On the Blaschke Conjecture for 3-Webs

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Abstract

We find relative differential invariants of orders eight and nine for a planar nonparallelizable 3-web such that their vanishing is necessary and sufficient for a 3-web to be linearizable. This solves the Blaschke conjecture for 3-webs. As a side result, we show that the number of linearizations in the Gronwall conjecture does not exceed fifteen and give criteria for rigidity of 3-webs.

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0 Introduction

Let W_d be a d-web given by d one-parameter foliations of curves on a two-dimensional manifold M^2. The web W_d is linearizable (rectifiable) if it is equivalent to a linear d-web, i.e., a d-web formed by d one-parameter foliations of straight lines on a projective plane.

The problem of finding a criterion of linearizability of webs was posed by Blaschke in the 1920s (see, for example, his book [4], §17 and §42) who claimed that it is hopeless to find such a criterion. Comparing the numbers of relative invariants for a general 3-web W_3 (and a general 4-web W_4) and a linear 3-web (and a linear 4-web), Blaschke made the conjectures that conditions of linearizability for a 3-web W_3 should consist of four relations for the ninth order web invariants (four PDEs of ninth order) and those for a 4-web W_4 should consist of two relations for the fourth order web invariants (two PDEs of fourth order) .

In [1] the authors proved that the Blaschke conjecture on linearizability conditions for 4-webs was correct: a 4-web W_4 is linearizable if and only if its two fourth order invariants vanish. In [1] a complete solution of the linearizability problem for d-webs, $d \geq 5$, was also presented. In [11] the linearizability conditions found in [1] were applied to check whether some known classes of 4-webs are linearizable.

In the present paper we continue to use the Akivis approach (see [1]) for establishing criteria of linearizability of 3-webs. In this approach, the linearizability problem is reduced to the solvability of the system of nonlinear partial differential equations on the components of the affine deformation tensor. This is the system of four nonlinear first-order PDEs on three functions defined on the plane. In the paper [10] the first obstruction for integrability of the system was found. In this paper we use results of [15] to investigate the integrability of the system and show that the obstruction found in [10] coincides with the Mayer bracket defined in [15].

We show that for nonparallelizable 3-webs, the solvability of the system indicated above is equivalent to the existence of real and smooth solutions of the system of five algebraic equations of degrees not exceeding $17, 18, 18$ and $24, 24$. This allows us:

(i) To find relative differential invariants whose vanishing leads to the linearizability of a 3-web W_3. This solves the *Blaschke problem* mentioned earlier on finding linearizability conditions in the form of invariants whose vanishing is necessary and sufficient for linearizability of a 3-web W_3. There are two types of invariants: 18 of them have order eight and 1040 have order nine. Note that the number of invariants can be different but there are always invariants of order eight. Note also that the Blaschke estimation of the "functional codimension" of the orbits of the linearizable 3-webs was correct, but the number of invariants was not. Moreover, the problem has invariants of order eight that do not match his prediction.

(ii) To establish the algorithm for determining whether a given 3-web W_3 is linearizable. This algorithm is based on investigation of the existence of a real solution of the five algebraic equations mentioned above.

We have checked that the differential invariants vanish for all linear 3-webs W_3 and apply the algorithm to two more examples (of nonlinear) 3-webs W_3.

As a side result, we obtain an estimation for the Gronwall conjecture. In 1912 Gronwall ([13]) made the following conjecture: *if a nonparallelizable 3-web W_3 in the plane is linearizable, then, up to a projective transformation, a diffeomorphism transforming W_3 into a linear 3-web is uniquely determined.* The Gronwall conjecture is also called the "fundamental theorem" of nomography. Note that for parallelizable 3-webs such uniqueness does not take place. In fact, such a 3-web is formed by the tangents to a curve of third degree, but curves of third degree have nontrivial projective invariants (see [4], §17).

Bol ([6], [7], 1938) and Borůvka ([8], 1938) proved that the number of projectively nonequivalent linearizations of a nonparallelizable, linearizable 3-web does not exceed 16. Grifone, Muzsnay and Saab ([12], 2001) proved that this number does not exceed 15. We also prove that this number does not exceed 15, and give criteria for rigidity of 3-webs, but our method is different from that in [12].

Note that Vaona ([20], 1961) and Smirnov ([18], [19]) considered the Gronwall conjecture from the point of view of nomography. Vaona claimed that the above mentioned number does not exceed 11, and Smirnov claimed that this number does not exceed one (i.e., that the Gronwall conjecture is right).

In addition, we find the linearity condition for 3-webs and establish the relationship of this to the condition that a plane curve consists of flexes and to the Euler equation in gas-dynamics.

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1 Basics Constructions

We recall main constructions for 3-webs on two-dimensional manifolds (see, for example, [5] or [4], [10]) in a form suitable for us.

Let M^2 be a two-dimensional manifold, and suppose that a 3-web W_3 is given on M^2 by three differential 1-forms ω_1, ω_2, and ω_3 such that any two of them are linearly independent.

Proposition 1.1 *The forms ω_1, ω_2, and ω_3 can be normalized in such a way that the normalization condition*

$$\omega_1 + \omega_2 + \omega_3 = 0 \tag{1}$$

holds.

Proof. In fact, if we take the forms ω_1 and ω_2 as co-basis forms of M^2, then the form ω_3 is a linear combination of the forms ω_1 and ω_2:

$$\omega_3 = \alpha\omega_1 + \beta\omega_2\,,$$

where $\alpha, \beta \neq 0$. After the substitution

$$\omega_1 \to \frac{1}{\alpha}\omega_1,\ \omega_2 \to \frac{1}{\beta}\omega_2,\ \omega_3 \to -\omega_3$$

the above equation becomes (1). ■

It is easy to see that any two of such normalized triplets $\omega_1, \omega_2, \omega_3$ and $\omega_1^s, \omega_2^s, \omega_3^s$ determine the same 3-web W_3 if and only if

$$\omega_1^s = s^{-1}\omega_1,\ \omega_2^s = s^{-1}\omega_2,\ \omega_3^s = s^{-1}\omega_3 \tag{2}$$

for a non-zero smooth function $s \in C^\infty\left(M^2\right)$.

1.1 Structure Equations

From now on we shall assume that a 3-web W_3 is given by differential 1-forms ω_1, ω_2, and ω_3 normalized by condition (1).

Because M^2 is a two-dimensional manifold, there is a unique differential 1-form γ such that

$$\begin{aligned} d\omega_1 &= \omega_1 \wedge \gamma, \\ d\omega_2 &= \omega_2 \wedge \gamma. \end{aligned} \tag{3}$$

Moreover, it follows from (1) that

$$d\omega_3 = \omega_3 \wedge \gamma.$$

We call γ the *connection form* and equations (3) the *web structure equations.*

Later on we shall see that γ determines the so-called Chern connection on M^2.

For other representations $(\omega_1^s, \omega_2^s, \omega_3^s)$ of the web, structure equations (3) take the form

$$\begin{aligned} d\omega_1^s &= \omega_1^s \wedge \gamma^s, \\ d\omega_2^s &= \omega_2^s \wedge \gamma^s, \end{aligned}$$

where

$$\gamma^s = \gamma + \frac{ds}{s}.$$

Note that the differential 2-form $d\gamma$ does not depend on the web representation and is an invariant of 3-webs.

Let

$$d\gamma^s = K_s \ \omega_1^s \wedge \omega_2^s$$

and

$$d\gamma = K \ \omega_1 \wedge \omega_2.$$

The function K is called the *web curvature.* It follows from the last two equations that

$$K_s = s^2 K.$$

This means that the web curvature K is a relative invariant of weight two.

Let ∂_1, ∂_2 be the dual basis of the vector field module: $\omega_i\left(\partial_j\right) = \delta_{ij}$, $i, j = 1, 2$. One has

$$df = \partial_1\left(f\right)\ \omega_1 + \partial_2\left(f\right)\ \omega_2$$

for smooth functions $f \in C^\infty\left(M^2\right)$.

If we decompose the connection forms γ and γ^s relative to the basis $\{\omega_1, \omega_2\}$:

$$\gamma = g_1\ \omega_1 + g_2\ \omega_2 \tag{4}$$

and

$$\gamma^s = g_{s1}\ \omega_1^s + g_{s2}\ \omega_2^s,$$

we get

$$\begin{aligned} g_{s1} &= s g_1 + \partial_1 s, \\ g_{s2} &= s g_2 + \partial_2 s. \end{aligned}$$

In addition, we find

$$[\partial_1, \partial_2] = -g_2\partial_1 + g_1\partial_2. \tag{5}$$

This follows from

$$\omega_1([\partial_1, \partial_2]) = -d\omega_1(\partial_1, \partial_2) = (\gamma \wedge \omega_1)(\partial_1, \partial_2) = -\gamma(\partial_2) = -g_2$$

and

$$\omega_2([\partial_1, \partial_2]) = -d\omega_2(\partial_1, \partial_2) = (\gamma \wedge \omega_2)(\partial_1, \partial_2) = \gamma(\partial_1) = g_1.$$

Remark that

$$\gamma([\partial_1, \partial_2]) = 0.$$

For the curvature function, one has

$$K = \partial_1(g_2) - \partial_2(g_1), \tag{6}$$

because

$$\begin{aligned} d\gamma &= dg_1 \wedge \omega_1 + dg_2 \wedge \omega_2 + g_1 d\omega_1 + g_2 d\omega_2 = \\ & -\partial_2(g_1)\ \omega_1 \wedge \omega_2 + -\partial_1(g_2)\ \omega_1 \wedge \omega_2 + g_1\omega_1 \wedge \gamma + g_2\omega_2 \wedge \gamma \\ &= -\partial_2(g_1)\ \omega_1 \wedge \omega_2 + -\partial_1(g_2)\ \omega_1 \wedge \omega_2 + g_1 g_2 \omega_1 \wedge \omega_2 - g_1 g_2 \omega_1 \wedge \omega_2 \\ &= (\partial_1(g_2) - \partial_2(g_1))\,\omega_1 \wedge \omega_2. \end{aligned}$$

In this paper we shall apply the following two normalizations: (i) $d\omega_3 = 0$, and (ii) $K = 1$.

The first one defines a 3-web up to gauge transformations: $f \to F(f)$, while the second one defines the e-structure on M^2.

Below we consider these two normalizations in detail.

1.2 Normalization $d\omega_3 = 0$

We assume that M^2 is a simply connected domain of $\mathbb{R}^2$, and therefore there exists a smooth function f such that ω_3 is proportional to df, that is, $\omega_3 \wedge df = 0$. The function f is called the *web function.*

Note that this function is defined up to a renormalization (gauge transformation) $f \longmapsto F(f)$.

We choose a representation of W_3 such that

$$\omega_3 = df. \tag{7}$$

Similarly, one finds smooth functions x and y for forms ω_1 and ω_2 such that

$$\omega_1 = a\ dx,\ \ \omega_2 = b\ dy$$

for some smooth functions a and b.

Moreover, the functions x and y are independent and therefore can be viewed as (local) coordinates. In these coordinates, the normalization condition gives

$$\omega_1 = -f_x\ dx,\ \ \omega_2 = -f_y\ dy,\ \ \omega_3 = df.$$

The vector fields ∂_1 and ∂_2 take the following form

$$\partial_1 = -\frac{1}{f_x}\frac{\partial}{\partial x}, \quad \partial_2 = -\frac{1}{f_y}\frac{\partial}{\partial y}.$$

In this case

$$0 = d\omega_3 = \omega_3 \wedge \gamma$$

and

$$\gamma = -H\omega_3 = H\left(\omega_1 + \omega_2\right)$$

for some function H.

Hence (see (4))

$$g_1 = g_2 = H.$$

In terms of the web function f, one has

$$H = \frac{f_{xy}}{f_x f_y},$$

and

$$\gamma = -\frac{f_{xy}}{f_x f_y}\omega_3.$$

For the curvature function K one gets the following expression:

$$K = -\frac{1}{f_x f_y}\left(\log\left(\frac{f_x}{f_y}\right)\right)_{xy} = \frac{f_{xyy}}{f_x f_y^2} - \frac{f_{xxy}}{f_x^2 f_y} + \frac{f_{xx} f_{xy}}{f_x^3 f_y} - \frac{f_{xy} f_{yy}}{f_x f_y^3}$$

(cf. [4], § 9, or [2], p. 43).

For the basis vector fields ∂_1 and ∂_2, the structure equations take the form

$$[\partial_1, \partial_2] = H\ (\partial_2 - \partial_1), \tag{8}$$

and

$$K = \partial_1\left(H\right) - \partial_2\left(H\right). \tag{9}$$

1.3 Normalization $K = 1$

In this section we assume that K is a nonvanishing function: $K \neq 0$. We can assume that $K > 0$ (changing the orientation if necessary), that is,

$$K = k^2$$

for some weight one smooth function k.

Let us take $s = k^{-1}$ and denote by θ_i the differential 1-forms ω_i^s with $s = k^{-1}$:

$$\theta_i = k\omega_i$$

for $i = 1, 2$.

We shall denote the corresponding connection form γ^s by α:

$$\alpha = \gamma - \frac{dk}{k}.$$

One has $k_t = tk$ for any positive smooth function t, and therefore $\theta_i = k\omega_i = k_t\omega_i^t$, $i = 1, 2$, are invariant differential 1-forms intrinsically connected with the web. They define the e-structure on M^2 and satisfy the structure equations

$$\begin{aligned} d\theta_1 &= \theta_1 \wedge \alpha, \\ d\theta_2 &= \theta_2 \wedge \alpha, \\ d\alpha &= \theta_1 \wedge \theta_2, \end{aligned} \tag{10}$$

because $K_{k^{-1}} = (k^{-1})^2 K = 1$.

Let $\{\nabla_1, \nabla_2\}$ be the basis dual to the co-basis $\{\theta_1, \theta_2\}$, and let

$$\alpha = a_1\ \theta_1 + a_2\ \theta_2.$$

Then (5) and (6) imply that

$$[\nabla_1, \nabla_2] = -a_2\ \nabla_1 + a_1\ \nabla_2 \tag{11}$$

and

$$\nabla_1 (a_2) - \nabla_2 (a_1) = 1, \tag{12}$$

where a_1 and a_2 are invariants of the web.

In terms of the web function f, one has

$$a_1 = \frac{H}{k} - \frac{\partial_1 k}{k^2}, \quad a_2 = \frac{H}{k} - \frac{\partial_2 k}{k^2}. \tag{13}$$

1.4 Linear 3-Webs

In this section we consider linear 3-webs. Let W_3 be a 3-web given by a web function $z = f(x, y)$. The following theorem gives us a criterion for W_3 to be linear.

Theorem 1.2 *Suppose that a 3-web W_3 is given locally by the function $z = f(x, y)$. Then W_3 is linear if and only if*

$$f_y^2\ f_{xx} - 2f_x f_y\ f_{xy} + f_x^2\ f_{yy} = 0. \tag{14}$$

Proof. Note that a 3-web W_3 can be also given by a nonvanishing function $f_x(x, y)/f_y(x, y)$. Namely, the horizontal and vertical leaves are given by $x =$ const and $y =$ const, respectively, and the transversal leaves are defined in such a way that $t = \tan\alpha$, where α is the angle of the normal to the transversal leaves with the horizontal leaves. So, the web W_3 is linear if and only if the function $f_x(x, y)/f_y(x, y)$ remains constant along the transversal leaves. Thus

$$d\left(\frac{f_x}{f_y}\right) = 0 \bmod (\omega_1 + \omega_2)$$

and

$$\partial_1\left(\frac{f_x}{f_y}\right)\omega_1+\partial_2\left(\frac{f_x}{f_y}\right)\omega_2=0 \mod (\omega_1+\omega_2)$$

or

$$\partial_1\left(\frac{f_x}{f_y}\right)-\partial_2\left(\frac{f_x}{f_y}\right)=0. \tag{15}$$

It is easy to see that equation (15) is equivalent to equation (14). ■

Remark. Note that linearity condition (14) of a 3-web W_3 can be written in the determinant form:

$$\det\begin{Vmatrix} f_{xx} & f_{xy} & f_x \\ f_{xy} & f_{yy} & f_y \\ f_x & f_y & 0 \end{Vmatrix}=0. \tag{16}$$

Note also that linearity condition (14) (or (16)) for a 3-web is also the necessary and sufficient condition for a point (x,y) to be a flex of the curve defined by the equation $f(x,y)=0$ (see, for example, [17], section 1.1.5). The difference is that here (14) is the equation for finding the function $z=f(x,y)$ (it should be satisfied for all points (x,y)) while in algebraic geometry (14) is the equation for finding the flexes (x,y) of the curve defined by the equation $f(x,y)=0$ provided that the function $f(x,y)$ is given.

Differential equation (14) can be integrated as follows. Let us rewrite this equation in form (15). Then

$$\partial_x\left(\frac{f_x}{f_y}\right)-\left(\frac{f_x}{f_y}\right)\partial_y\left(\frac{f_x}{f_y}\right)=0,$$

or setting

$$w=\frac{f_x}{f_y},$$

we can rewrite (14) as the following system:

$$\begin{aligned}\partial_x w - w\partial_y w &= 0,\\ \partial_x f - w\partial_y f &= 0.\end{aligned}$$

The first equation

$$\partial_x w - w\partial_y w = 0$$

is the Euler equation in gas-dynamics (see, for example, [16], p. 3).

Solutions of this equation are well-known. Namely, if $w_0(y)=w|_{x=0}$ gives a Cauchy data, then the solution $w(x,y)$ can be found from the system

$$\begin{aligned} y+w_0(\lambda)\,x-\lambda &= 0,\\ w(x,y)-w_0(\lambda) &= 0\end{aligned} \tag{17}$$

by elimination of the parameter λ.

Further, if w is a solution of the Euler equation, then the functions w and f are first integrals of the vector field

$$\partial_x - w\partial_y,$$

and therefore there is the relation $f = F(w)$ for some smooth function F.

Summarizing we get the following description of linear 3-webs.

Proposition 1.3 *The web functions $f(x,y)$ of linear 3-webs have the form*

$$f(x,y) = F(w(x,y)),$$

where $w(x,y)$ is a solution of the Euler equation, and F is some smooth function.

As we saw earlier, the web functions are defined up to gauge transformations $f \longmapsto F(f)$. Therefore, the above proposition yields the following description of linear 3-webs.

Theorem 1.4 *Web functions of linear 3-webs can be chosen as solutions of the Euler equation.*

Example 1 Taking $w_0(y) = y$, we get the linear 3-web with the web function $w = y/(1-x)$. This 3-web is generated by two families of coordinate lines $\{x = \text{const}\}$, $\{y = \text{const}\}$ and the straight lines of the pencil with the center $(1,0)$. This 3-web is parallelizable.

