $\varpi([N_2]^l_a - [N_1]^l_a) = 0$.* □

The requirement $l > 1$ is related to remark 1. For $l = 1$ there is difference between symbolic Lie groups and algebras: In the first case one gets orbits in the $\mathrm{Grass}_r(T_aM)$, while in the latter one gets affine subspaces in its tangent space at a_1. Thus 1-jets require a separate treatment.

2.2. Differential invariants and equivalence

Let $\mathcal{I}$ be the algebra of differential invariants of the pseudogroup G, i.e. functions constant on the orbits of G-action on $\mathfrak{N}$. Denote by $\mathcal{I}_k$ be the subalgebra consisting of order $k \le l$ invariants (lifted from the space of k-jets). Fix a point $a_l \in J^l_r(M)$ and define the increasing filtration of $T^*_{a_l}J^l_r(M)$ by

$$\Theta_k(a_l) = \{d_{a_l}f \mid f \in \mathcal{I}_k\} \subset T^*_{a_l}J^l_r(M), \quad k = 0, \dots, l.$$

Note that Θ_l is the 1st order equation defining G^l-differential invariants on $J^l_r(M)$. Near singular orbits the differential invariants have bad behavior, and there we define the filtration as follows (the definitions at regular points coincide):

$$\Theta_k(a_l) = \pi^*_{l,k} \operatorname{Ann} T_{a_k}(G^k \cdot a_k).$$

Proposition 3. *For $0 < k \le l$: $\mathfrak{O}^k_a = (\Theta_k/\Theta_{k-1})^*$.*

Proof. In fact, $\mathfrak{O}^k_a = T_{a_k}(\pi_{k,k-1})^{-1}(G^{k-1} \cdot a_{k-1})/T_{a_k}(G^k \cdot a_k)$, and the claim follows. □

Thus we get the a solution to the formal equivalence problem by the following inductive procedure. We start with a pseudogroup G and all submanifolds, i.e. $\mathfrak{N} = J^l_r(M)$. Suppose the first nontrivial space of l-covariants is $\mathfrak{O}^l$. To settle a point in it we fix l-variants from $(\mathfrak{O}^l_a)^* = \Theta^l$, which is equivalent to fixing values of order l differential invariants. In other words, we fix the type of interior geometry for submanifolds. This yields a smaller equation $\mathfrak{N} \subset J^l_r(M)$ on submanifolds and we continue. At regular points the procedure stops in a finite number of steps by the Cartan-Kuranishi prolongation theorem.

Remark 2. *The last statement equivalently means that the space of differential invariants is finitely generated w.r.t. linear combinations and Tresse derivatives.*

2.3. Transitivity and transversality

Now we need a criterion to check absence of l-variants or an effective method to calculate differential invariants.

Definition 5. *A pseudogroup G is said to act* l-transitively *near $a_l \in \mathfrak{N}$, if for any other jet $b_l \in \mathfrak{N}$, close to a_l, there exists an element $\varphi_l \in G^l_{a,b}$ such that $\varphi_l(a_l) = b_l$. In other words, the orbit $G^l \cdot a_l$ is open.*

Definition 6. *An action of a pseudogroup G is said to be* l-transversal *near a_l, if the conclusion of definition 5 holds whenever $a_{l-1} = b_{l-1}$. In other words, $\mathfrak{G}^l_a$ acts transitively on $\mathfrak{F}(a_{l-1})$.*

To explain the name, consider the map $\lambda : \theta \mapsto \bar{\theta}$ from § 2.1. The space $\lambda^{-1}(h^l_a) \subset S^l T^*_a M \otimes T_a M$ contains two subspaces St^l_a and $\mathfrak{g}^l_a$.

Let $l > 1$. The following statement follows from (2), (3) and definitions:

Proposition 4. *l-transversality of G on $\mathfrak{N}$ is equivalent to any of the conditions:*

- St^l_a *is transversal to* $\mathfrak{g}^l_a$ *in* $\lambda^{-1}(h^l_a)$: $\mathrm{St}^l_a + \mathfrak{g}^l_a = \lambda^{-1}(h^l_a)$.
- *There are no l-covariants:* $\mathfrak{O}^l_a = 0$. □

Certainly l-transversality is an inductive step to get l-transitivity (the 1st jets a_1, b_1 are considered separately due to remark 1). Namely, we have:

Theorem 5. *Let $G^1 \cdot a_1$ be open and G acts j-transversally on a_j for $1 < j \le l$. Then G acts l-transitively near a_l.* □

3. Basic examples

In this section we check the transitivity condition for the automorphisms pseudogroups of some basic geometric structures. Irreducible Lie pseudogroups were classified by E. Cartan ([C]). For the general pseudogroup $G = \mathrm{Diff}_{\mathrm{loc}}(M)$ and the volume preserving pseudogroup $G = \mathrm{Diff}_{\mathrm{loc}}(M, \Omega)$ all submanifolds of codimension r are locally G-equivalent. Let's consider some other cases.

Note that for the transformation pseudogroup of some non-integrable structure, the l-pseudogroup G^l consists of the jets of diffeomorphisms preserving the structure to order l. Thus, by abuse of notations, the sub-pseudogroup G^j for $j < \min(k, l)$ can be different as embedded into G^k and G^l.

3.1. Complex pseudogroup

Let G be a pseudogroup of local holomorphic transformations of a complex manifold (M, J) of complex dimension n. Then $\mathfrak{g}_a^l = S^l T_a^* M \otimes_{\mathbb{C}} T_a M$.

