Finite Dimensional Dynamics for Kolmogorov-Petrovsky-Piskunov Equation

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December 25, 2005

Abstract

We construct new finite-dimsnional submanifolds in the solutiuon space of Kolmogorov-Petrovsky-Piskunov equation. We descrie the corresponding evolutionary dynamics and exact solutions.

1 Introduction

In this paper we consider finite dimensional dynamics for the classical Kolmogorov-Petrovsky-Piskunov equation (or a non-linear reaction-diffusion equation)

$$u_t = u_{xx} + f(u), \tag{1}$$

which first appeared in the context of genetics model for the spread of an advantageous gene through a population ([10]). It has been applied since to a number of biological and chemical models.

Usually one requires a special form of the non-linearity: $f(0) = f(1) = 0$, $f(u) > 0$ for $0 < u < 1$ (and a condition like $f''(u) < 0$). There are different constraints for other types of reaction-diffusion equation. We will not restrict to a special form of $f(u)$, but note that the above conditions are satisfied for a certain range of parameters of our solutions.

Special attention has been devoted to the convergence to the travelling waves and the stability of these waves ([2], [7],[14]). Such solutions naturally appear with our approach.

It is not much known about existence of entire or meromorphic families of solutions of the Kolmogorov-Petrovsky-Piskunov (KPP) equation for general $f(u)$ (we will be interested in non-linear function $f(u)$ from (1)).

In addition to the travelling waves and x-independent solutions there are only few examples of finite-dimensional submanifolds in the solutions space ([5]).

For some particular $f(u)$ exact solutions of the KPP equation were obtained via Painleve expansion method ([1], [6]), bi-linear method ([9]), symmetry methods ([3]) and others ([15]).

We find finite dimensional dynamics for the Kolmogorov-Petrovsky-Piskunov equation by the method developed in [13]. We investigate 1, 2 and 3-dimensional dynamics and this allows us to find new classes of solutions.

Indeed, the solutions are identified with the trajectories within these dynamics (so that we essentially find 0, 1 and 2-dimensional spaces of solutions, if we fix parameters), so they are obtained via integration of a pair of ODEs. Moreover the dynamical approach allows to understand which solutions are stable or attracting within the considered family.

2 The Method

Finite dimensional dynamics for evolutionary equations

$$u_t = \varphi\left(x, u, u_x, ..., \frac{\partial^m u}{\partial x^m}\right) \tag{2}$$

are finite dimensional submanifolds in the space of functions $u(x)$, on which equation (2) defines a dynamical system.

These submanifolds can be described as spaces of solutions of ODE

$$g(x, y, y', y'', ..., y^{(n-1)}, y^{(n)}) = 0 \tag{3}$$

with function φ being a symmetry. Here $y = u(t, \cdot)$ with "frozen" dependent coordinate t. This gives an n-dimensional dynamics in equation (2).

Let J^k be the space of k-jets of functions $y = y(x)$ with canonical coordinates $(x, p_0, ..., p_{k-1}, p_k)$, see [11]. ODE (3) corresponds in the jet-space to the surface

$$g(x, p_0, \dots, p_{n-1}, p_n) = 0$$

and a function $\varphi(x, p_0, ..., p_{m-1}, p_m)$ is a generating function of a symmetry iff φ satisfies the following equation

$$X_\varphi(g) = 0 \tag{4}$$

on the $(m-1)^{\text{st}}$ prolongation $\mathcal{E}_n^{(m-1)} \subset J^{m+n-1}$ of the equation $\{g = 0\}$. Here the symmetry vector field is

$$X_\varphi = \bar{\varphi}\,\partial_{p_0} + \mathcal{D}(\bar{\varphi})\partial_{p_1} + ... + \mathcal{D}^m(\bar{\varphi})\partial_{p_m},$$

where we denote

$$\mathcal{D} = \frac{\partial}{\partial x} + p_1 \frac{\partial}{\partial p_0} + \cdots + p_m \frac{\partial}{\partial p_{m-1}} + \cdots$$

the total derivative and $\bar{\varphi}$ the restriction of φ to the prolonged equation (3). In terms of the total derivative the prolongation can be written as

$$\mathcal{E}_n^{(k)} = \{g = 0, \mathcal{D}g = 0, \dots, \mathcal{D}^{(k)} g = 0\}.$$

We call function $g(x, p_0, ..., p_{n-1}, p_n)$ *dynamics of order* n for evolutionary equation (2) if g satisfies ODE (4) for the function φ. This equivalently means that equations $u_t = \varphi$ and $g = 0$ are compatible. The compatibility condition can be written via the bracket criterion of [12] and this yields (4) again.

