A compatibility criterion for systems of PDEs and generalized Lagrange-Charpit method

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Abstract. In this paper we give a general compatibility theorem for overdetermined systems of scalar partial differential equations of complete intersection type in terms of generalized Mayer brackets. As an applications we propose a generalization of the classical Lagrange-Charpit method for integration of a single scalar PDE.

INTRODUCTION

In this paper we give an efficient compatibility criterion for a certain class of overdetermined systems $\mathscr{E} \subset J^k M$ of PDEs. Namely we consider the systems with the characteristic symbolic ideal $I(g) \subset STM$ being *complete intersection*. This means that it can be generated by elements $f_1, \dots, f_r$ (one can think of them as of homogeneous polynomials), the only relations between which are the Koszul relations: $f_j f_j = f_j f_i$.

We also call such systems of differential equations *complete intersections*. By the above arguments they can be represented by (in general non-linear) PDEs $F_1 = 0, \dots, F_r = 0$ with the symbols f_i of differential operators F_i satisfying the above requirements.

Compatibility (or formal integrability) of the system $\mathscr{E}$ is an algebraic problem, which is studied via the Spencer theory: the obstruction to integrability belongs to the second Spencer δ-cohomology group $H^{*,2}(\mathscr{E})$ (see [16, 3] or the main text). Moreover, the precise obstruction is a kind of curvature (structural) tensor and its component $W_l(\mathscr{E}) \in H^{l-1,2}$ corresponding to l-jets is called the Weyl tensor of order l ([13, 9]).

Leaving the precise definitions of the basic notions to the main text let us formulate the two main results already now.

Theorem A. *Let $\mathscr{E}$ be a system of scalar differential equations on a manifold M, which is of complete intersection type. Let its generating PDEs (without Koszul relations on the symbolic level) have orders $k_1, \dots, k_r$ (clearly $r \le n = \dim M$). Then the only non-zero Spencer δ-cohomologies are:*

$$H^{*,0}(\mathscr{E}) = \mathbb{R}^1,\ H^{*,1}(\mathscr{E}) = \mathbb{R}^r,\ H^{*,2}(\mathscr{E}) = \mathbb{R}^{r(r-1)/2}, \dots, H^{*,r}(\mathscr{E}) = \mathbb{R}^1.$$

More precisely, the generators of $H^{*,s}(\mathscr{E}) = \oplus_k H^{k,s}(\mathscr{E}) = \mathbb{R}^{\binom{r}{s}}$ correspond to $k = k_{i_1} + \dots + k_{i_s} - s$ for various choices of $1 \le i_1 < \dots < i_s \le r$.

In order to evaluate the Weyl tensors $W_l(\mathscr{E})$ we introduce the special brackets. Mayer bracket $[F, G]$ for a pair of differential operators generalizes the classical Mayer bracket known in the first order case ([10]). Its restriction to the equation prolongation $[,]_{\mathscr{E}}$ is defined canonically, whence the bracket for functions on $J^\infty M$, see details below. Note that for linear operators F,G the bracket $[F,G]_{\mathscr{E}}$ is the usual (reduced) commutator.

Theorem B. *For a complete intersection scalar system $\mathscr{E}$ the only non-zero Weyl tensors $W_k = W^{(i,j)}(\mathscr{E}) \in H^{k-1,2}(\mathscr{E})$ occur at orders $k = k_i + k_j - 1$ for various choices of $1 \le i < j \le r$. Let the system $\mathscr{E}$ be represented in the form*

$$\left\{\begin{array}{rcl} F_1\bigl(x,u(x),\dots,\frac{\partial^{|\varsigma|}u}{\partial x^\varsigma}\bigr) &=& 0, \quad |\varsigma| \le k_1, \\ \dots\dots\dots\dots\dots && \\ F_r\bigl(x,u(x),\dots,\frac{\partial^{|\tau|}u}{\partial x^\tau}\bigr) &=& 0, \quad |\tau| \le k_r. \end{array}\right. \tag{1}$$

Then the Weyl tensors express via the restricted Mayer brackets as follows:

$$W^{(i,j)}(\mathscr{E}) = [F_i,F_j]\big|_{\mathscr{E}_{k_i+k_j-1}} \cdot \Omega^{(i,j)},$$

where the classes $\Omega^{(i,j)} = [\omega^{(i,j)}] \in H^{k_i+k_j-2,2}(\mathscr{E})$ form a basis of $H^{*,2}(\mathscr{E})$. Moreover, they can be represented locally via the forms $\omega^{(i,j)} = \sum_{\alpha,\beta} e^{(i,j)}_{\alpha,\beta} \otimes dx^\alpha \wedge dx^\beta$, where $e^{(i,j)}_{\alpha,\beta}$ are some sections of the symbol bundle g associated to $\mathscr{E}$, which depend only on pair-wise resultants $R(f_i, f_j)$ of the symbols $f_i = \sigma(F_i)$, $f_j = \sigma(F_j)$ of the generating differential operators.

Notice that the condition of complete intersection can be reformulated as follows: The characteristic varieties $\mathrm{Char}^{\mathbb{C}}(F_i) \subset P^{\mathbb{C}}T^*M$ are jointly transversal, i.e. the codimension of $\cap_{i=1}^r \mathrm{Char}^{\mathbb{C}}(F_i)$ is exactly r.

Corollary. *System $\mathscr{E}$ is formally integrable if and only if all the pair-wise Mayer brackets vanish due to the system:*

$$[F_i,F_j]\big|_{\mathscr{E}_{k_i+k_j-1}} = 0 \quad \forall\, 1 \le i < j \le r.$$

If in addition $r = n$, then the system is locally smoothly integrable.

Particular cases of the above theorem were proved in [10, 11].

Using the above formula for compatibility we generalize the known (in the first order) Lagrange-Charpit method for integration of partial differential equations: One imposes on the system an overdetermination of a special kind.

This resembles the other known method of differential constraints. The difference is however that in the former case we require the new system to be compatible, while in the latter it needs to be only solvable, so that compatibility conditions are to be imposed on the system additionally.

We dot not assume any restriction on the order of the equations in the system and check compatibility via our criterion.

