Abstract

We define, for wave turbulence, probability density functions ρ, (pdf's) on a suitably chosen phase space. We derive the Liouville equation for their evolution and identify their long time behaviors corresponding to equipartition and finite flux Kolmogorov-Zakharov (KZ) spectra. We demonstrate that, even in nonisolated systems, entropy production $-\frac{d}{dt}\int \rho \ln \rho dV$ is well defined, plays an important role in the system's evolution and find its representation in the wave turbulence approximation.

Invariant measures and entropy production in wave turbulence

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1 Introduction

We begin by briefly reviewing existing theory .Wave (or weak) turbulence deals with zero mean, bounded random fields $u_j(x,t)$ of weakly coupled dispersive waves (the sea surface without whitecaps) on infinite domains in dimension d. Present theories are formulated in terms of the BBGKY hiearchy of equations for the Fourier transforms of the physical space cumulants. The moment $\langle u_1(x_1)...u_n(x_n)\rangle$ decomposes into a sum of cumulants and their products, the sum containing one decomposition for every partition of $1...n$. Cumulants have two important properties. They decay as all differences $x_r - x_1$ (we assume translational invariance), $r = 2,..,n$ become large and all cumulants of order three or higher are zero on the joint Gaussian state. The only assumption is that, at some initial time, cumulants decay sufficiently fast to permit ordinary Fourier Tranforms. Fourier space cumulants are related to averages of products of Fourier amplitudes (generalized functions) $A^{\sigma}_{\underline{k}}$ which diagonalize the linear system by the formulas

$$\begin{aligned}
\langle A^{\sigma}_{k} A^{\sigma'}_{k'}\rangle &= \delta(\sigma k + \sigma' k')Q^{\sigma,\sigma'}(k,k') \qquad (1)\\
\langle A^{\sigma}_{k} A^{\sigma'}_{k'} A^{\sigma''}_{k''}\rangle &= \delta(\sigma k + \sigma' k' + \sigma'' k'')Q^{\sigma,\sigma',\sigma''}(k,k',k'')\\
\langle A^{\sigma}_{k} A^{\sigma'}_{k'} A^{\sigma''}_{k''} A^{\sigma'''}_{k'''}\rangle &= \delta(\sigma k + \sigma' k' + \sigma'' k'' + \sigma^{'''} k^{'''})Q^{\sigma,\sigma',\sigma''\sigma'''}(k,k',k'',k^{'''})\\
&+ \mathcal{P}^{0',0'',0'''}(\delta(\sigma k + \sigma' k')\delta(\sigma'' k'' + \sigma''' k''')\\
&Q^{\sigma,\sigma'}(k,k')Q^{\sigma'',\sigma'''}(k'',k'''))
\end{aligned}$$

and so on. Where $\mathcal{P}^{0',0'',0'''}$ ennumerates a permutaton over all possible pairings. The symbol σ enumerate degeneracy, the number of frequencies $\omega^{\sigma}(\overrightarrow{k})$ associated with the wavevector $\overrightarrow{k}$; here we take $\sigma = \pm 1$ and $\omega^{\sigma}(k) = \sigma\omega(|k|)$. The physical variables (e.g. sea surface elevation) are k and σ dependent combinations of A_k^σ. We also define complex physical observables $u^\sigma(x) = \int A_k^\sigma e^{i\sigma k\cdot x}dk$ [1]. The A_k^σ satisfy (we only include quadratic interactions)

$$\begin{aligned}\frac{dA_k^\sigma}{dt} &= i\sigma\frac{\delta H}{\delta A_k^{-\sigma}} \qquad (2)\\ &= i\sigma\omega_k A_k^\sigma + \varepsilon\sum_{\sigma_1,\sigma_2}\int L_{k,k_1,k_2}^{\sigma,\sigma_1,\sigma_2}A_{k_1}^{\sigma_1}A_{k_2}^{\sigma_2}\delta(\sigma_1 k_1+\sigma_2 k_2-\sigma k)dk_1dk_2\end{aligned}$$

where $0 < \varepsilon << 1$ is small. The coupling coefficient is related to the interaction Hamiltonian by $L_{k_1,k_2,k_3}^{\sigma_1,\sigma_2,\sigma_3} = 3i\sigma_1 H_{k_1,k_2,k_3}^{-\sigma_1,\sigma_2,\sigma_3}$. From this equation the BBGKY hierarchy for the Fourier space cumulants can easily be found. The strategy is to solve the hierarchy iteratively

$$Q^{(N)} = q_0^{\sigma,\sigma',\cdots}e^{i(\sigma\omega+\sigma'\omega'+..)t} + \varepsilon Q_1^{(N)} + \varepsilon^2 Q_2^{(N)} + ... \qquad (3)$$