Thus finite-dimensional dynamics can be obtained by the integration of equation (4) with respect to the function g. Since in our case (1) the function $\varphi = p_2 + f(p_0)$ does not involve x explicitly, it is natural to look for a dynamics not involving x explicitly (see Sec. 6 for justification), i.e. $g = g(p_0, \ldots, p_n)$.

In this paper we find solutions of equation (4) for functions g of order ≤ 3 that have polynomial form with respect to some group of variables. This reduces the problem to an ODE system for the coefficients of the polynomials. We will omit the straightforward analysis of these systems and just formulate the results.

3 Dynamics of the first order

We start with the case, leading to the known solutions, but which simply demonstrates our method.

Theorem 1 *One-dimsnional dynamics $g(p_{0,}p_1)$ of the KPP equation is: Either $g(p_0, p_1) = p_1$ with $f(p_0)$ arbitrary or*

$$g(p_0, p_1) = p_1 - a(p_0) \text{ with } f(p_0) = ka(p_0) - a'(p_0)a(p_0),$$

where $a(p_0)$ is an arbitrary smooth function and $k \in \mathbb{R}$.

Remark 1 *Linear in p_1 dynamics $g_1, \ldots, g_k$ produce dynamics $g = g_1 \cdots g_k$ of order 1 and degree k and locally, outside singularities, all algebraic in p_1 dynamics are such. However equivalence problem can be more complicated: Note that equations $g(p_0, p_1)$ of order k in p_1, which have k different real roots, can be geometrically interpreted as a k-webs in the plane (x, p_0), i.e. k different foliations. The KPP equation represents a symmetry of this k-web.*

If $g(p_0, p_1) = p_1$, then we get x-independent solutions.

In the second case we assume that $a(p_0) \geq 0$. Then $\bar{\varphi} = p_2 + f(p_0)|_{\{g=0\}} = ka(p_0)$. The corresponding dynamics is described by the vector field

$$X_\varphi = ka(p_0)\frac{\partial}{\partial p_0} + kp_1 a'(p_0)\frac{\partial}{\partial p_1}$$

on $\{g = 0\}$. To integrate the vector field we introduce the following function:

$$B(p_0) = \int \frac{dp_0}{a(p_0)}.$$

Then $X_\varphi(B) = k$ and a trajectory $p_0(t)$ can be found from the relation

$$B(p_0(t)) = kt + B(p_0(0)).$$

Since $x = B(y)$ is the inverse function to the solution $y = y(x)$ of the equation $y' = a(p_0)$, we have $p_0(t) = B^{-1}(kt + x)$. Applying this formula to a solution $u_0(x)$ of the equation $\{g = 0\}$ we get the travelling wave solution of the Kolmogorov-Petrovsky-Piskunov equation

$$u(x,t) = B^{-1}(kt + x), \quad \text{with } u(x,0) = u_0(x) = B^{-1}(x).$$

Example 1 *The quadratic dynamics $g(p_0, p_1) = p_1^2 - b(p_0)$ is equivalent to a linear dynamics. However particular forms of the function $b(p_0)$ can provide interesting phenomena. Consider, for example,*

$$b(p_0) = ap_0 + c.$$

Then the function $f(p_0)$ has the following form

$$f(p_0) = k\sqrt{ap_0 + c} + \frac{a}{2}.$$

The equation corresponding to $g = 0$ is

$$(y')^2 - ay - c = 0 \tag{5}$$

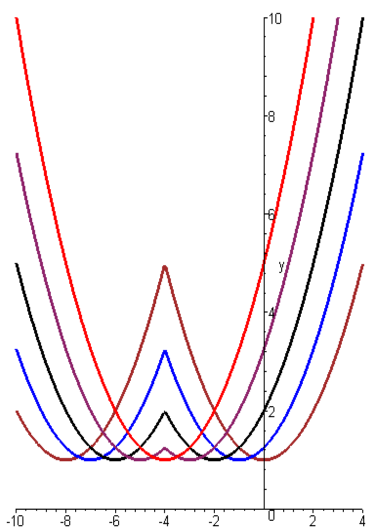
with solutions

$$y(x) = -\frac{c}{a} + \frac{1}{a}\left(\frac{ax}{2} \pm \sqrt{ay_0 + c}\right)^2.$$

Calculating the trajectories of the vector field X_φ we get the solutions:

$$u_\pm(x,t) = -\frac{c}{a} + \frac{a}{4}\left(kt - sign(a) \mid x \pm 2\sqrt{au_0(0) + c}/a \mid\right)^2.$$

These are the travelling waves with a reflection. A typical graph of it is shown on the picture below.