BASICS FROM THE DIFFERENTIAL EQUATIONS THEORY

Jet bundles and prolongations

Let M be a smooth manifold and J^kM the corresponding jet-bundle. Recall that its fiber J^k_xM over a point $x \in M$ consists of classes $[f]^k_x$ of functions $f \in C^\infty_{\text{loc}}(M)$ by the following equivalence relation: $f_1 \sim f_2$ iff $f_1(z) - f_2(z) = o(z^k)$, where z is a local coordinate centered at x. The equivalence class is called the k-jet $x_k = [f]^k_x = j_k f(x)$.

The jet space has a bundle structure given by the natural projection $\pi_k : J^kM \to M$. There are also natural bundle morphisms $\pi_{k,l} : J^kM \to J^lM$. Any smooth function $f \in C^\infty_{\text{loc}}(M)$ induces a section $j_k f : x \mapsto [f]^k_x$ of the bundle π_k.

The fiber $F(x_{k-1}) = \pi^{-1}_{k,k-1}(x_{k-1})$ has a natural affine structure associated to the vector space $S^k\tau^*_x$, where $\tau_x = T_xM$. We write $F(x_{k-1}) \simeq S^k\tau^*_x$.

An order k scalar differential equation (or system) is represented as a submanifold $\mathscr{E} \subset J^kM$ ([8]). The i^{th} prolongation of the equation $\mathscr{E}$ is defined by the formula

$$\mathscr{E}^{(i)} = \{x_{k+i} = [f]^{k+i}_x \in J^{k+i}M | j_k f(M) \text{ is tangent to } \mathscr{E} \text{ at } x_k \text{ with order} \geq i\}.$$

In order to cover the case of several equations of different orders we modify the usual definition. By a differential equation/system of (maximal) order k we mean a sequence $\mathscr{E} = \{\mathscr{E}_l\}_{-1 \leq l \leq k}$ of submanifolds $\mathscr{E}_l \subset J^l(\pi)$ with $\mathscr{E}_{-1} = M$, $\mathscr{E}_0 = J^0M = M \times \mathbb{R}$ and such that for all $0 < l \leq k$ the following conditions hold:

1. $\pi^{\mathscr{E}}_{l,l-1} : \mathscr{E}_l \to \mathscr{E}_{l-1}$ are smooth fiber bundles.
2. The first prolongations $\mathscr{E}^{(1)}_{l-1}$ are smooth subbundles of π_l and $\mathscr{E}_l \subset \mathscr{E}^{(1)}_{l-1}$.

Such a system $\mathscr{E}$ is called *formally integrable* (compatible) if $\pi_{k+i,k+i-1} : \mathscr{E}^{(i)}_k \to \mathscr{E}^{(i-1)}_k$ are smooth bundle projections for all $i > 0$. It is called *solvable* if the fiber $\mathscr{E}_\infty = \mathscr{E}^{(\infty)}_k$ over every $x \in M$ is non-empty.

Due to Cartan-Kuranishi theorem on prolongations there exists a (minimal) number l_1 such that $\mathscr{E}^{(1)}_{l-1} = \mathscr{E}_l$ for all $l \geq l_1$. Each number l, where the previous equality fails is called an *order* of the system. The codimension of $\mathscr{E}_l$ in $\mathscr{E}^{(1)}_{l-1}$ is called the multiplicity $m(l)$ of the order. Thus Cartan-Kuranishi theorem can be reformulated as finiteness of the set of orders and multiplicities. In addition the theorem says that the problems of formal integrability and solvability can be resolved in a finite number of steps.

Denote the set of all orders of the system by $\text{ord}(\mathscr{E})$. Their totality counted with multiplicity is called *formal codimension*. This number $r = \text{codim}(\mathscr{E})$ of involved PDEs is an important invariant of the system. If $r > 1$ the system is overdetermined, while in the case $r = 1$ we have a single (determined) equation.

Cartan and metasymplectic structures

Fix a point $x_k \in J^kM$. Then the tangent planes to jet-sections span a subspace

$$\mathscr{C}(x_k) = \langle \cup (j_k f)_* \tau_x \,|\, [f]_x^k = x_k \rangle,$$

which is called the *Cartan subspace* ([8]). The corresponding distribution $\mathscr{C}(x_k) \subset T_{x_k}J^kM$ on J^kM is called the *Cartan distribution*.

In J^1M this distribution is nothing else but the standard contact structure. For general k we can describe $\mathscr{C}(x_k)$ in local coordinates as follows.

Every (local) coordinate system (x^i) on M induces coordinates (x^i, p_σ) on J^kM (multiindex σ has length $|\sigma| \le k$), where $p_\sigma\Big([f]_x^k\Big) = \dfrac{\partial^{|\sigma|} f}{\partial x^\sigma}(x)$. Then the differential forms (Cartan forms)

$$\omega_\sigma = dp_\sigma - \sum \omega_{\sigma+1_i} dx^i$$

span the annulator of the distribution $\mathscr{C}$, i.e. $\mathscr{C} = \operatorname{Ker}\{\omega_\sigma\}_{0\le|\sigma|<k}$.

To describe a basis of sections of $\mathscr{C}$ we recall ([8]) the *operator of total derivative* $\mathscr{D} : C^\infty_{\text{loc}}(J^kM) \to \Omega^1(M) \otimes_{C^\infty_{\text{loc}}(M)} C^\infty_{\text{loc}}(J^{k+1}M)$. To define $\mathscr{D}$ we note that every function on J^kM is a differential operator of order k. Composing it with a vector field $X \in \mathscr{D}(M)$ we get a differential operator of order $k+1$ producing the needed operator $\mathscr{D}_X : C^\infty_{\text{loc}}(J^kM) \to C^\infty_{\text{loc}}(J^{k+1}M)$.

In coordinates $\mathscr{D}_i = \mathscr{D}_{\partial_{x^i}}$ is given by infinite series

$$\mathscr{D}_i = \partial_{x^i} + \sum p_{\sigma+1_i} \partial_{p_\sigma}. \tag{2}$$

If in the above sum we restrict $|\sigma| < k$ we get vector fields $\mathscr{D}_i^{(k)}$ on J^kM. In terms of them the Cartan distribution on J^kM is given by

$$\mathscr{C} = \langle \mathscr{D}_i^{(k)}, \partial_{p_\sigma} \rangle_{1\le i\le n, |\sigma|=k} \tag{3}$$

Recall that for any distribution Π its curvature $\Xi_\Pi \in \Lambda^2\Pi^* \otimes \nu$ is the 2-form on Π with values in the normal bundle $\nu = TM/\Pi$, defined as $\Xi_\Pi(X,Y) = [X,Y] \mod \Pi$ (although the formula uses vector fields, the tensor Ξ_Π depends only on their values at the considered point). We will however consider a smaller space of values — the linear span of the $\operatorname{Im}\Xi_\Pi \subset \nu$ — and denote it by ν.

