ABELIAN EQUATIONS AND DIFFERENTIAL INVARIANTS OF WEBS

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I. INTRODUCTION

- [Pantazi, 1938, 1940]

  [Mihăileanu, 1941]
- [Dou, 1953, 1955]
- [Bol, 1932]
- [Blaschke, 1955]

  "Find invariant conditions for a planar 4-web to be of rank 1 or 2 or 3."
- [Pirio, 2005]
- [Akivis, Goldberg, Lychagin, 2004]

II. ABELIAN EQUATIONS

- $W_d =< \omega_1, ..., \omega_d >$ = a smooth planar d-web in $D \subset \mathbb{R}^2$ and $\pi : E \to D$ = a subbundle of the trivial bundle $\mathbb{R}^d \times D \to D$ consisting of points $(x_1, \dots, x_d, a)$, where $(x_1, \dots, x_d) \in \mathbb{R}^d$, $a \in D$, such that $\sum_1^d x_i \omega_{i,a} = 0.$

Definition 1 By the **abelian equation** associated with the d-web W_d we mean a system of 1-st order differential equations for sections $(\lambda_1, ..., \lambda_d)$ (i.e. $\sum_1^d \lambda_i \omega_i = 0)$ of the bundle π such that (cf. [Griffiths, 1977]):

$$d(\lambda_1 \omega_1) = \cdots = d(\lambda_d \omega_d) = 0.$$

An abelian equation can be obtained by differentiation from an abelian relation

$$\varphi_1(x) + \varphi_2(y) + \varphi_3(f_3(x,y)) + \cdots + \varphi_d(f_d(x,y)) = 0.$$

Let $\mathfrak{A}_1 \subset J^1(\pi)$ be the subbundle of the 1-jet bundle corresponding to the abelian equation, and $\mathfrak{A}_k \subset J^k(\pi)$ be the $(k-1)$-prolongation of $\mathfrak{A}_1$. Denote by $\pi_{k,k-1} : \mathfrak{A}_k \to \mathfrak{A}_{k-1}$ the restrictions of the natural projections $J^k(\pi) \to J^{k-1}(\pi)$.

- **Proposition 2** Let $k \leq d-2$. Then $\mathfrak{A}_k$ are vector bundles and the maps $\pi_{k,k-1} : \mathfrak{A}_k \leftarrow \mathfrak{A}_{k-1}$ are projections. Moreover, $\dim \ker \pi_{k,k-1} = d-k-2$.
- In other words, one has the following tower of vector bundles:

$$D \overset{\pi}{\leftarrow} E \overset{\pi_{1,0}}{\leftarrow} \mathfrak{A}_1 \overset{\pi_{2,1}}{\leftarrow} \mathfrak{A}_2 \overset{\pi_{3,1}}{\leftarrow} \ldots \mathfrak{A}_{d-3} \overset{\pi_{d-2,d-3}}{\leftarrow} \mathfrak{A}_{d-2}.$$

The last projection $\mathfrak{A}_{d-2} \leftarrow \mathfrak{A}_{d-3}$ is an isomorphism, and geometrically can be viewed as a linear connection in the vector bundle $\pi_{d-3} : \mathfrak{A}_{d-3} \longrightarrow D$. Remark that the abelian equation is formally integrable if and only if this linear connection is flat.

- The dimension of this bundle is equal to

$$(d-2)+\cdots+1=(d-2)(d-1)/2. \rightarrow$$

The solution space Sol $(\mathfrak{A})$ of the abelian equation $\mathfrak{A}$ is finite-dimensional and

$$\dim \textit{Sol } (\mathfrak{A}) \leq (d-1)(d-2)/2.$$

Definition 3 The dimension $\dim$ Sol $(\mathfrak{A})$ is called the **rank** of the corresponding d-web W_d.

Proposition 4 [Bol, 1932] The rank does not exceed $(d-1)(d-2)/2$.