Proposition 6. *The pseudogroup G is l-transversal near $a_l = [N]_a^l$ iff:*

1. $\mathfrak{t}_a \cap J\mathfrak{t}_a = \{0\}$ *or* $\mathfrak{t}_a + J\mathfrak{t}_a = T_a M$, *when* $l = 1$, *or*
2. $\mathfrak{t}_a \cap J\mathfrak{t}_a = \{0\}$, *when* $l > 1$.

Proof. The conditions of the first case mean that there do not exist simultaneously non-trivial complex subspaces $L' \subset \mathfrak{t}_a$, $L'' \cap \mathfrak{t}_a = \{0\}$. If this fails, the image of $\lambda : \mathfrak{g}_a^1 \to h_a^1 = \mathfrak{t}_a^* \otimes \mathfrak{v}_a$ consists of maps $A : \mathfrak{t}_a \to \mathfrak{v}_a$ with the induced $\tilde{A} : L' \to L''$ being complex linear: $\tilde{A}(J\xi) = J\tilde{A}(\xi)$. Thus not every tensor in h_a^1 is in the image of λ. Similarly for $l > 1$ if there exists a complex subspace $L' \subset \mathfrak{t}_a$, the restriction of every $A \in \lambda(\mathfrak{g}_a^l)$ to it satisfies: $\tilde{A}(\xi_1, \xi_2, \xi_3, \dots) + \tilde{A}(J\xi_1, J\xi_2, \xi_3, \dots) = 0$ and so is not arbitrary. This proves necessity of the conditions.

Consider the sufficiency. If the intersection $J\mathfrak{t}_a \cap \mathfrak{t}_a = \{0\}$, then every tensor $R \in S^l \mathfrak{t}_a^* \otimes \mathfrak{v}_a$ can be extended to a tensor $\hat{R} \in S^l T_a^* M \otimes_{\mathbb{C}} T_a M$. In fact, we decompose $T_a M = \mathfrak{t}_a \oplus J\mathfrak{t}_a \oplus W$ with $JW = W$ and let $\hat{R}(J\xi, \cdot) = J\hat{R}(\xi, \cdot)$, $\xi \in \mathfrak{t}_a$ and $\hat{R}(w, \cdot) = 0$, $w \in W$. Similarly one proves that if $\mathfrak{t}_a + J\mathfrak{t}_a = T_a M$, we have 1-transversality. $\square$

In particular, the jet of a generic at $a \in N$ submanifold of dimension $\dim_{\mathbb{R}} N \le n$ is l-transversal for every $l \ge 0$ and thus the complex pseudogroup G acts transitively on such local (analytic) submanifolds. On the other hand, a submanifold of dimension $\dim_{\mathbb{R}} N > n$ is never transversal. Namely, the intersection $\Pi_a = \mathfrak{t}_a \cap J\mathfrak{t}_a \neq \{0\}$ and so N possesses an intrinsic geometry.

Investigation of manifolds N equipped with a complex structure on a distribution Π is the subject of Cauchy-Riemann geometry. Its invariants are the curvatures of Cartan-Chern-Moser [CM]. Fixing the curvatures we get a smaller class $\mathfrak{N}$ of submanifolds, on which the action is transversal and so transitive.

Another important class $\mathfrak{N}$ consists of all complex submanifolds $N \subset M$ of $\mathbb{C}$-codimension r. This class is l-transversal for every l, and so is transitive.

3.2. Almost complex pseudogroup

Consider now the case of non-integrable almost complex structure J, $J^2 = -\mathbf{1}$. The pseudogroup G^l consists of all J-holomorphic l-jets: $J \circ d\varphi_l = d\varphi_l \circ J$.

In the case $l = 1$ we have: $G_a^1 = \mathrm{GL}_{\mathbb{C}}(T_a M)$, as in the complex case.

Let $N_J \in \mathrm{Hom}_{\bar{\mathbb{C}}}(\Lambda^2 TM, TM)$ be the Nijenhuis tensor of the structure J (recall that it is the obstruction to integrability). For $l = 2$ we have:

$$G_a^1 = \rho_{2,1}(G_a^2) = \{\Phi \in T_a^* M \otimes T_a M \mid J \circ \Phi = \Phi \circ J,\ N_J \circ (\Phi \wedge \Phi) = \Phi \circ N_J\}.$$

A symmetric torsion-free connection ∇ gives a decomposition of the 2-jet $\varphi_2 \in G^2$ into components $(a, \Phi, \Phi^{(2)})$. The last terms $\Phi^{(2)} \in \mathfrak{F}(\varphi_1)$ for $\varphi_1 = (a, \Phi)$ are jointly described by the formula:

$$\{\Phi^{(2)} \in S^2 T_a^* M \otimes T_a M,\ J\Phi^{(2)}(\xi, \eta) - \Phi^{(2)}(J\xi, \eta) = \Phi \circ \nabla_\eta(J)(\xi) - \nabla_{\Phi\eta}(J)(\Phi\xi)\},$$

Thus $\mathfrak{g}_a^2 = S^2 T_a^* M \otimes_{\mathbb{C}} T_a M$ as in the complex case, but for a smaller set of φ_1. The 2-pseudogroup G^2 is not 2-integrable in general. Proof of these facts, as well as a description of the projection $\rho_{l,l-1} : G_a^l \to G_a^{l-1}$ are contained in [Kr1].