The curvature of the Cartan distribution is called *metasymplectic structure*. In this case $\nu \simeq S^{k-1}\tau^* (= F(x_{k-2}))$ and so the metasymplectic structure can be viewed as the following 2-form

$$\Omega_k = \Omega_{x_k} \in \Lambda^2(\mathscr{C}^*(x_k)) \otimes S^{k-1}\tau_x^*.$$

Thus for every $\lambda \in S^{k-1}\tau_x$ the evaluation $\Omega_k(\lambda) = \langle \Omega_{x_k}, \lambda \rangle$ is an ordinary 2-form on the Cartan space $\mathscr{C}(x_k)$. In particular for $k=1$ we have the standard symplectic form Ω_1 on $\mathscr{C}(x_1)$.

Fix some $f \in C^\infty_{\text{loc}}(M)$ with $[f]_x^k = x_k$. Denote $H_f(x_k) = (j_k f)_* \tau_x$ (this subspace does not depend on a particular choice of f, but only on $x_{k+1} = [f]_x^{k+1}$; it is often denoted by

$L(x_{k+1})$). Then we have a decomposition

$$\mathscr{C}(x_k) = H_f(x_k) \oplus F(x_{k-1}) \simeq \tau_x \oplus S^k \tau_x^*. \tag{4}$$

Note that $\Omega_k|_H = 0$ for this choice of $H = H_f$ because $j_k f(M)$ is an integral manifold of the Cartan distribution. On the other hand it is obvious that $\Omega_k|_F = 0$.

So to calculate Ω_k it is enough to know the value on the bivector $X \wedge \theta$ with $\theta \in S^k \tau_x^*$, $X \in H(x_k) \simeq \tau_x$. This value can be expressed in terms of the Spencer operator $\delta : S^k \tau^* \to \tau^* \otimes S^{k-1} \tau^*$ as follows (cf. [13]):

$$\Omega_k(X, \theta) = \delta_X \theta \in S^{k-1} \tau^*, \tag{5}$$

where $\delta_X = i_X \circ \delta$ is the differentiation along X. The introduced structure does not depend on the point x_{k+1} determining the decomposition because the subspace $H(x_k) = L(x_{k+1})$ is Ω_k-isotropic.

Let us write the metasymplectic structure is coordinates. The commutators of the basic fields (3) are:

$$[\mathscr{D}_i, \mathscr{D}_j] = 0, \ [\partial_{p_\sigma}, \partial_{p_\gamma}] = 0 \ \text{ and } \ [\partial_{p_\sigma}, \mathscr{D}_i] = \partial_{p_\theta} \text{ if } \sigma = \theta + 1_i \text{ and 0 otherwise.}$$

So we can take $\nu \simeq \langle \partial_{p_\theta} \rangle_{|\theta| = k-1}$ and the above relations represent the metasymplectic structure $\Omega_k = \Xi_{\mathscr{C}}$ on $J^k M$. Another form of it is

$$\Omega_k = \sum_{|\sigma| = k-1} d\omega_\sigma \otimes \partial_{p_\sigma}. \tag{6}$$

If $\mathscr{E}$ is a PDEs system represented by a submanifold $\mathscr{E}_k \subset J^k M$ then the restriction of the Cartan distribution on $\mathscr{E}_k$

$$\mathscr{C}_{\mathscr{E}_k}(x_k) = \mathscr{C}(x_k) \cap T_{x_k}(\mathscr{E}_k)$$

is called Cartan distribution on $\mathscr{E}_k$. The restriction of the metasymplectic structure to $\mathscr{C}_{\mathscr{E}_k}$ will be denoted by $\Omega_{\mathscr{E}_k}$.

Spencer cohomology and Weyl tensor

The *symbol* of differential equation $\mathscr{E}$ at a point $x_k \in \mathscr{E}_k$ is the vertical tangent space to $\mathscr{E}_k$:

$$g_k(x_k) = T_{x_k}(F(x_{k-1})) \cap T_{x_k}(\mathscr{E}) \subset S^k \tau_x^*.$$

Note that the metasymplectic structure of equation $\mathscr{E}$ takes values in symbols:

$$\Omega_{\mathscr{E}_k} \in \Lambda^2 \left(\mathscr{C}^*_{\mathscr{E}_k}(x_k) \right) \otimes g_{k-1}(x_{k-1}).$$

The symbol of prolongations $\mathscr{E}_k^{(1)}$ at the point $x_{k+1} \in F(x_k)$ is a subspace

$$g_k^{(1)}(x_k) = \{ \theta \in S^{k+1} \tau_x^* \, \big| \, \delta\theta \in \tau_x^* \otimes g_k(x_k) \},$$

where $\delta : \Lambda^i \tau_x^* \otimes S^{j+1} \tau_x^* \to \Lambda^{i+1} \tau_x^* \otimes S^j \tau_x^*$ is the Spencer δ-operator (one can think of δ as of de Rham operator on forms with polynomial coefficients). The space $g_k^{(1)}$ is called the algebraic prolongation of g_k.

Thus any sequence $\{x_l \in \mathscr{E}_l \,|\, \pi_{l,l-1}(x_l) = x_{l-1}\}$ determines the maps $\delta : g_l(x_l) \to g_{l-1}(x_{l-1}) \otimes \tau_x^*$ and therefore the sequence of symbols $\{g_l(x_l)\}$ defines the Spencer δ-complex

$$0 \to g_k(x_k) \xrightarrow{\delta} g_{k-1}(x_{k-1}) \otimes \tau_x^* \xrightarrow{\delta} \cdots \xrightarrow{\delta} g_{k-n}(x_{k-n}) \otimes \Lambda^n \tau_x^* \to 0,$$

where $n = \dim M$.

The cohomology group at the term $g_i(x_i) \otimes \Lambda^j \tau_x^*$ is called the *Spencer δ-cohomology group* at the point x_k and shall be denoted by $H^{i,j}(\mathscr{E}; x_k)$.

Though the system $\mathscr{E}$ of maximal order k determines only $\{g_l(x_k)\}_{l \le k}$, we can study higher cohomology as well by setting $g_l(x_k) = g_k^{(l-k)}(x_k)$ for $l > k$.