and choose asymptotic expansions for the slow evolution of $q_0^{\sigma,\sigma',\cdots}$ in order to remove secular growths from the higher iterates. This leads to (i) the kinetic equation (we assume $q_0^{\sigma,-\sigma}(k,k') = q_0^{-\sigma,\sigma}(k,k') = n_k$) for the leading order approximation to the energy density $e_k = \omega_k n_k$

$$\begin{aligned}\frac{dn_k}{dt} = 4\pi\varepsilon^2\sum_{\sigma_1,\sigma_2}\int\left|L_{k,k_1,k_2}^{+,\sigma_1,\sigma_2}\right|^2 n_k n_{k_1}n_{k_2}\left(\frac{1}{n_k}-\sigma_1\frac{1}{n_{k_1}}-\sigma_2\frac{1}{n_{k_2}}\right) \qquad (4)\\ \delta(\sigma_1\omega_{k_1}+\sigma_2\omega_{k_2}-\omega_k)\delta(\sigma_1\overrightarrow{k_1}+\sigma_2\overrightarrow{k_2}-\overrightarrow{k})d\overrightarrow{k_1}d\overrightarrow{k_2}\end{aligned}$$

(ii) the frequency renormalization

$$\begin{aligned}&\sigma\omega_k \to \sigma\omega_k + \varepsilon^2\Omega_{2k}^\sigma + ... \qquad (5)\\ &\Omega_{2k}^\sigma = 4\sum_{\sigma_1,\sigma_2}\int L_{k,k_1,k_2}^{\sigma,\sigma_1,\sigma_2}L_{k,k_1,k_2}^{\sigma,\sigma_1,-\sigma_2}\Big(P\Big(\frac{1}{\sigma_1\omega_{k_1}+\sigma_2\omega_{k_2}-\sigma\omega_k}\Big)\\ &\qquad - i\pi\delta(\sigma_1\omega_{k_1}+\sigma_2\omega_{k_2}-\sigma\omega_k)\Big)\delta(\sigma_1 k_1+\sigma_2 k_2-\sigma k)dk_1dk_2\end{aligned}$$

where P is the Cauchy Principal value; and (iii) the first asymptotic survivor

$$Q^{\sigma,\sigma',\sigma''}(k,k',k'') = 2\varepsilon\mathcal{P}^{0',0'',0'''}(L_{k,k_1,k_2}^{\sigma,-\sigma_1,-\sigma_2}n_{k'}n_{k''}\Delta(\sigma\omega_k+\sigma'\omega_{k'}+\sigma''\omega_{k''}) \qquad (6)$$

are $\Delta(x) = \frac{e^{ixt}-1}{ix}$. We note from (3) that, for a sea of linear waves, the fields $u^\sigma(x)$ relax to a exact joint Gaussian state as the oscillatory factor leads to a decay (Riemann-Lebesgue lemma) in all cumulants except for $\langle u^\sigma(x)u^{-\sigma}(x+r)\rangle$ as $t\to\pm\infty$.The asymptotic survivor (which leads to a weakly decaying third order

[1] We use this definition of the FT because it is convenient for us here to use A_k^σ and $A_k^{-\sigma}$ as conjugate variables. In [1], the σ does not appear in the transform and then the $\sigma's$ do not appear in either the exponent of the transform or in the argument of the delta function in (1).