Example 2 Taking $w_0(y) = y^2/4$, we get the linear 3-web with the web function $\left(\frac{1+\sqrt{1-xy}}{x}\right)^2$, or simply

$$f = \frac{1+\sqrt{1-xy}}{x}.$$

It is easy to prove that this 3-web is generated by two families of coordinate lines $\{x = \text{const}\}$, $\{y = \text{const}\}$ and the tangents to the hyperbola $y = \frac{1}{x}$. In fact, the leaves of the third foliation of this web are level sets of the above web function, i.e., they are determined by the equation

$$\frac{1+\sqrt{1-xy}}{x} = C,$$

where C is a constant. The latter equation is equivalent to the equation

$$y = -C^2 x + 2C.$$

Thus the leaves of the the third foliation are straight lines. To find the envelope of these leaves, we differentiate the above equation with respect to C. This gives $C = \frac{1}{x}$. Therefore, the envelope is defined by the equation $y = \frac{1}{x}$.

Example 3 Taking $w_0(y) = -2\sqrt{-y},$ we get the linear 3-web with the web function

$$f = x + \sqrt{x^2 - y}.$$

Using the same approach as in Example 2, we can prove that the leaves of the third foliation are straight lines defined by the equation

$$y = 2Cx - C^2,$$

and these straight lines are tangent to the parabola $y = x^2$.

2 The Chern connection

Recall that a connection ∇ in a vector bundle $\pi : E(\pi) \to B$ over a manifold B can be defined by a covariant differential $d_\nabla : \Gamma(\pi) \to \Gamma(\pi) \otimes \Omega^1(B)$, where $\Gamma(\pi)$ is the module of smooth sections of the bundle π, and $\Omega^1(B)$ is the module of smooth differential 1-forms on the manifold B. The covariant differential can be extended in a natural way to the following sequence:

$$\Gamma(\pi) \stackrel{d_\nabla}{\to} \Gamma(\pi) \otimes \Omega^1(B) \stackrel{d_\nabla}{\to} \Gamma(\pi) \otimes \Omega^2(B) \stackrel{d_\nabla}{\to} \cdots$$

The square of the covariant differential is the module homomorphism

$$d_\nabla^2 \stackrel{\text{def}}{=} R_\nabla : \Gamma(\pi) \to \Gamma(\pi) \otimes \Omega^2(B).$$

This homomorphism R_∇ is called the *curvature* of the connection ∇.

We shall apply this construction to 3-webs on a two-dimensional manifold M. Let $\pi = \tau^* : T^*(M) \to M$ be the cotangent bundle, and let W_3 be a 3-web defined by the differential 1-forms $\{\omega_1, \omega_2, \omega_3\}$ normalized by (1).

We use the differential 1-form γ to define a connection in the cotangent bundle by the following covariant differential:

$$d_\gamma : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M),$$

where

$$\begin{aligned} d_\gamma(\omega_1) &= -\omega_1 \otimes \gamma, \\ d_\gamma(\omega_2) &= -\omega_2 \otimes \gamma; \end{aligned}$$

and $\otimes$ denotes the tensor product.

Note that in the tensor product $\Omega^1(M) \otimes \Omega^1(M)$ the first factor plays the role of coefficients and should be differentiated due to the connection, and the second one is differentiated by the de Rham differential.

It is easy to check that the curvature form of the above connection is equal to $-d\gamma$, that is, $d_\gamma^2 : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^2(M)$ is the multiplication by $-d\gamma$:

$$d_\gamma^2(\omega) = -\omega \otimes d\gamma$$

for any differential form $\omega \in \Omega^1(M)$.

This connection is called the *Chern connection* of the web.

It is also easy to check that the Chern connection satisfies the relations

$$d_{\gamma^s}(\omega_i^s) = -\omega_i^s \otimes \gamma^s$$

for $i = 1, 2$, and any non-zero smooth function s.

The straightforward computation shows also that d_γ is a torsion-free connection.

Note that in the case $K \neq 0$ the second normalization ($K = 1$) leads us to the invariant 1-forms θ_1 and θ_2 and to the unique Chern connection d_α.

Recall that for the covariant differential $d_\nabla : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M)$ of any torsion-free connection ∇, one has $d_\nabla = d_\gamma - T$, where

$$T : \Omega^1(M) \to S^2\left(\Omega^1(M)\right) \subset \Omega^1(M) \otimes \Omega^1(M)$$

is the *affine deformation tensor* of the connection, and $S^2\left(\Omega^1(M)\right)$ is the module of the symmetric $(0,2)$-tensors on M.

In what follows, we shall use the notation $\nabla_X(\theta) \overset{\text{def}}{=} (d_\nabla \theta)(X)$ for the covariant derivative of a differential 1-form θ along a vector field X with respect to the connection ∇.

Proposition 2.1 *Let $d_\nabla : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M)$ be the covariant differential of a connection ∇ in the cotangent bundle of M. Then the foliation $\{\theta = 0\}$ on M given by the differential 1-form $\theta \in \Omega^1(M)$ consists of geodesics of ∇ if and only if*

$$d_\nabla(\theta) = \alpha \otimes \theta + \theta \otimes \beta$$

for some differential 1-forms $\alpha, \beta \in \Omega^1(M)$.

Proof. Let θ' be a differential 1-form such that θ and θ' are linearly independent. Then

$$d_\nabla(\theta) = \alpha \otimes \theta + \theta \otimes \beta + h\theta' \otimes \theta'.$$

Assume that X is a geodesic vector field on M such that $\theta(X) = 0$. Then $\nabla_X(\theta)$ must be equal to zero on X. But

$$d_\nabla \theta(X) = \beta(X)\theta + h\theta'(X)\theta'.$$

Therefore, $h = 0$. ■

Corollary 2.2 *The foliations $\{\omega_1 = 0\}, \{\omega_2 = 0\}$, and $\{\omega_3 = 0\}$ are geodesic with respect to the Chern connection.*

The problem of linearization of webs can be reformulated as follows: *find a torsion-free flat connection such that the foliations of the web are geodesic with respect to this connection.*

Proposition 2.3 *Let* $d_\nabla = d_\gamma - T : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M)$ *be the covariant differential of a torsion-free connection* ∇ *such that the foliations* $\{\omega_p = 0\}$, $p = 1, 2, 3$, *are geodesic with respect to the connection* ∇. *Then*

$$\begin{aligned} T &= (T^1_{11}\omega_1 \otimes \omega_1 + T^1_{12}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)) \otimes \partial_1 \\ &\quad + (T^2_{22}\omega_2 \otimes \omega_2 + T^2_{12}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)) \otimes \partial_2, \end{aligned} \tag{18}$$

where the components of the affine deformation tensor have the form

$$T^2_{12} = \lambda_1, \quad T^1_{12} = \lambda_2, \; T^1_{11} = 2\lambda_1 + \mu, \; T^2_{22} = 2\lambda_2 - \mu \tag{19}$$

for some smooth functions λ_1, λ_2, *and* μ.

Proof. Due to Proposition 2.1 and the requirement that the foliations $\{\omega_1 = 0\}$ and $\{\omega_2 = 0\}$ are geodesic, one gets (18). The same requirement for the foliation $\{\omega_3 = 0\}$ gives the following relation for the components of the affine deformation tensor T:

$$T^1_{11} + T^2_{22} = 2(T^1_{12} + T^2_{12}),$$

and this implies (19). ■

Therefore, in order to linearize a 3-web, one should find functions λ_1, λ_2 and μ in such a way that the connection corresponding to $d_T = d_\gamma - T$, where the affine deformation tensor T has form (19), is flat.

The covariant differential d_T has the following form:

$$\begin{aligned} d_T\omega_1 &= -\omega_1 \otimes \sigma_{11} - \omega_2 \otimes \sigma_{12}, \\ d_T\omega_2 &= -\omega_1 \otimes \sigma_{21} - \omega_2 \otimes \sigma_{22}, \end{aligned}$$

where

$$\begin{aligned} \sigma_{11} &= \gamma + (2\lambda_1 + \mu)\,\omega_1 + \lambda_2\omega_2, \\ \sigma_{12} &= \lambda_2\omega_1, \\ \sigma_{21} &= \lambda_1\omega_2, \\ \sigma_{22} &= \gamma + \lambda_1\omega_1 + (2\lambda_2 - \mu)\,\omega_2. \end{aligned}$$

Using structure equations (3), we get

$$\begin{aligned} d_T^2\omega_1 &= \omega_1 \otimes (\sigma_{21} \wedge \sigma_{12} - d\sigma_{11}) + \omega_2 \otimes (\sigma_{12} \wedge \sigma_{11} + \sigma_{21} \wedge \sigma_{12} - d\sigma_{12}), \\ d_T^2\omega_2 &= \omega_1 \otimes (\sigma_{11} \wedge \sigma_{21} + \sigma_{21} \wedge \sigma_{22} - d\sigma_{21}) + \omega_2 \otimes (\sigma_{12} \wedge \sigma_{21} - d\sigma_{22}). \end{aligned}$$

Therefore, in order to obtain a flat torsion-free connection, components of the affine deformation tensor must satisfy the following *Akivis–Goldberg equations*:

$$\begin{aligned} d\sigma_{11} &= \sigma_{21} \wedge \sigma_{12}, \\ d\sigma_{12} &= \sigma_{12} \wedge \sigma_{11} + \sigma_{21} \wedge \sigma_{12}, \\ d\sigma_{21} &= \sigma_{11} \wedge \sigma_{21} + \sigma_{21} \wedge \sigma_{22}, \\ d\sigma_{22} &= \sigma_{12} \wedge \sigma_{21}. \end{aligned} \tag{20}$$

Because ω_1 and ω_2 are linearly independent, equations (20) imply that

$$\begin{aligned}
&2\partial_2(\lambda_1) - \partial_1(\lambda_2) + \partial_2(\mu) = K + \lambda_1\lambda_2 + g_2(2\lambda_1 + \mu) - g_1\lambda_2,\\
&\partial_2(\lambda_2) = \lambda_2(g_2 + \lambda_2 - \mu),\\
&\partial_1(\lambda_1) = \lambda_1(g_1 + \lambda_1 + \mu),\\
&\partial_2(\lambda_1) - 2\partial_1(\lambda_2) + \partial_1(\mu) = K - \lambda_1\lambda_2 + \lambda_1 g_2 - g_1(2\lambda_2 - \mu)
\end{aligned} \tag{21}$$

3 Calculus of Covariant Derivatives

Let $d_\gamma : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M)$ be the covariant differential with respect to the Chern connection. It induces the connection $d_\gamma^* : \mathcal{D}(M) \to \mathcal{D}(M) \otimes \Omega^1(M)$ in the tangent bundle, where

$$\begin{aligned}
d_\gamma^* &: \partial_1 \to \partial_1 \otimes \gamma,\\
d_\gamma^* &: \partial_2 \to \partial_2 \otimes \gamma.
\end{aligned}$$

Denote by $\Theta^{p,q}(M) = (\mathcal{D}(M))^{\otimes p} \otimes \left(\Omega^1(M)\right)^{\otimes q}$ the module of tensors of type (p,q). Then the Chern connection induces the covariant differential

$$d_\gamma^{(p,q)} : \Theta^{p,q}(M) \to \Theta^{p+1,q}(M),$$

where

$$d_\gamma^{(p,q)} : u\partial_{j_1}\otimes\cdots\otimes\partial_{j_p}\otimes\omega_{i_1}\otimes\cdots\otimes\omega_{i_q} \longmapsto \partial_{j_1}\otimes\cdots\otimes\partial_{j_p}\otimes\omega_{i_1}\otimes\cdots\otimes\omega_{i_q}\otimes(du + (p-q)\gamma u)$$

and $u \in C^\infty(M)$.

We say that u is of weight $q-p$ and call the form

$$\delta^{(p,q)}(u) \overset{\text{def}}{=} \delta^{(q-p)}(u) = du - (q-p)u\gamma \tag{22}$$

the *covariant differential* of u.

Decomposing the form $\delta^{(q-p)}(u)$ in the basis $\{\omega_1, \omega_2\}$, we obtain

$$\delta^{(q-p)}(u) = \delta_1^{(q-p)}(u)\ \omega_1 + \delta_2^{(q-p)}(u)\ \omega_2,$$

where

$$\begin{aligned}
\delta_1^{(q-p)}(u) &= \partial_1(u) - (q-p)g_1 u,\\
\delta_2^{(q-p)}(u) &= \partial_2(u) - (q-p)g_2 u
\end{aligned} \tag{23}$$

are the covariant derivatives of u with respect to the Chern connection.

Note that $\delta_1^{(q-p)}(u)$ and $\delta_2^{(q-p)}(u)$ are of weight $q-p+1$.

Lemma 3.1 *For any $s = 0, \pm1, \pm2, \ldots,$ the relation*

$$\delta_2^{(s+1)} \circ \delta_1^{(s)} - \delta_1^{(s+1)} \circ \delta_2^{(s)} = sK \tag{24}$$

holds for the commutator.

Proof. We have

$$\delta_2^{(s+1)} \circ \delta_1^{(s)} = \partial_2\partial_1 - sg_1\partial_2 - (s+1)\, g_2\partial_1 + s\,(s+1)\, g_1g_2 - s\partial_2(g_1)$$

and

$$\delta_1^{(s+1)} \circ \delta_2^{(s)} = \partial_1\partial_2 - sg_2\partial_1 - (s+1)\, g_1\partial_2 + s\,(s+1)\, g_1g_2 - s\partial_1(g_2).$$

The statement follows now from (9). ■

Note that the curvature K is of weight two, while λ_1, λ_2 and μ are of weight one.

The classical Leibnitz rule leads to the corresponding rule for weighted functions.

Lemma 3.2 (Leibnitz rule) *Let u be of weight k and v be of weight l. Then*

$$\delta_i^{(k+l)}\,(uv) = \delta_i^{(k)}\,(u)\;\; v + u\;\delta_i^{(l)}\,(v)\,.$$

In what follows, we shall omit the superscript indicating the weight in the cases when the weight is known. For example, we shall write $\delta_1 K$ instead of $\delta_1^{(2)}K$, or $\delta_1\mu$ instead of $\delta_1^{(1)}\mu$.

4 Differential Invariants and Rigidity of 3-Webs

As we have noted above, the curvature K is a relative invariant of weight two of a 3-web W. The covariant derivatives of K are relative invariants of weight three. The invariants (13) can be written in terms of the curvature K as follows:

$$a_1 = \frac{-\delta_1 K}{2K^{\frac{3}{2}}}, \qquad a_2 = -\frac{\delta_2 K}{2K^{\frac{3}{2}}}.$$

They are absolute invariants of a 3-web W with nonvanishing curvature K.

Hence all the derivatives

$$a_1^{i,j} = \nabla_1^i \nabla_2^j(a_1) \quad \text{and} \quad a_2^{i,j} = \nabla_1^i \nabla_2^j(a_2)$$

are absolute invariants too; here $i, j = 0, 1, 2, ...$

It is easy to see that they are differential operators with respect to the web function f of order $i + j + 4$.

Note also that condition (12),

$$\nabla_1\,(a_2) - \nabla_2\,(a_1) = 1,$$

gives the differential relations between the invariants $a_1^{i,j}$ and $a_2^{i,j}$.

In particular, it follows that there are no 3-webs with constant invariants a_1 and a_2.

The following theorem is valid (cf. [4], §13 and [5], §20).

Theorem 4.1 *The differential invariants* $a_1^{i,j}$ *and* $a_2^{i,j}$ *form a complete system of differential invariants of 3-webs with nonvanishing curvature, that is, any differential invariant of such 3-webs is a function of a finite number of invariants from the system* $\left\{ a_1^{i,j}, a_2^{i,j} \right\}$, $i, j = 0, 1, 2, ...$

We say that a 3-web W is *locally rigid in a domain* $D \subset M$ if for any two distinct points $p, q \in D$ there is no local diffeomorphism ϕ sending p to q and transforming the web W in a neighborhood of p into the web W in a neighborhood of q.

The problem of local rigidity can be viewed as a generalized Gronwall conjecture (see the description of the Gronwall conjecture for linearizable webs in Section 8 or in [4], §17).

It is easy to see that locally rigid webs do not have nontrivial (infinitesimal) automorphisms.

Let W be a 3-web defined in some neighborhood D of the point p, let θ_1, θ_2 and α be its invariant differential 1-forms, and let a_1, a_2 be its absolute differential invariants. Denote by $\overline{W}$ a copy of W with corresponding forms $\overline{\theta}_1, \overline{\theta}_2$, $\overline{\alpha}$ and invariants $\overline{a}_1, \overline{a}_2$.

On the product $D \times D$, we consider the 1-forms

$$\Theta_1 = \overline{\theta}_1 - \theta_1, \ \Theta_2 = \overline{\theta}_2 - \theta_2, \ \ \aleph = \overline{\alpha} - \alpha$$

and the functions

$$A_1 = \overline{a_1} - a_1, \ A_2 = \overline{a_2} - a_2.$$

Then the graph $G_\phi \subset D \times D$ of a local diffeomorphism $\phi : D \to D$, $\phi(p) = q$, transforming W in a neighborhood of p into W in a neighborhood of q is an integral surface of the differential system

$$\Theta_1 = 0, \ \Theta_2 = 0, \ \ \aleph = 0 \tag{25}$$

such that

$$A_1|_{G_\phi} = 0, \ A_2|_{G_\phi} = 0. \tag{26}$$

Assume that the functions a_1 and a_2 are functionally independent in D, and D is sufficiently small. Then the invariants a_1 and a_2 can be viewed as coordinates on D, and therefore the distinct points p and q have distinct coordinates. This means that the web W is locally rigid.

Let us assume that there is a functional dependence between the invariants a_1 and a_2, say, $a_2 = F(a_1)$. Then (26) determines a 3-dimensional manifold N such that the graphs G_ϕ are integral surfaces of differential system (25) on N.

For the system

$$\Theta_1|_N = 0, \Theta_2|_N = 0, \ \ \aleph|_N = 0$$

to have two-dimensional integral manifolds, it is necessary and sufficient that the forms $\Theta_1|_N$, $\Theta_2|_N$ and $\aleph|_N$ are proportional. In fact, the distribution defined by the above system should be two-dimensional and completely integrable.

This follows from the fact that proportionality of these forms implies complete integrability of the system.

Indeed, let $\Theta_1|_N \wedge \Theta_2|_N = 0$. Then

$$\aleph|_N = a_1\ \Theta_1|_N + a_2\ \Theta_2|_N\ ,$$

and therefore $\Theta_1|_N \wedge \aleph|_N = \Theta_2|_N \wedge \aleph|_N = 0$.

Moreover,

$$d\ \Theta_i|_N = \Theta_i|_N \wedge \overline{\alpha}|_N + \theta_i|_N \wedge \aleph|_N\ ,$$

and hence the system is completely integrable.

Summarizing, we arrive at the following theorem.