We define *regular system of PDEs* of maximal order k as a submanifold $\mathscr{E} = \mathscr{E}_k \subset J^k M$ filtered by $\mathscr{E}_l$ and such that the symbolic system and the Spencer cohomology form graded bundles over it. Thus we often omit reference to the point.

A subspace $H \subset T_{x_k}\mathscr{E}_k$ is called *horizontal* if $(\pi_k)_* : H \to T_x M$ is an isomorphism.

Definition 1. *A horizontal distribution $H(x_k)$ on $\mathscr{E}_k$ is called a* Cartan connection *if $H(x_k) \subset \mathscr{C}_{\mathscr{E}_k}(x_k)$.*

One can prove that near every point x_k there exists a Cartan connection H. Any such connection determines the splitting similar to (4):

$$\mathscr{C}_{\mathscr{E}_k}(x_k) = H(x_k) \oplus g_k(x_k). \tag{7}$$

Definition 2. *The tensor $\Omega_H(x_k) = \Omega_{x_k}|_{H(x_k)}$ is called the* curvature *of the Cartan connection H at the point $x_k \in \mathscr{E}_k$.*

This curvature due to isomorphism $(\pi_k)_* : H(x_k) \simeq \tau_x$ can be viewed as a 2-form

$$\Omega_H(x_k) \in \Lambda^2(\tau_x^*) \otimes g_{k-1}(x_{k-1}).$$

Note that calculation of Ω_k in decomposition (7) is different of that in a *special* decomposition (4), where $H(x_k) = L(x_{k+1})$. However (5) produces a way to calculate Ω_H.

The curvature form of the Cartan connection enjoys the following properties: It is δ-closed, $\Omega_H \in \Lambda^2(\tau^*) \otimes g_{k-1}(x_{k-1})$. In addition if H' is another Cartan connection then $\Omega_{H'} = \Omega_H + \delta\sigma$, where $\sigma \in \tau_x^* \otimes g_k(x_k)$ represents H' in decomposition (7): $H' = \mathrm{graph}\{\sigma : H \to g_k\}$.

Thus we get a canonical cohomology class:

Definition 3. *The* Weyl tensor *of differential equation $\mathscr{E}$ at a point $x_k \in \mathscr{E}_k$ is the δ-cohomology class*

$$W_k(\mathscr{E}; x_k) = \Omega_H \bmod \delta(\tau_x^* \otimes g_k(x_k)) \in H^{k-1,2}(\mathscr{E}; x_k).$$

For geometric structures represented as PDEs systems this tensor coincides with the classical structural function.

Theorem 1 ([13]). *Let $\pi_{k,k-1} : \mathcal{E}_k \to \mathcal{E}_{k-1}$ be a smooth bundle. Then differential equation $\mathcal{E}_k \subset J^k M$ admits the first prolongation $\mathcal{E}_k^{(1)}$ at a point x_k if and only if $W_k(\mathcal{E}_k; x_k) = 0$.*

So if $W_k(\mathcal{E}) = 0$ we consider the Weyl tensor $W_{k+1}(\mathcal{E}) \in H^{k,2}(\mathcal{E})$ for the equation $\mathcal{E}^{(1)}$, study the equation $W_{k+1}(\mathcal{E}) = 0$ etc. Due to Poincaré δ-lemma after some number t of prolongations the second Spencer δ-cohomologies vanish, $H^{k-1+i,2}(\mathcal{E}) = 0$, $i \geq t$. Therefore the number of conditions $W_l = 0$ is finite.

A system of different orders should be investigated for formal integrability successively by the maximal order k. If some prolongation $\mathcal{E}_k^{(1)}$ is not regular, its projections $\{\pi_{k+1,l}(\mathcal{E}_{k+1})\}_{l \leq k}$, form a new system of maximal order k. Taking the regular part one continues with prolongations. The process stops in a finite number of steps (by the Cartan-Kuranishi theorem).

Remark 1. *If $g_k(x_k) = 0$ the equation $W_k(\mathcal{E}) = 0$ is exactly the Frobenius theorem for the horizontal (in this case) distribution $\mathcal{C}_{\mathcal{E}}$. More generally if $\mathcal{E}$ is a regular equation of finite type, i.e. $g_k^{(i)} = 0$ for some i, then the conditions $W_k = 0, \dots, W_{k+i-1} = 0$ guarantee the existence of local smooth solutions.*

Characteristic variety

Consider the symbolic system $\{g_l(x_k) \subset S^l \tau^*\}$. Let $g^*(x_k) = \oplus g_l^*$ be its $\mathbb{R}$-dual. It bears the structure of an $S\tau$-module given by

$$(v \cdot \varkappa)p = \varkappa(\delta_v p),\ v \in S\tau,\ \varkappa \in g^*,\ p \in g.$$

Call g^* the *symbolic module*. It is Noetherian and the Spencer cohomology of g dualizes to the Koszul homology of g^*.

Define the *characteristic ideal* by $I(g) = \operatorname{ann}(g^*) \subset S\tau$. It can be also described as follows.

Let $\delta_u : S^{l+1}\tau^* \to S^l \tau^*$ be the differentiation along $u \in \tau$ as above. Extend it to symmetric multi-vectors by the formula $\delta_v = \delta_{u_1} \cdots \delta_{u_k} : S^{k+l}\tau^* \to S^l \tau^*$ on decomposable $v = u_1 \cdots u_k \in S^k \tau$.

Define $I_k^{(l)} = \operatorname{ann} g_{k+l} \subset S^k \tau$ via the pairing $S^k \tau \otimes S^{k+l}\tau^* \to S^l \tau^*$ for $l \geq 0$: $(v, p) \mapsto \delta_v p$. Let $I^{(l)} = \oplus_k I_k^{(l)} \subset S\tau$ be the corresponding ideal. Then (by Noetherian property the intersection is actually finite):

Proposition 2 ([11]). *The characteristic ideal satisfies $I(g) = \cap_{l=0}^{\infty} I^{(l)}$.*

Define the *characteristic variety* as the set of $v \in T^* \setminus \{0\}$ such that for every k there exists a $w \in N \setminus \{0\}$ with $v^k \otimes w \in g_k$. This is a conical affine variety. We projectivize its complexification and denote the result by $\operatorname{Char}(g) \subset P^{\mathbb{C}} T^*$. Since only complex characteristics will be used, we omit the $\mathbb{C}$-superscript. Also for the characteristic variety we will denote by (co)dim its complex (co)dimension.