physical space cumulant) is the non Gaussian contribution which leads to energy transfer. Its asymptotic limit ($t \to \infty$) can only be taken in physical space where $\Delta(x) \to \pi\delta(x) + iP(\frac{1}{x})$. The N^{th} order cumulant has its first asymptotic survivor at $O(\varepsilon^{(N-2)})$. The results are the same as if we had assumed the phases of A_{0k}^{σ} (only the leading order approximation) were uniformly distributed (random phase approximation) but we stress that the phases of A_k^{σ} are correlated (at order ε and by the resonant nonlinearities). Stationary solutions of the kinetic equation are the fluxless equipartition spectrum $n_k = T\omega_k^{-1}$(by inspection) and, with more difficulty, the finite flux (P) KZ spectrum $n_k = cP^{\frac{1}{2}k^{-(\beta+d)}}$ where β is the degree of homogeneity of $H_{k_1,k_2,k_3}^{\sigma_1,\sigma_2,\sigma_3}$;ie $H_{\lambda k} = \lambda^{\beta} H_k$. Assuming $\omega_k = k^{\alpha}$, we note that the KZ spectrum has finite (infinite) capacity (i.e $\int \omega_k n_k dk < \infty$ $(= \infty)$) when $\beta > \alpha$ and $(\beta \leq \alpha)$. For capillary waves $\beta = \frac{9}{4}$, $\alpha = \frac{3}{2}$. When fed at a finite rate at large scales, the KZ spectra in the finite and infinite capacity cases are realized differently. Both spectra evolve in a self similar way $n_k(\omega = k^{\alpha}) = \omega^*(t)F(\frac{\omega}{\omega^*(t)})$ where $F(\eta) \to 0$, $\eta > 1$, $F(\eta) \to \eta^{-x}$ for $\eta < 1$. In the infinite capacity case ,$\omega^*(t) \to t^b$ where $b = \frac{\alpha}{\alpha-\beta}$ and $x = x_0 = \frac{\beta+d}{\alpha}$. The KZ spectrum is left in the wake. In the finite capacity case, $\omega^*(t) = (t^* - t)^b$, $0 < t < t^*, x_0 < x < x_0 + \frac{\beta-\alpha}{\alpha}, b = \frac{1}{x - x_0 - \frac{\beta-\alpha}{\alpha}} < 0$. The value x is chosen by a nonlinear eigenvalue argument but is always greater than x_0. For $t > t^*$, the KZ spectrum is realized as a backward moving front from $k = \infty$. Energy is no longer conserved for $t \geq t^*$. Throughout the process the entropy production functional $\frac{d}{dt}\int \ln n_k d\overrightarrow{k}$ is always positive but finally becomes zero on the KZ spectrum. Locally in ω we can compute $\frac{dS_k(\omega)}{dt} = \frac{1}{n_k}\frac{dN(\omega)}{dt}$, $\int n_k dk = \int N_{\omega} d\omega$ as the sum of the negative gradient of a function $(-\frac{\partial}{\partial\omega}(\frac{1}{\omega n_k}P(\omega))$) and a bulk production $P\frac{\partial}{\partial\omega}(\frac{1}{\omega n_k})$. The latter is always positive. Relaxation to the KZ spectrum and zero local entropy production rate is achieved by having the flux difference between ω and $\omega + d\omega$ match exactly the bulk production term.

Our goals in this letter are (i) to refocus attention on the pdf of wave turbulence and the original work [5] of Sagdeev and Zaslavski (SZ) who found the Brout-Prigogine [8] equation in this context,a work somewhat overshadowed by Zakharov's discovery of the KZ spectrum,(ii) to find the form of that pdf which correspond to both the equipartition and KZ spectra,and (iii) to show how the usual Gibbs entropy $-\int \rho \ln \rho dV$ connects with the entropy functional defined above.

2 Discrete, finite dimensional Hamiltonian formulation.

In order to avoid difficulties connected with infinite dimensional phase spaces, we assume our random functions $u^{\sigma}(\overrightarrow{x})$ to be spatially periodic over a box of size L^d. The continuous space of wave turbulence is thereby transformed to the Abelian group $G = \mathbb{Z}_{m_1} \times ... \times \mathbb{Z}_{m_d}$ and the random fields are elements in a finite dimensional Hilbert space V with a fixed orthonormal basis $\{\xi_j\}$ enumerated by the elements in the group G. The point values of the fields are given by projections $u_j^{\sigma} : V \to \mathbb{C}$ onto the orthonormal basis.Thus $u_j^{\sigma}(\xi) = \langle \xi, \xi_j \rangle^{\sigma}$where $z^{+1} = z$ and $z^{-1} = \overline{z}$ for any complex number z.The wave amplitudes are functions $A_k^{\sigma} : V \to \mathbb{C}$ related to the point observables by the discrete Fourier transform $A_k^{\sigma} = \frac{1}{\sqrt{N}} \sum_j u_j^{\sigma} \lambda_{\sigma k}^j$ where $N = |G|$ and $\lambda_k^j = \Pi_q e^{\{-2\pi i \frac{j_q k_q}{m_q}\}}$. The group G acts on the space of random fields V. The induced action on the observables is $j'(u_j^{\sigma}) = u_{j-j'}^{\sigma}$ and $j(A_k^{\sigma}) = \lambda_{\sigma k}^j A_k^{\sigma}$. We will assume that the dynamical system and measures underlying wave turbulence are invariant with respect to this action. This is the basic requirement of translational invariance underlying much of the work in wave turbulence. In terms of the observables A_k^{σ} the Hamiltonian is

$$H = H_0 + \delta H = \frac{1}{2} \sum_{\sigma,k} \omega_k A_k^{\sigma} A_k^{-\sigma} + \sum_{\underline{\sigma},\underline{k}} H_{\underline{k}}^{\underline{\sigma}} A_{\underline{k}}^{\underline{\sigma}} \delta(\sigma \cdot \underline{k}) \tag{7}$$

where $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\underline{k} = (k_1, k_2, k_3)$ and where ω_k is a discretized version of the continous wave dispersion relation. The interaction coefficients $H_{\underline{k}}^{\underline{\sigma}}$ satisfy $H_{\underline{k}}^{-\underline{\sigma}} = H_{\underline{k}}^{\underline{\sigma}}$ and $H_{\pi(\underline{k})}^{\pi(\underline{\sigma})} = H_{\underline{k}}^{\underline{\sigma}}$ for any permutation $\pi \in S_3$. The canonically conjugate pairs of variables are A_k^{σ} and $A_k^{-\sigma}$ and the Hamiltonian dynamical system, the discretized version of (2), is

$$\frac{dA_k^{\sigma}}{dt} = i\sigma \frac{\partial H}{\partial A_k^{-\sigma}}. \tag{8}$$