Theorem 4.2 (i) *Let W be a 3-web defined in a domain D in which the invariants a_1 and a_2 are functionally independent and form a coordinate system. Then W is locally rigid in D.*

(ii) *Let the invariants a_1 and a_2 be functionally dependent in some domain D, say, $a_2 = F(a_1)$, for a smooth function F, but the differential 3-form*

$$\Theta_1 \wedge \Theta_2 \wedge dA_1 \neq 0 \tag{27}$$

at points of the manifold $\{(p,q)|\ a_1(p) = \overline{a_1}(q),\ p \neq q\} \subset D \times D$. Then W is locally rigid in this domain.

We say that a vector field X is an *infinitesimal automorphism* of a 3-web W if the one-parameter group of shifts along X consists of diffeomorphisms preserving W. A 3-web W is said to be *infinitesimally rigid* if W has the trivial infinitesimal automorphism ($X = 0$) only.

In terms of the invariant forms θ_1 and θ_2, this means that the following Lie equations

$$L_X(\theta_1) = 0,\ L_X(\theta_1) = 0$$

hold. Here L_X is the Lie derivative along X.

Let

$$X = X_1\nabla_1 + X_2\nabla_2$$

be the decomposition of X in the basis $\{\nabla_1, \nabla_2\}$. Using structure equations (10), one can rewrite the Lie equations as follows:

$$\begin{aligned} dX_1 &= a_2X_2\theta_1 - a_2X_1\theta_2, \\ dX_2 &= -a_1X_2\theta_1 + a_1X_1\theta_2, \end{aligned}$$

or

$$\begin{aligned} &\nabla_1(X_1) = a_2X_2, \quad \nabla_2(X_1) = -a_2X_1, \\ &\nabla_1(X_2) = -a_1X_2, \quad \nabla_2(X_2) = a_1X_1. \end{aligned} \tag{28}$$

The compatibility conditions for these equations follow from (11). Namely, applying the operators from the left- and right-hand sides of (11) to X_1 and X_2, we get

$$\begin{aligned} \nabla_1(a_2)\,X_1 + \nabla_2(a_2)\,X_2 &= 0, \\ \nabla_1(a_1)\,X_1 + \nabla_2(a_1)\,X_2 &= 0. \end{aligned} \tag{29}$$

This implies the following theorem.

Theorem 4.3 (Infinitesimal Rigidity of 3-Webs) *Let W be a 3-web given in a domain D, and let the invariant*

$$J = \det \begin{Vmatrix} \nabla_1(a_1) & \nabla_1(a_2) \\ \nabla_2(a_1) & \nabla_2(a_2) \end{Vmatrix}$$

be nonvanishing in D. Then W is infinitesimally rigid in D.

Let us assume now that J identically equals zero in D. As we have seen earlier, the entries of the above matrix do not vanish simultaneously, that is, the rank of the matrix equals one.

Hence system (29) has solutions of the form

$$X = s\,(\nabla_2(a_2)\ \nabla_1 - \nabla_1(a_2)\ \nabla_2)$$

for some smooth function s.

Substituting this expression into system (28), we get

$$\begin{aligned} \nabla_1(s) &= -\frac{a_2\nabla_1(a_2) + \nabla_1\nabla_2(a_2)}{\nabla_2(a_2)}s, \\ \nabla_1(s) &= -\frac{a_1\nabla_1(a_2) + \nabla_1^2(a_2)}{\nabla_1(a_2)}s, \\ \nabla_2(s) &= -\frac{a_2\nabla_2(a_2) + \nabla_2^2(a_2)}{\nabla_2(a_2)}s, \\ \nabla_2(s) &= -\frac{a_1\nabla_2(a_2) + \nabla_2\nabla_1(a_2)}{\nabla_1(a_2)}s. \end{aligned} \tag{30}$$

It follows that

$$\begin{aligned} &a_2(\nabla_1(a_2))^2 + \nabla_1\nabla_2(a_2)\ \nabla_1(a_2) = a_1\nabla_1(a_2)\ \nabla_2(a_2) + \nabla_1^2(a_2)\ \nabla_2(a_2), \\ &a_2\nabla_2(a_2)\,\nabla_1(a_2) + \nabla_2^2(a_2)\,\nabla_1(a_2) = a_1(\nabla_2(a_2))^2 + \nabla_2\nabla_1(a_2)\,\nabla_2(a_2). \end{aligned} \tag{31}$$

The compatibility conditions for the above system take the form:

$$\begin{aligned} &\nabla_2\left(\frac{a_2\nabla_1(a_2) + \nabla_1\nabla_2(a_2)}{\nabla_2(a_2)}\right) - \nabla_1\left(\frac{a_2\nabla_2(a_2) + \nabla_2^2(a_2)}{\nabla_2(a_2)}\right) \\ = \quad &-a_2\frac{a_2\nabla_1(a_2) + \nabla_1\nabla_2(a_2)}{\nabla_2(a_2)} + a_1\frac{a_2\nabla_2(a_2) + \nabla_2^2(a_2)}{\nabla_2(a_2)} \end{aligned}$$

or

$$\nabla_2\nabla_1\nabla_2(a_2) + a_2\nabla_1\nabla_2(a_2) = a_1\nabla_2^2(a_2) + a_2\nabla_1(a_2)\,\nabla_2(a_2) + \nabla_1\nabla_2^2(a_2).$$

Theorem 4.4 *Let W be a 3-web such that $J = 0$, and suppose that the invariants a_1 and a_2 satisfy the relations*

$$\begin{aligned} a_2(\nabla_1(a_2))^2 + \nabla_1\nabla_2(a_2)\ \nabla_1(a_2) &= a_1\nabla_1(a_2)\ \nabla_2(a_2) + \nabla_1^2(a_2)\ \nabla_2(a_2), \\ a_2\nabla_2(a_2)\nabla_1(a_2) + \nabla_2^2(a_2)\nabla_1(a_2) &= a_1(\nabla_2(a_2))^2 + \nabla_2\nabla_1(a_2)\nabla_2(a_2), \\ \nabla_2\nabla_1\nabla_2(a_2) + a_2\nabla_1\nabla_2(a_2) &= a_1\nabla_2^2(a_2) + a_2\nabla_1(a_2)\nabla_2(a_2) + \nabla_1\nabla_2^2(a_2). \end{aligned}$$

Then there is a nontrivial infinitesimal automorphism of W which is unique up to a factor and has the form

$$X = s\left(\nabla_2(a_2)\ \nabla_1 - \nabla_1(a_2)\ \nabla_2\right),$$

where the function s is a solution of (30).

4.1 Examples

Example 4 Consider the 3-web W given by the web function

$$f = x + \sqrt{x^2 - y}$$

in the domain $\left\{x > 0,\ y > 0,\ y < x^2\right\}$ (cf. Example 3).

As we saw in Example 3, this web is generated by two families of coordinate lines $\{x = \text{const}\}$, $\{y = \text{const}\}$ and the tangents to the parabola $y = x^2$.

For this web, we have

$$\begin{aligned} \omega_1 &= -\frac{f\ dx}{f-x},\ \ \omega_2 = \frac{dy}{2(f-x)},\ \ \gamma = \frac{x(-2fdx+dy)}{2(f-x)(y-xf)}, \\ H &= \frac{x}{y-xf},\ \ K = \frac{2x^2f - y(f+x)}{f(xf-y)^2}, \\ \theta_1 &= -\frac{\sqrt{f}dx}{f-x},\ \theta_2 = \frac{dy}{2\sqrt{f}(f-x)},\ \ \alpha = \frac{(f+2x)\ dx}{2(f-x)^{3/2}} - \frac{(2f+x)\,dy}{4\sqrt{f(f-x)}}, \\ a_1 &= -\frac{f+2x}{2\sqrt{f(f-x)}},\ \ a_2 = -\frac{2f+x}{2\sqrt{f(f-x)}} \end{aligned}$$

Note that $da_1 \wedge da_2 = 0$. Hence the invariants a_1 and a_2 are functionally dependent. The dependence is

$$8a_1^2 - 5a_2^2 + 4a_1\left(a_2^2 - 1\right)\sqrt{a_1^2+6} + a_2\left(4a_1^2 - 1\right)\sqrt{a_2^2+3} + 3 = 0.$$

Conditions (26) mean that

$$a_1(x,y) = a_1(\overline{x},\overline{y})$$

or

$$\frac{y}{x^2} = \frac{\overline{y}}{\overline{x}^2}.$$

Then

$$\Theta_1 = \frac{\sqrt{f}\ (\sqrt{x}d\overline{x} - \sqrt{\overline{x}}dx)}{(f-x)\sqrt{\overline{x}}},$$

and

$$\Theta_2 = \frac{x\sqrt{x}}{2\sqrt{f}\,(f-x)}\left[\frac{\sqrt{\overline{x}}-\sqrt{x}}{x^2}dy + \frac{2y}{x^3\sqrt{\overline{x}}}(xd\overline{x} - \overline{x}dx)\right].$$

It is easy to check that on the manifold N, the condition $\Theta_1 \wedge \Theta_2 = 0$ holds if and only if $x = \overline{x}$ and consequently $y = \overline{y}$.

In other words, this web is locally rigid.

Example 5 Consider the 3-web W given by the web function

$$f = (x+y)e^{-x}. \tag{32}$$

This web is generated by two families of coordinate lines $\{x = \text{const}\}$, $\{y = \text{const}\}$ and the level sets of the function f.

Let $t = 1 - x - y$. Then for web (32) one has

$$\begin{aligned}
\omega_1 &= -te^{-x}dx, \ \ \omega_2 = -e^{-x}dy, \ \gamma = dx + \frac{dy}{t}, \\
H &= -\frac{e^x}{t}, \ \ K = \frac{e^{2x}}{t^3}, \\
\theta_1 &= -\frac{dx}{\sqrt{t}}, \ \theta_2 = -\frac{dy}{t^{3/2}}, \ \alpha = -\frac{3dx+dy}{2t}, \\
a_1 &= \frac{3}{2\sqrt{t}}, \ \ a_2 = \frac{\sqrt{t}}{2}.
\end{aligned}$$

Note that $da_1 \wedge da_2 = 0$. Hence the invariants a_1 and a_2 are functionally dependent:

$$a_1 a_2 = \frac{3}{4}.$$

The three-dimensional manifold N is defined by

$$x + y = \overline{x} + \overline{y},$$

and the differential 1-forms are

$$\Theta_1|_N = \frac{dx - d\overline{x}}{\sqrt{t}}, \Theta_2|_N = -\frac{dx - d\overline{x}}{t^{3/2}}, \quad \aleph|_N = \frac{dx - d\overline{x}}{t}.$$

Therefore the integral surfaces are given by the equations:

$$\overline{x} = x + c, \ \overline{y} = y - c,$$

and the requirement $\phi(p) = p$ implies $c = 0$.

Therefore web (32) is not locally rigid. Note that the vector field

$$X = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

is the infinitesimal symmetry of web (32).

5 Akivis–Goldberg Equations

Using the covariant derivatives instead of the partial derivatives, we write equations (21) as follows:

$$\begin{aligned} 2\delta_2(\lambda_1) - \delta_1(\lambda_2) + \delta_2(\mu) &= K + \lambda_1\lambda_2, \\ \delta_2(\lambda_2) &= \lambda_2(\lambda_2 - \mu), \\ \delta_1(\lambda_1) &= \lambda_1(\lambda_1 + \mu), \\ \delta_2(\lambda_1) - 2\delta_1(\lambda_2) + \delta_1(\mu) &= K - \lambda_1\lambda_2. \end{aligned}$$

Solving this system with respect to the covariant derivatives of λ_1 and λ_2, we obtain the following system of PDEs:

$$\begin{aligned} \delta_1(\lambda_1) &= \lambda_1(\lambda_1 + \mu), \\ \delta_2(\lambda_1) &= \lambda_1\lambda_2 + \frac{K}{3} + \frac{1}{3}\delta_1(\mu) - \frac{2}{3}\delta_2(\mu), \\ \delta_1(\lambda_2) &= \lambda_1\lambda_2 - \frac{K}{3} + \frac{2}{3}\delta_1(\mu) - \frac{1}{3}\delta_2(\mu), \\ \delta_2(\lambda_2) &= \lambda_2(\lambda_2 - \mu). \end{aligned}$$

We shall look at the above system as a system of partial differential equations with respect to the functions λ_1 and λ_2 provided that μ is given.

From(24) we get the compatibility conditions for this system:

$$\delta_1(\delta_2(\lambda_i)) - \delta_2(\delta_1(\lambda_i)) + K\lambda_i = 0,$$

where $i = 1, 2$.

After a series of straightforward computations, we obtain the following two compatibility equations:

$$I_1(\mu) = 0, \; I_2(\mu) = 0, \tag{33}$$

where $I_1(\mu)$ and $I_1(\mu)$ have the form

$$I_1(\mu) = \delta_1^2(\mu) - 2\delta_1\delta_2(\mu) - \mu\delta_1(\mu) + 2\mu\delta_2(\mu) - \mu K + \delta_1(K)$$

and

$$I_2(\mu) = \delta_2^2(\mu) - 2\delta_1\delta_2(\mu) - 2\mu\delta_1(\mu) + \mu\delta_2(\mu) - \mu K + \delta_2(K).$$

We shall use the symmetrized derivatives. Namely, let

$$\delta_{ij} = \frac{1}{2}(\delta_i\delta_j + \delta_j\delta_i)$$

be the symmetrized mixed second derivatives.

Then for functions of weight one, we have

$$\begin{aligned} \delta_{12} &= \delta_1\delta_2 + \frac{K}{2}, \\ \delta_{21} &= \delta_1\delta_2 - \frac{K}{2}, \end{aligned}$$

and the expressions for $I_1(\mu)$ and $I_1(\mu)$ can be written as follows:

$$\begin{aligned} I_1(\mu) &= \delta_{11}(\mu) - 2\delta_{12}(\mu) - \mu\delta_1(\mu) + 2\mu\delta_2(\mu) + \delta_1(K), \\ I_2(\mu) &= \delta_{22}(\mu) - 2\delta_{12}(\mu) - 2\mu\delta_1(\mu) + \mu\delta_2(\mu) + \delta_2(K). \end{aligned} \tag{34}$$

We summarize these results in the following theorem.

Theorem 5.1 ([1]) *The Akivis–Goldberg equations as differential equations with respect to the components $T^1_{12} = \lambda_2$ and $T^2_{12} = \lambda_1$ of the affine deformation tensor T are compatible if and only if the function μ satisfies the following differential equations:*

$$I_1(\mu) = 0, \ I_2(\mu) = 0. \tag{35}$$

If conditions (35) are valid, then system (20) of PDEs is a Frobenius-type system, and for given values $\lambda_1(x_0)$ and $\lambda_2(x_0)$ at a point $x_0 \in M^2$, there is (a unique) smooth solution of the system in some neighborhood of x_0.

Let us denote by τ the following involution:

$$\tau : (x, y, \mu, K) \to (y, x, -\mu, -K).$$

Then one can check that

$$\tau(I_1) = I_2.$$

6 Calculus in Jet Spaces of Weighted Functions

6.1 Cartan's Forms in Nonholonomic Coordinates

Let $\mathbb{J}^r(s)$ be the space of r-jets of weight s functions in the plane $\mathbb{R}^2$. We shall use the coordinates $(x, y, u, p_1, p_2, ..., p_{i_1...i_l}, ...)$ in this space corresponding to the symmetrized covariant derivatives, that is,

$$\begin{aligned} u(j_r(h)) &= h, \ p_1(j_r(h)) = \delta_1(h), \ p_2(j_r(h)) = \delta_2(h), \\ & p_{i_1...i_l}(j_r(h)) = \delta_{i_1...i_l}(h), \ ... \end{aligned}$$

Here $j_r(h)$ is the r-jet of the function h. The function u is of weight s, and $\delta_{i_1...i_l}$ is its symmetrized covariant derivative of order $i_1 + \cdots + i_l$.

In what follows, we shall denote the symmetrized covariant derivatives of the curvature function K by

$$K_{i_1....i_l} \overset{\text{def}}{=} \delta_{i_1...i_l}(K).$$

We describe now the Cartan distribution (see [14] or [3]) in $\mathbb{J}^r(s)$ in these coordinates. Let us begin with $\mathbb{J}^1(s)$. The formula

$$df = (\delta_1 f + sg_1 f)\,\omega_1 + (\delta_2 f + sg_2 f)\,\omega_2,$$

where f is a function of weight s, shows that the contact form on $\mathbb{J}^1(s)$ can be expressed as

$$\begin{aligned}\varepsilon_0 &= du - (p_1 + sg_1 u)\,\omega_1 - (p_2 + sg_2 u)\,\omega_2 \\ &= du - su\gamma - p_1\omega_1 - p_2\omega_2.\end{aligned}$$

To find the Cartan forms on $\mathbb{J}^2(s)$, we shall use the relations

$$\delta_1\delta_2 - \delta_2\delta_1 = -wK,$$
$$\delta_{12} = \frac{1}{2}(\delta_1\delta_2 + \delta_2\delta_1),$$

which hold for functions of weight w.

These formulae imply that

$$\begin{aligned}\delta_1\delta_2 &= \delta_{12} - \tfrac{1}{2}wK, \\ \delta_2\delta_1 &= \delta_{12} + \tfrac{1}{2}wK\end{aligned} \tag{36}$$

and give the following representation of the second-order Cartan forms:

$$\begin{aligned}\varepsilon_1 &= dp_1 - (s+1)\,p_1\gamma - p_{11}\omega_1 - (p_{12} + \frac{1}{2}sKu)\omega_2, \\ \varepsilon_2 &= dp_2 - (s+1)\,p_2\gamma - (p_{12} - \frac{1}{2}sKu)\;\omega_1 - p_{22}\omega_2.\end{aligned}$$

To obtain the Cartan forms on the next jet space $\mathbb{J}^3(s)$, we need the following relations:

$$\begin{aligned}\delta_1\delta_{12} &= \delta_{112} - \tfrac{1}{6}(3s+2)\,K\delta_1 - \tfrac{1}{6}sK_1, \\ \delta_2\delta_{12} &= \delta_{122} + \tfrac{1}{6}(3s+2)\,K\delta_2 + \tfrac{1}{6}sK_2, \\ \delta_1\delta_{22} &= \delta_{122} - \tfrac{1}{3}(3s+2)\,K\delta_2 - \tfrac{1}{3}sK_2, \\ \delta_2\delta_{11} &= \delta_{112} + \tfrac{1}{3}(3s+2)\,K\delta_1 + \tfrac{1}{3}sK_1,\end{aligned} \tag{37}$$

which follow from (36).