It can be shown, see [Kr2], that for a generic structure J the set G^2 consists of identity for $n > 3$, G^3 consists of identity for $n > 2$ and G^4 is the identity for $n = 2$ (we ignore the case $n = 1$ corresponding to always integrable J). The analysis of pseudoholomorphic invariants for jets of submanifolds based on the classification of Nijenhuis tensors ([Kr2]) results in:

Proposition 7. *Let (M, J) be an almost complex manifold with a generic non-integrable structure J and $n > 1$. For $l = 1$ the transversality is described by condition 1 of proposition 6. For $l = 2$ no 2-jet is transversal save for the case $n = 2$ and $\dim_{\mathbb{R}} N = 1$. The transversality is absent for $l = 3$ and higher.* □

3.3. Riemannian pseudogroup

Consider at first the isometry pseudogroup of the Euclidean space $\mathbb{R}^n$. It integrates to the group $G = O(n) \rightthreetimes \mathbb{R}^n$. The pseudogroup is of finite type and $\mathfrak{g}_a^l = 0$ for $l \geq 2$ ([Ko]). Thus the transversality is absent for $l \geq 2$, but the action is 1-transversal near each 1-jet φ_1.

Consider now a Riemannian manifold (M^n, q) and let G be the isometry pseudogroup. For $l = 1, 2$ the group G^l is the same as in Euclidean case. Consider $l = 3$. Then $G_a^1 = \rho_{3,1}(G_a^3)$ consists of linear isometries from $O(T_a M)$ preserving the Riemannian curvature R_q.

As in the almost complex case for a generic structure q the pseudogroups G^l consists of identity only, when $l > 3$ or $l = 3, n > 2$.

Proposition 8. *The action of G is not transversal near any l-jet for $l > 1$.* □

In fact, various intrinsic and extrinsic curvatures are l-variants. Fixing them we obtain transversality and thus equivalence.

3.4. Symplectic pseudogroup

Consider a symplectic manifold (M, ω) of dimension $2n$ and let G be its (pseudo) group of symplectomorphisms. Using the identification $TM \overset{\omega}{\simeq} T^*M$ we write the symbols $\mathfrak{g}_a^l = S^{l+1} T_a^* M \subset S^l T_a^* M \otimes T_a^* M$, understood as homogeneous generating functions (Hamiltonians) of degree $l + 1$.

Proposition 9. *G acts l-transversally near $a_l \in J_r^l(M)$ for all $l \geq 1$ iff the restriction of ω to $\mathfrak{t}_a = a_1$ is of maximal rank.*

Note that dimension $\dim N$ of a submanifold can be arbitrary.

Proof. Necessity of the conditions is obvious. To prove the statement into the other side consider at first the case $\dim \mathfrak{t}_a \in 2\mathbb{Z}$. Let us decompose symplectically $TM = \mathfrak{t} \oplus \mathfrak{v}$. The transversality condition due to proposition 4 reads:

$$\mathfrak{v}^* \circ S^{l-1}T^*M \otimes T^*M + S^lT^*M \otimes \mathfrak{t}^* + S^{l+1}T^*M = S^lT^*M \otimes T^*M,$$

which obviously holds.

For $\dim \mathfrak{t}_a \in 2\mathbb{Z}+1$ let us decompose symplectically $TM = \mathfrak{u} \oplus (\mathfrak{l} \oplus \mathfrak{r}) \oplus \mathfrak{v}$, where $\mathfrak{t} = \mathfrak{u} \oplus \mathfrak{l}$ and $\mathfrak{u}$, $\mathfrak{v}$ are symplectic, $\dim \mathfrak{l} = \mathfrak{r} = 1$. Now the transversality

$$(\mathfrak{r}^* \oplus \mathfrak{v}^*) \circ S^{l-1}T^*M \otimes T^*M + S^lT^*M \otimes (\mathfrak{u}^* \oplus \mathfrak{r}^*) + S^{l+1}T^*M = S^lT^*M \otimes T^*M,$$

holds due to the identity $S^i\mathfrak{l}^* \otimes \mathfrak{l}^* = S^{i+1}\mathfrak{l}^*$. □

The obtained fact is equivalent to a particular case of Weinstein-Givental theorem ([AG]). To obtain the more general case we should allow various ranks for the restrictions $\omega|_N$. Then the transversality fails and we get 1-variant, which is obviously the rank (or dimension of $\mathrm{Ker}(\omega|_N)$). Fixing it we obtain the transversality for the corresponding equation $\mathfrak{N}$ on submanifolds.

Finally consider the class $\mathfrak{N}$ of isotropic or co-isotropic submanifolds. Similar calculations show that G acts on it l-transversally for every l.

3.5. Contact pseudogroup

Consider a contact manifold (M, Π^{2n}), $\dim M = 2n+1$, and denote by $\nu = TM/\Pi$ the normal. Let G be the (pseudo) group of contact transformations. Its Lie algebra consists of contact vector fields X_f, which are determined by generating functions (Hamiltonians) $f \in C^\infty(M) \otimes \nu$.