The function $u_+(x, t_0)$ for $a = k = 1 = -c, u_0(0) = 5$ and various t_0.

Notice that there are two solutions of equation (5), corresponding to $y(0) = -c/a$. One is the constant solution $y(x) = -\frac{c}{a}$ and it is isolated and irrelevant for the dynamics. The other is the solution $y(x) = -\frac{c}{a} + \frac{a}{4}x^2$ and the dynamics converts it into the travelling wave.
Here we see the influence of the singularity: the vector field X_φ on $\mathbb{R}^1(p_0)$ has a zero corresponding to $p_0 = -c/a$, while there is no stationary solution of the dynamics. The explanation is that the flow dynamics of the KPP equation enters into the singular point in a finite time and then momentally exits.

4 Dynamics of the second order

If we consider the general case of KPP dynamics of (1) given by the 2nd order ODE $g = p_2 - F(p_0, p_1)$, its defining equation is

$$\left(\nabla^2 - F_{p_1}\nabla - F_{p_0}\right)(F - f) = 0, \text{ where } \nabla = F\partial_{p_1} + p_1\partial_{p_0}.$$

Thus we have plenty of solutions (at least analytic in the domain, where Cauchy-Kovalevskaya theorem holds) for every function $f(p_0)$.

To get more explicit solutions of (1) let us study some special forms of the function $F(p_0, p_1)$. If it does not depend on p_0, then $f(p_0)$ is linear, which is not much an interesting case. Consider the case, when $F(p_0, p_1)$ is linear (we always mean non-homogeneous, i.e. just of degree 1) in p_1:

Theorem 2 *Second order dynamics of the form $g = p_2 + a(p_0)p_1 + b(p_0)$ is either:*

$$g = p_2 - (\alpha p_0 + \beta)p_1 - \frac{1}{2}f(p_0)$$

with $f(p_0) = \frac{\alpha^2}{9}p_0^3 + \frac{\alpha\beta}{3}p_0^2 + \gamma p_0 + \delta$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ arbitrary, or

$$g = p_2 - \beta p_1 + \gamma f(p_0)$$

with $\beta, \gamma \in \mathbb{R}$ arbitrary if $f(p_0)$ is linear and $\gamma = 1$ if $f(p_0)$ is arbitrary.

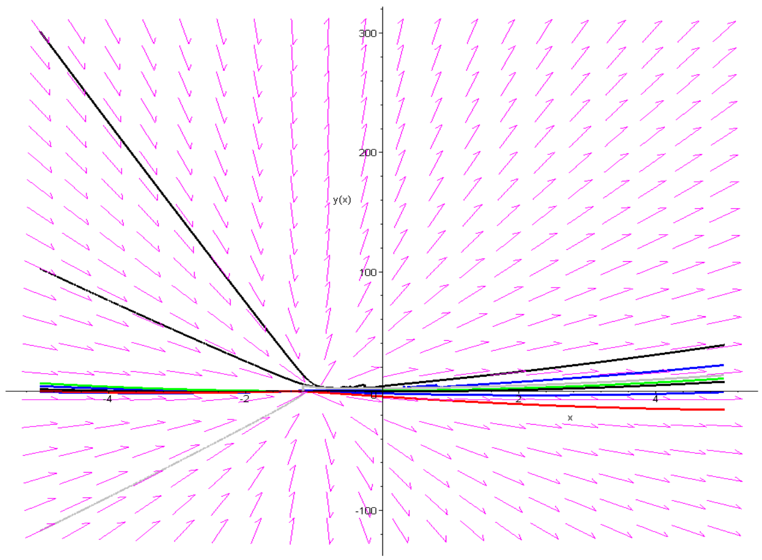
Remark 2 *In the first case (mainly the only non-trivial) the parameters can be chosen to satisfy the requirements for non-linearity in KPP. Note also that for $\alpha \neq 0$ the function $f(u)$ is cubic, as it is in the Fitzhugh-Nagumno equation.*