- Let us write down the abelian equation in the explicit form. To do this, we choose a 3-subweb, say $\langle \omega_1, \omega_2, \omega_3 \rangle$, and normalize d-web as it was done in [Akivis, Goldberg, Lychagin, 2004]:

$$a_1\omega_1+\omega_2+\omega_3 = 0, \dots, a_{d-2}\omega_1+\omega_2+\omega_d = 0,$$

with $a_1 = 1.$

Then $\lambda_1 = \sum_1^{d-2} a_i u_i$, $\lambda_2 = \sum_1^{d-2} u_i,$ and the system of differential equations

$$d\left(\lambda_i \omega_i\right) = 0, \quad i = 1, .., d$$

is equivalent to

$$\Delta_1\left(u_1\right) = \cdots = \Delta_{d-2}\left(u_{d-2}\right) = 0,$$
$$\delta_1\left(u_1\right) + \cdots + \delta_1\left(u_{d-2}\right) = 0,$$

where $\Delta_i = \delta_1 - \delta_2 \circ a_i, i = 1, .., d-2$ and δ_i are the covariant derivatives with respect to the Chern connection. Moreover, the functions u_i are assumed to be of weight one.

- The obstruction for compatibility of the abelian system is given by the multi-bracket [Kruglikov, Lychagin, 2006]. Computing the multi-bracket, we get the following compatibility condition for the abelian system:

$$\varkappa = \square_1 u_1 + \cdots + \square_{d-2} u_{d-2} = 0,$$

where

$$\square_i = \Delta_1 \cdots \Delta_{d-2} \cdot \delta_1 \\ -\Delta_1 \cdots \Delta_{i-1} \cdot \delta_1 \cdot \Delta_{i+1} \cdots \Delta_{d-2} \cdot \Delta_i$$

are linear differential operators of order $\leq d-2.$

Theorem 5 A d-web W_d is of maximum rank $(d-1)(d-2)/2$ if and only if $\varkappa = 0$ on $\mathfrak{A}_{d-2}.$

- Remark that $\varkappa$ can be viewed as a linear function on the vector bundle $\mathfrak{A}_{d-2},$ and therefore the above theorem imposes $(d-1)(d-2)/2$ conditions on the web W_d in order the web has the maximum rank. A calculation of these conditions is pure algebraic, and we shall illustrate this calculation for planar 4-webs.

III. PLANAR 4-WEBS OF DIFFERENT RANKS

Obstruction

- For 4-webs W_4, we use the following normalization:

  $$\omega_1 + \omega_2 + \omega_3 = 0, \; a\,\omega_1 + \omega_2 + \omega_4 = 0,$$

  where $a_1 = 1$, and $a_2 = a$ is the basic invariant of W_4 of weight 0. For webs W_4, the tower of prolongations of the abelian equation is

  $$\mathbb{D} \xleftarrow{\pi} E \xleftarrow{\pi_{1,0}} \mathfrak{A}_1 \xleftarrow{\pi_{2,1}} \mathfrak{A}_2,$$

  where $\pi_{2,1} : \mathfrak{A}_2 \to \mathfrak{A}_1$ defines a linear connection on the 3-dimensional vector bundle $\pi_1 \colon \mathfrak{A}_1 \to \mathbb{D}$.

- To write down the obstruction

$$\varkappa = (\Delta_1\Delta_2\delta_1 - \delta_1\Delta_1\Delta_2)u + (\Delta_1\Delta_2\delta_1 - \Delta_1\delta_1\Delta_2)v$$

on $\mathfrak{A}_2$, we use the standard coordinates in the bundle: u, v and v_2, where v_2 corresponds to the covariant derivative δ_2 of v. In these coordinates, the restriction $\varkappa$ to $\mathfrak{A}_2$ is

$$\varkappa = cv_2 + c_1 v + c_2 u,$$

$$c_0 = K + \frac{a_{11} - aa_{22} - 2(1-a)a_{12}}{4a(1-a)} + \frac{(-1+2a)a_1^2 - a^2 a_2^2 + 2(1-a)^2 a_1 a_2}{4(1-a)^2 a^2},$$

$$c_1 = \frac{K_2 - K_1}{4(1-a)} + \frac{(a-4)a_1 + \left(11 - 20a + 12a^2\right)a_2}{12(1-a)^2 a} K + \frac{a_{112} - a_{122}}{4a(1-a)} + \frac{a_1 - aa_2}{4a^2(1-a)} a_{22} + \frac{(2a-1)(a_1 - aa_2)}{4(1-a)^2 a^2} a_{12} - \frac{a_2^2((1-2a)a_1 + aa_2)}{4(1-a)^2 a^2},$$

$$c_2 = \frac{aK_2 - K_1}{4a(1-a)} + \frac{(1-2a)a_1 - (a-2)aa_2}{4(1-a)^2 a^2} K.$$

Here $K\omega_1 \wedge \omega_2$ is the curvature 2-form of 3-subweb $[1,2,3]$ and the low indices mean the corresponding covariant derivatives with respect to the Chern connection.