A choice of a non-zero section of ν is equivalent to a choice of a contact form $\alpha \in C^\infty(\mathrm{Ann}(\Pi) \setminus 0)$, $\alpha \wedge d\alpha^n \neq 0$. Then the Hamiltonian is scalar-valued, $f \in C^\infty(M)$, and the contact field is uniquely given by

$$\alpha(X_f) = f,\ d\alpha(\cdot, X_f) = df|_\Pi\,.$$

In Darboux coordinates (q, u, p), $\alpha = du - p_i dq^i$, we have:

$$X_f = \mathcal{D}_{q^i}(f)\partial_{p_i} - \partial_{p_i}(f)\mathcal{D}_{q^i} + f\partial_u, \qquad \text{where } \mathcal{D}_{q^i} = \partial_{q^i} + p_i\partial_u.$$

Note that fixing α is equivalent to the splitting $T_aM = \Pi_a \oplus \nu_a$, where the first summand is symplectic and the second is Euclidean 1-dimensional. To describe the symbol $\mathfrak{g}^l_a$, we identify $TM \simeq T^*M$ summand-wise via the symplectic structure on Π and the Euclidean structure on ν. Then we get:

$$\mathfrak{g}^l \simeq S^l\nu^* \oplus \sum_{i>0} S^i\Pi^* \otimes S^{l+1-i}\nu^* \simeq S^{l+1}T^*M.$$

In fact, order l contact fields X_f are determined by Hamiltonians f of order $(l+1)$ in all variables except the pure power of u, where the degree is l.

Proposition 10. *The action of G is l-transversal near $\varphi_l = [N]_a^l$ for all $l \geq 1$ iff $\mathfrak{t}_a = T_a N$ is transversal to the contact plane Π_a and the induced structure on $\Pi_a^N = \Pi_a \cap T_a N$ from the canonical conformally-symplectic structure on Π is maximally nondegenerate. It means that*

- *if* $\dim \Pi_a^N = 2r$, *then* $(d\alpha)^r|_{\Pi_a^N} \neq 0$.
- *if* $\dim \Pi_a^N = 2r+1$, *then* $\operatorname{rank}(d\alpha|_{\Pi_a^N}) = 2r$.

The conditions are equivalent to the claim that through every point close to $a \in N$ there passes an isotropic submanifold of dimension no greater than r.

Proof. Since the symbol is the same as in the symplectic case, the statement is proved similarly to proposition 9. Another approach to uncover the symbol $\mathfrak{g}^l$ and show the claim is via the natural filtration of $S^l T^*M \otimes TM$ by the degrees of $\operatorname{Ann}(\Pi)$ in the first factor and Π in the second. Our representation shows the corresponding grading. □

As in the symplectic case we note that $\operatorname{rank}(\,(d\alpha)^r|_{\Pi_a^N})$ is 1-variant, fixing which we get transversality. The contact analog of Weinstein-Givental theorem holds true, though we have not found a reference. It is proved by the standard isotopy method. For simplicity we present only the local version (the global one with a requirement on the normal bundle holds as well):

Theorem 11. *If two local submanifolds of codimension r of a contact manifold (M, Π) have isomorphic restrictions of the contact structure (N, Π^N), they have contactomorphic neighborhoods.* □

At last, as in § 3.4, a particular case says that restricting to the class $\mathfrak{N}$ of isotropic submanifolds of fixed dimension, we get transversality of the G-action.

4. Lie transformations pseudogroup

Let π be a vector bundle and G a pseudogroup of Lie transformations of $M = J^k\pi$. It consists of local diffeomorphisms of the jet-bundle, preserving the Cartan distribution $\mathcal{C}_k$ ([KLV]). Equivalently, a Lie transformation preserves the class of isotropic submanifolds of the metasymplectic structure. Lie-Bäcklund theorem ([KLV]) states that such a transformation is lifted from a diffeomorphism of $J^0\pi$ in the case $r := \operatorname{rank}\pi > 1$ or from a contact transformation of $J^1\pi$ for $r = 1$. So Lie transformations are lifted from $J^\epsilon\pi$, where $\epsilon = \max(0, 2-r)$.

The pseudogroup G is integrable. We will calculate its symbols below.

A vector field is called an infinitesimal Lie transformation if its flow is a local Lie transformation. To describe these Lie fields, choose a coordinate system (x, u) on the bundle π (we refer to [KLV] for the coordinate-free description). It produces the coordinates $(x^i, p_\sigma^j)_{0 \leq |\sigma| \leq k}$ on $J^k\pi$, where $p_\sigma^j([u]_x^k) = \dfrac{\partial^{|\sigma|} u^j}{\partial x^\sigma}$.

Recall that the operator of total derivative $D_i = \partial_{x^i} + \sum_{j;\sigma} p_{\sigma+1_i}^j \partial_{p_\sigma^j}$ maps $C^\infty(J^k\pi)$ to $C^\infty(J^{k+1}\pi)$. The total derivative corresponding to a multi-index $\sigma = (i_1, \dots, i_s)$ is denoted by $D_\sigma = D_{i_1} \circ \dots \circ D_{i_s}$.

Denote by τ_x the tangent space to the base of π at the point $x = \pi_k(x_k)$ and by ν_{x_0} the tangent to the fiber at $x_0 = \pi_{k,0}(x_k)$. Let also $F(x_k)$ be the $\pi_{k+1,k}$-fiber and $\upsilon_{x_1} = T_{x_1}\bigl(F(x_0)\bigr)$.