Another description of this variety is given via the characteristic ideal $I(g) = \oplus I_k$:

Proposition 3 ([16]). $\mathrm{Char}(g) = \{p \in P^{\mathbb{C}}T^* \mid f(p^k) = 0 \,\forall f \in I_k, \forall k\}$.

A symbolic system g is called an *(algebraic) complete intersection* if the algebra $S\tau/I(g)$ is such, i.e. if the ideal $I(g)$ is generated by $r = \operatorname{codim}\mathrm{Char}(g)$ elements (we mean codimension at the non-singular stratum).

For a system $\mathscr{E}$ of PDEs we denote by $\mathrm{Char}(\mathscr{E})$ the characteristic variety of the corresponding symbolic system.

Definition 4. *Call a regular PDEs system $\mathscr{E} \subset J^kM$ a complete intersection if the corresponding symbolic system g is such at each point $x_k \in \mathscr{E}_k$.*

A scalar system $\mathscr{E}$ is a complete intersection if it can be represented by differential equations $F_1 = 0, \dots, F_r = 0$ and the characteristic varieties, given by the symbols $f_i = 0$, $1 \le i \le r$, are jointly transversal.

Consider an example. The system $\{u_{xx} = 0, u_{yy} = 0, u_{zz} = 0\}$ is a complete intersection, while $\{u_{xy} = 0, u_{yz} = 0, u_{zy} = 0\}$ is not.

COMPATIBILITY RESULT

Mayer and Jacobi brackets

Define the higher total derivative operators $\mathscr{D}_\sigma : C^\infty(J^lM) \to C^\infty(J^{l+|\sigma|}M)$ by the formula $\mathscr{D}_\sigma = \mathscr{D}_1^{i_1} \dots \mathscr{D}_n^{i_n}$, where $\mathscr{D}_i$ is the total derivative (2) by x^i and $\sigma = (i_1, \dots, i_n)$ is a multiindex. Denote $F_\sigma = \partial_{p_\sigma}(F)$.

The *Jacobi brackets* of $F \in C^\infty(J^kM)$ and $G \in C^\infty(J^lM)$ is the following function:

$$\{F, G\} = \sum_\sigma \mathscr{D}_\sigma(F)G_\sigma - \sum_\tau \mathscr{D}_\tau(G)F_\tau \in C^\infty(J^{k+l}M).$$

This bracket is canonical (independent of coordinates [8]), its $(k+l)$-symbol vanishes and so $\{F, G\}$ is a scalar (non-linear) differential operator of order $k + l - 1$. The same concerns the following bracket (shrunk summation), which is however not canonical:

$$[F, G] = \sum_{|\sigma|=l} \mathscr{D}_\sigma(F)G_\sigma - \sum_{|\tau|=k} \mathscr{D}_\tau(G)F_\tau \in C^\infty(J^{k+l-1}M).$$

But the difference between the brackets belongs to the $(k+l-1)$-st order ideal generated by F and G:

$$[F, G] - \{F, G\} \in \mathscr{J}^{(k,l)}(F, G) = \Big\langle \mathscr{D}_\sigma(F), \mathscr{D}_\tau(G) \Big\rangle_{0 \le |\tau| < k, 0 \le |\sigma| < l}.$$

Definition 5. *The* Mayer brackets *of functions $F \in C^\infty(J^kM)$ and $G \in C^\infty(J^lM)$ is the restriction $[F, G]_{\mathscr{E}}$ of any of the above brackets to the prolongation $\mathscr{E}_{k+l-1}$ (that always exists!) of the system $\mathscr{E} = \{F = G = 0\}$.*

Now this bracket is canonical ([10]), but by abuse of notations its representative $[F, G]$ will be also called the Mayer bracket.

Proof of theorem A

Consider at first the case when the codimension of the system equals the dimension of the base: $r = n$. This is precisely the case, when the characteristic variety is empty: $\mathrm{Char}(\mathscr{E}) = 0$.

As we have noted before, the Spencer cohomology of $\mathscr{E}$ (or g) is $\mathbb{R}$-dual to the Koszul homology of the symbolic module g^*, which in the scalar case becomes the algebra $S\tau/I(g)$.

For the complete intersection $S\tau/I(g)$ the Koszul homology are known. They are given by the following characterization due to Tate and Assmus ([1]): a module g^* is a complete intersection iff $H_i(g^*) = \Lambda^i H_1(g^*)$, i.e. $H_*(g^*)$ is the exteriour algebra generated by the first homology group.

But $H_1(g^*) \simeq H^{*,1}(g)$ has the rank equal to the formal codimension $\mathrm{codim}(\mathscr{E})$, because the first Spencer cohomology counts the number of relations ([11]). Thus in the considered case the statement is proved.

In the general case $r \le n$ it follows from the reduction theorem proved in [11], which we cite in the simplified for our purposes form.

Consider the scalar symbolic system $g = \{g_l \subset S^l \tau^*\}$ and let $V^* \subset \tau^*$ be a subspace. Then we can define another scalar symbolic system $\tilde{g} = \{g_l \cap S^l V^*\} \subset SV^*$. It is called the V^**-reduction* of g.

Theorem 4 ([11]). *Let g be a symbolic system of complete intersection type and the subspace $V^* \subset \tau^*$ be transversal to the characteristic variety of g:* $\mathrm{codim}(\mathrm{Char}(g) \cap P^{\mathbb{C}} V^*) = r =: \mathrm{codim}\,\mathrm{Char}(g)$*. For instance this is so if V^* is a non-characteristic subspace of dimension r. Then the Spencer cohomology of the system g and of its V^*-reduction $\tilde{g}$ are isomorphic:*

$$H^{i,j}(g) \simeq H^{i,j}(\tilde{g}).$$

Remark 2. *The theorem was proved for more general Cohen-Macaulay modules (not necessary scalar case) and complete intersections are Cohen-Macaulay.*

By the Noether normalization lemma ([15]) we can always choose a transversal to $\mathrm{Char}(g)$ subspace V^* of dimension r, which intersects it only at zero and so is non-characteristic and transversal simultaneously. The corresponding V^*-reduction $\tilde{g}$ has the same Spencer cohomology and is a complete intersection of finite type. The result follows.

Reduction and transversality

In this section we wish to discuss the reduction procedure used in the previous theorem. Since only symbolic systems are concerned we use the notation g, not $\mathscr{E}$.