In the first case $\bar{\varphi} = (\alpha p_0 + \beta)p_1 + \frac{3}{2}(\frac{\alpha^2}{9}p_0^3 + \frac{\alpha\beta}{3}p_0^2 + \gamma p_0 + \delta)$ on $\{g = 0\} \simeq \mathbb{R}^2(p_0, p_1)$. The dynamics is described by the equation

$$\begin{aligned}\dot{p}_0 &= (\alpha p_0 + \beta)p_1 + \frac{3}{2}\left(\frac{\alpha^2}{9}p_0^3 + \frac{\alpha\beta}{3}p_0^2 + \gamma p_0 + \delta\right),\\ \dot{p}_1 &= \alpha p_1^2 + \left(\frac{3}{2}\alpha^2 p_0^2 + 3\alpha\beta p_0 + \beta^2 + \frac{3}{2}\gamma\right)p_1\\ &\qquad + \frac{(\alpha p_0 + \beta)}{2}\left(\frac{\alpha^2}{9}p_0^3 + \frac{\alpha\beta}{3}p_0^2 + \gamma p_0 + \delta\right).\end{aligned}$$

This equation can possess up to 4 critical points and for generic values of parameters they are non-degenerate. One can check that all possible signatures (sourse, saddle and sink) can be realized. In particular, for the parameters with sinks we have a stable solution and so the dynamics (within the considered family) does not converge to travelling waves.

Analysis of this polinomial system of ODEs shows that it has quite complicated phase portrait and we illustrate this on the pictures below.

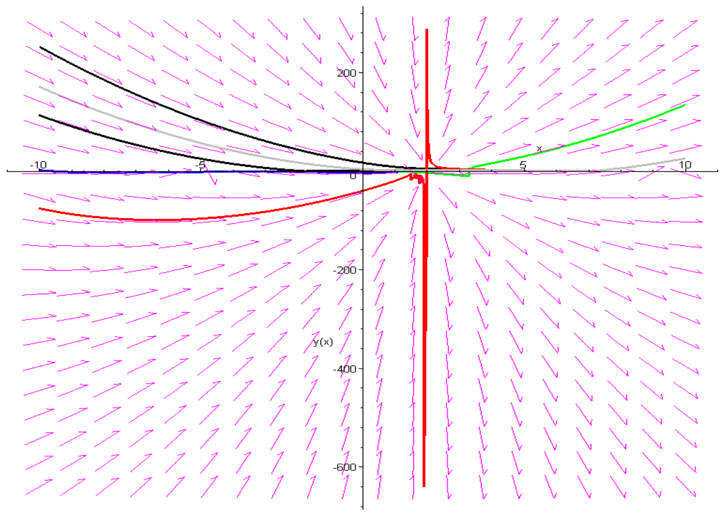


The phase portrait for $\alpha = \beta = \gamma = \delta = 1$.

In the second case $\bar{\varphi} = \beta p_1 + (1-\gamma) f(p_0)$ and so for linear $f(p_0)$ we have a general linear system on the plane $\mathbb{R}^2(p_0, p_1)$. For non-linear $f(p_0)$ we have $\gamma = 1$, so the system becomes ($\beta \neq 1$, otherwise it is trivial)

$$\ddot{p}_0 - \beta^2 \dot{p}_0 + \beta\gamma\, f(p_0) = 0$$

($p_1 = \dot{p}_0/\beta$), which by a substitution $p_0 = v\, e^{\beta^2 x/2}$ reduces to the system $v'' = \Psi(x, v)$ with $\Psi = \frac{1}{4}\beta^4 v - \beta\gamma\, e^{-\beta^2 x/2} f(e^{\beta^2 x/2} v)$.



The picture shows sensitivity to initial conditions.

The equation $g = 0$ has two symmetries: p_1 – shift by x and the above $\bar{\varphi}$. They commute and hence by the Lie-Bianchi theorem ODE $g = 0$ is integrable in quadratures, see [4]. This gives exact solutions of the KPP. Indeed, the first integrals I_1, I_2 in the first case of the above theorem are found by the formula

$$(I_1, I_2) = \int (dp_0, dp_1) \cdot \begin{pmatrix} p_1 & \bar{\varphi} - f \\ \bar{\varphi} & \mathcal{D}_g \bar{\varphi} \end{pmatrix}^{-1},$$

where $\mathcal{D}_g = p_1 \partial_{p_0} + (\bar{\varphi} - f)\partial_{p_1}$ and similarly for the second case.

5 Dynamics of the third order

We will study dynamics g, which is quasi-linear in p_2 and p_3. In any of the dynamics below one can obtain a more general form by substitution $p_0 \mapsto p_0 + c$, where $c = \text{const}$, but we do not write it for the sake of brevity.