The delta function in the interaction Hamiltonian will ensure that it is translational invariant. We introduce new variables $\{J_k, \varphi_k\}_{k \in G}$ through $A_k^{\sigma} = \sqrt{J_k} e^{i\sigma\varphi_k}$. These are action-angle variables for the Hamiltonian H_0 and using these variables our full Hamiltonian can be written as

$$H = \langle \omega, J \rangle + \sum_{\underline{\sigma},\underline{k}} H_{\underline{k}}^{\underline{\sigma}} \sqrt{J_{\underline{k}}} e^{i\underline{\sigma} \cdot \varphi_{\underline{k}}} \delta(\underline{\sigma} \cdot \underline{k}) \tag{9}$$

where $\langle \omega, J \rangle = \sum_k \omega_k J_k$. The point observables u_j^{σ} induce a measure, $d\xi$, on the space of random fields by pullback of the standard measure on $\mathbb{C}$. The probability measures underlying the expectations of observables used in wave turbulence is assumed to be described by a density ρ relative to the measure

$d\xi$. The density ρ will always be assumed to be G invariant $j(\rho) = \rho \circ j^{-1} = \rho$. The expectation of observables is taken with respect to the density ρ

$$\langle u^{\underline{\sigma}}_{\underline{j}} \rangle = \int u^{\underline{\sigma}}_{\underline{j}} \rho d\xi \tag{10}$$

where for any vectors $\underline{n} = (n_1, .., n_r), \underline{j} = (j_1, ..., j_r)$ and symbol S we have $S^{\underline{n}}_{\underline{j}} = S^{n_1}_{j_1} .... S.^{n_r}_{j_r}$ and $S_{\underline{j}} = S_{j_1} ... S_{j_r}$. We will also use the notation $\underline{n} \cdot S_{\underline{j}} = n_1 S_{j_1} + ... + n_r S_{j_r}$. The size of r is usually clear from the context and often equal to two or three. Finally for any symbol S define $S^{+1} = S$ and $S^{-1} = \overline{S}$.

Since our dynamical system is Hamiltonian, the corresponding flow will preserve the measure $d\xi$. The Liouville equation for ρ induced by the flow is then

$$\partial_t \rho = L\rho. \tag{11}$$

Since the Hamiltonian is G invariant,the linear Liouville operator L is also invariant and this ensures that if $\rho(0)$ is G invariant then $\rho(t)$ is also G invariant for all t. It is thus consistent to assume, as we do in wave turbulence, that the density is G invariant.

3 The Liouville hierarchy.

We next develop the equations for the pdf ρ. We use the weak coupling property of H and write $L = L_0 + \delta L$ where

$$\begin{aligned} L_0 &= -i \sum \omega_k \partial_{\varphi_k}, \qquad (12) \\ \delta L &= \sum_{\underline{\sigma}, \underline{k}} i H^{\underline{\sigma}}_{\underline{k}} \sqrt{J_{\underline{k}}} e^{i\underline{\sigma} \cdot \varphi_{\underline{k}}} (i\underline{\sigma} \cdot \partial_{J_{\underline{k}}} - \frac{1}{2J_{\underline{k}}} \cdot \partial_{\varphi_{\underline{k}}}) \delta(\underline{\sigma} \cdot \underline{k}). \end{aligned}$$

The following notation is useful when we expand the density in eigenfunctions for the unperturbed Liouville operator L_0. Let $\mathcal{P} = \{f : G \to \mathbb{Z}\}$ be the space of integer valued function on G. If $\delta_k \in \mathcal{P}$ is the function $\delta_k(k') = 1$ when $k = k'$ and $\delta_k(k') = 0$ otherwise, then any element $p \in \mathcal{P}$ can be written as $p = \sum_k n_k \delta_k$. For any pair of r-tuples $\underline{n} = (n_1, ..., n_r)$ and $\underline{k} = (k_1, ..., k_r)$ we define a element $p^{\underline{n}}_{\underline{k}} = \sum_{i=1}^{r} n_i \delta_{k_i} \in \mathcal{P}$. We also use the functions $p^{\underline{n} \pm \underline{m}}_{\underline{k} \pm \underline{l}} = p^{\underline{n}}_{\underline{k}} \pm p^{\underline{m}}_{\underline{l}}$. The eigenfunctions of L_0 are labeled by the elements of $\mathcal{P}$ and are given by