4.1. Higher point transformations

In the case $r > 1$, denote the projection of the Lie field to $J^\epsilon\pi = J^0\pi$ by $X = \sum_i a^i(x,u)\partial_{x^i} + \sum_j b^j(x,u)\partial_{u^j}$. Then the prolongation to $J^k\pi$ is

$$X^{(k)} = \sum_i a^i(x,u)D_i^{(k+1)} + \sum_{j;|\sigma|\le k} D_\sigma(\varphi^j)\partial_{p^j_\sigma}, \tag{4}$$

where $\varphi^j = b^j - \sum_{i=1}^n a^i p^j_i$ are components of the so-called *generating function* $\varphi = (\varphi^1,\dots,\varphi^r)$ and $D_i^{(k+1)} = \partial_{x^i} + \sum_{j;|\sigma|\le k} p^j_{\sigma+1_i}\partial_{p^j_\sigma}$ is the operator of total derivative restricted to $J^k\pi$. Though the coefficients of (4) depend seemingly on the $(k+1)$-jets, the Lie field is in fact on $J^k\pi$.

Formula (4) follows from the claim the Lie field preserves the co-distribution

$$\mathrm{Ann}(\mathcal{C}_k) = \Bigl\langle \omega^j_\sigma = dp^j_\sigma - \sum_i p^j_{\sigma+1_i}dx^i \,\Big|\, 1\le j\le r,\ |\sigma|<k\Bigr\rangle$$

and the formula $d = \sum_i dx^i \otimes D_i^{(k+1)} + \sum_{j;|\sigma|\le k}\omega^j_\sigma \otimes \partial_{p^j_\sigma}$ on $J^k\pi$.

Proposition 12. *The l-symbol $\mathfrak{g}^l(x_k)$ of the pseudogroup G at a point $x_k \in J^k\pi$ admits the splitting $\mathfrak{g}^l = \mathfrak{g}^l_H \oplus \mathfrak{g}^l_V$ depending on a point $x_{k+1}\in F(x_k)$. The horizontal part is isomorphic to*

$$\mathfrak{g}^l_H(x_k) \simeq \Bigl[S^l\nu^*_{x_0} \oplus \sum_{0<i<k}(S^i\tau^*_x \otimes S^{l-1}\nu^*_{x_0}) \oplus \sum_{i\ge k}(S^i\tau^*_x \otimes S^{k+l-i-1}\nu^*_{x_0})\Bigr]\otimes \tau_x,$$

while the vertical (evolutionary) parts is represented as

$$\mathfrak{g}^l_V(x_k) \simeq \Bigl[\sum_{0\le i<k}(S^i\tau^*_x \otimes S^l\nu^*_{x_0}) \oplus \sum_{i\ge k}(S^i\tau^*_x\otimes S^{k+l-i}\nu^*_{x_0})\Bigr]\otimes \nu_{x_0}.$$

Proof. The space $T_{x_k}J^k\pi$ is decomposed into direct sum of the horizontal $L(x_{k+1}) \subset \mathcal{C}_k(x_k)$ and the vertical $T^v_{x_k} = \mathrm{Ker}(\pi_k)_*$ components. Thus we have:

$$\mathfrak{g}^l(x_k) \subset S^lT^*_{x_k}J^k\pi \otimes T_{x_k}J^k\pi = \bigl[S^lT^*_{x_k}J^k\pi\otimes L(x_{k+1})\bigr] \oplus \bigl[S^lT^*_{x_k}J^k\pi \otimes T^v_{x_k}J^k\pi\bigr],$$

whence the required splitting. In formula (4) the horizontal and vertical components correspond to the first and the second summands respectively.

Denote by μ_a the ideal in $C^\infty(M)$ generated by functions vanishing at $a\in M$, and by μ^l_a its degree. Let $\mu^l_a(\mathfrak{Lie})$ be the space of Lie fields vanishing at a to the order l. Then $\mathfrak{g}^l(x_k) = \mu^l_{x_k}(\mathfrak{Lie})/\mu^{l+1}_{x_k}(\mathfrak{Lie})$.

As in the contact and symplectic cases we represent the symbol via the jets of generating functions. It embeds into the space $S^lT^*_{x_k}J^k\pi \otimes T_{x_k}J^k\pi$ by (4).

Let us choose a coordinate system such that the point x_k becomes the origin. If $x_k = [s]^k_a$ for some section s, this is achieved by making it the zero-section: $s = \{u^j = 0\}$. Then the condition $X^{(k)} \in \mu^l_{x_k}$ is expressed via the components of the generating function as follows:

$$a^i \in \mu^l_{x_0},\ \partial_{x^\sigma}(a^i) \in \mu^{l-1}_{x_0}, \qquad b^j \in \mu^l_{x_0},\ \partial_{x^\sigma}(b^j) \in \mu^l_{x_0}, \qquad 0 \le |\sigma| \le k.$$

This yields the claim. Note that the decomposition $T^*_{x_0} J^0\pi = \tau^*_x \oplus \nu^*_{x_0}$ is induced by the point x_1 and so the representation in the statement is canonical. □

4.2. Higher contact transformations

A Lie transformation for $r = 1$ is determined by a contact transformation $X^{(1)} = X_\varphi$ on $J^1\pi$ with a generating scalar-valued function $\varphi = \varphi(x^i, u, p_i)$:

$$X^{(1)} = \sum_i \Bigl[D^{(1)}_i(\varphi)\partial_{p_i} - \partial_{p_i}(\varphi) D^{(1)}_i \Bigr] + \varphi \partial_u.$$

The prolongation of this field to $J^k\pi$ is given by the formula similar to (4):

$$X^{(k)} = -\sum_i \partial_{p_i}(\varphi) D^{(k+1)}_i + \sum_{|\sigma|\le k} D^{(k)}_\sigma(\varphi)\partial_{p_\sigma}. \tag{5}$$

Again a calculation shows this is a field on $J^k\pi$, coinciding with X_φ for $k = 1$.