Definition 6. *We call a symbolic system $\{g_k\}$ on τ* reductive *from dimension n to dimension $m < n$ if there is a subspace $W \subset \tau$ of codimension m and a system $\{\tilde{g}_k\}$ (V^*-reduction) on the quotient $V = \tau/W$ with order multiplicities $\tilde{m}(r)$ (zero for $r \notin \mathrm{ord}(\tilde{g})$) such that*

$$g_k \cap S^k V^* = \tilde{g}_k \text{ and } \dim H^{k-1,1}(g) = \tilde{m}(k).$$

The second condition means $g_k = g_{k-1}^{(1)}$ until k is an order of $\tilde{g}$, in which case the quotient $g_{k-1}^{(1)}/g_k$ has dimension equal to the multiplicity $\tilde{m}(k)$.

In the statement below we do not assume g to be a complete intersection.

Proposition 5. *If the system is reductive, then for a generic $V^* = \operatorname{ann}(W)$ of $\dim = r$ and any $k > 0$ the subspace g_k is transversal to $g_{k-1}^{(1)} \cap S^k V^*$ in $g_{k-1}^{(1)}$.*

Proof. Choose V^* to be non-characteristic, which means that $L = P^{\mathbb{C}}V^*$ does not intersect $X = \operatorname{Char}(g)$. By the Noether normalization lemma ([15]) such subspaces L are generic. Let $f_1, \dots, f_m$ be homogeneous (complex) polynomials defining the characteristic variety X.

Consider at first the case $\deg f_1 = \dots = \deg f_m = k$. The condition we need to prove is that $g_k = \{f_1 = \dots = f_m = 0\}$ is transversal to $S^k V^*$ in $S^k \tau^*$. If it is not the case, then intersection $g_k \cap S^k V^*$ has codimension less than m and so there is a non-trivial linear relation $\lambda^1 \, f_1|_V + \dots + \lambda^m \, f_m|_V = 0$. Since $m-1$ curves of degree k on $\mathbb{C}P^{m-1}$ have a common point (generically k^{m-1} points), the subspace L intersects X that contradicts our assumption.

Consider the general case. Let homogeneous generators of the ideal I of the equation g be $f_1^j, \dots, f_{m_j}^j$ of degrees k_j $(1 \le j \le s)$ and $k_1 < \dots < k_s$. By a modification of the above argument we obtain a flag $V_1^* \subset \dots V_s^* = V^*$ of subspaces of dimensions $m_1, m_1 + m_2, \dots, m_1 + \dots + m_s = m$ such that the functions $\{f_i^j\}_{i=1,\dots,m_j}^{j=1,\dots,l}$ have no common zeros on V_l^*.

Now we prove the claim $g_k + g_{k-1}^{(1)} \cap S^k V^* = g_{k-1}^{(1)}$ using induction by k and starting from obvious $k = 0$. If the number $k \neq m_j$ the statement holds because $g_k = g_{k-1}^{(1)}$. So we should study only the cases $k = m_l$. We are going to prove by induction a more general transversality:

$$g_{m_l} + g_{m_l-1}^{(1)} \cap S^{m_l} V_l^* = g_{m_l-1}^{(1)}.$$

If this fails, then equations $f_1^l = \dots = f_{m_l}^l = 0$ are dependent on $g_{m_l-1}^{(1)} \cap S^{m_l} V_l^*$ and define variety of codimension $< m_l$.

This in turn means that we can exclude one of the functions $\{f_i^j|_{V_l}\}_{i=1,\dots,m_j}^{j=1,\dots,l}$ that have degree k_l (for example $f_1^l = \sum_{t=2}^{m_l} \lambda^t f_t^l + \sum_{j=1}^{l-1} \sum_{i=1}^{m_j} p_j^i f_i^j$ for some numbers λ^t and polynomials p_j^i) and the resulting set will have the same zeros. Since the number of functions in the resulting set is $m_1 + \dots + m_l - 1 = \dim V_l - 1$ there is a zero. This contradicts our assumptions about V_j^*. $\square$

Note that the statement is equivalent to the surjectivity of the map $\delta_w : g_k \to \delta(g_{k-1}^{(1)}) \subset g_{k-1}$ for all nonzero $w \in W$. Actually in [11] we proved more, namely the surjectivity of $\delta_w : g_k \to g_{k-1}$ for $w \in W \setminus \{0\}$. But maybe this weaker condition is in fact equivalent to the reducibility.

Remark 3. *It is interesting to compare this with a criteria of involutivity ([5]) for the first order PDEs systems: $g_1 \subset T^* \otimes N$ is involutive iff there exists a filtration*

$\{0\} = W_0 \subset \dots W_i \subset \dots W_n = T$ with $\dim W_i = i$ such that for $g[i] = g_1 \cap \mathrm{ann}(W_i) \otimes N$ the map $\delta_w : g[i]^{(1)} \to g[i]$ is epimorphic for some $w \in W_{i+1} \setminus W_i$.

Proof of theorem B

We give here only a sketch. A detailed exposition will appear elsewhere.

Again we restrict to the case $r = n$, the general situation is to be treated via the reduction theorem (using a transversal non-characteristic subspace $V^* \subset \tau^*$) and the ideas developed in the proof of the case $r = 2$, n arbitrary, from [11].

Let $k_i \in \mathrm{ord}(\mathscr{E})$ be orders of the operators F_i, generating the system (1). Recall that the Weyl tensor $W_{k_i+k_j-1}(\mathscr{E})$ of a system $\mathscr{E}$ is the δ-cohomology class of the metasymplectic structure restriction $(\Omega_{k_i+k_j-1})\big|_H$, where $H = \langle \nabla_1, \dots, \nabla_n \rangle$ is a horizontal subspace generated by

$$\nabla_t = \mathscr{D}_t^{(k_i+k_j-1)} + \textstyle\sum_{|\sigma|=k_i+k_j-1} a_t^\sigma \partial_{p_\sigma}, \quad 1 \le t \le n,$$

and $\mathscr{D}_i^{(s)}$ is the total derivative restricted to $J^s(M)$. However the specified space H is not unique and has $\dim g_{k_i+k_j-1}$-parametric freedom.