$$\psi_p = (\frac{1}{2\pi})^N e^{i\langle p, \varphi - \omega t \rangle} \tag{13}$$

and we write

$$\rho(J, \varphi) = \sum_p \rho_p(J) \psi_p(\varphi). \tag{14}$$

The density component ρ_0 plays a particularly important role in wave turbulence and will here be called the vacuum component. For $p = p^{\underline{n}}_{\underline{k}}$ we have

$$\langle A^{\underline{n}}_{\underline{k}} \rangle = \int \sqrt{J_{\underline{k}}} \rho_{p^{-\underline{n}}_{\underline{k}}} e^{i(\underline{n}\cdot\omega_{\underline{k}})t} dJ. \tag{15}$$

The normalization condition for ρ gives $\int \rho_0 dJ = 1$ and no conditions on the complex Fourier amplitudes ρ_p when $p \neq 0$. The expansion leads to the following coupled system of linear partial differential equations for the coefficients ρ_p.

$$i\partial_t \rho_p = \sum_{p'} \langle p|\delta L|p'\rangle e^{i\langle p-p',\omega\rangle t} \rho_{p'}$$

where the matrix elements $\langle p|\delta L|p'\rangle = \int \psi_p \delta L \psi_{p'} d\varphi$ are given by

$$\begin{aligned}\langle p|\delta L|p'\rangle &= \sum_{\underline{\sigma},\underline{k}} L^{\underline{\sigma}}_{\underline{k},p} \delta_{p',p-p^{\underline{\sigma}}_{\underline{k}}}, \\ L^{\underline{\sigma}}_{\underline{k},p} &= H^{\underline{\sigma}}_{\underline{k}} \sqrt{J_{\underline{k}}} \delta(\underline{\sigma}\cdot\underline{k})(-\underline{\sigma}\cdot\partial_{J_{\underline{k}}} + \frac{1}{2J_{\underline{k}}}\cdot(p - p^{\underline{\sigma}}_{\underline{k}})).\end{aligned}$$

Explicitly we have

$$i\partial_t \rho_p = \sum_{\underline{\sigma},\underline{k}} L^{\underline{\sigma}}_{\underline{k},p} e^{i\underline{\sigma}\cdot\omega_{\underline{k}} t} \rho_{p-p^{\underline{\sigma}}_{\underline{k}}}. \tag{16}$$

It is easy to verify that the substitution $\rho_p \to \rho_p \delta(\sum_j jp(j))$ in all the above formulas will ensure that ρ is G invariant. For the G invariant densities of wave turbulence the formula (15) will contain a factor $\delta(\underline{n}\cdot\underline{k})$ and therefore $\langle A^{\underline{n}}_{\underline{k}}\rangle = 0$ for any choise of $\underline{n}$ and $\underline{k}$ with $\underline{n}\cdot\underline{k} \neq 0$. If any such quantity is nonzero at $t = 0$ the density $\rho(0)$ is not translation invariant and the theory described here must be modified . Note that to this point everything is exact, there are no conditions and no use of a small ε has been made.

4 Asymptotic expansions

Just as we did for the BBGKY hierarchy for the Fourier space cumulants, we now solve the Liouville hierarchy iteratively in powers of ε and choose the slow time behavior of the leading order approximation in order to keep the asymptotic expansion well ordered in time. Details are given in a longer paper. Here we give the main ideas and the principal results. For times of order ε^{-2}, the vacuum component satisfies the equation

$$\partial_t \rho_0 = 6\pi \sum_{\underline{\sigma},\underline{k}} |H^{\underline{\sigma}}_{\underline{k}}|^2 \delta(\underline{\sigma}\cdot\underline{k})\delta(\underline{\sigma}\cdot\omega_{\underline{k}})(\underline{\sigma}\cdot\partial_{J_{\underline{k}}}) J_{\underline{k}} (\underline{\sigma}\cdot\partial_{J_{\underline{k}}})\rho_0 = B\rho_0. \tag{17}$$