Theorem 3 *Third order dynamics of the KPP of the form*

$$g(p_0, p_1, p_2, p_3) = p_3 + p_2 A(p_0, p_1) + B(p_0, p_1)$$

is either

$$g(p_0, p_1, p_2, p_3) = p_3 - \frac{p_2 p_1}{p_0} + a \frac{(p_1^2 - p_2 p_0)}{p_0} - b\, p_1$$

with

$$f(p_0) = c p_0 - b\, p_0 \log p_0,$$

or

$$g(p_0, p_1, p_2, p_3) = p_3 - 3\frac{p_2 p_1}{p_0} + 2\frac{p_1^3}{p_0^2} + a\, p_1$$

with
$$f(p_0) = cp_0 - b\, p_0 \log p_0 + a\, p_0 (\log p_0)^2,$$
or
$$g(p_0, p_1, p_2, p_3) = p_3 - \frac{p_2 p_1}{p_0} + a\frac{(p_1^2 - p_2 p_0)}{p_0} + b\frac{p_1}{p_0}$$
with $f(p_0) = 0$. *Here* $a, b, c \in \mathbb{R}$ *are arbitrary.*

Remark 3 *Note that for* $c = 0$ *and* $b > 0$ *in the first case, or* $b > 2a > 0$ *in the second case, the function* $f(u)$ *is concave for* $u \in (0, 1)$, *i.e. has the form important for biological applications of KPP.*

From now on we will restrict to investigate only the first case from the theorem. The second is similar and the last one is not an interesting case.

For the dynamics from the first case the vector field X_φ restricted to the equation $\{g = 0\} \simeq \mathbb{R}^3(p_0, p_1, p_2)$ has the form

$$X_\varphi = (p_2 + p_0(c - b\log p_0))\, \partial_{p_0} + \left(\frac{p_1(p_2 - ap_1)}{p_0} + ap_2 + p_1(c - b\log p_0) \right) \partial_{p_1}$$
$$+ \left(\frac{p_2^2 - a^2 p_1^2}{p_0} + abp_1 + p_2(a^2 + c - b\log p_0) \right) \partial_{p_2}$$

If $a, b \neq 0$, this vector field has the only singularity at the point

$$p_0 = e^{c/b}, p_1 = p_2 = 0.$$

For $a = 0$, there is a plane if singularities, corresponding to KPP:

$$p_2 + p_0(c - b\log p_0) = 0.$$

In the case $b = 0, a \neq 0$ we have two lines of singular points for $c < 0$ and one for $c = 0$:
$$p_1 = \pm\sqrt{-c}p_0, p_2 = ap_1 - (c \pm a\sqrt{-c})p_0.$$

At the singular point $(p_0 = e^{c/b}, p_1 = p_2 = 0)$ the linear part of X_φ has the following spectrum

$$\lambda_1 = -b,$$
$$\lambda_2 = (a^2 - a\sqrt{a^2 + 4b})/2,$$
$$\lambda_3 = (a^2 + a\sqrt{a^2 + 4b})/2.$$

This singular point is hyperbolic for $a \neq 0$ and $b > 0$ with signature $(--+)$ and it is elliptic repelling (source) for $a \neq 0$ and $b < 0$. For $a = 0$ or $b = 0$ the singular points are all degenerate.

In order to find trajectories of X_φ let us introduce new coordinates

$$u = \log p_0,$$
$$v = \frac{p_1}{p_0},$$
$$w = \frac{p_2 - ap_1}{p_0}.$$

Then the system for trajectories is the following

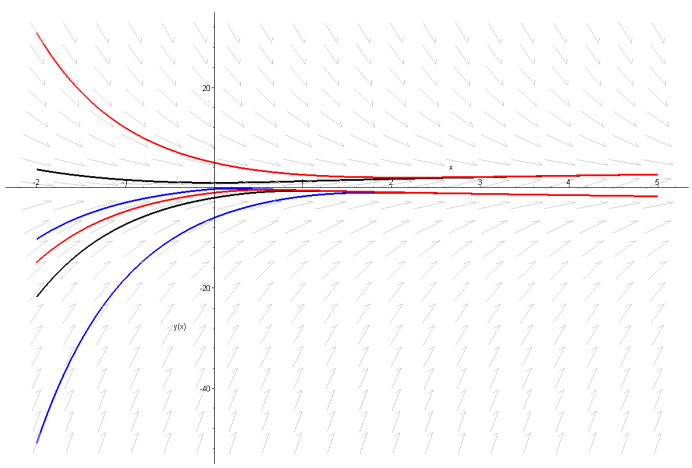
$$\dot{u} = c + w + av - bu, \tag{6}$$
$$\dot{v} = a(w + av - v^2), \tag{7}$$
$$\dot{w} = abv. \tag{8}$$

From this system we get

$$\ddot{w} = a^2bw + a^2\dot{w} - (\dot{w})^2/b$$

and
$$\frac{dv}{dw} = \frac{1}{b}\left(\frac{w}{v(w)} + a - v(w)\right). \tag{9}$$

This equation describes the phase portrait of the system (7-8) in the non-degenerate case $b \neq 0$ (for $b = 0$ we get 1-dimensional logistic equation). We picture it below.