We will need below a decomposition $T_{x_1} J^1\pi = \tau_x \oplus \nu_{x_0} \oplus \upsilon_{x_1}$, which is not canonical. Though the point x_2 determines the splitting $T_{x_1} J^1\pi = L(x_1) \oplus T^v_{x_1}$, the last summand is further decomposed by a connection in the bundle $\pi_{1,0}$.

Proposition 13. *The l-symbol of the pseudogroup G at a point $x_k \in J^k\pi$ is*

$$\mathfrak{g}^l(x_k) \simeq \sum_{0\le j\le l} (S^j \nu^*_{x_0} \otimes S^{l+1-j} \upsilon^*_{x_1}) \oplus S^l \nu^*_{x_0} \oplus \sum_{1\le i<k;j} (S^i \tau^*_x \otimes S^j \nu^*_{x_0} \otimes S^{l-j} \upsilon^*_{x_1})$$
$$\oplus \sum_{k\le i;j} (S^i \tau^*_x \otimes S^j \nu^*_{x_0} \otimes S^{k+l-i-j} \upsilon^*_{x_1}).$$

Proof. As in proposition 12, due to (5), in the coordinate system (x^i, u) such that $p_\sigma(x_k) = 0$ for $|\sigma| \le k$, the condition $X^{(k)} \in \mu^l_{x_k}(\mathfrak{Lie})$ is equivalent to:

$$\varphi \in \mu^l_{x_1},\ \partial_{p_i}(\varphi) \in \mu^l_{x_1},\ \partial_{x^\sigma}(\varphi) \in \mu^l_{x_1}, \qquad 0 \le |\sigma| \le k.$$

The claim follows. □

Remark 3. *Another representation of the symbol of Lie transformations, different from that of propositions 12–13, may be obtained via the natural filtration of the space $S^l T^*_{x_k} J^k\pi \otimes T_{x_k} J^k\pi$ by the $\pi_{j,j-1}$-projections.*

4.3. Differential equations

A submanifold in $J^k\pi$ is identified with a (regular) differential equation if it is a subbundle w.r.t. all $\pi_{j,j-1}$-projections. This is a generic case (locally).

Consider two such differential equations $\mathcal{E} \subset J^k\pi$ and $\mathcal{E}' \subset J^k\pi'$ and two points x_k, x'_k. There exists a Lie transformation $\varphi : J^\epsilon\pi \to J^\epsilon\pi'$, for which $\varphi^{(k)}(x_k) = x'_k$. So we reduce the problem to the case, when both equations $\mathcal{E}$ and $\mathcal{E}'_\varphi = \left(\varphi^{-1}\right)^{(k)}(\mathcal{E}')$ live in one space $J^k\pi$. Then we try to identify $\mathcal{E}$ to $\mathcal{E}'_\varphi$ by means of a Lie transformation. Correspondingly we get point or contact equivalence of differential equations.

These were extrinsic symmetries. There are also intrinsic symmetries, which in many cases coincide with the extrinsic ones (rigid and, in particular, normal equations [KLV]). We won't consider them in this paper.

5. Dimensional obstruction for transversality

The l-transversality condition, by proposition 4, impose the following inequality on the symbol h^l_a of the class of submanifolds $\mathfrak{N}$ (equation) we consider:

$$\dim \mathfrak{g}^l_a \geq \dim h^l_a. \tag{6}$$

This easy-to-check condition is often helpful. Namely, we show in the examples below, that its fulfilment implies transversality for *generic* submanifolds N.

Therefore in this section we consider arbitrary submanifolds $N \subset M$ of codimension r. In particular, $h^l_a = S^l\mathfrak{t}^*_a \otimes \mathfrak{v}_a$.

5.1. Calculation for the basic examples

1°. Let M be a complex manifold of $\dim_{\mathbb{C}} M = n$ and G the pseudogroup from §3.1. In this case condition (6) reads:

$$\tbinom{l+n-1}{n-1} \cdot 2n \geq \tbinom{l+2n-r-1}{l} \cdot r.$$

This holds true when $r \geq n$, but for $r < n$ it is wrong when $l > 1$. Note though that the for $l = 1$ the above inequality holds for all $0 < r < n$. In this case (6) is not sufficient for 1-transversality of all N, but it is sufficient for submanifolds N of general type at the point a.

2. In the case of generic almost complex or Riemannian pseudogroup we have: $\mathfrak{g}^l = 0$ for big l, whence no transversality.

3. For the symplectic and contact pseudogroups $\mathfrak{g}^l \simeq S^{l+1}T^*M$. So denoting $\dim M = n$, we write condition (6) as:

$$\tbinom{l+n}{n-1} \geq \tbinom{l+n-r-1}{n-r-1} \cdot r,$$

which always holds. Thus we get no restrictions on the dimension of a submanifold and a generic $N \subset M$ is l-transversal for any l.