The RHS arises from resubstituting the order ε component of $\rho_{p_{\underline{k}}^{-\sigma}}$

$$\rho_{p_{\underline{l}}^{-\underline{\sigma}}}(t) = -6iL^{-\underline{\sigma}}_{\underline{l},p_{\underline{l}}^{-\underline{\sigma}}} \int\limits_0^t e^{-i\underline{\sigma}\cdot\omega_{\underline{l}}t'}\rho_0(t')dt' \tag{18}$$

back into the equation for ρ_0. Only ρ_0 terms give secular growth and thus the equation for ρ_0 is closed. This equation was originally derived by Brout and Prigogine (BP)[8] in the context of anharmonic crystals and applied by SZ [5] to study wave phase randomization. From (17) we derive the kinetic equation (4). From the secular behavior of ρ_p for $p \neq 0$, we derive the frequency renormalization (5). From the order ε component of $\rho_{p_{\underline{k}}^{\underline{\sigma}}}$, we derive the asymptotic survivor relevant for three wave interaction. In what we do, the following remark is important. The Dirac delta distribution $\delta(\underline{\sigma}\cdot\omega_{\underline{k}}) = \delta(\sigma\omega_k + \sigma'\omega_{k'} + \sigma''\omega_{k''})$ arises from the asymptotics of $\sum f(x)\Delta(x) = \lim_{t\to\infty}\sum f(x)\frac{e^{ixt}-1}{ix}$ being evaluated in the continuous limit. But since ω_k is discrete and $\omega_k = k^\alpha$,$\alpha > 1$ (otherwise triad resonances do not occur), the discrete points at which $\sigma\cdot\omega_{\underline{k}} = 0$ for large k may become sparse. Therefore in order that the asymptotics are dominated by the $\delta(\underline{\sigma}\cdot\omega_{\underline{k}})$ contribution, it is necessary to have N very large and imagine there is a large cutoff wavenumber k_c so that everywhere the set of points $\underline{\sigma}\cdot\omega_{\underline{k}} = 0$ is dense.

5 Relaxation to equipartition equlibrium

The BP equation has a family of equlibrium solutions given by

$$\rho_0 = \rho_0(\langle\omega, J\rangle). \tag{19}$$

In wave turbulence, a basic statistical quantity is the expectation of the wave-action or particle number $n_k = \langle J_k\rangle = \int J_k\rho_0 dJ$. It is readily seen that any such solution (19) gives $n_k = a\omega_k^{-1}$ where the constant a depends on the particular equilibrium solution used. The corresponding energy spectrum is $e_k = \omega_k n_k = a$. Thus all these solutions correspond to the situation where the energy is distributed equally among all the modes. Note that although the different equipartition equlibria $\rho_0 = \rho_0(\langle\omega, J\rangle)$ have the same energy spectrum, they will differ in their dependence of higher moments on k.

It is easy to verify that the spectrum of the operator B is selfadjoint with nonpositive spectrum. From this it follows that any initial vacuum density $\rho_0(0)$ will relax to some particular equipartition equlibrium. The statistics of the equilibrium state will thus depend on the initial distribution $\rho_0(0)$. The BP equation was derived for $t \to +\infty$. It is straight forward to show that we get exactly the same equation for $t \to -\infty$. Thus for the vacuum we get asymptotic reversibility of the dynamics. Any initial condition relaxes to equilibrium in both forward and backward time in exactly the same way.

6 The kinetic equation

The main object of interest in wave turbulence is the kinetic equation (4). This we derive from (17). Let $\rho_k(J_k)$ be the marginal distribution we get from ρ_0 by integrating over all variables different from J_k. Let us now assume that the vacuum stays close to a product of the marginals. This means that the wave amplitudes $\sqrt{J_k}$,are nearly independent random variables. It is then easy to show from (17) that the marginal distributions satisfy the following closed equation

$$\partial_t \rho_k = \partial_{J_k} J_k (F_k + D_k \partial_{J_k}) \rho_k \tag{20}$$

where

$$\begin{aligned} F_k &= 36\pi \sum_{\underline{\sigma},\underline{k}} |H^{-;\underline{\sigma}}_{k,\underline{k}}|^2 \delta(\underline{\sigma}\cdot\underline{k} - k)\delta(\underline{\sigma}\cdot\omega_{\underline{k}} - \omega_k) n_{\underline{k}} \left(\underline{\sigma}\cdot\frac{1}{n_{\underline{k}}}\right), \\ D_k &= 36\pi \sum_{\underline{\sigma},\underline{k}} |H^{-;\underline{\sigma}}_{k,\underline{k}}|^2 \delta(\underline{\sigma}\cdot\underline{k} - k)\delta(\underline{\sigma}\cdot\omega_{\underline{k}} - \omega_k) n_{\underline{k}}. \end{aligned} \tag{21}$$