A typical phase portrait of the system (7-8).

Equation (9) is an Abel's ODE of the second kind (class A [8, p.26]). It has exact formula for solutions in the case $a = 0$

$$v(w) = \pm\sqrt{w - \frac{b}{2} + C_0 e^{-2w/b}},$$

but generally it is not integrable in quadratures. We however can desribe the behaviour of the solutions of this equation for arbitrary a.

Namely, in the left half-plane for the "time" $w \to -\infty$ the integral curves exponentially diverge, so that we have hyperbolic non-stability (this easily follows from the form of ODE). On the contrary, in the right half-plane when "time" $w \to +\infty$ we have the following stability property: All the integral curves asymptotically approach the curve $v_0(w) = \frac{a}{2} \pm \sqrt{w + \frac{a^2}{4}}$ (we assume without loss of generality that $a, b > 0$, because for $b < 0$ we need to reverse "time" w – see below, while for $a < 0$ it suffices to reflect the plane with respect to the line $\{w = 0\}$).

To demonstrate this last claim, we formulate it more formally in the first quadrant of the plane $\mathbb{R}^2(w, v)$: For every $\varepsilon > 0$ the integral curve eventually enters the strip between the curves

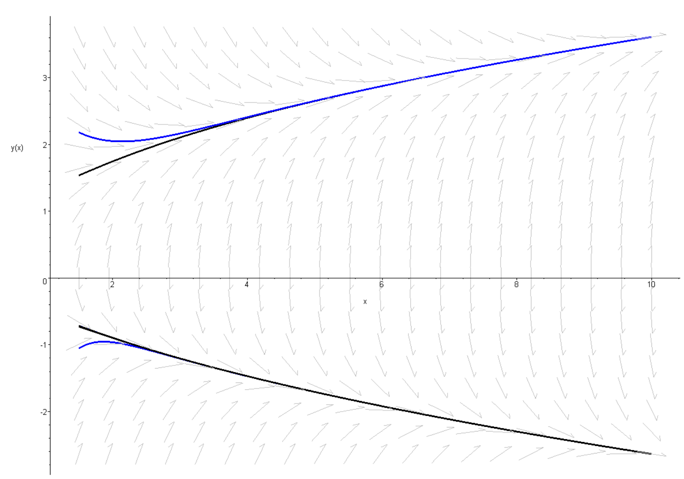
$$v_0(w) = \frac{a}{2} + \sqrt{w + \frac{a^2}{4}} \text{ and } v_\varepsilon(w) = \frac{a-\varepsilon}{2} + \sqrt{w + \frac{(a-\varepsilon)^2}{4}}.$$

Indeed, the vector field

$$\xi = b\frac{\partial}{\partial w} + \left(\frac{w}{v} + a - v\right)\frac{\partial}{\partial v}$$

along the upper boundary $v = v_0(w)$ is horizontal, $\xi = (b, 0)$, while the tangent vector is $\tau = (1, v_0'(w))$ with $v_0'(w) > 0$. Thus the flow of ξ enters the strip along the upper boundary. On the lower boundary $v = v_\varepsilon(w)$ the vector field is $\xi \approx (b, \varepsilon)$, while the tangent vector is $\tau = (1, v_\varepsilon'(w))$ with $v_\varepsilon'(w) < \frac{\varepsilon}{2b}$ for $w \gg 1$. Thus the flow of ξ enters the strip along the lower boundary as well.

Since ε is arbitrary small, the curve $v = v_0(w)$ (though not precisely invariant by the dynamics) attracts asymptotically all the integral curves of the vector field ξ (i.e. of our Abel's ODE). We demonstrate this effect on the picture, where we magnify the piece of the right half-plane to see the attraction.



Attracting parabola with several attracted integral curves for the Abel's ODE.