These relations allow us to represent the third-order Cartan forms:

$$\begin{aligned}\varepsilon_{11} &= dp_{11} - (s+2)\,p_{11}\;\gamma - p_{111}\omega_1 - \left(p_{112} + \frac{1}{3}(3s+2)\,Kp_1 + \frac{1}{3}sK_1u\right)\omega_2, \\ \varepsilon_{12} &= dp_{12} - (s+2)\,p_{12}\;\gamma - \left(p_{112} - \frac{1}{6}(3s+2)\,Kp_1 - \frac{1}{6}sK_1u\right)\omega_1 \\ &\quad - \left(p_{122} + \frac{1}{6}(3s+2)\,Kp_2 + \frac{1}{6}sK_2u\right)\omega_2, \\ \varepsilon_{22} &= dp_{22} - (s+2)\,p_{22}\;\gamma - \left(p_{122} - \frac{1}{3}(3s+2)\,Kp_2 - \frac{1}{3}sK_2u\right)\omega_1 - p_{222}\omega_2.\end{aligned}$$

In a similar way, from the relations

$$\begin{aligned}
\delta_1\delta_{112} &= \delta_{1112} - \frac{1}{6}(3s+4)K\delta_{11} - \frac{1}{6}(2s+1)K_1\delta_1 - \frac{1}{12}sK_{11},\\
\delta_1\delta_{122} &= \delta_{1122} - \frac{1}{3}(3s+4)K\delta_{12} - \frac{1}{6}(2s+1)K_2\delta_1 - \frac{1}{6}(2s+1)K_1\delta_2 - \frac{1}{6}sK_{12},\\
\delta_1\delta_{222} &= \delta_{1222} - \frac{1}{2}(3s+4)K\delta_{22} - \frac{1}{2}(2s+1)K_2\delta_2 - \frac{1}{4}sK_{22},\\
\delta_2\delta_{111} &= \delta_{1112} + \frac{1}{2}(3s+4)K\delta_{11} + \frac{1}{2}(2s+1)K_1\delta_1 + \frac{1}{4}sK_{11},\\
\delta_2\delta_{112} &= \delta_{1122} + \frac{1}{3}(3s+4)K\delta_{12} + \frac{1}{6}(2s+1)K_2\delta_1 + \frac{1}{6}(2s+1)K_1\delta_2 + \frac{1}{6}sK_{12},\\
\delta_2\delta_{122} &= \delta_{1222} + \frac{1}{6}(3s+4)K\delta_{22} + \frac{1}{6}(2s+1)K_2\delta_2 + \frac{1}{12}sK_{22},
\end{aligned}$$

we get the following representation for the fourth-order Cartan forms:

$$\begin{aligned}
&\varepsilon_{111} = dp_{111} - (s+3)\,p_{111}\gamma - p_{1111}\omega_1\\
&-\left(p_{1112} + \frac{1}{2}(3s+4)Kp_{11} + \frac{1}{2}(2s+1)K_1p_1 + \frac{1}{4}sK_{11}u\right)\omega_2,\\
&\varepsilon_{112} = dp_{112} - (s+3)\,p_{112}\gamma\\
&-\left(p_{1112} - \frac{1}{6}(3s+4)Kp_{11} - \frac{1}{6}(2s+1)K_1p_1 - \frac{1}{12}sK_{11}u\right)\omega_1\\
&-\left(p_{1122} + \frac{1}{3}(3s+4)Kp_{12} + \frac{1}{6}(2s+1)K_2p_1 + \frac{1}{6}(2s+1)K_1p_2 + \frac{1}{6}sK_{12}u\right)\omega_2,\\
&\varepsilon_{122} = dp_{122} - (s+3)\,p_{122}\gamma\\
&-\left(p_{1122} - \frac{1}{3}(3s+4)Kp_{12} - \frac{1}{6}(2s+1)K_2p_1 - \frac{1}{6}(2s+1)K_1p_2 - \frac{1}{6}sK_{12}u\right)\omega_1\\
&-\left(p_{1222} + \frac{1}{6}(3s+4)Kp_{22} + \frac{1}{6}(2s+1)K_2p_2 + \frac{1}{12}sK_{22}u\right)\omega_2,\\
&\varepsilon_{222} = dp_{222} - (s+3)\,p_{222}\gamma\\
&-\left(p_{1222} - \frac{1}{2}(3s+4)Kp_{22} - \frac{1}{2}(2s+1)K_2p_2 - \frac{1}{4}sK_{22}u\right)\omega_1 - p_{2222}\omega_2.
\end{aligned}$$

6.2 The Total Derivative and the Mayer Bracket

We shall denote by $\widehat{X}$ the total derivative corresponding to a vector field X on the manifold M^2 (see, for example, [14] or [3]). Using the representations of

Cartan's forms, we get the following expressions for the vector fields $\widehat{\partial_1}$ and $\widehat{\partial_2}$:

$$\widehat{\partial_1} = \partial_1 + (sg_1u + p_1)\frac{\partial}{\partial u} + ((s+1)\,g_1p_1 + p_{11})\frac{\partial}{\partial p_1} + \left((s+1)\,g_1p_2 + p_{12} - \frac{s}{2}Ku\right)\frac{\partial}{\partial p_2}$$
$$+ ((s+2)\,g_1p_{11} + p_{111})\frac{\partial}{\partial p_{11}} + \left((s+2)\,g_1p_{12} + p_{112} - \frac{3s+4}{6}Kp_1 - \frac{s}{6}K_1u\right)\frac{\partial}{\partial p_{12}}$$
$$+ \left((s+2)\,g_1p_{22} + p_{122} - \frac{3s+4}{3}Kp_2 - \frac{s}{3}K_2u\right)\frac{\partial}{\partial p_{22}}$$
$$+ ((s+3)\,g_1p_{111} + p_{1111})\frac{\partial}{\partial p_{111}}$$
$$+ \left((s+3)\,g_1p_{112} + p_{1112} - \frac{3s+4}{6}Kp_{11} - \frac{2s+1}{6}K_1p_1 - \frac{s}{12}K_{11}u\right)\frac{\partial}{\partial p_{112}}$$
$$+ \left((s+3)\,g_1p_{122} + p_{1122} - \frac{3s+4}{3}Kp_{12} - \frac{2s+1}{6}K_2p_1 - \frac{2s+1}{6}K_1p_2 - \frac{s}{6}K_{12}u\right)\frac{\partial}{\partial p_{122}}$$
$$+ \left((s+3)\,g_1p_{222} + p_{1222} - \frac{3s+4}{2}Kp_{22} - \frac{2s+1}{2}K_2p_2 - \frac{s}{4}K_{22}u\right)\frac{\partial}{\partial p_{222}} + \cdots$$

and

$$\widehat{\partial_2} = \partial_2 + (sg_2u + p_2)\frac{\partial}{\partial u} + \left((s+1)\,g_2p_1 + p_{12} + \frac{s}{2}Ku\right)\frac{\partial}{\partial p_1} + ((s+1)\,g_2p_2 + p_{22})\frac{\partial}{\partial p_2}$$
$$+ \left((s+2)\,g_2p_{11} + p_{112} + \frac{3s+4}{3}Kp_1 + \frac{s}{3}K_1u\right)\frac{\partial}{\partial p_{11}}$$
$$+ \left((s+2)\,g_2p_{12} + p_{122} + \frac{3s+4}{6}Kp_2 + \frac{s}{6}K_2u\right)\frac{\partial}{\partial p_{12}}$$
$$+ ((s+2)\,g_2p_{22} + p_{222})\frac{\partial}{\partial p_{22}}$$
$$+ \left((s+3)\,g_2p_{111} + p_{1112} + \frac{3s+4}{2}Kp_{11} + \frac{2s+1}{2}K_1p_1 + \frac{s}{4}K_{11}u\right)\frac{\partial}{\partial p_{111}}$$
$$+ \left((s+3)\,g_2p_{112} + p_{1122} + \frac{3s+4}{3}Kp_{12} + \frac{2s+1}{6}K_2p_1 + \frac{2s+1}{6}K_1p_2 + \frac{s}{6}K_{12}u\right)\frac{\partial}{\partial p_{112}}$$
$$+ \left((s+3)\,g_2p_{122} + p_{1222} + \frac{3s+4}{6}Kp_{22} + \frac{2s+1}{6}K_2p_2 + \frac{s}{12}K_{22}u\right)\frac{\partial}{\partial p_{122}}$$
$$+ ((s+3)\,g_2p_{222} + p_{2222})\frac{\partial}{\partial p_{222}} + \cdots$$

We shall denote by

$$\widehat{\delta_i}\,(h) = \widehat{\partial_i}(h) - wg_ih$$

the covariant derivatives of a function h of weight w on the jet space $\mathbb{J}^r\,(s)$ and call it the *total covariant derivative* of h along ∂_i. As earlier, we shall denote the symmetrized total derivatives by $\widehat{\delta}_{i_1\ldots i_l}$.

In these notations, the linearization of a function h of weight w on $\mathbb{J}^r(s)$ (cf. [14] and [3]) has the form

$$l_h = \sum_{(i_1 \ldots i_l)} \frac{\partial^l h}{\partial p_{i_1 \ldots i_l}} \widehat{\delta}_{i_1 \ldots i_l},$$

and the Mayer bracket (see [15]) of functions f and g of weights w_1 and w_2 defined correspondingly on $\mathbb{J}^n(s)$ and $\mathbb{J}^m(s)$ has the form

$$[f,g] = \sum_{(i_1 \ldots i_n)} \frac{\partial^n f}{\partial p_{i_1 \ldots i_n}} \widehat{\delta}_{i_1 \ldots i_n}(g) - \sum_{(j_1, \ldots, j_m)} \frac{\partial^m f}{\partial p_{j_1 \ldots j_m}} \widehat{\delta}_{j_1 \ldots j_m}(g).$$

7 The Mayer Bracket and the First Obstruction for Linearizability

Let us rewrite equations (34) symbolically. The functions on $\mathbb{J}^2(1)$, that correspond to these equations, are

$$\begin{aligned} I_1 &= p_{11} - 2p_{12} - up_1 + 2up_2 + K_1, \\ I_2 &= p_{22} - 2p_{12} - 2up_1 + up_2 + K_2. \end{aligned}$$

Equations (35) are compatible if and only if the Mayer bracket of I_1 and I_2 vanishes (see [15]). In our case,

$$I_{12} = [I_1, I_2] = \widehat{\delta}_{11}(I_2) - \widehat{\delta}_{22}(I_1) + 2\widehat{\delta}_{12}(I_1 - I_2)$$

or

$$\begin{aligned} &I_{12} = u(-2p_{111} + 3p_{112} + 3p_{122} - 2p_{222}) \\ &+3(p_2 - 2p_1)p_{11} + 6(p_1 + p_2)p_{12} + 3(p_1 - 2p_2)p_{22} + 8K(p_{11} - p_{12} + p_{22}) \\ &+3(2K_1 - K_2)p_1 + 3(2K_2 - K_1)p_2 + u(K_{11} - K_{12} + K_{22}) \\ &+3(K_{112} - K_{122}). \end{aligned}$$

Solving the first prolongation of the system

$$\begin{aligned} \widehat{\delta}_1(I_1) &= 0, \quad \widehat{\delta}_2(I_1) = 0, \\ \widehat{\delta}_1(I_2) &= 0, \quad \widehat{\delta}_2(I_2) = 0 \end{aligned}$$

with respect to p_{111}, $p_{112,}$ p_{122} and p_{222} and substituting the result into I_{12}, we get

$$\begin{aligned} I_{12} &= 24Kp_{12} + 6(2K_1 - K_2)p_1 + 6(2K_2 - K_1)p_2 + 24Ku(p_1 - p_2) \\ &\quad +3u(K_{11} - K_{12} + K_{22}) - 8K(K_1 + K_2) + 3(K_{112} - K_{122}) - 3Ku^3. \end{aligned}$$

Note that

$$\tau(I_{12}) = I_{12}.$$

Solving the equations

$$I_1 = 0,\ I_2 = 0,\ I_{12} = 0$$

with respect to p_{ij}, we obtain

$$\begin{aligned}
12Kp_{11} &= 3Ku^3 + 12Kup_1 + 6(K_2 - 2K_1)p_1 + 6(K_1 - 2K_2)p_2 \\
&\quad -3u\,(K_{11} - K_{12} + K_{22}) + 3\,(K_{122} - K_{112}) + 4K(2K_2 - K_1), \\
12Kp_{22} &= 3Ku^3 + 12Kup_2 + 6(K_2 - 2K_1)p_1 + 6(K_1 - 2K_2)p_2 \\
&\quad -3u\,(K_{11} - K_{12} + K_{22}) + 3\,(K_{122} - K_{112}) + 4K(2K_1 - K_2), \\
24Kp_{12} &= 3Ku^3 + 24Ku(p_2 - p_1) + 6(K_2 - 2K_1)p_1 + 6(K_1 - 2K_2)p_2 \\
&\quad -3u\,(K_{11} - K_{12} + K_{22}) + 3\,(K_{122} - K_{112}) + 8K(K_1 + K_2).
\end{aligned} \tag{38}$$

The expressions for the symmetric covariant derivatives of the curvature function K are given in section 10.1.

We write down the above equations in the form

$$\begin{aligned}
p_{11} &= P_{11}\,(u, p_1, p_2, K)\,, \\
p_{12} &= P_{12}\,(u, p_1, p_2, K)\,, \\
p_{22} &= P_{22}\,(u, p_1, p_2, K)\,.
\end{aligned}$$

In order to find their compatibility, first, taking $s = 1$, we derive from (37) that

$$\begin{aligned}
\delta_2\delta_{11} - \delta_1\delta_{12} &= \frac{5}{2}K\delta_1 + \frac{1}{2}K_1, \\
\delta_2\delta_{12} - \delta_1\delta_{22} &= \frac{5}{2}K\delta_2 + \frac{1}{2}K_2.
\end{aligned}$$

It follows that the equations

$$\begin{aligned}
\widehat{\delta}_2\,(P_{11}) - \widehat{\delta}_1\,(P_{12}) - \frac{5}{2}Kp_1 - \frac{1}{2}K_1u &= 0, \\
\widehat{\delta}_2\,(P_{12}) - \widehat{\delta}_1\,(P_{22}) - \frac{5}{2}Kp_2 - \frac{1}{2}K_2u &= 0
\end{aligned}$$

are the compatibility conditions for (38).

Let us denote by G_1 and G_2 the left-hand sides of the above equations into which the values of p_{ij} taken from (38) are substituted.

These functions are polynomials in p_i and u of the form

$$\begin{aligned}
G_1 &= (p_1^2 - 2p_1p_2) + A_{11}p_1 + A_{12}p_2 + A_{10}, \\
G_2 &= (p_2^2 - 2p_1p_2) + A_{21}p_1 + A_{22}p_2 + A_{20},
\end{aligned}$$

where all coefficients are functions of the curvature K and its covariant derivatives up to order four:

$$
\begin{aligned}
A_{11} &= \frac{5u^2}{8} + \frac{3u\,(K_1 - K_2)}{4K} - \frac{13K}{4} - \frac{7(K_1 - 2K_2)(2K_1 - K_2)}{16K^2} \\
&\quad + \frac{7(5K_{11} + 5K_{22} - 11K_{12})}{8K}, \\
A_{12} &= -\frac{5u^2}{4} - \frac{3u\ K_1}{4K} + \frac{7(K_1 - 2K_2)^2}{16K^2} + \frac{5K_{12} - 2K_{11} - 5K_{22}}{4K}, \\
A_{10} &= \frac{9u^3(K_1 - 2K_2)}{96K} + u\left(-\frac{5K_1}{2} + \frac{21(2K_2 - K_1)(K_{11} - K_{12} + K_{22})}{96K^2}\right) \\
&\quad + \frac{u(K_{111} - 2K_{222} - 3K_{112} + 3K_{122})}{8K} + \frac{9(2K_{22} - 2K_{12} - K_{11})}{16} \\
&\quad - \frac{21(2K_2 - K_1)(K_{122} - K_{112})}{96K^2} + \frac{17(2K_2^2 - 2K_1K_2 - K_1^2)}{48K} \\
&\quad + \frac{2K_{1222} - 3K_{1122} + K_{1112}}{8K},
\end{aligned}
$$

and

$$
A_{21} = -\tau\,(A_{12})\,, A_{22} = -\tau\,(A_{11})\,, A_{20} = \tau\,(A_{10})
$$

because

$$
\tau\,(G_1) = G_2.
$$

The following theorem outlines the successive steps in the investigation of solvability for main equations (21).

Theorem 7.1

1. *Differential equations* (*21*) *are solvable with respect to the functions* λ_1 *and* λ_2 *if and only if the function* μ *satisfies differential equations* (*35*).
2. *For the system of differential equations* (*35*) *be solvable, one needs to add the compatibility condition* $I_{12} = 0$ *to this system.*
3. *The compatibility conditions for the resulting system* (*38*) *have the form*

$$
G_1 = 0,\ \ G_2 = 0. \tag{39}
$$

8 The Second Obstruction for Linearizability

In this section, we investigate the solvability of the system of equations (38) and (39). To this end, we differentiate the left-hand sides of (39),

$$
\begin{aligned}
G_{11} &= \widehat{\delta_1}\,(G_1)\,,\ G_{12}^s = \frac{1}{2}\left(\widehat{\delta_1}\,(G_2) + \widehat{\delta_2}\,(G_1)\right), \\
G_{22} &= \widehat{\delta_2}\,(G_2)\,,\ G_{12}^a = \frac{1}{2}\left(\widehat{\delta_1}\,(G_2) - \widehat{\delta_2}\,(G_1)\right)
\end{aligned}
$$

and substitute the second covariant derivatives taken from (38) into the result of differentiation.

Finally, we arrive at the system

$$G_1 = 0,\, G_2 = 0,\; G_{11} = 0,\; G^s_{12} = 0,\; G^a_{12} = 0,\; G_{22} = 0, \tag{40}$$

which is equivalent to system (38)–(39).