This equation was also derived in a different way in [10]. The kinetic equation follows by multiplying (20) by J_k and integrating over J_k. We do this and get

$$\partial_t n_k = 36\pi \sum_{\underline{\sigma},\underline{k}} |H^{-;\underline{\sigma}}_{k,\underline{k}}|^2 \delta(\underline{\sigma}\cdot\underline{k} - k)\delta(\underline{\sigma}\cdot\omega_{\underline{k}} - \omega_k) n_k n_{\underline{k}} (\frac{1}{n_k} - \underline{\sigma}\cdot\frac{1}{n_{\underline{k}}}) \tag{22}$$

which is (4). Note that in this formula $\underline{\sigma} = (\sigma_1, \sigma_2)$ and $\underline{k} = (k_1, k_2)$.This equation has two types of equilibria. The first is the equipartition spectrum $n_k \sim \omega_k^{-1}$ and the other is the Kolmogorov spectra $n_k \sim \omega_k^{-\frac{\beta+d}{\alpha}}$. Both of these spectra corresponds to stationary solutions for the equations for the marginals ρ_k. These stationary marginal densities turns both out to be Gaussians

$$\rho_k(J_k) = \omega_k^y e^{-\omega_k^y J_k} \tag{23}$$

with $y = 1$ for the equipartition case and $y = \frac{\beta+d}{\alpha}$ for the Kolmogorov case. With $\rho_y = \Pi_k \rho_k = \Pi \omega_k^y e^{-\omega_k^y J_k}$ we observe that whereas ρ_1 is a equipartition equilibrium for the full BP equation the density $\rho_{\frac{\beta+d}{\alpha}}$ is not a equilibrium for the BP equation. It is well known in wave turbulence that the Kolmogorov spectrum requires a flux of energy through the system in order to sustain itself. If this flux is not provided by sources and sinks of energy in the wavespectrum the spectrum will relax to equipartition.

7 Asymptotic survivors

The vacuum will generate higher ρ_p's when the coupling coefficient with the vacuum is nonzero. The only such modes that become significant for large times are the ones that resonate with the vacuum. In wave turbulence the first asymptotic survivor is the cumulant $Q^{\underline{\sigma}}_{\underline{k}}$ where by definition $\langle A^{\underline{\sigma}}_{\underline{k}}\rangle = \delta(\underline{\sigma}\cdot\underline{k})Q^{\underline{\sigma}}_{\underline{k}}$. Here $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\underline{k} = (k_1, k_2, k_3)$. From the Fourier expansion for the density we find that $Q^{\underline{\sigma}}_{\underline{k}} = \int \sqrt{J_{\underline{k}}}\rho_{p^{-\underline{\sigma}}_{\underline{k}}} e^{i\underline{\sigma}\cdot\omega_{\underline{k}}t}dJ$. From the Liouville hierarchy we have the following expression for the density $\rho_{p^{-\underline{\sigma}}_{\underline{k}}}$

$$\rho_{p^{-\underline{\sigma}}_{\underline{k}}} = -6iH^{-\underline{\sigma}}_{\underline{k}}\sqrt{J_{\underline{k}}}(\underline{\sigma}\cdot\partial_{\underline{k}})\int_0^t e^{-i\underline{\sigma}\cdot\omega_{\underline{k}}t}\rho_0 dt'. \tag{24}$$

Combining these expressions, we find,after integration by parts and assuming $t \to \infty$, that

$$Q^{\underline{\sigma}}_{\underline{k}} = 6iH^{-\underline{\sigma}}_{\underline{k}}\widetilde{\Delta}(\underline{\sigma}\cdot\omega_{\underline{k}})n_{\underline{k}}(\underline{\sigma}\cdot\frac{1}{n_{\underline{k}}}) \tag{25}$$

which is (6). But the Liouville approach gives us more. The full density ρ in the limit $t \to +\infty$ relaxes to

$$\rho = (1 + 6i\sum_{\underline{\sigma},\underline{k}} H^{\underline{\sigma}}_{\underline{k}}\delta(\underline{\sigma}\cdot\underline{k})\widetilde{\Delta}(\underline{\sigma}\cdot\omega_{\underline{k}})(\frac{1}{2\pi})^N e^{i\underline{\sigma}\cdot\varphi_{\underline{k}}}\sqrt{J_{\underline{k}}}(\underline{\sigma}\cdot\partial_{J_{\underline{k}}}))\rho_0. \tag{26}$$

8 Phase relaxation and frequency renormalization

The long time asymptotic equation for the mode $\rho_{p^{\underline{n}}_{\underline{k}}}$ is found to be

$$\partial_t\rho_{p^{\underline{n}}_{\underline{k}}} = 6\sum_{\underline{\sigma},\underline{l}}|H^{\underline{\sigma}}_{\underline{l}}|^2\delta(\underline{\sigma}\cdot\underline{l})\widetilde{\Delta}(\underline{\sigma}\cdot\omega_{\underline{l}})(\underline{\sigma}\cdot\partial_{J_{\underline{l}}} - \frac{1}{2J_{\underline{l}}}\cdot p^{\underline{n}}_{\underline{k}})J_{\underline{k}}(\underline{\sigma}\cdot\partial_{J_{\underline{l}}} + \frac{1}{2J_{\underline{l}}}\cdot p^{\underline{n}}_{\underline{k}})\rho_{p^{\underline{n}}_{\underline{k}}} \tag{27}$$

for $t \to +\infty$. It is straightforward to show that any initial condition $\rho_{p^{\underline{n}}_{\underline{k}}}(0)$ will relax to zero in forward time. The dynamics is not asymptotically reversible since the asymptotic equation for $t \to -\infty$ includes the distribution $\widetilde{\Delta}_-(x) = -\pi\delta(x) + iP(\frac{1}{x})$ rather than $\widetilde{\Delta}(x)$, but any initial density $\rho_{p^{\underline{n}}_{\underline{k}}}(0)$ will still relax to zero as $t \to -\infty$. This asymmetric relaxation to equilibrium in forward and backward time is a well known result [2] of applying coarse graining to reversible dynamical systems.