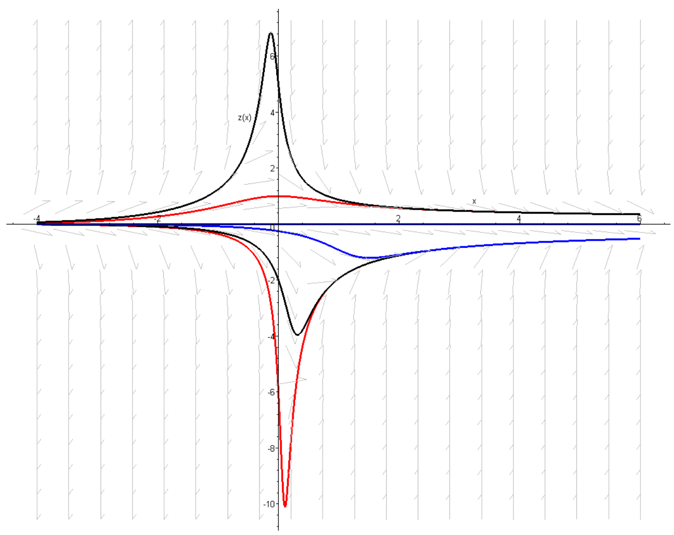
The global dynamics is thus exponentially diverging at the left-half plane and exponentially converging at the right one. However, the dynamics is more complicated than just going from some infinity from the left to close-to-parabola on the right. There is another piece of sensitive dependence on initial data near the axis $\{w = 0\}$.

To see this let us change the variables: $\{w \to x, v \to z^{-1}\}$. This transforms Abel's ODE of the second kind to the following Abel's ODE of the first kind:

$$\frac{dz}{dx} = -\frac{x}{b}z^3 - \frac{a}{b}z^2 + \frac{1}{b}z.$$

This equation has vanishing at $x = 0$ main term in the right-hand-side and this leads to a certain blow-up of solutions.

For $b < 0$ the above described parabola as well as the ray $\{v = 0, w < 0\}$ from the origin to $-\infty$ are repelling (unstable).



The solutions repell, when they pass the origin.

Thus we have described the 2-dimensional dynamics of equations (7-8). Given functions $v(t), w(t)$ we can solve the remaining equation (6), which is a linear non-homogeneous ODE in $u(t)$. The solution $u(t) = u_0(t) + Ke^{-bt}$ will converge to a particular solution $u_0(t)$ as $t \to +\infty$ when $b > 0$ and will diverge from it in all directions when $b < 0$.

In particular, for $b > 0$ the dynamics $(u(t), v(t), w(t))$ asymptotically converges to one curve over the above parabola and so for $t \to +\infty$ our 3-dimensional dynamics becomes 1-dimensional, while for $t \to -\infty$ we have exponential instability. But for $b < 0$ we have exponential instability in all directions for $t \to +\infty$, but stability for $t \to -\infty$. Thus the dynamics exibits sensitive dependence on initial conditions.

To find a solution space for ODE $g(p_0, p_1, p_2, p_3) = 0$ we remark that this equation has three symmetries

$$\varphi_1 = p_0,$$
$$\varphi_2 = p_1,$$
$$\varphi_3 = cp_0 - bp_0 \log(p_0) + p_2.$$

These symmetries are linearly dependent and

$$\alpha_1 X_{\varphi_1} + \alpha_2 X_{\varphi_2} + X_{\varphi_3} = 0$$

for

$$\alpha_1 = \frac{ap_1 - p_2 - cp_0 + bp_0 \log p_0}{p_0}$$

and

$$\alpha_2 = -a.$$

Therefore

$$H = b \log p_0 + a\frac{p_1}{p_0} - \frac{p_2}{p_0},$$

is a first integral of the ODE $\{g = 0\}$, and the order of this equation can be reduced. Thus we get an ODE of the second order instead of $g = 0$:

$$p_2 = ap_1 + bp_0 \log p_0 - Cp_0.$$

If $a = 0$ this equation becomes a stationary Shrödinger equation with logarithmic non-linearity and it can be reduced to the following 1-st order ODE:

$$p_1^2 + p_0^2(b + C) - 2bp_0^2 \log p_0 = \mathrm{const}\,.$$

6 Conclusion

We have constructed some new explicit solutions of the KPP equation and studied the corresponding dynamics. If we do not specify the overdetermination $g = 0$, then the non-linearity in (1) can be arbitrary and we can find a compatible g.

If first order $g = g(p_0, p_1)$, then as we showed the corresponding solutions are standard: x-independent or travelling waves. For a more general form $g = g(x, p_0, p_1)$ all the solutions $u = u(t, x)$ can appear as invariant dynamics of (1) obtained via ODE $g = 0$.