By the construction, we get the symmetry

$$\tau\left(G_{11}\right) = G_{22},\; \tau\left(G^s_{12}\right) = G^s_{12},\; \tau\left(G^a_{12}\right) = -G^a_{12},\; \tau\left(G_{22}\right) = G_{11}.$$

In the coordinates, these functions have the form

$$\begin{aligned}
G_{11} &= -\frac{(K_1+K_2)}{4K}p_1^2 + \frac{7K_1-8K_2}{4K}p_1p_2 + \frac{2K_2-K_1}{K}p_2^2 \\
&\quad + A_{111}\; p_1 + A_{112}\; p_2 + A_{110} + \frac{5u}{4}G_1, \\
G^s_{12} &= \frac{8K_1-7K_2}{8K}p_1^2 + \frac{K_1+K_2}{2K}p_1p_2 - \frac{7K_1-8K_2}{8K}p_2^2 \\
&\quad + A_{121}p_1 + A_{122}p_2 + A_{120} + \frac{5u}{4}G_1 - \frac{5u}{4}G_2, \\
G^a_{12} &= \frac{39}{4}up_1p_2 + B_{121}p_1 + B_{122}p_2 + B_{120} \\
&\quad + \left(\frac{13}{4}u - \frac{3K_2}{8K}\right)G_1 + \left(\tfrac{13}{4}u + \tfrac{3K_1}{8K}\right)G_2, \\
G_{22} &= \frac{2K_1-K_2}{K}p_1^2 + \frac{7K_2-8K_1}{4K}p_1p_2 - \frac{K_1+K_2}{4K}p_2^2 \\
&\quad + A_{221}\; p_1 + A_{222}\; p_2 + A_{220} - \frac{5u}{4}G_2,
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
A_{111} &= \frac{3}{32}u^3 - \frac{3\left(7K_1-12K_2\right)}{32K}u^2 + \cdots, \\
A_{112} &= -\frac{3}{16}u^3 + \frac{3K_1}{16}u^2 + \cdots, \\
A_{110} &= -\frac{3\left(K_1-2K_2\right)}{128K}u^4 + \frac{33K_1\left(2K_2-K_1\right)}{128K^2}u^3 + \frac{3\left(2K_{12}-K_{11}\right)}{16K}u^3 + \cdots,
\end{aligned}$$

and

$$\begin{aligned}
A_{121} &= -\frac{3}{32}u^3 + \frac{3\left(5K_2-14K_1\right)}{64K}u^2 + \cdots, \\
A_{122} &= -\frac{3}{64}u^3 + \frac{3\left(14K_2-5K_1\right)}{128K}u^2 + \cdots, \\
A_{120} &= \frac{3\left(K_1+K_2\right)}{128K}u^4 + \frac{33\left(K_2^2-K_1^2\right)}{128K^2}u^3 + \frac{3\left(K_{22}-K_{11}\right)}{16K}u^2 + \cdots
\end{aligned}$$

and

$$\begin{aligned} B_{121} &= -\frac{195}{16}u^3 + \frac{9\,(9K_2 - 5K_1)}{8K}u^2 + \cdots, \\ B_{122} &= -\frac{195}{16}u^3 + \frac{9\,(9K_1 - 5K_2)}{8K}u^2 + \cdots, \\ B_{120} &= \frac{15}{32}u^5 + \frac{117\,(K_2 - K_1)}{64K}u^4 + \cdots \end{aligned}$$

Moreover,

$$A_{222} = -\tau\,(A_{111})\,,\;\; A_{221} = -\tau\,(A_{112})\,,\;\; A_{220} = \tau\,(A_{110})\,.$$

The detailed expressions for these coefficients can be found in Section 10.2.

Summarizing, we get the following system of first-order PDEs on the function μ:

$$G_1 = 0,\; G_2 = 0,\; G_{11} = 0,\; G_{12}^s = 0,\; G_{12}^a = 0, G_{22} = 0, \tag{42}$$

which is equivalent to system (35).

We remark that this system is symmetric with respect to the involution τ.

Next we note that equations (42) contain only linear combinations of the functions p_1, p_2, p_1^2, p_1p_2, p_2^2 with coefficients depending on u, K and the covariant derivatives of K up to order five.

We solve the equations $G_1 = 0,\; G_{12}^a = 0,\; G_2 = 0$ with respect to $p_1^2,\; p_1p_2,\; p_2^2$.

The determinant of the system is equal to $39u/4$.

Note that $\mu = 0$ implies $K_1 = K_2 = 0$ due to (35), and it is impossible for nonparallelizable 3-webs.

Indeed, if $K_1 = K_2 = 0$, then $\partial_1\,(K) = \partial_1\,(K) = 2HK$, and

$$0 = H\,(\partial_2 - \partial_1)\,(K) = [\partial_1, \partial_2]\,(K) = 2K\,(\partial_1\,(H) - \partial_2\,(H)) = -2K^2.$$

Solving the equations $G_1 = 0,\; G_{12}^a = 0,\; G_2 = 0$ with respect to p_1^2, p_1p_2, p_2^2, we get the expressions for p_ip_j in the form of linear combinations of p_1 and p_2.

Substituting these expressions into the system $G_{11} = 0,\; G_{22} = 0$ and solving the resulting system of linear equations with respect to p_1 and p_2, we find that

$$p_1 = \frac{V_1}{V_0},\; p_2 = \frac{V_2}{V_0},$$

where V_1 and V_2 are polynomials of degree eight with respect to u, and their coefficients depend on the curvature function K and its covariant derivatives up to order five. The leading terms of V_1 and V_2 are

$$\begin{aligned} V_1 &= -\frac{3^4}{2^8}KK_1u^8 + \frac{3^2}{2^5}[7(K_1^2 + 2K_1K_2 - 2K_2^2) + 13K(-K_{11} - 2K_{12} + 2K_{22})]u^7 + \cdots, \\ V_2 &= -\frac{3^4}{2^8}KK_2u^8 + \frac{3^2}{2^5}[7(2K_1^2 - 2K_1K_2 - K_2^2) + 13K(K_{22} + 2K_{12} - 2K_{11})]u^7 + \cdots, \end{aligned}$$

and the denominator V_0 is the seven-degree polynomial (see section 10.3)

$$V_0 = -\frac{13 \cdot 3^3}{2^6}K^2u^7 + \frac{3^3}{2^5}[15(K_1^2 - K_1K_2 + K_2^2) + 13K(K_{11} - K_{12} + K_{22})]u^5 + \cdots$$

As we have seen, the functions $p_i p_j$ are linear combinations of p_1 and p_2. Substituting the above expressions for p_1 and p_2 into the expressions for $p_i p_j$, we get

$$p_1^2 = \frac{V_{11}}{V_0},\ p_1 p_2 = \frac{V_{12}}{V_0},\ p_2^2 = \frac{V_{22}}{V_0},$$

where V_{ij} are polynomials of degree 11 with respect to u and their coefficients depend on the curvature function K and its covariant derivatives up to order five. The leading terms of V_{ij} are

$$\begin{aligned}
V_{11} &= \frac{5\cdot 3^3}{2^9}K^2u^{11} - \frac{3^6}{2^{10}}KK_1u^{10}\\
&+\frac{3^2}{2^{10}}[35K_1^2 + 412K_1K_2 - 412K_2^2 + 20K(-16K_{11} - 23K_{12} + 23K_{22})]u^9 + \cdots,\\
V_{12} &= \frac{5\cdot 3^3}{2^{10}}K^2u^{11} + \frac{3^6}{2^{10}}K\left(K_2 - K_1\right)u^{10}\\
&+\frac{3^2}{2^{10}}[206(K_1^2 - K_2^2) + 653K_1K_2 + 10K(23K_{11} - 101K_{12} + 23K_{22})]u^9 + \cdots,\\
V_{22} &= \frac{5\cdot 3^3}{2^9}K^2u^{11} + \frac{3^6}{2^{10}}KK_2u^{10}\\
&-\frac{3^2}{2^{10}}[412K_1^2 - 412K_1K_2 - 35K_2^2 + 20K(-23K_{11} + 23K_{12} + 16K_{22})]u^9 + \cdots
\end{aligned}$$

Note that the equation $G_{12}^s = 0$ holds automatically.

The resulting system

$$\begin{aligned}
p_1 &= \frac{V_1}{V_0},\quad p_2 = \frac{V_2}{V_0},\\
p_1^2 &= \frac{V_{11}}{V_0},\ p_1p_2 = \frac{V_{12}}{V_0},\quad p_2^2 = \frac{V_{22}}{V_0}
\end{aligned} \tag{43}$$

is τ-symmetric:

$$\begin{aligned}
\tau\left(V_0\right) &= -V_0,\ \tau\left(V_1\right) = V_2,\ \tau\left(V_2\right) = V_1,\\
\tau\left(V_{11}\right) &= -V_{22},\ \tau\left(V_{12}\right) = -V_{12},\ \tau\left(V_{22}\right) = -V_{11}.
\end{aligned}$$

This system gives us the following polynomial equations on u:

$$V_0\ V_{11} - V_1^2 = 0,\ V_0\ V_{22} - V_2^2 = 0,\ V_0\ V_{12} - V_1V_2 = 0. \tag{44}$$

Let us denote the left-hand sides of the above equations by Q_{ij} and Q_a and Q_s symmetrizations of Q_{11} and Q_{22}. We consider the polynomials

$$\begin{aligned}
2Q_a &= Q_{11} + Q_{22} = V_0\ \left(V_{11} - V_{22}\right) - V_1^2 + V_2^2,\\
2Q_s &= Q_{11} - Q_{22} = V_0\ \left(V_{11} + V_{22}\right) - V_1^2 - V_2^2\\
Q_{12} &= V_0\ V_{12} - V_1V_2.
\end{aligned}$$

The degree of each of the polynomials Q_s and Q_{12} equals 18 while the degree of Q_a does not exceed 17:

$$
\begin{aligned}
Q_a &= \frac{13\cdot 3^9}{2^{17}}K^3\left(K_1+K_2\right)\ u^{17}-\frac{3^6K^2}{2^{16}}[973\left(K_1^2-K_2^2\right)+1690K\left(K_{22}-K_{11}\right)]\ u^{16}+\cdots,\\
Q_s &= -\frac{65\cdot 3^6}{2^{15}}K^4u^{18}+\frac{13\cdot 3^9}{2^{17}}K^3\left(K_1-K_2\right)\ u^{17}\\
&\quad -\frac{3^5}{2^{16}}K^2[-3337(K_1^2+K_2^2)+6256K_1K_2+130K\left(K_{11}-40K_{12}+K_{22}\right)]\ u^{16}+\cdots,\\
Q_{12} &= -\frac{65\cdot 3^6}{2^{16}}K^4u^{18}+\frac{13\cdot 3^9}{2^{16}}K^3(K_1-K_2)\ u^{17}\\
&\quad -\frac{243}{2^{15}}[K^2(-1564(K_1^2+K_2^2)+4483K_1K_2+65K(10K_{11}-49K_{12}+10K_{22})]\ u^{16}+\cdots,
\end{aligned}
$$

In order to complete integration of system (43), we differentiate one of equations (44), say, the first one,

$$
\begin{aligned}
\frac{\partial Q_a}{\partial u}p_1+\widehat{\delta_1^K}\left(Q_a\right) &= 0,\\
\frac{\partial Q_a}{\partial u}p_2+\widehat{\delta_2^K}\left(Q_a\right) &= 0,
\end{aligned}
\qquad (45)
$$

where $\widehat{\delta_i^K}$ are the total derivatives relative to K (see 10.4 for the expressions of $\widehat{\delta_i^K}$):

$$
\begin{aligned}
\widehat{\delta_1^K} &= K_1\frac{\partial}{\partial K}+K_{11}\frac{\partial}{\partial K_1}+\left(K_{12}-K^2\right)\frac{\partial}{\partial K_2}+\cdots\\
\widehat{\delta_2^K} &= K_2\frac{\partial}{\partial K}+(K_{12}+K^2)\frac{\partial}{\partial K_1}+K_{22}\frac{\partial}{\partial K_2}+\cdots
\end{aligned}
$$

Substituting the covariant derivatives p_1 and p_2 taken from the first two equations of (43) into (45), we get the new system of polynomial equations on u:

$$
\begin{aligned}
\frac{\partial Q_a}{\partial u}V_1+V_0\ \widehat{\delta_1^K}\left(Q_a\right) &= 0,\\
\frac{\partial Q_a}{\partial u}V_2+V_0\ \widehat{\delta_2^K}\left(Q_a\right) &= 0.
\end{aligned}
$$

The polynomials

$$Q_1=\frac{\partial Q_a}{\partial u}V_1+V_0\ \widehat{\delta_1^K}\left(Q_a\right)$$

and

$$Q_2=\frac{\partial Q_a}{\partial u}V_2+V_0\ \widehat{\delta_2^K}\left(Q_a\right)$$

are of degree 24, and their coefficients depend on the curvature function K and its covariant derivatives up to order six:

$$\begin{aligned} Q_1 &= \frac{131 \cdot 65 \cdot 3^9}{2^{23}} K^5 K_1 u^{24} + \cdots, \\ Q_2 &= \frac{131 \cdot 65 \cdot 3^9}{2^{23}} K^5 K_2 u^{24} + \cdots \end{aligned}$$

The next result follows from the above consideration and is basic for finding linearizability conditions for 3-webs.

Theorem 8.1 *Let W be a nonparallelizable 3-web. Then the smooth solvability of the system of nonlinear partial differential equations*

$$I_1(\mu) = 0, \ I_2(\mu) = 0$$

is equivalent to the existence of real and smooth solutions of the following system of algebraic equations:

$$Q_a = 0, \ Q_s = 0, \ Q_{12} = 0, \ Q_1 = 0, \ Q_2 = 0.$$

In 1912 Gronwall ([13]) made the following conjecture: *if a nonparallelizable 3-web W_3 in the plane is linearizable, then, up to a projective transformation, a diffeomorphism transforming W_3 into a linear 3-web, is uniquely determined.*

Bol ([6],[7], 1938) and Borůvka ([8], 1938) proved that the number of projectively nonequivalent linearizations of a nonparallelizable linearizable 3-web does not exceed 16. Vaona ([20], 1961) reduced this number to 11. Grifone, Muzsnay and Saab ([12], 2001) proved that this number does not exceed 15.

The above theorem implies the following result.

Corollary 8.2 *Let W be a nonparallelizable, linearizable 3-web. Then the number of projectively nonequivalent linearizations of such a web does not exceed 15.*

Proof. Observe, that if μ satisfies the system $I_1(\mu) = 0, \ I_2(\mu) = 0$, then system (21) is completely integrable, and its solutions (λ_1, λ_2) are determined by values $\lambda_1(a_0)$ and $\lambda_2(a_0)$ at some fixed point $a_0 \in M$. Moreover, it is easy to check that the projective transformations act transitively on the set of $(\lambda_1(a_0), \lambda_2(a_0))$. So, up to a projective transformation, the values $(\lambda_1(a_0), \lambda_2(a_0))$ are nonessential.

As we showed earlier, the polynomials Q_a, Q_s and Q_{12} are of degrees 17, 18, and 18, and each of the polynomials Q_1 and Q_2 is of degree 24. Hence, there is a linear combination L of Q_s and Q_{12} having degree ≤ 17, and there is a linear combination S of Q_a and L having degree ≤ 16.

In fact, we can take as L the polynomial

$$\begin{aligned} L &= Q_s - 2Q_{12} = \frac{13 \cdot 3^{10}}{2^{17}} K^3 (K_2 - K_1) u^{17} \\ &\quad + \frac{3^6}{2^{15}} K^2 [973(4K_1K_2 - K_1^2 - K_2^2) + 1690K(K_{11} - 4K_{12} + K_{22})] u^{16} + \cdots \end{aligned}$$

If $K_2 - K_1 \neq 0$ and $K_1 + K_2 \neq 0$, then as S we can take the polynomial

$$\begin{aligned} S &= (K_1 + K_2)L - 3(K_2 - K_1)Q_a \\ &= \frac{3^6}{2^{15}} K^2[-1946(K_1^3 + K_2^3) + 2919K_1K_2(K_1 + K_2) + 1690K(2K_1 - K_2)K_{11} \\ &\quad -3380K(K_1 + K_2)K_{12} + 1690K(2K_2 - K_1)K_{22}]u^{16} + \cdots \end{aligned}$$

If $K_2 - K_1 = 0$ (or $K_1 + K_2 = 0$), then the polynomial L (resp. Q_a) is already of degree 16.

Thus the polynomials $Q_a,\ Q_s, Q_{12}$ and $Q_1,\ Q_2$ can have at most 16 common roots. One of these roots gives μ for the 3-web under consideration. Therefore, the number of projectively nonequivalent linearizations of the web W does not exceed 15. ■

Remark. In the paper [1] we have proved that μ is uniquely determined by the basic invariant of linearizable d-webs, if $d \geq 4$. The above proof shows that the Gronwall conjecture is correct for such webs. Namely, up to a projective transformation, for linearizable d-webs, $d \geq 4$, there exists a unique linearization.

9 Differential Invariants for Linearizability and the Blaschke Conjecture

In this section we consider the case of nonparallelizable, linearizable 3-webs. We will need some new algebraic constructions.

9.1 Resultant and Its Generalizations

Let $T, S_1, ..., S_n$ be polynomials over an algebraically closed field $\mathbb{F}$, $T, S_1, ..., S_n \in \mathbb{F}[u]$, and $\operatorname{char} \mathbb{F} = 0$. Denote by $\mathbf{R}(f, g)$ the resultant of polynomials f and g. Recall that $\mathbf{R}(f, g)$ as a function in g given f is homogeneous of degree $\deg f$. Hence $\mathbf{R}(T, x_1S_1 + x_2S_2 + \cdots + x_nS_n)$ is a homogeneous polynomial of degree $\deg T$ in $x_1, ..., x_n$:

$$\mathbf{R}(T, \sum_{i=1}^{n} x_iS_i) = \sum_{\sigma} x^{\sigma}\mathbf{R}_{\sigma}(T, S_1, ..., S_n),$$

where σ runs over all multi-indices of the length $\deg T$,i.e.,

$$\mathbf{R}(T, \sum_{i=1}^{n} x_iS_i) = x_1^{i_1}x_2^{i_2}...x_n^{i_n}\mathbf{R}_{i_1i_2...i_n}(T, S_1, ..., S_n).$$

We call the coefficients $\mathbf{R}_{\sigma}(T, S_1, ..., S_n)$ *(generalized) resultants* of the system of polynomials $T, S_1, ..., S_n$.

Theorem 9.1 *The polynomials $T, S_1, ..., S_n$ have a common root if and only if all resultants $\mathbf{R}_{\sigma}(T, S_1, ..., S_n)$ are equal to zero.*

Proof. To illustrate the idea of the proof and to avoid unnecessary technicalities, we consider only the case $n = 2$. Assume also that the leading coefficient of T is equal to 1.

Let $\lambda_1, .., \lambda_t$ be roots of T, and $t = \deg T$. Then

$$\mathbf{R}(T, x_1 S_1 + x_2 S_2) = \prod_{i=1}^{t} (x_1 S_1 (\lambda_i) + x_2 S_2 (\lambda_i)) = \sum_{a=0}^{t} x_1^a x_2^{t-a} \mathbf{R}_{a,t-a}(T, S_1, S_2),$$

where $\mathbf{R}_{t,0}(T, S_1, S_2) = \mathbf{R}(T, S_1), \mathbf{R}_{0,t}(T, S_1, S_2) = \mathbf{R}(T, S_2)$, and for $1 \leq a \leq t - 1$, we get

$$\mathbf{R}_{a,t-a}(T, S_1, S_2) = \sum_I S_1 (\lambda_{i_1}) \cdots S_1 (\lambda_{i_a}) S_2 (\lambda_{j_1}) \cdots S_2 \left(\lambda_{j_{t-a}}\right).$$

Here we have denoted by $(j_1, ..., j_{t-a})$ the multi-index complementary to $I = (i_1, ..., i_a)$.