From this equation for the density we can derive dynamical equations for the cumulants discussed in wave turbulence. For the cumulant $q^{+,+}_{k,-k}$ defined through $\langle A^+_k A^+_{k'}\rangle = \delta(k+k')q^{++}_{k,-k}e^{-i(\omega_k+\omega_{-k})t}$ we get,by assuming independence between amplitude and phase that

$$\partial_t q^{+,+}_{k,-k} = iq^{+,+}_{k,-k}(\Omega_{2k} + \Omega_{-2k}) \tag{28}$$

where the frequency correction is given by

$$\Omega_{2k} = -36\sum_{\underline{\sigma},\underline{k}}\sigma_1|H^{-,\underline{\sigma}}_{k,\underline{k}}|^2\delta(\underline{\sigma}\cdot\underline{k} - k)\widetilde{\Delta}(\underline{\sigma}\cdot\omega_{\underline{k}} - \omega_k)n_{k_2} \tag{29}$$

where $\underline{\sigma} = (\sigma_1, \sigma_2)$ and $\underline{k} = (k_1, k_2)$. This is (5).

9 Entropy production

A natural candidate for entropy in our discrete Liouvillian approach to wave turbulence is the Gibbs entropy $S = -\int \rho \ln \rho dJ$. Whether the Gibbs expression is the right one for nonequilibrium systems in general is of course an open question.Nevertheless it is interesting that we can make the connection between this and the expression natural to the kinetic equation.We also remind the reader that while there is no H-theorem for the Liouville equation,the BP and kinetic equations arise from a form of coarse graining in which we take a limit of time large with respect to an original wave period but small with respect to the nonlinear interaction time,a limit incidentally that is only possible for a range of wavenumbers [1].We also remark that the kinetic equation is derived under the assumption that the density ρ is dominated by the vacuum and the vacuum is a product of marginals. These assumptions lead to the following expression for the entropy that should apply in the context of wave turbulence $S_G = -\sum_k \int \rho_k \ln \rho_k dJ_k$. From this formula and the BP equation we can derive a expression for the entropy production

$$\frac{dS_G}{dt} = \sum_k \left[F_k - D_k \int J_k (\partial_{J_k} \ln(\rho_k))^2 \rho_k dJ_k \right].$$

In wave turbulence, the formula $S_w = -\int \ln n_k dk$ is used to define the entropy. Discretization of this formula leads to the following alternative expression for the entropy production

$$\frac{dS_w}{dt} = \sum_k \left[F_k - D_k \frac{1}{n_k} \right]$$

where we have used the kinetic equation. These expressions for the entropy production coincide only if

$$\frac{1}{\int J_k \rho_k dJ_k} = \int J_k (\partial_{J_k} \ln(\rho_k))^2 \rho_k dJ_k$$

This is a integral equation for the marginal densities ρ_k. In our approach the Gibbs entropy appears as a natural candidate for the entropy and we must therefore conclude that the entropy S_w commonly used in wave turbulence should only be applied for marginal densities satisfying the integral equation. It is easy to see that any Gaussian $\rho_k = a_k e^{-a_k J_k}$ solves the integral equation, so a sufficient condition for the validity of S_w is that the marginal densities stay close to a Gaussian.

In the refereeing process,we learned of a just published paper of Falkovich and and Fouxon [7] which also addressed entropy production in nonisolated systems.While they also noted that the wave turbulence entropy is the expression given in the introduction,they do not show, as we do, how it relates to the classical Gibbs entropy.We also note that in other works [4],[6] which address entropy and entropy production in general dynamical systems,there are more generalized notions of entropy.

The purpose of this letter was to announce several new results: the form of the pdf's corresponding to the equipartition and KZ spectra of wave turbulence;the connections with the kinetic equation,the frequency renormalization and the asymptotic survivors;the connection between the Gibbs entropy functional and the one natural to the kinetic equation.In a longer paper we will include more details and more in depth discussions of the notions of relative entropy (usually with respect to the equipartition distribution), the slight difference in initial ansatz required in the continuous and discrete cases and the connection with the so called random phase approximation.

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