5.2. Lie pseudogroup of point transformations

Now we calculate dimensional restrictions for the pseudogroup of Lie transformations. We start with transformations lifted from $J^0\pi$. Let $n=\dim\tau$ be dimension of the base of π and $r=\dim\nu$ be the rank of π. By proposition 12:

$$\dim\mathfrak{g}^l(x_k)=n\Bigl(\tbinom{l+r-1}{r-1}+\sum_{1\le i<k}\tbinom{i+n-1}{n-1}\tbinom{l+r-2}{r-1}+\sum_{i=k}^{k+l-1}\tbinom{i+n-1}{n-1}\tbinom{l+k+r-i-2}{r-1}\Bigr)$$
$$+r\Bigl(\sum_{0\le i<k}\tbinom{i+n-1}{n-1}\tbinom{l+r-1}{r-1}+\sum_{i=k}^{l+k}\tbinom{i+n-1}{n-1}\tbinom{l+k+r-i-1}{r-1}\Bigr).$$

Using the asymptotic $\binom{l+s}{s}\sim\dfrac{l^s}{s!}$ as $l\to\infty$ we conclude that

$$\dim\mathfrak{g}^l\sim(n+r)\cdot\frac{l^{n+r-1}}{(n+r-1)!}. \tag{7}$$

Proposition 14. *The only equations that satisfy condition (6) have dimension $d=n+r$ and order $k=1$. Moreover, either $n=1$ or $r=1$ or $n=r=2$.*

Proof. Recall that dimension of the jet-space $J^k\pi$ is $d_k=n+r\binom{n+k}{k}$. Denote $d=\dim\mathcal{E}$. We have:

$$\dim h^l_{x_k}=\dim(S^lT^*_{x_k}\mathcal{E}\otimes\mathfrak{v}_{x_k})=(d_k-d)\cdot\tbinom{d+l-1}{l}\sim(d_k-d)\cdot\frac{l^{d-1}}{(d-1)!}.$$

Comparing this with (7) and (6) we obtain $d\le n+r$ and since $\pi_{k,0}:\mathcal{E}\to J^0\pi$ is surjective, we have actually the equality. In addition we have the inequality $n+r\ge r[\binom{n+k}{k}-1]$, which for $k>1$ has the only solution: $k=2$, $n=r=1$. Since $\mathcal{E}\subset J^2(1,1)$ has dimension 2, it is given by one equation of the first order and one of the second, which is the prolongation of the first. Thus actually the equation is of the first order.

Now we get $k=1$ and so $d_k-d=nr$. The additional inequality $n+r\ge nr$ has only the solutions $n=1$ or $r=1$ or $n=r=2$. For these values inequality (6) holds for all l (not only asymptotically). $\square$

Consider now the obtained three cases.

1. $n=k=1$. A submanifold $\mathcal{E}\subset J^1(1,r)$ of codimension r is a determined system of ODEs. By the well-known existence and uniqueness theorem locally all such systems are equivalent, i.e. we have transversality.

2. $r=k=1$. Here $\mathcal{E}\subset J^1(n,1)$ of codimension n is diffeomorphically projected by π_0 to $J^0\pi$. Through every point $x_0=\pi_0(x_1)$ an n-plane $L(x_1)$ passes. The obtained rank n distribution on the manifold $\mathcal{E}$ of dimension $n+1$ is generically non-integrable (this correspons to non-integrability of $\mathcal{E}$) and is either contact or even-contact. In both cases we get transversality and local equivalence of all such equations.

3. $k = 1$, $n = r = 2$. In complex notations we consider an ordinary, not necessary holomorphic, differential equation $w'_z = \psi(z, \bar{z}, w, \bar{w})$. Again here we obtain transversality and local equivalence.

Remark 4. *Since* $\pi_{1,0} : \mathcal{E} \to J^0\pi$ *is a diffeomorphism, we have the distribution* $L(\pi_{1,0}^{-1}(\cdot))$ *on* $J^0\pi$*. Thus the described cases correspond to the known distributions without moduli: 1) Line field of* rank $= 1$*; 2) (Even-)contact distribution of* corank $= 1$*; 3) The Engel distribution of* rank $=$ corank $= 2$*.*

Note that if we restrict to the equivalences induced by bundle morphisms of π (with the generating fields of the form $X = \sum a^i(x)\partial_{x^i} + b^j(x,u)\partial_{u^j}$), the corresponding dimension has the asymptotic: $\dim \mathfrak{g}^l \sim r \cdot \frac{l^{n+r-1}}{(n+r-1)!}$. Thus the only candidate for inequality (6) is the case $n = 1$ of first order ODE systems. In fact in this case we have transversality w.r.t. the smaller group as well.

5.3. Lie pseudogroup of contact transformations

In the case $\operatorname{rank} \pi = 1$, the calculations are similar and proposition 13 yields:

$$\dim \mathfrak{g}^l \sim \frac{l^{2n}}{(2n)!}. \tag{8}$$

Proposition 15. *The only equations that satisfy condition (6) have dimension* $n + 1 \le d \le 2n + 1$ *and either* $\pi_1(\mathcal{E}) \subset J^1\pi$ *is proper or* $n = 1$, $k = 2$*.*

Proof. The first claim follows by substitution of (8) into (6) and the requirement that $\pi_{k,0}|_{\mathcal{E}}$ is surjective. However if $d = 2n + 1$, as in the case $\pi_{k,1}(\mathcal{E}) = J^1\pi$, we have an additional equality $d_k - d = 1$, which has the only solution $n = 1$, $k = 2$. $\square$

Let's discuss the two obtained cases.

1. $k = 1$ and $\mathcal{E} \subset J^1(n, 1)$. As in §3.5 we see that PDE $\mathcal{E} \subset J^1\pi$ of $\dim \mathcal{E} = d$ is transversal w.r.t. the Lie pseudogroup at $x_1 \in \mathcal{E}$ iff there are no integral manifolds of dimension greater than $\left[\frac{d-1}{2}\right]$. Note that the induced distribution $\Pi \cap T\mathcal{E}$ on $\mathcal{E}$ has always integral submanifolds L of dimension $\left[\frac{d-1}{2}\right]$. If $\pi_1 : L \to L_0$ is a diffeomorphism, the submanifold has the form $j_1 s(L_0)$ for some section s of the bundle π. So transversality of $\mathcal{E}$ means there are no "partial solutions" $s : L_0 \to J^0\pi$, $j_1 s(L_0) \subset \mathcal{E}$, of dimension greater than the minimal possible.