The coefficients a_t^σ can be found from the following condition: $H \subset T\mathscr{E}_{k_i+k_j-1}$ is equivalent to the linear algebraic system

$$\sum_{|\tau|=k_s} a_t^{\sigma+\tau} \cdot (F_s)_\tau = -\mathscr{D}_{\sigma+1_t}^{(k_i+k_j-1)} F_s, \quad |\sigma| = k_i + k_j - 1 - k_s,\ 1 \le s,t \le n. \qquad (8)$$

This system is underdetermined until $k = 1$ or $n = 2$, in which case it is determined (the first case is classically known and the second was considered in details in [10, 11]). In fact, for a fixed t it consists of $\binom{2k+n-2}{n-1}$ unknowns a_t^σ (they are symmetric in multiindices σ) and $n \cdot \binom{k+n-1}{n-1}$ equations.

Whenever the symbols of $F_1, \dots, F_n$ are independent (form complete intersection as in the assumptions of the theorem), linear system (8) has full rank and so is compatible. Each solution determines a subspace $H \subset \mathscr{C}(x_{k_i+k_j-1})$.

The metasymplectic structure has in coordinates form (6). Therefore for such a horizontal space H we have:

$$(\Omega_{k_i+k_j-1})\big|_H = \sum_{|\sigma|=k_i+k_j-1} \sum_{\alpha,\beta} d\omega_\sigma(\nabla_\alpha, \nabla_\beta)\, \partial_{p_\sigma} \otimes dx^\alpha \wedge dx^\beta. \qquad (9)$$

Let us evaluate the coefficients:

$$d\omega_\sigma(\nabla_\alpha, \nabla_\beta) = a_\beta^{\sigma+1_\alpha} - a_\alpha^{\sigma+1_\beta}.$$

It turns out that they do not depend on a particular choice of a solution a_t^σ of (8) and substitution of this value into formula (9) for the curvature shows that the Weyl tensors $W_{k_i+k_j-1}$ are proportional to the Mayer brackets as indicated in the statement.

Remark on the space of solutions

Let the system $\mathscr{E} = \{F_1 = 0, \dots, F_n = 0\}$ satisfy the hypotheses of theorem B. Denote by $\mathscr{R}_{\mathscr{E}}$ the space of (germs of) its solutions. It is finite-dimensional if and only if $r = n$. Let $\operatorname{ord}(\mathscr{E}) = \{k_1, \dots, k_r\}$.

Proposition 6. *If $r = n$, then $\mathscr{R}_{\mathscr{E}}$ is smooth and* $\dim \mathscr{R}_{\mathscr{E}} = k_1 \cdots k_n$.

Proof. Since the considered case is of finite type, Frobenius theorem applied to $\mathscr{E}_\infty \simeq \mathscr{E}_{k_1+\dots+k_n-n+1}$ implies smoothness. For the dimension we have: $\dim \mathscr{R}_{\mathscr{E}} = \dim \mathscr{E}_\infty - n = \sum_0^\infty \dim g_k$.

We can deform the symbolic system g preserving the complete intersection requirement to the product of 1-dimensional systems $g^{[i]} \subset SV^1_{[i]}$ (i.e. ODEs, $T = \oplus V^1_{[i]}$), $1 \le i \le n$, of orders $k_1, \dots, k_n$. But for an ODE of order k_i the space of solutions depends on k_i parameters. The dimension formula follows. □

GENERALIZATION OF THE LAGRANGE-CHARPIT METHOD

Auxiliary integrals

Consider a compatible (i.e. formally integrable) system $\mathscr{E}$ of PDEs.

Definition 7. *Call a system $\tilde{\mathscr{E}}$ an* auxiliary integral *(or a set of integrals) for the system $\mathscr{E}$ if the joint system $\mathscr{E} \cap \tilde{\mathscr{E}}$ is also compatible.*

It was noted in [11] that classical objects, such as point symmetries, contact symmetries and intermediate integrals ([4, 6, 8, 12]) as well as higher symmetries are partial cases of this notion. Moreover, some of the newly introduced generalized symmetries are also auxiliary integrals.

Consider the particular case of one single scalar PDE $\mathscr{E} = \{F = 0\}$. We wish to add to it a set of PDEs $\tilde{\mathscr{E}} = \{F_2 = 0 \dots, F_r = 0\}$. Then by the corollary of theorem B $\tilde{\mathscr{E}}$ is the set of auxiliary integrals iff $[F_i, F_j]_{\mathscr{E} \cap \tilde{\mathscr{E}}} = 0$ (where we denoted $F_1 = F$). Equivalently we can write

$$\{F_i, F_j\} = \sum_l \lambda^l_{i,j} \circ F_l, \tag{10}$$

where $\lambda^l_{i,j}$ are some differential operators of orders $\operatorname{ord}(\lambda^l_{i,j}) \le k_i + k_j - k_l - 1$.

Traditional methods of solving PDEs are based on usage of a certain kind of auxiliary integral ([4, 5, 17]). The Lagrange-Charpit method consists of finding an overdetermination of a special form for a given PDE to solve it.

Given a single PDE $F = 0$ we will search for the set of auxiliary integrals $F_2 = 0, \dots, F_r = 0$ with $r = n$. Moreover we will suppose that the symbols have jointly transversal characteristic varieties, so that the total system is a complete intersection. In this case our compatibility criterion is applicable. Moreover, the system is of finite type and thus has a finite dimensional space of solutions.

They can be found as follows. Whenever the integrability conditions are fulfilled, the system has zero symbol at order $k = k_1 + \cdots + k_n - n + 1$. Thus $\mathscr{E}_k$ is equipped with the integrable horizontal distribution $\mathscr{C}_{\mathscr{E}_k}$. It can be integrated to a foliation formed by the jet-extension of solutions. The latter step means solution of a system of ODEs, which is convenient to solve via a symmetry algebra by the method of S.Lie ([2]).

Thus we obtain the solution space with parametrized solutions. We call this procedure the *generalized Lagrange-Charpit method*.

Let us notice that even on symbolic level the proposed method imposes certain obstructions. Actually, consider a scalar PDE $F = 0$. Denote the symbol of an operator H by $\sigma(H)$. If $F_2, \dots, F_n$ is a set of auxiliary integrals for F, then we have an identity for the symbols:

$$\{\sigma(F_i), \sigma(F_i)\} = \sum_l \lambda_l^{i,j} \cdot \sigma(F_l), \tag{11}$$

where $\{,\}$ is the standard Poisson bracket on T^*M. Thus the ideal $\langle \sigma(F_i) \,|\, 1 \le i \le n \rangle$ is Poisson.