First, let T, S_1 and S_2 have a common root. Then the polynomials T and $x_1 S_1 + x_2 S_2$ have a common root for all x_1, x_2, and therefore $\mathbf{R}_{a,t-a}(T, S_1, S_2) = 0$ for all a. Conversely, let $\mathbf{R}_{a,t-a}(T, S_1, S_2) = 0$ for all a. Then $\mathbf{R}(T, S_1) = \mathbf{R}_{t,0}(T, S_1, S_2) = 0$, $\mathbf{R}(T, S_2) = \mathbf{R}_{0,t}(T, S_1, S_2) = 0$, and therefore T and S_1 have a common root, say ν, and T and S_2 have a common root, say μ. Assume that they have no more common roots, and consider, for example, $\mathbf{R}_{1,t-1}(T, S_1, S_2)$.

One has

$$\mathbf{R}_{1,t-1}(T, S_1, S_2) = S_1 (\mu) \cdot S_2 (\nu) \cdot S_2 (\lambda_{j_1}) \cdot ... \cdot S_2 \left(\lambda_{j_{t-2}}\right) = 0,$$

where $(\lambda_1, .., \lambda_t) = \nu \cup \mu \cup (\lambda_1, .., \lambda_{t-2})$ is the disjoint union.

Hence, either $S_1 (\mu) = 0$ or $S_2 (\nu) = 0$, and therefore T, S_1 and S_2 have a common root.

In the case when the polynomials have common roots of multiplicity two or higher, $\mathbf{R}_{1,t-1}(T, S_1, S_2) = \mathbf{R}_{1-t,1}(T, S_1, S_2) = 0$, and vanishing of $\mathbf{R}_{2,t-2}(T, S_1, S_2)$ shows that T, S_1 and S_2 have a common root, etc. ■

Remark. The number of resultants $\mathbf{R}_\sigma(T, S_1, ..., S_n)$ equals the dimension of homogeneous polynomials of degree $t = \deg T$ in n variables, and therefore equals

$$\binom{n + t - 1}{t}.$$

9.2 Differential Invariants for Linearizability

As we have seen earlier, the solvability of the system of differential equations (35) is equivalent to the existence of real roots of the system of algebraic equations

$$Q_a = 0,\ Q_s = 0,\ Q_{12} = 0,\ Q_1 = 0,\ Q_2 = 0. \tag{46}$$

We apply the above theorem and get the following result.

Theorem 9.2 *Let W be a nonparallelizable 3-web. If the 3-web W is linearizable, then the following differential invariants*

$$\mathbf{R}_{i_1 i_2 i_3 i_4}\left(Q_a, Q_s, Q_{12}, Q_1, Q_2\right)$$

vanish, and algebraic system (46) has at least one real smooth solution.

Conversely, if the differential invariants vanish and algebraic system (46) has at least one real smooth solution, then the 3-web is linearizable.

Note that all the differential invariants depend on the curvature function K and its covariant derivatives up to order six, but $\mathbf{R}_{i_1 i_2 i_3 i_4}\left(Q_a, Q_s, Q_{12}, Q_1, Q_2\right)$ with $i_3 = i_4 = 0$ depend on the curvature function K and its covariant derivatives up to order five. Since for a nonparallelizable 3-web, we have deg $Q_a = 17$, the total number of invariants equals $1040 = \binom{4+17-1}{17}$, and among them there are $18 = \binom{2+17-1}{17}$ invariants of order five in K. In terms of the web function $f\left(x, y\right)$, the corresponding orders are nine and eight.

Note also that the number of invariants is not invariant: it depends which of the polynomials $Q_a, Q_s, Q_{12}, Q_1, Q_2$ we take as the first one. In our considerations we took the polynomial Q_a of the least degree 17 as the first polynomial. Moreover, the number of invariants can be reduced if we find a linear combination of the above five polynomials whose degree is less than 17, replace one of five polynomials by this linear combination and take this combination as the first polynomial (see our earlier considerations where we found a polynomial of degree not exceeding 16).

Remark. In the book [4] (§17) Blaschke made the following conjecture: The linearizability conditions for a nonparallelizable 3-web are expressed in terms of the web function $f\left(x, y\right)$ and its covariant derivatives up to order nine, and the table in §17 shows that the number of differential invariants equals four. As we have seen, Blaschke's estimate of the "functional codimension" of the orbits of the linearizable 3-webs was correct while the number of algebraic conditions is much greater than four. Moreover, not all linearizability invariants are of order nine: eighteen of them are of order eight.

To find out whether algebraic system (46) has real solutions, we consider the greatest common divisor $\mathbf{G} = \mathbf{GCD}[Q_a, Q_s, Q_{12}, Q_1, Q_2]$ of the polynomials $Q_a, Q_s, Q_{12}, Q_1, Q_2$.

The following theorem, which is important when one is testing a 3-web for linearizability, is obvious.

Theorem 9.3 *If* $\deg \mathbf{G} = 0$, *then there are no common solutions, and the 3-web is nonlinearizable. If* $\deg \mathbf{G} > 1$, *but* $\mathbf{G}$ *has no real roots, then the 3-web is also nonlinearizable. In the case when* $\deg \mathbf{G} = 1$, *or* $\deg \mathbf{G} > 1$ *but* $\mathbf{G}$ *has a real root, a 3-web is linearizable.*

Note that in the latter case, the number of real roots can give us an improvement of our estimate of the Gronwall conjecture: if the number of real roots of $\mathbf{G}$ equals s, then the number of projectively nonequivalent linearizations of a nonparallelizable, nonexceptional linearizable 3-web W does not exceed s. If $s < 15$, then this will be an improvement of our estimate.

9.3 Linear 3-Webs

To test our systems of equations and the differential invariants, we consider them for linear 3-webs.

Let us assume that the web function $f(x, y)$ defines a linear 3-web, and let d_{st} be the covariant differential of the flat connection in coordinates x and y.

Then $d_{\mathrm{st}}(a\ dx) = dx \otimes da$ and $d_{\mathrm{st}}(a\ dy) = dy \otimes da$. Therefore,

$$d_{\mathrm{st}}(\omega_1) = \omega_1 \otimes \frac{df_x}{f_x}, \quad d_{\mathrm{st}}(\omega_2) = \omega_2 \otimes \frac{df_y}{f_y},$$

and the affine deformation tensor $T = d_\gamma - d_{\mathrm{st}}$ between the Chern and the flat connections equals to

$$\begin{aligned} T(\omega_1) &= \left(\frac{f_{xx}}{f_x^2} - H\right)\omega_1 \otimes \omega_1, \\ T(\omega_2) &= \left(\frac{f_{yy}}{f_y^2} - H\right)\omega_2 \otimes \omega_2. \end{aligned}$$

Therefore, for linear 3-webs we have

$$\begin{aligned} \lambda_1 &= \lambda_2 = 0, \\ \mu &= \frac{f_{xx}}{f_x^2} - H = -\left(\frac{f_{yy}}{f_y^2} - H\right). \end{aligned}$$

If we assume that $f(x, y)$ is a solution of the Euler equation, i.e., $f_x = f\ f_y$, then we get

$$\mu = \frac{1}{f}.$$

Moreover, in this case

$$H = \frac{1}{f} + \frac{f_{yy}}{f_y^2},$$

and the curvature function is

$$K = -\frac{f_{yy}}{f\ f_y^2}.$$

The first covariant derivatives of the curvature function are

$$\begin{aligned} K_1 &= -\frac{2K}{f} + \frac{f_{yyy}}{f f_y^3}, \\ K_2 &= -\frac{K}{f} + \frac{f_{yyy}}{f f_y^3} \end{aligned}$$

and

$$K_2 - K_1 = \frac{K}{f}.$$

Note that the covariant derivatives of the function μ are

$$\delta_1(\mu) = \delta_2(\mu) = K.$$

One can check that equations (21) and (35) have the common solution $\mu = 1/f$, and the same is true for equations (42).

Moreover $1/f$ is a common real root for algebraic system (46).

9.4 Procedure for Applying the Linearizability Criterion

Now we can outline a procedure which can be applied to determine whether a 3-web W_3 given by a web function $z = f(x, y)$ is linearizable:

1. Compute the curvature K and its covariant derivatives up to order five (see formula (9) and formulas in Section 10.1).
2. Compute $A_{ij},\ A_{ijk,}$ and $B_{ijk,}\ i, j, k = 0, 1, 2$ (see formulas in Sections 8 and 10.2).
3. Compute the polynomial V_0 (see Sections 8 and 10.3).
4. Compute the polynomials $V_{ij},\ V_i$ and $Q_a, Q_s, Q_{12}, Q_1, Q_2$ (see Sections 8 and 10.3).
5. Compute $\mathbf{G} = \mathbf{GCD}[Q_a, Q_s, Q_{12}, Q_1, Q_2]$ and apply the linearizability condition outlined in Theorem 9.3.

9.5 Examples

Example 6 We consider the 3-web in the plane with the web function

$$f(x, y) = x^2 + xy + y^2.$$

For this web we have:

$$\begin{aligned} H &= \frac{1}{(2x+y)(x+2y)},\ K = -\frac{6(x^2-y^2)}{(2x+y)^3(x+2y)^3}, \\ K_1 &= -\frac{6(4x^3+3x^2y-12xy^2-13y^3)}{(2x+y)^5(x+2y)^4}, \\ K_2 &= -\frac{6(13x^3+12x^2y-3xy^2-4y^3)}{(2x+y)^4(x+2y)^5}, ..., \end{aligned}$$

and

$$
\begin{aligned}
Q_a &= \frac{13\cdot 3^{14}(x^2-y^2)^4(5x^2-8xy+5y^2)}{2^{12}(2x+y)^{14}(x+2y)^{14}}\mu^{17}+...,\\
Q_s &= -\frac{65\cdot 3^{10}(x^2-y^2)^4}{2^{11}(2x+y)^{12}(x+2y)^{12}}\mu^{18}+...,\\
Q_{12} &= -\frac{65\cdot 3^{10}(x^2-y^2)^4}{2^{12}(2x+y)^{12}(x+2y)^{12}}\mu^{18}+...,\\
Q_1 &= \frac{13\cdot 3^{14}(x^2-y^2)^5}{2^{12}(2x+y)^{22}(x+2y)^{21}}\cdot(5700x^5+13577x^4y-2480x^3y^2\\
&\quad -37710x^2y^3-44660xy^4-18343y^5)\mu^{24}+...,\\
Q_2 &= \frac{13\cdot 3^{14}(x^2-y^2)^5}{2^{12}(2x+y)^{21}(x+2y)^{22}}\cdot(18343x^5+44660x^4y+37710x^3y^2\\
&\quad +2480x^2y^3-13577xy^4-5700y^5)\mu^{24}+...
\end{aligned}
$$

Evaluating the polynomials $Q_a,\ Q_s,\ Q_{12}$ and $Q_1,\ Q_2$ at the point $(0.1, 1)$, we find that

$$
\begin{aligned}
F_a &= Q_a(0.1,1) = 0.204819\mu^{17}+...,\\
F_s &= Q_s(0.1,1) = -0.0274492\mu^{18}+...,\\
F_{12} &= Q_{12}(0.1,1) = -0.0137246\mu^{18}+...,\\
F_1 &= Q_1(0.1,1) = -3.94038\mu^{24}+...,\\
F_2 &= Q_1(0.1,1) = -0.678834\mu^{24}+...
\end{aligned}
$$

We calculate now the resultant of the polynomials F_a and F_{12}:

$$\mathbf{R}(F_a, F_{12}) = -1.046\cdot 10^{185} \neq 0.$$

Since the resultant of F_a and F_{12} does not vanish, the polynomials Q_a and Q_{12} (and therefore the polynomials $Q_a, Q_s, Q_{12}, Q_1, Q_2$) have no common roots, and as a result, *the* 3*-web under consideration is not linearizable.*

Remark. Note that even if the resultants of all pairs of the polynomials $Q_a, Q_s, Q_{12}, Q_1, Q_2$ were vanished, we could not make any conclusion—the further investigation involving the generalized resultants or finding the greatest common divisor $\mathbf{G} = \mathbf{GCD}[Q_a, Q_s, Q_{12}, Q_1, Q_2]$ would be necessary to answer the question whether the 3-web under consideration is linearizable or not linearizable.

Example 7 We consider the 3-web in the plane with the web function

$$f(x,y) = (x+y)e^{-x}$$

(see Example 5).

For this web we have:

$$
\begin{aligned}
H &= \frac{e^x}{x+y-1},\quad K = \frac{e^{2x}}{(1-x-y)^3},\\
K_1 &= \frac{3e^{3x}}{(x+y-1)^5},\quad K_2 = -\frac{e^{3x}}{(x+y-1)^4},...,
\end{aligned}
$$

and

$$
\begin{aligned}
Q_a &= \frac{13 \cdot 3^9 e^{9x}(x+y-4)}{2^{17}(x+y-1)^{12}}\mu^{17} + ..., \\
Q_s &= -\frac{65 \cdot 3^6 e^{8x}}{2^{15}(x+y-1)^{12}}\mu^{18} + ..., \\
Q_{12} &= -\frac{65 \cdot 3^6 e^{8x}}{2^{16}(x+y-1)^{12}}\mu^{18} + ..., \\
Q_1 &= \frac{13 \cdot 3^{12} e^{14x}(829x+829y-3472)}{41 \cdot 31 \cdot 11 \cdot 3 \cdot 2^3(x+y-1)^{22}}\mu^{24} + ..., \\
Q_2 &= -\frac{13 \cdot 3^{12} e^{14x}(259x+259y-1192)}{41 \cdot 31 \cdot 11 \cdot 3 \cdot 2^3(x+y-1)^{21}}\mu^{24} + ...
\end{aligned}
$$

Evaluating the polynomials $Q_a,\ Q_s,\ Q_{12}$ and $Q_1,\ Q_2$ at the point $(0, 0.1)$, we find that

$$
\begin{aligned}
F_a &= Q_a(0,0.1) = -33.2808\mu^{17} + ..., \\
F_s &= Q_s(0,0.1) = -5.12013\mu^{18} + ..., \\
F_{12} &= Q_{12}(0,0.1) = -2.56006\mu^{18} + ..., \\
F_1 &= Q_1(0,0.1) = -7085.94\mu^{24} + ..., \\
F_2 &= Q_1(0,0.1) = -2194.28\mu^{24} + ...
\end{aligned}
$$

We calculate now the resultant of the polynomials F_{11} and F_{22}:

$$
\mathbf{R}(F_a, F_{12}) = 1.23007 \cdot 10^{272} \neq 0.
$$

Since the resultant of F_a and F_{12} does not vanish, the polynomials Q_a and Q_{12} (and therefore the polynomials $Q_a, Q_s, Q_{12}, Q_1, Q_2$) have no common roots, and as a result, *the 3-web under consideration is not linearizable.*

10 Appendix. Computational Formulae

10.1 Symmetrized Covariant Derivatives of the Curvature

$$
K_{12} = \delta_1(K_2) + K^2.
$$

$$
\begin{aligned}
K_{112} &= \delta_1(K_{12}) + \frac{5}{3}KK_1; \\
K_{122} &= \delta_1(K_{22}) + \frac{10}{3}KK_2.
\end{aligned}
$$

$$
\begin{aligned}
K_{1112} &= \delta_1\left(K_{112}\right) + \frac{11}{6} K K_{11} + \frac{5}{6} K_1^2; \\
K_{1122} &= \delta_1\left(K_{122}\right) + \frac{11}{3} K K_{12} + \frac{5}{3} K_1 K_2; \\
K_{1222} &= \delta_1\left(K_{222}\right) + \frac{11}{2} K K_{22} + \frac{5}{2} K_2^2.
\end{aligned}
$$

$$
\begin{aligned}
K_{11112} &= \delta_1\left(K_{1112}\right) + \frac{21}{10}\left(K K_{111} + K_1 K_{11}\right); \\
K_{11122} &= \delta_1\left(K_{1122}\right) + \frac{7}{5}\left(3 K K_{112} + K_2 K_{11} + 2 K_1 K_{12}\right); \\
K_{11222} &= \delta_1\left(K_{1222}\right) + \frac{21}{10}\left(3 K K_{122} + K_1 K_{22} + 2 K_2 K_{12}\right); \\
K_{12222} &= \delta_1\left(K_{2222}\right) + \frac{42}{5}\left(K K_{222} + K_2 K_{22}\right).
\end{aligned}
$$

$$
\begin{aligned}
K_{111112} &= \delta_1\left(K_{11112}\right) + \frac{12}{5} K K_{1111} + \frac{14}{5} K_1 K_{111} + \frac{7}{5} K_{11}^2; \\
K_{111122} &= \delta_1\left(K_{11122}\right) + \frac{1}{5}\left(24 K K_{1112} + 7 K_2 K_{111} + 21 K_1 K_{112} + 14 K_{12} K_{11}\right); \\
K_{111222} &= \delta_1\left(K_{11222}\right) \\
&\quad + \frac{1}{5}\left(36 K K_{1122} + 21 K_1 K_{122} + 21 K_2 K_{112} + 7 K_{11} K_{22} + 14 K_{12}^2\right); \\
K_{112222} &= \delta_1\left(K_{12222}\right) + \frac{2}{5}\left(24 K K_{1222} + 7 K_1 K_{222} + 21 K_2 K_{122} + 14 K_{12} K_{22}\right); \\
K_{122222} &= \delta_1\left(K_{22222}\right) + 12 K K_{2222} + 14 K_2 K_{222} + 7 K_{22}^2.
\end{aligned}
$$

10.2 Coefficients A_{ijk} and B_{ijk} in (41)

Here we give the expressions of the coefficients A_{ijk} and B_{ijk} in formulas (41) for $G_{11},\ G_{12}^s,\ G_{12}^a$, and G_{22} (see Section 8):

$$
\begin{aligned}
&A_{111} = \\
&\frac{3u^3}{32} + \frac{3\left(-7K_1 + 12K_2\right)u^2}{32K} + \frac{145Ku}{16} + \frac{\left(26K_1^2 - 95K_1K_2 - 10K_2^2\right)u}{64K^2} \\
&- \frac{\left(13K_{11} - 43K_{12} + 13K_{22}\right)u}{32K} + \frac{1}{48}\left(-179K_1 - 62K_2\right) \\
&+ \frac{77K_1\left(2K_1^2 - 5K_1K_2 + 2K_2^2\right)}{64K^3} + \frac{K_2\left(45K_{11} - 41K_{12} + 7K_{22}\right)}{16K^2} \\
&- \frac{K_1\left(95K_{11} - 145K_{12} + 27K_{22}\right)}{32K^2} + \frac{3K_{111} - 8K_{112} + 5K_{122} - K_{222}}{4K};
\end{aligned}
$$

$A_{112} =$

$$-\frac{3u^3}{16} + \frac{3K_1u^2}{16K} + \frac{\left(53K_1^2 - 20K_1K_2 + 20K_2^2\right)u}{64K^2} - \frac{(2K_{11} + 13K_{12} - 13K_{22})\,u}{16K}$$
$$-\frac{31}{24}(K_1 - 2K_2) - \frac{77K_1(K_1 - 2K_2)^2}{64K^3} - \frac{17K_2(K_{11} - 2K_{12})}{8K^2}$$
$$+\frac{K_1(25K_{11} - 54K_{12} + 20K_{22})}{16K^2} - \frac{2K_{111} - 7K_{112} + 7K_{122}}{4K};$$