Since x does not enter the KPP equation, the dynamics preserves the class of differential equations $g = 0$ not involving x. Let us explain why for $n > 1$ almost every solution of (1) belongs to some dynamics $g = g(p_0, \dots, p_{n-1}, p_n)$.

Indeed, let $\psi_t(x) = u(t, x)$ be a solution with t considered as a parameter. Then we have 3 functions $\psi_t(x), \psi_t'(x), \psi_t''(x)$, from which we can generically exclude t, x (consider two of the functions as "parameters") and get a relation $h'' = \Phi(h, h')$, where $h = \psi_t$. In other words, given function ψ_0 the evolution determines a 1-parameter family ψ_t, shifts of x-parameter (along the symmetry p_1) makes this family 2-dimensional, and it is given as a solutions space of some 2^{nd} order ODE. Thus we justify usage of dynamics g of order $n > 1$ not involving x.

Note however that generalizing equation (1) to have non-linearity $f(x, u)$ involving x explicitely, the usage of x can be essential. For instance, let us consider evolutionary PDE

$$u_t = u_{xx} + b(x)f(u).$$

Then it has linear second order dynamics

$$g = p_2 + \alpha(x)p_1 + \beta(x)p_0 \tag{10}$$

iff $f(u)$ is linear. Thus we consider instead the evolutionary equation (2) with $\varphi = p_2 + b(x)u + c(x)$. Then (10) is an invariant dynamics iff $\alpha(x) = y(x)$, $\beta(x) = w(x) + \frac{1}{4}y(x)^2 + \frac{1}{2}y'(x)$, $b(x) = w(x) - \frac{1}{4}y(x)^2 + y'(x)$, where $w(x)$ is arbitrary and $y(x)$ is the solution of the equation

$$y''' + 4wy' + 2w'y = 0. \tag{11}$$

The function $c(x)$ can then be found from the equation $c'' + \alpha(x)c' + \beta(x)c = 0$.

Remark 4 *Equation (11) coincides with Eq.(9) from §2.4.1 of [13], which arise in relation to the spectral problem for the Shrödinger equation.*

Compatibility equation (4) for the ODE $g = 0$ is a determined equation and every its solution yields a family of solutions for initial problem (1). However it can be difficult to find such solutions explicitly. That's why we fix an ansatz for g and classify the KPP equations, admiting solutions with this g.

The usual symmetry can be also thought of as an ansatz and there exists a classification of KPP (non-linear reaction-diffusion) equations, admitting classical and generalized symmetries, see [3].

In this paper we consider dynamics of order $n \leq 3$, which are quasilinear in p_{n-1}, p_n. We finish with a general result about such systems with arbitrary n:

Theorem 4 *Let $g = p_n + p_{n-1}R(p_0, \dots, p_{n-2}) + S(p_0, \dots, p_{n-2})$ be a dynamics for the KPP equation (1). Then R is quadratic and S is cubic in p_{n-2}.*

Indeed, if we calculate $G = X_{\bar{\varphi}}(g) \,\mathrm{mod}(g = 0)$ as a function on J^{n-1}, then we find that

$$\frac{\partial^3 G}{\partial p_{n-1}^3} = -6\frac{\partial^2 R}{\partial p_{n-2}^2} \qquad \text{and} \qquad \frac{\partial^4 G}{\partial p_{n-1}^2 \partial p_{n-2}^2} = -2\frac{\partial^4 S}{\partial p_{n-2}^4},$$

whence the claim follows from (4). Moreover, if $n > 2$, then

$$-\frac{1}{4}\frac{\partial^3 G}{\partial p_{n-1}^2 \partial p_{n-2}} = \left(\frac{\partial R}{\partial p_{n-2}}\right)^2 - \frac{\partial^2 R}{\partial p_{n-2}\partial p_{n-3}} - \frac{1}{2}\frac{\partial^3 S}{\partial p_{n-2}^3}.$$

Thus the dynamics has the form:

$$g = p_n + p_{n-1}p_{n-2}R_0 + p_{n-1}T + \frac{1}{3}p_{n-2}^3(R_0^2 - \frac{\partial R_0}{\partial p_{n-3}}) + \sum_{k=0}^{2} S_k p_{n-2}^k,$$

where R_0, T, S_k depend on $p_0, \dots, p_{n-3}$. Exploring further the homogeneous terms we will find the normal form of g and then of f for every n.

Acknowledgement We are thankful to Valentin Lychagin for helpful discussions.

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