So at the beginning we search for an extension of $\sigma(F)$ to a Poisson ideal, solving symbolic equation (11). By a parameter count, generically there are obstructions for such an extension. Then we try to extend the symbols $\sigma(F)$ to differential operators and to satisfy the next approximation to the system (10). Thus we adjust symbols of order $k_i - 1$ of F_i etc.

Lagrange complete integrals and variation of constants

By results of the previous section the generalized Lagrange-Charpit method, i.e. an overdetermination of a single PDE $F = 0$ by auxiliary integrals $F_2, \dots, F_n$, is equivalent to finding a k-parametric family of solutions to $F_1 = 0$, with $k = \dim \mathscr{R}_{\mathscr{E}}$ being found via proposition 6.

Another choice is to use a complete overdetermination on the level of $m = \operatorname{ord}(F)$ jets, i.e. to impose $\binom{n+m-1}{m} - 1$ additional auxiliary integrals, so that the m-th symbol of the joint system vanishes. Let's restrict for simplicity to this case, when $k = \binom{n+m}{m} - 1$. The former case is treated quite similarly.

Definition 8. *Call a k-parametric family of solutions $u = V(x_1, \dots, x_n; a_1, \dots, a_k)$ a* complete integral *if the following non-degeneracy condition is fulfilled: The lift of these solutions to an appropriate jet-space form a submanifold of dimension $n + k$.*

Thus a complete integral corresponds to a parametrization (chart) of a domain in the solutions space of $\{F = 0\}$. Notice that for the case of first order equations, when $k = n$, this coincides with the classical Lagrange complete integral.

Knowing a complete integral (which is the case, when we applied our generalization of the Lagrange-Charpit method) one can fix constants a_i and obtain a solutions. But in fact one can more: Take some functions $a_i(x)$ instead of them and obtain an infinite-dimensional family of solutions. This method of *variation of constants* extends the classical relation between complete and general integrals ([6, 7, 17]). The functions are not arbitrary, but depend on a choice of the complete integral.

Theorem 7. *The functions $a_i(x)$ give a solution upon substitution into the complete integral $u = V(x;a)$ if and only if*

$$\begin{bmatrix} V_{a_1} & \dots & V_{a_k} \\ \dots & \dots & \dots \\ V_{x_\sigma a_1} & \dots & V_{x_\sigma a_k} \\ \dots & \dots & \dots \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \dots & \frac{\partial a_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_k}{\partial x_1} & \dots & \frac{\partial a_k}{\partial x_n} \end{bmatrix} = 0, \qquad |\sigma| < k. \tag{12}$$

This system is determined (NB: don't be mislead by a mere equations count) and thus is formally integrable. In particular, if the integral $V(x;a)$ is analytic it possesses an analytic solution (with arbitrariness in initial conditions as in the Cauchy problem). The same concerns the elliptic case: If the systems is elliptic and regular it is solvable.

Proof. The family keeps parametrizing solutions upon substitution of $a_i(x)$ iff the Cartan forms ω_σ vanish on its lift. This means that the equations $p_\sigma = \partial_{x^\sigma} V$ preserve their form after the substitution $a_i = a_i(x)$. This condition is equivalent to the following determined system of PDEs:

$$\sum_i V_{a_i} \frac{\partial a_i}{\partial x_s} = 0,$$

$$\sum_i \Bigl(V_{x_s a_i} \frac{\partial a_i}{\partial x_t} + V_{x_t a_i} \frac{\partial a_i}{\partial x_s} \Bigr) + \sum_{i,j} V_{a_i a_j} \frac{\partial a_i}{\partial x_s} \frac{\partial a_j}{\partial x_t} + \sum_i V_{a_i} \frac{\partial^2 a_i}{\partial x_s \partial x_t} = 0,$$

$$\dots\dots$$

First order in V equations are exactly those from (12). If we differentiate them by x^t and subtract from the equations having the second order in V we obtain the next equations from (12) etc. □

Thus choosing different overdeterminations (auxiliary integrals) we cover the whole space of solutions to $F = 0$.

Remark 4. *The vector-rows of the first matrix in (12) are orthogonal in $\mathbb{R}^k$ to the vectors-columns of the second matrix. If we additionally assume they are linearly independent a.e., we get the following restriction: $\binom{n+m-1}{m-1} + n \le \binom{n+m}{m} - 1$. This inequality is strict for $n > 2$, $m \ne 1$ or $n \ne 1$, $m > 2$. If $n = m = 2$ we have equality and for $m = 1$ the inequality fails to hold. In this latter case $k = n$ and whenever $\nabla V \ne 0$ we get $\det \| \frac{\partial a_i}{\partial x_j} \| = 0$, i.e. there are functional relations between a_i: $\Phi_s(a_1, \dots, a_n) = 0$, $1 \le s \le r$. This is an essential difference between the first and higher order cases.*

Examples

1. For a PDE $F(u_{xx}, u_{xy}, u_{yy}) = 0$ we have the following complete integral

$$V(x_1, x_2; a_1, a_2, a_3, a_4, a_5) = a_1 + a_2 x_1 + a_3 x_2 + a_4 x_1^2 + a_5 x_1 x_2 + \tilde{a}_{45} x_2^2,$$

where the last constant is found from the equation $F(2a_4, a_5, 2\tilde{a}_{45}) = 0$. For instance, if we study the Laplace equation $\Delta u = 0$, then $\tilde{a}_{45} = -a_4$. Variation of constants for this choice of V yields all polynomial solutions (and others too).

2. Let (M^n, g) be a Riemannian manifold and $F = \Delta_g$ the corresponding Beltrami-Laplace equation. If the metric g is geodesically equivalent to another metric $\bar{g}$, non-proportional to g at least at one point, then by a result of Matveev and Topalov ([14]) there are commuting independent second order differential operators $F_2, \dots, F_n$. They allow to integrate the Laplace equation and moreover the equations $\Delta_g u = \lambda u$ for eigenvalues completely using our approach.

3. If a PDE $F = 0$ is linear and has constant coefficients along some subspace W, we can exclude them via the Fourier transform. Alternatively we can impose auxiliary integrals $G_\sigma = p_\sigma$ with the multi-index corresponding to the W-direction: They will definitely commute with F. Thus we need to establish a smaller set of auxiliary integrals to integrate F. For instance, if the coefficient depend only on one variable, it is reducible to an ODE and so is integrable.

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