2. $k > 1$. If $\mathcal{E}_1 = \pi_{k,1}(\mathcal{E}) \subset J^1\pi$ is proper then either there does not exist the prolongation $\mathcal{E}_1^{(1)}$ or the equation $\mathcal{E}_1$ and hence $\mathcal{E}$ is not transversal. So we consider $\mathcal{E}_1 = J^1\pi$ and then $\pi_{2,1} : \mathcal{E}_2 \to J^1\pi$ is a diffeomorphism. As proved in the proposition $n = 1$ then. We have l-transversality in this case for every l.

In fact, this is a known result of S. Lie: All the scalar ordinary differential equations of the second order are contact equivalent.

Thus we arrive at the following statement:

Theorem 16. *The only transversal (and equivalent) equations $\mathcal{E} \subset J^k\pi$ w.r.t. the Lie transformation pseudogroup are the following:*

1. $u'_x = F(x,u)$, $x \in \mathbb{R}$, $u \in \mathbb{R}^n$.
2. $u'_{x^i} = \varphi_i(x,u)$, $i = 1,\dots,n$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$.
3. $w'_z = \varphi(z,\bar{z},w,\bar{w})$, $w'_{\bar{z}} = \psi(z,\bar{z},w,\bar{w})$, $z,w \in \mathbb{C}$.
4. $u'_{x^i} = \varphi_i(x,u,u'_{x^{s+1}},\dots,u'_{x^n})$, $1 \le i \le s < n$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$.
5. $u''_{xx} = F(x,u,u'_x)$, $x \in \mathbb{R}$, $u \in \mathbb{R}$. □

The first three cases correspond, by remark 4, to open orbits in the germs of distributions. The next one is described as a submanifold in a contact manifold. The last case is equivalent to a Legendrian foliation of the contact $J^1(1,1)$. In fact, locally all Legendrian foliations of a contact manifold $J^1(n,1)$ are equivalent, but only for $n = 1$ (corresponding to S. Lie's theorem) the corresponding equation $\mathcal{E}$ is generic. Otherwise, an additional assumption of integrability should be imposed on $\mathcal{E}$. For such class of equations we get the following result:

Proposition 17. *The Lie transformations pseudogroup acts transitively on the class $\mathfrak{N}$ of equations $\mathcal{E} \subset J^{1+\epsilon}\pi$ (with $\epsilon = \max(0, 2-r)$ as before), which are integrable and project diffeomorphically $\pi_{1+\epsilon,\epsilon} : \mathcal{E} \xrightarrow{\sim} J^\epsilon\pi$.*

Proof. In fact, for $r = \operatorname{rank}\pi = 1$ we get Legendrian foliation of the contact manifold J^π and for $r > 1$ simply a foliation of $J^0\pi$. □

6. Conclusion

We have described the obstructions to transversality and hence equivalence with respect to pseudogroup G action. It turns out that these obstructions have homological nature. We explain this for the spaces $\mathfrak{O}^l$ appearing in the transversality criteria $\mathfrak{O}^l_a = 0$ of proposition 4. The exposition is short and the details will appear elsewhere.

Denote $H^{l-j,j}(\mathfrak{h})$ the cohomology group of the complex

$$0 \to \mathfrak{h}^l_a \to \mathfrak{h}^{l-1}_a \otimes \mathfrak{t}^*_a \xrightarrow{\delta} \mathfrak{h}^{l-2}_a \otimes \Lambda^2\mathfrak{t}^*_a \xrightarrow{\delta} \dots \to \mathfrak{h}^{l-j}_a \otimes \Lambda^j\mathfrak{t}^*_a \xrightarrow{\delta} \dots \tag{9}$$

Note that the δ-differential acts along $\mathfrak{t}$, not along TM, so that the zero cohomology $H^{l,0}(\mathfrak{h}) = \mathfrak{h}_l \cap S^l(\operatorname{Ann}\mathfrak{t}) \otimes TM$ can be non-zero. It however vanishes if the subspace $\mathfrak{t}_a$ is (weakly) non-characteristic.

If the pseudogroup G of order k is 2-acyclic, then the first cohomology in the above complex vanish and so $\mathfrak{h}^l_a$ prolongs $\mathfrak{h}^{l-1}_a$ along $\mathfrak{t}$ for $l > k$. Assume also that $l > m$ – the order of the equation on submanifold $\mathfrak{N}$, so that we have: $h^l_a = (h^{l-1}_a)^{(1)}$, $\mathfrak{g}^l_a = (\mathfrak{g}^{l-1}_a)^{(1)}$ for $l > k$.

By the standard tools of homological algebra we obtain:

Theorem 18. *Suppose that the pseudogroup G is 3-acyclic. Then if $\mathfrak{O}_a^{l-1}=0$, we get $\mathfrak{O}_a^l = H^{l-2,2}(\mathfrak{h}_a)$. In particular, if the action of G is transversal to order l and $\mathfrak{h}$ is 2-acyclic, the action is formally transversal.* □

This is the case, for example, with complex, symplectic and some other important pseudogroups, where we obtain the transversality.

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