$A_{110} =$

$$\frac{3(K_1 - 2K_2)u^4}{128K} - \frac{33K_1(K_1 - 2K_2)u^3}{128K^2} + \frac{3(K_{11} - 2K_{12})u^3}{16K}$$
$$-\frac{23K_2(K_{11} - K_{12} + K_{22})u^2}{64K^2} + \frac{23K_1\left(16K^2 + K_{11} - K_{12} + K_{22}\right)u^2}{128K^2}$$
$$-\frac{5(K_{111} - 3K_{112} + 3K_{122} - 2K_{222})u^2}{32K} - \frac{31}{64}(K_{11} + 2K_{12} - 2K_{22})\,u$$
$$+\frac{77K_1(K_1 - 2K_2)(K_{11} - K_{12} + K_{22})u}{128K^3} - \frac{5(K_{11} - 2K_{12})(K_{11} - K_{12} + K_{22})u}{16K^2}$$
$$+\frac{K_2(28K_{111} - 51K_{112} + 51K_{122})u}{64K^2} - \frac{21K_1^2 - 198K_1K_2 + 198K_2^2}{64K}u$$
$$+\frac{K_1(-44K_{111} + 99K_{112} - 99K_{122} + 32K_{222})u}{128K^2}$$
$$+\frac{8K_{1111} - 34K_{1112} + 54K_{1122} - 36K_{1222}}{64K}u + \frac{11}{3}K(K_1 - 2K_2) +$$
$$\frac{17K_1\left(11K_1^2 - 20K_1K_2 + 20K_2^2\right)}{192K^2} - \frac{353K_2(K_{11} - 2K_{12})}{480K}$$
$$+\frac{K_1(-907K_{11} + 398K_{12} - 1104K_{22})}{960K} - \frac{5(K_{11} - 2K_{12})(K_{112} - K_{122})}{16K^2}$$
$$+\frac{77K_1(K_1 - 2K_2)(K_{112} - K_{122})}{128K^3} - \frac{33}{40}(K_{111} - K_{112} + K_{122})$$
$$+\frac{7K_2(K_{1112} - K_{1122})}{16K^2} + \frac{K_1(-11K_{1112} + 19K_{1122} - 8K_{1222})}{32K^2}$$
$$+\frac{K_{11112} - 3K_{11122} + 2K_{11222}}{8K};$$

$A_{221} =$

$$-\frac{3u^3}{16} - \frac{3K_2u^2}{16K} + \frac{20K_1^2 - 20K_1K_2 + 53K_2^2}{64K^2}u + \frac{13K_{11} - 13K_{12} - 2K_{22}}{16K}u$$
$$-\frac{31}{24}(-2K_1 + K_2) + \frac{77K_2(-2K_1 + K_2)^2}{64K^3} + \frac{17K_1(-2K_{12} + K_{22})}{8K^2}$$
$$-\frac{K_2(20K_{11} - 54K_{12} + 25K_{22})}{16K^2} + \frac{7K_{112} - 7K_{122} + 2K_{222}}{4K};$$

$$
\begin{aligned}
&A_{222} = \\
&\frac{3u^3}{32} + \frac{3\left(-12K_1 + 7K_2\right)}{32K}u^2 - \frac{10K_1^2 + 95K_1K_2 - 26K_2^2}{64K^2}u \\
&-\frac{145Ku}{16} - \frac{13K_{11} - 43K_{12} + 13K_{22}}{32K}u + \frac{1}{48}\left(-62K_1 - 179K_2\right) \\
&-\frac{77K_2\left(2K_1^2 - 5K_1K_2 + 2K_2^2\right)}{64K^3} - \frac{K_1\left(7K_{11} - 41K_{12} + 45K_{22}\right)}{16K^2} \\
&+\frac{K_2\left(27K_{11} - 145K_{12} + 95K_{22}\right)}{32K^2} + \frac{K_{111} - 5K_{112} + 8K_{122} - 3K_{222}}{4K};
\end{aligned}
$$

$$
\begin{aligned}
&A_{220} = \\
&-\frac{3\left(-2K_1 + K_2\right)u^4}{128K} + \frac{33K_2\left(-2K_1 + K_2\right)u^3}{128K^2} + \frac{6K_{12} - 3K_{22}}{16K}u^3 \\
&-\frac{23K_2u^2}{8} + \frac{23\left(-2K_1 + K_2\right)\left(K_{11} - K_{12} + K_{22}\right)u^2}{128K^2} \\
&-\frac{5\left(-2K_{111} + 3K_{112} - 3K_{122} + K_{222}\right)u^2}{32K} + \frac{31}{64}\left(2K_{11} - 2K_{12} - K_{22}\right)u \\
&-\frac{3\left(66K_1^2 - 66K_1K_2 + 7K_2^2\right)u}{64K} - \frac{5\left(2K_{12} - K_{22}\right)\left(K_{11} - K_{12} + K_{22}\right)u}{16K^2} \\
&-\frac{77K_2\left(2K_1\left(K_{12} - K_{22}\right) + K_2\left(K_{11} + K_{22}\right)\right)u}{128K^3} + \frac{7^2 11^2 K_1K_2^3K_{11}K_{12}u}{2^{13}K^6} \\
&-\frac{K_1\left(51K_{112} - 51K_{122} + 28K_{222}\right)u}{64K^2} + \frac{18K_{1112} - 27K_{1122} + 17K_{1222} - 4K_{2222}}{32K}u \\
&+\frac{K_2\left(-32K_{111} + 99K_{112} - 99K_{122} + 44K_{222}\right)u}{128K^2} + \frac{11}{3}K\left(-2K_1 + K_2\right) \\
&-\frac{17K_2\left(20K_1^2 - 20K_1K_2 + 11K_2^2\right)}{192K^2} + \frac{353K_1\left(-2K_{12} + K_{22}\right)}{480K} \\
&+\frac{K_2\left(1104K_{11} - 398K_{12} + 907K_{22}\right)}{960K} + \frac{77\left(2K_1 - K_2\right)K_2\left(K_{112} - K_{122}\right)}{128K^3} \\
&-\frac{5\left(2K_{12}K_{112} - 2K_{12}K_{122} + K_{22}K_{122}\right)}{16K^2} + \frac{33\left(K_{112} - K_{122} + K_{222}\right)}{40} \\
&+\frac{5K_{22}K_{112}}{16K^2} - \frac{K_2\left(8K_{1112} - 19K_{1122} + 11K_{1222}\right)}{32K^2} + \frac{14K_1\left(-K_{1122} + K_{1222}\right)}{32K^2} \\
&+\frac{2K_{11122} - 3K_{11222} + K_{12222}}{8K};
\end{aligned}
$$

$$
\begin{aligned}
&A_{121} = -\frac{3u^3}{32} + \frac{3\left(-14K_1 + 5K_2\right)}{64K}u^2 + \frac{145Ku}{16} + \frac{10K_1^2 - 79K_1K_2 + 61K_2^2}{64K^2}u \\
&+\frac{13K_{11} + 17K_{12} - 17K_{22}}{32K}u + \frac{77\left(2K_1^3 - K_1^2K_2 - 2K_1K_2^2 + K_2^3\right)}{64K^3} \\
&+\frac{1}{96}\left(124K_1 - 365K_2\right) + \frac{K_2\left(55K_{11} + 79K_{12} - 81K_{22}\right)}{64K^2} \\
&+\frac{K_1\left(-190K_{11} + 54K_{12} + 64K_{22}\right)}{64K^2} + \frac{3K_{111} - K_{112} - 2K_{122} + K_{222}}{4K};
\end{aligned}
$$

$$A_{122} = -\frac{3u^3}{32} + \frac{3(-5K_1+14K_2)u^2}{64K} + -\frac{145Ku}{16} + \frac{61K_1^2 - 79K_1K_2 + 10K_2^2}{64K^2}u + \frac{-17K_{11}+17K_{12}+13K_{22}}{32K}u - \frac{77\left(K_1^3 - 2K_1^2K_2 - K_1K_2^2 + 2K_2^3\right)}{64K^3} + \frac{1}{96}(-365K_1+124K_2) - \frac{K_2(32K_{11}+27K_{12}-95K_{22})}{32K^2} + \frac{K_1(81K_{11}-79K_{12}-55K_{22})}{64K^2} + \frac{-K_{111}+2K_{112}+K_{122}-3K_{222}}{4K};$$

$$A_{120} = \frac{3(K_1+K_2)u^4}{128K} - \frac{33\left(K_1^2-K_2^2\right)u^3}{128K^2} + \frac{3(K_{11}-K_{22})u^3}{16K} + \frac{23}{8}(K_1-K_2)u^2 - \frac{23(K_1+K_2)(K_{11}-K_{12}+K_{22})u^2}{128K^2} + \frac{5(K_{111}+K_{222})u^2}{32K} - \frac{3\left(33K_1^2-92K_1K_2+33K_2^2\right)u}{64K} - \frac{77(K_2^2K_{11}+K_1^2K_{12})}{128K^3}u + \frac{5929K_1^2K_2^2K_{11}K_{12}u}{16384K^6} + \frac{31}{64}(K_{11}-4K_{12}+K_{22})u - \frac{5(K_{11}-K_{22})(K_{11}-K_{12}+K_{22})u}{16K^2} + \frac{77\left(K_1^2-K_2^2\right)K_{22}u}{128K^3} - \frac{K_1(44K_{111}-15K_{112}+15K_{122}+6K_{222})u}{128K^2} + \frac{K_2(6K_{111}+15K_{112}-15K_{122}+44K_{222})}{128K^2}u + \frac{4K_{1111}+K_{1112}-K_{1222}-4K_{2222}}{32K}u - \frac{11}{3}K(K_1+K_2) + \frac{17\left(22K_1^3-53K_1^2K_2+53K_1K_2^2-22K_2^3\right)}{384K^2} + \frac{K_1(-1814K_{11}+402K_{12}-2011K_{22})}{1920K} + \frac{K_2(2011K_{11}-402K_{12}+1814K_{22})}{1920K} + \frac{77\left(K_1^2-K_2^2\right)(K_{112}-K_{122})}{128K^3} - \frac{5(K_{11}-K_{22})(K_{112}-K_{122})}{16K^2} - \frac{33}{40}(K_{111}-2K_{112}+2K_{122}-K_{222}) + \frac{K_2(3K_{1112}+19K_{1122}-22K_{1222})}{64K^2} + \frac{K_1(-22K_{1112}+19K_{1122}+3K_{1222})}{64K^2} + \frac{K_{11112}-K_{11122}-K_{11222}+K_{12222}}{8K},$$

$$B_{121} = -\frac{195u^3}{32} + \frac{9(-5K_1+9K_2)u^2}{16K} + \frac{169Ku}{16} + \frac{21\left(26K_1^2-31K_1K_2+9K_2^2\right)u}{64K^2} - \frac{3(65K_{11}-75K_{12}+31K_{22})u}{32K} - \frac{35K_2}{16} + \frac{49\left(2K_1^3-3K_1^2K_2+3K_1K_2^2-K_2^3\right)}{32K^3} + \frac{K_1(-58K_{11}+58K_{12}-37K_{22})}{16K^2} + \frac{K_2(29K_{11}-50K_{12}+29K_{22})}{16K^2} + \frac{6K_{111}-9K_{112}+9K_{122}-3K_{222}}{8K};$$

$$
\begin{aligned}
&B_{122} = \\
&\frac{195u^3}{32} + \frac{9\,(9K_1 - 5K_2)\,u^2}{16K} + \frac{169Ku}{16} - \frac{21\left(9K_1^2 - 31K_1K_2 + 26K_2^2\right)u}{64K^2} \\
&+ \frac{3\,(31K_{11} - 75K_{12} + 65K_{22})\,u}{32K} + \frac{35K_1}{16} - \frac{49\left(K_1^3 - 3K_1^2K_2 + 3K_1K_2^2 - 2K_2^3\right)}{32K^3} \\
&- \frac{K_2\,(37K_{11} - 58K_{12} + 58K_{22})}{16K^2} + \frac{K_1\,(29K_{11} - 50K_{12} + 29K_{22})}{16K^2} \\
&- \frac{3\,(K_{111} - 3K_{112} + 3K_{122} - 2K_{222})}{8K};
\end{aligned}
$$

$$
\begin{aligned}
&B_{120} = \\
&\frac{15u^5}{64} + \frac{117\,(-K_1 + K_2)\,u^4}{128K} - \frac{21\left(K_1^2 - K_1K_2 + K_2^2\right)u^3}{64K^2} \\
&+ \frac{65}{8}\,(K_1 + K_2)\,u^2 + \frac{273\,(K_1 - K_2)\,(K_{11} - K_{12} + K_{22})\,u^2}{128K^2} \\
&- \frac{3\,(26K_{111} - 47K_{112} + 47K_{122} - 26K_{222})\,u^2}{64K} + \frac{33K^2u}{8} \\
&+ \frac{221\left(K_1^2 - K_2^2\right)u}{64K} + \frac{351}{64}\,(K_{11} - K_{22})\,u - \frac{23\,(K_{11} - K_{12} + K_{22})^2\,u}{64K^2} \\
&- \frac{7^2\left(K_1^2\,(K_{12} - K_{22}) - K_2^2\,(K_{11} + K_{22}) + K_1K_2\,(K_{11} - K_{12} + K_{22})\right)}{2^6K^3}u \\
&- \frac{7^4K_1^2K_2^2K_{11}K_{12}}{2^{12}K^6}u - \frac{7K_1\,(8K_{111} - 51K_{112} + 51K_{122} - 4K_{222})}{128K^2}u \\
&- \frac{7K_2\,(-4K_{111} + 51K_{112} - 51K_{122} + 8K_{222})}{128K^2}u \\
&+ \frac{4K_{1111} - 47K_{1112} + 90K_{1122} - 47K_{1222} + 4K_{2222}}{32K}u \\
&+ \frac{119\left(2K_1^3 - 3K_1^2K_2 - 3K_1K_2^2 + 2K_2^3\right)}{192K^2} + \frac{191K_1\,(-2K_{11} + 2K_{12} + K_{22})}{960K} \\
&+ \frac{191K_2\,(K_{11} + 2K_{12} - 2K_{22})}{960K} + \frac{49\left(K_1^2 - K_1K_2 + K_2^2\right)(K_{112} - K_{122})}{64K^3} \\
&- \frac{23\,(K_{11} - K_{12} + K_{22})\,(K_{112} - K_{122})}{64K^2} - \frac{33}{80}\,(2K_{111} - 3K_{112} - 3K_{122} + 2K_{222}) \\
&- \frac{7\,(K_1\,(2K_{1112} - 3K_{1122} + K_{1222})}{32K^2} + \frac{K_2\,(K_{1112} - 3K_{1122} + 2K_{1222})}{32K^2} \\
&+ \frac{K_{11112} - 2K_{11122} + 2K_{11222} - K_{12222}}{8K}.
\end{aligned}
$$

10.3 The Polynomials V_i and V_{ij}

Here we give the expressions of the polynomials $V_0,\ V_1,\ V_2,\ V_{11},\ V_{22}$, and V_{12} (see Section 8) in terms of $A_{ij},\ A_{ijk},\ B_{ijk}$, the curvature K and its first covariant derivatives K_1 and K_2.

$V_0 =$
$-12\left(K_1-2K_2\right)\left(3K_1-K_2\right)A_{21}B_{122}$
$+12\left(2K_1^2-7K_1K_2+3K_2^2\right)\left(A_{11}B_{122}+A_{22}B_{121}-A_{12}B_{121}\right)$
$+39u\left(11K_1^2-26K_1K_2+11K_2^2\right)\left(A_{11}A_{22}-A_{12}A_{21}\right)$
$+16K\left(-2K_1+K_2\right)\left(A_{111}B_{122}-A_{112}B_{121}\right)$
$+16K\left(K_1-2K_2\right)\left(A_{222}B_{121}-A_{221}B_{122}\right)$
$+208Ku\left(K_1-2K_2\right)\left(A_{21}A_{222}-A_{22}A_{221}\right)$
$+208Ku\left(2K_1-K_2\right)\left(A_{11}A_{112}-A_{12}A_{111}\right)$
$+52Ku\left(K_1+K_2\right)\left(A_{11}A_{222}-A_{12}A_{221}-A_{21}A_{112}+A_{22}A_{111}\right)$
$+208K^2u\left(A_{111}A_{222}-A_{112}A_{221}\right);$

$V_1 =$
$12\left(3K_1^2-7K_1K_2+2K_2^2\right)\left(A_{20}B_{122}-A_{22}B_{120}\right)$
$+12\left(-2K_1^2+7K_1K_2-3K_2^2\right)\left(A_{10}B_{122}-A_{12}B_{120}\right)$
$-39u\left(11K_1^2-26K_1K_2+11K_2^2\right)\left(A_{10}A_{22}-A_{12}A_{20}\right)$
$+16K\left(K_1-2K_2\right)\left(A_{220}B_{122}-A_{222}B_{120}\right)$
$+16K\left(2K_1-K_2\right)\left(A_{110}B_{122}-A_{112}B_{120}\right)$
$-208Ku\left(K_1-2K_2\right)\left(A_{20}A_{222}-A_{22}A_{220}\right)$
$-52Ku\left(K_1+K_2\right)\left(A_{10}A_{222}-A_{12}A_{220}-A_{20}A_{112}+A_{22}A_{110}\right)$
$-208Ku\left(2K_1-K_2\right)\left(A_{10}A_{112}-A_{12}A_{110}\right)$
$-208K^2u\left(A_{110}A_{222}-A_{112}A_{220}\right);$

$V_2 =$
$-12\left(3K_1^2-7K_1K_2+2K_2^2\right)\left(A_{20}B_{121}-A_{21}B_{120}\right)$
$-12\left(2K_1^2-7K_1K_2+3K_2^2\right)\left(A_{11}B_{120}-A_{10}B_{121}\right)$
$+39u\left(11K_1^2-26K_1K_2+11K_2^2\right)\left(A_{10}A_{21}-A_{11}A_{20}\right)$
$-16K\left(K_1-2K_2\right)\left(A_{220}B_{121}-A_{221}B_{120}\right)$
$-16K\left(2K_1-K_2\right)\left(A_{110}B_{121}-A_{111}B_{120}\right)$
$+208Ku\left(K_1-2K_2\right)\left(A_{20}A_{221}-A_{21}A_{220}\right)$
$+52Ku\left(K_1+K_2\right)\left(A_{10}A_{221}-A_{11}A_{220}-A_{20}A_{111}+A_{21}A_{110}\right)$
$+208Ku\left(2K_1-K_2\right)\left(A_{10}A_{111}-A_{11}A_{110}\right)$
$+208K^2u\left(A_{110}A_{221}-A_{111}A_{220}\right);$

$$
\begin{aligned}
V_{11} = \\
&-\left(52K_1^2 - 124K_1K_2 + 64K_2^2\right)\left(A_{10}A_{21}B_{122} - A_{11}A_{20}B_{122} - A_{10}A_{22}B_{121}\right)\\
&-\left(52K_1^2 - 124K_1K_2 + 64K_2^2\right)\left(A_{12}A_{20}B_{121} + A_{11}A_{22}B_{120} - A_{12}A_{21}B_{120}\right)\\
&+\frac{16}{3}K\left(K_1 - 8K_2\right)\left(A_{10}A_{221}B_{122} - A_{11}A_{220}B_{122} - A_{10}A_{222}B_{121}\right)\\
&+\frac{16}{3}K\left(K_1 - 8K_2\right)\left(A_{12}A_{220}B_{121} + A_{11}A_{222}B_{120} - A_{12}A_{221}B_{120}\right)\\
&+\frac{32}{3}K\left(K_1 + K_2\right)\left(A_{20}A_{111}B_{122} - A_{21}A_{110}B_{122} - A_{20}A_{112}B_{121}\right)\\
&+\frac{32}{3}K\left(K_1 + K_2\right)\left(A_{22}A_{110}B_{121} + A_{21}A_{112}B_{120} - A_{22}A_{111}B_{120}\right)\\
&-\frac{80}{3}K\left(2K_1 - K_2\right)\left(A_{10}A_{111}B_{122} - A_{11}A_{110}B_{122} - A_{10}A_{112}B_{121}\right)\\
&-\frac{80}{3}K\left(2K_1 - K_2\right)\left(+A_{12}A_{110}B_{121} + A_{11}A_{112}B_{120} - A_{12}A_{111}B_{120}\right)\\
&+\frac{128}{3}K\left(K_1 - 2K_2\right)\left(A_{20}A_{222}B_{121} - A_{22}A_{220}B_{121} - A_{21}A_{222}B_{120}\right)\\
&-\frac{128}{3}K\left(K_1 - 2K_2\right)\left(A_{20}A_{221}B_{122} - A_{21}A_{220}B_{122} - A_{22}A_{221}B_{120}\right)\\
&-208Ku\left(K_1 - 2K_2\right)\left(A_{10}A_{21}A_{222} - A_{11}A_{20}A_{222} - A_{10}A_{22}A_{221}\right)\\
&-208Ku\left(K_1 - 2K_2\right)\left(A_{12}A_{20}A_{221} + A_{11}A_{22}A_{220} - A_{12}A_{21}A_{220}\right)\\
&+52Ku\left(K_1 + K_2\right)\left(A_{10}A_{21}A_{112} - A_{11}A_{20}A_{112} - A_{10}A_{22}A_{111}\right)\\
&+52Ku\left(K_1 + K_2\right)\left(+A_{12}A_{20}A_{111} + A_{11}A_{22}A_{110} - A_{12}A_{21}A_{110}\right)\\
&-\frac{128K^2}{3}\left(A_{110}A_{221}B_{122} - A_{111}A_{220}B_{122} - A_{110}A_{222}B_{121}\right)\\
&-\frac{128K^2}{3}(A_{112}A_{220}B_{121} + A_{111}A_{222}B_{120} - A_{112}A_{221}B_{120})\\
&-208K^2u(-A_{12}A_{111}A_{220} + A_{11}A_{112}A_{220} + A_{12}A_{110}A_{221})\\
&+208K^2u(A_{10}A_{112}A_{221} + A_{11}A_{110}A_{222} - A_{10}A_{111}A_{222});
\end{aligned}
$$

$$
\begin{aligned}
&V_{22} = \\
&-\left(64K_1^2 - 124K_1K_2 + 52K_2^2\right)\left(A_{10}A_{21}B_{122} - A_{11}A_{20}B_{122} - A_{10}A_{22}B_{121}\right) \\
&-\left(64K_1^2 - 124K_1K_2 + 52K_2^2\right)\left(A_{12}A_{20}B_{121} + A_{11}A_{22}B_{120} - A_{12}A_{21}B_{120}\right) \\
&-\frac{80}{3}K\left(K_1 - 2K_2\right)\left(A_{20}A_{221}B_{122} - A_{21}A_{220}B_{122} - A_{20}A_{222}B_{121}\right) \\
&-\frac{80}{3}K\left(K_1 - 2K_2\right)\left(A_{22}A_{220}B_{121} + A_{21}A_{222}B_{120} - A_{22}A_{221}B_{120}\right) \\
&-\frac{80}{3}K\left(K_1 - 2K_2\right)\left(A_{12}A_{220}B_{121} + A_{11}A_{222}B_{120} - A_{12}A_{221}B_{120}\right) \\
&-\frac{32}{3}K\left(K_1 + K_2\right)\left(A_{10}A_{221}B_{122} - A_{11}A_{220}B_{122} - A_{10}A_{222}B_{121}\right) \\
&+\frac{16}{3}K\left(8K_1 - K_2\right)\left(A_{20}A_{111}B_{122} - A_{21}A_{110}B_{122} - A_{20}A_{112}B_{121}\right) \\
&+\frac{16}{3}K\left(8K_1 - K_2\right)\left(+A_{22}A_{110}B_{121} + A_{21}A_{112}B_{120} - A_{22}A_{111}B_{120}\right) \\
&-\frac{128}{3}K\left(2K_1 - K_2\right)\left(A_{10}A_{111}B_{122} - A_{11}A_{110}B_{122} - A_{10}A_{112}B_{121}\right) \\
&-\frac{128}{3}K\left(2K_1 - K_2\right)\left(A_{12}A_{110}B_{121} + A_{11}A_{112}B_{120} - A_{12}A_{111}B_{120}\right) \\
&+52Ku\left(K_1 + K_2\right)\left(A_{10}A_{21}A_{222} - A_{11}A_{20}A_{222} - A_{10}A_{22}A_{221}\right) \\
&+52Ku\left(K_1 + K_2\right)\left(A_{12}A_{20}A_{221} + A_{11}A_{22}A_{220} - A_{12}A_{21}A_{220}\right) \\
&+208Ku\left(2K_1 - K_2\right)\left(A_{10}A_{21}A_{112} - A_{11}A_{20}A_{112} - A_{10}A_{22}A_{111}\right) \\
&+208Ku\left(2K_1 - K_2\right)\left(A_{12}A_{20}A_{111} + A_{11}A_{22}A_{110} - A_{12}A_{21}A_{110}\right) \\
&-\frac{128K^2}{3}\left(A_{110}A_{221}B_{122} - A_{111}A_{220}B_{122} - A_{110}A_{222}B_{121}\right) \\
&-\frac{128K^2}{3}(A_{112}A_{220}B_{121} + A_{111}A_{222}B_{120} - A_{112}A_{221}B_{120}) \\
&+208K^2u(A_{22}A_{111}A_{220} - A_{21}A_{112}A_{220} - A_{22}A_{110}A_{221}) \\
&+208K^2u(+A_{20}A_{112}A_{221} + A_{21}A_{110}A_{222} - A_{20}A_{111}A_{222});
\end{aligned}
$$

$$
\begin{aligned}
&V_{12} = \\
&-\left(44K_1^2 - 104K_1K_2 + 44K_2^2\right)\left(A_{10}A_{21}B_{122} - A_{11}A_{20}B_{122} - A_{10}A_{22}B_{121}\right) \\
&-\left(44K_1^2 - 104K_1K_2 + 44K_2^2\right)\left(A_{12}A_{20}B_{121} + A_{11}A_{22}B_{120} - A_{12}A_{21}B_{120}\right) \\
&-\frac{64}{3}K\left(K_1 - 2K_2\right)\left(A_{20}A_{221}B_{122} - A_{21}A_{220}B_{122} - A_{20}A_{222}B_{121}\right) \\
&-\frac{64}{3}K\left(K_1 - 2K_2\right)\left(A_{22}A_{220}B_{121} + A_{21}A_{222}B_{120} - A_{22}A_{221}B_{120}\right) \\
&-\frac{64}{3}K\left(2K_1 - K_2\right)\left(A_{10}A_{111}B_{122} - A_{11}A_{110}B_{122} - A_{10}A_{112}B_{121}\right) \\
&-\frac{64}{3}K\left(2K_1 - K_2\right)\left(A_{12}A_{110}B_{121} + A_{11}A_{112}B_{120} - A_{12}A_{111}B_{120}\right) \\
&-\frac{16}{3}K\left(K_1 + K_2\right)\left(A_{10}A_{221}B_{122} - A_{11}A_{220}B_{122} - A_{20}A_{111}B_{122}\right) \\
&-\frac{16}{3}K\left(K_1 + K_2\right)\left(A_{21}A_{110}B_{122} - A_{10}A_{222}B_{121} + A_{12}A_{220}B_{121}\right) \\
&-\frac{16}{3}K\left(K_1 + K_2\right)\left(A_{20}A_{112}B_{121} - A_{22}A_{110}B_{121} + A_{11}A_{222}B_{120}\right) \\
&+\frac{16}{3}K\left(K_1 + K_2\right)\left(A_{12}A_{221}B_{120} + A_{21}A_{112}B_{120} - A_{22}A_{111}B_{120}\right) \\
&+\frac{64}{3}K^2(A_{112}A_{221}B_{120} - A_{111}A_{222}B_{120} - A_{112}A_{220}B_{121}) \\
&+\frac{64}{3}K^2(+A_{110}A_{222}B_{121} + A_{111}A_{220}B_{122} - A_{110}A_{221}B_{122}).
\end{aligned}
$$

10.4 Total Covariant Derivatives

$$
\begin{aligned}
&\delta_1 = \\
&p_1\frac{\partial}{\partial u} + p_{11}\frac{\partial}{\partial p_1} + \left(-\frac{Ku}{2} + p_{12}\right)\frac{\partial}{\partial p_2} + p_{111}\frac{\partial}{\partial p_{11}} \\
&+\left(p_{112} - \frac{uK_1}{6} - \frac{5Kp_1}{6}\right)\frac{\partial}{\partial p_{12}} + \left(p_{122} - \frac{uK_2}{3} - \frac{5Kp_2}{3}\right)\frac{\partial}{\partial p_{22}} \\
&+p_{1111}\frac{\partial}{\partial p_{111}} + \left(p_{1112} - \frac{uK_{11}}{12} - \frac{K_1p_1}{2} - \frac{7Kp_{11}}{6}\right)\frac{\partial}{\partial p_{112}} \\
&+\left(p_{1122} - \frac{uK_{12}}{6} - \frac{K_2p_1}{2} - \frac{K_1p_2}{2} - \frac{7Kp_{12}}{3}\right)\frac{\partial}{\partial p_{122}} \\
&+\left(p_{1222} - \frac{uK_{22}}{4} - \frac{3K_2p_2}{2} - \frac{7Kp_{22}}{2}\right)\frac{\partial}{\partial p_{222}};
\end{aligned}
$$

$$
\begin{aligned}
&\delta_2 = \\
&p_2 \frac{\partial}{\partial u} + \left(\frac{Ku}{2} + p_{12}\right)\frac{\partial}{\partial p_1} + p_{22}\frac{\partial}{\partial p_2} + \left(p_{112} + \frac{uK_1}{3} + \frac{5Kp_1}{3}\right)\frac{\partial}{\partial p_{11}} \\
&+ \left(p_{122} + \frac{uK_2}{6} + \frac{5Kp_2}{6}\right)\frac{\partial}{\partial p_{12}} + p_{222}\frac{\partial}{\partial p_{22}} + \\
&\left(p_{1112} + \frac{uK_{11}}{4} + \frac{3K_1p_1}{2} + \frac{7Kp_{11}}{2}\right)\frac{\partial}{\partial p_{111}} \\
&+ \left(p_{1122} + \frac{uK_{12}}{6} + \frac{K_2p_1}{2} + \frac{K_1p_2}{2} + \frac{7Kp_{12}}{3}\right)\frac{\partial}{\partial p_{112}} \\
&+ \left(p_{1222} + \frac{uK_{22}}{12} + \frac{K_2p_2}{2} + \frac{7Kp_{22}}{6}\right)\frac{\partial}{\partial p_{122}} + p_{2222}\frac{\partial}{\partial p_{222}};
\end{aligned}
$$

$$
\begin{aligned}
&\delta_1^K = \\
&K_1\frac{\partial}{\partial K} + K_{11}\frac{\partial}{\partial K_1} + \left(-K^2 + K_{12}\right)\frac{\partial}{\partial K_2} + K_{111}\frac{\partial}{\partial K_{11}} \\
&+ \left(K_{112} - \frac{5KK_1}{3}\right)\frac{\partial}{\partial K_{12}} + \left(K_{122} - \frac{10KK_2}{3}\right)\frac{\partial}{\partial K_{22}} \\
&+ K_{1111}\frac{\partial}{\partial K_{111}} + \left(-\frac{5K_1^2}{6} - \frac{11KK_{11}}{6} + K_{1112}\right)\frac{\partial}{\partial K_{112}} \\
&+ \left(K_{1122} - \frac{5}{3}K_1K_2 - \frac{11KK_{12}}{3}\right)\frac{\partial}{\partial K_{122}} + \left(K_{1222} - \frac{5K_2^2}{2} - \frac{11KK_{22}}{2}\right)\frac{\partial}{\partial K_{222}} \\
&+ K_{11111}\frac{\partial}{\partial K_{1111}} + \left(K_{11112} - \frac{21K_1K_{11}}{10} - \frac{21KK_{111}}{10}\right)\frac{\partial}{\partial K_{1112}} \\
&+ \left(K_{11122} - \frac{7K_2K_{11}}{5} - \frac{14K_1K_{12}}{5} - \frac{21KK_{112}}{5}\right)\frac{\partial}{\partial K_{1122}} \\
&+ \left(K_{11222} - \frac{21K_2K_{12}}{5} - \frac{21K_1K_{22}}{10} - \frac{63KK_{122}}{10}\right)\frac{\partial}{\partial K_{1222}} \\
&+ \left(K_{12222} - \frac{42K_2K_{22}}{5} - \frac{42KK_{222}}{5}\right)\frac{\partial}{\partial K_{2222}} + K_{111111}\frac{\partial}{\partial K_{11111}} \\
&+ \left(K_{111112} - \frac{7K_{11}^2}{5} - \frac{14K_1K_{111}}{5} - \frac{12KK_{1111}}{5}\right)\frac{\partial}{\partial K_{11112}} \\
&+ \left(K_{111122} - \frac{14K_{11}K_{12}}{5} - \frac{7K_2K_{111}}{5} - \frac{21K_1K_{112}}{5} - \frac{24KK_{1112}}{5}\right)\frac{\partial}{\partial K_{11122}} \\
&+ \left(K_{111222} - \frac{14K_{12}^2}{5} - \frac{7K_{11}K_{22}}{5} - \frac{21K_2K_{112}}{5} - \frac{21K_1K_{122}}{5} - \frac{36KK_{1122}}{5}\right)\frac{\partial}{\partial K_{11222}} \\
&+ \left(K_{112222} - \frac{28K_{12}K_{22}}{5} - \frac{42K_2K_{122}}{5} - \frac{14K_1K_{222}}{5} - \frac{48KK_{1222}}{5}\right)\frac{\partial}{\partial K_{12222}} \\
&+ \left(K_{122222} - 7K_{22}^2 - 14K_2K_{222} - 12KK_{2222}\right)\frac{\partial}{\partial K_{22222}};
\end{aligned}
$$

$$
\begin{aligned}
&\delta_2^K = \\
&K_2\frac{\partial}{\partial K} + \left(K_{12}+K^2\right)\frac{\partial}{\partial K_1} + K_{22}\frac{\partial}{\partial K_2} + \left(K_{112}+\frac{10KK_1}{3}\right)\frac{\partial}{\partial K_{11}} \\
&+\left(K_{122}+\frac{5KK_2}{3}\right)\frac{\partial}{\partial K_{12}} + \left(K_{1112}+\frac{5K_1^2}{2}+\frac{11KK_{11}}{2}\right)\frac{\partial}{\partial K_{111}} \\
&+K_{222}\frac{\partial}{\partial K_{22}} + \left(K_{1122}+\frac{5K_1K_2}{3}+\frac{11KK_{12}}{3}\right)\frac{\partial}{\partial K_{112}} + K_{2222}\frac{\partial}{\partial K_{222}} \\
&+\left(K_{1222}+\frac{5K_2^2}{6}+\frac{11KK_{22}}{6}\right)\frac{\partial}{\partial K_{122}} + K_{22222}\frac{\partial}{\partial K_{2222}} \\
&+\left(K_{12222}+\frac{21K_2K_{22}}{10}+\frac{21KK_{222}}{10}\right)\frac{\partial}{\partial K_{1222}} \\
&+\left(K_{11222}+\frac{14K_2K_{12}}{5}+\frac{7K_1K_{22}}{5}+\frac{21KK_{122}}{5}\right)\frac{\partial}{\partial K_{1122}} \\
&+\left(K_{11122}+\frac{21K_2K_{11}}{10}+\frac{21K_1K_{12}}{5}+\frac{63KK_{112}}{10}\right)\frac{\partial}{\partial K_{1112}} \\
&+\left(K_{11112}+\frac{42K_1K_{11}}{5}+\frac{42KK_{111}}{5}\right)\frac{\partial}{\partial K_{1111}} \\
&+K_{222222}\frac{\partial}{\partial K_{22222}} + \left(K_{122222}+\frac{7K_{22}^2}{5}+\frac{14K_2K_{222}}{5}+\frac{12KK_{2222}}{5}\right)\frac{\partial}{\partial K_{12222}} \\
&+\left(K_{112222}+\frac{14K_{12}K_{22}}{5}+\frac{21K_2K_{122}}{5}+\frac{7K_1K_{222}}{5}+\frac{24KK_{1222}}{5}\right)\frac{\partial}{\partial K_{11222}} \\
&+\left(K_{111222}+\frac{14K_{12}^2}{5}+\frac{7K_{11}K_{22}}{5}+\frac{21K_2K_{112}}{5}+\frac{21K_1K_{122}}{5}+\frac{36KK_{1122}}{5}\right)\frac{\partial}{\partial K_{11122}} \\
&+\left(K_{111122}+\frac{28K_{11}K_{12}}{5}+\frac{14K_2K_{111}}{5}+\frac{42K_1K_{112}}{5}+\frac{48KK_{1112}}{5}\right)\frac{\partial}{\partial K_{11112}} \\
&+\left(K_{111112}+7K_{11}^2+14K_1K_{111}+12KK_{1111}\right)\frac{\partial}{\partial K_{11111}}.
\end{aligned}
$$

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[1] In the bibliography we will use the following abbreviations for the review journals: JFM for Jahrbuch für die Fortschritte der Mathematik, MR for Mathematical Reviews, and Zbl for Zentralblatt für Mathematik und ihren Grenzgebiete.

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