IV. 4-WEBS OF MAXIMUM RANK

- **Definition 6** The sum of the curvature 2-forms of all four 3-subwebs $[1, 2, 3]$, $[1, 2, 4]$, $[1, 3, 4]$ and $[2, 3, 4]$ of a 4-web W_4 is called the **curvature 2-form** of W_4.

$$\boxed{L\omega_1 \wedge \omega_2 = 4c_0\omega_1 \wedge \omega_2}$$

- **Theorem 7** A planar 4-web is of maximum rank three if and only if the invariants c_0, c_1 and c_2 vanish: $\boxed{c_0 = c_1 = c_2 = 0.}$

- $c_0 = 0 \longrightarrow$

$$\boxed{K = -\frac{a_{11} - aa_{22} - 2\,(1-a)\,a_{12}}{4a(1-a)} - \frac{(-1+2a)a_1^2 - a^2a_2^2 + 2(1-a)^2a_1a_2}{4(1-a)^2a^2}}$$

$\longrightarrow K_1 = \ldots; K_2 = \ldots$

$c_1 = c_2 = 0 \longrightarrow K_1 = \ldots; K_2 = \ldots$

Comparing the two values of K_1 and K_2, we arrive at two conditions which, if $c_0 = 0$, are equivalent to the linearizability conditions for a 4-web W_4.

Theorem 8 [Akivis, Goldberg, Lychagin, 2004] A 4-web $W(4,2)$ is linearizable if and only if its curvature K along with its covariant derivatives K_1, K_2 and its basic invariant a, along with its covariant derivatives a_i, a_{ij}, a_{ijk} satisfy the following conditions:

$$
\begin{aligned}
K_1 \;=\; & \frac{1}{(a-a^2)}\Big[(a_1 + aa_2)K - a_{111} + (2+a)a_{112} - 2a_{12} \\
& + \frac{1}{(a-a^2)^2}\Bigg\{\Big[(4-6a)a_1 + (-2+3a+a^2)a_2\Big]a_{11} \\
& + \Big[(-6+7a+2a^2)a_1 + (2a-3a^2)a_2\Big]a_{12} \\
& + \Big[(2a(1-a)a_1 - 2a^2a_2\Big]a_{22}\Bigg\} \\
& + \frac{1}{(a-a^2)^3}\Bigg\{(-3+8a-6a^2))(a_1)^3 - 2a^3(a_2)^3 \\
& + (6-15a+9a^2+2a^3)(a_1)^2a_2 \\
& + (-2a+6a^2-3a^3))a_1(a_2)^2\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
K_2 = \; & \frac{1}{(a-a^2)}\Big[(a_1 + a a_2)K + 2a_{112} \\
& -(2a+1)a_{122} + a a_{222}\Big] \\
& + \frac{1}{(a-a^2)^2}\Bigg\{\Big[(2a_1 + (2a-2)a_2\Big]a_{11} \\
& + \Big[(-5+6a)a_1 + (2-3a-2a^2)a_2\Big]a_{12} \\
& + \Big[(1-a-2a^2)a_1 + 2a^2 a_2\Big]a_{22}\Bigg\} \\
& + \frac{1}{(a-a^2)^3}\Bigg\{(4a-2)(a_1)^3 + a^3(a_2)^3 \\
& + (5-12a+6a^2)(a_1)^2 a_2 \\
& + (-2+5a-3a^2-2a^3)a_1(a_2)^2\Bigg\}
\end{aligned}
$$

Theorem 9 A planar 4-web is of maximum rank three if and only if this web is linearizable and its curvature 2-form vanishes.

- **Corollary 10** A linear planar 4-web is of maximum rank three if and only if its curvature 2-form vanishes.

  Corollary 11 (Theorem of Poincaré) A planar 4-web of maximum rank three is linearizable.

- **Definition 12** 4-webs all 3-subwebs of which are parallelizable (hexagonal) are called **Mayrhofer 4-webs.**

  Corollary 13 The Mayrhofer 4-webs are of maximum rank three.

- **Example 14** Consider the planar 4-web W_4 formed by the coordinate lines $y = const., \ x = const.$ and by the level sets of the functions

$$f(x,y) = \frac{x}{y} \text{ and } g(x,y) = \frac{x-xy}{y-xy}.$$

This web is a 4-subweb of the famous Bol's 5-web. It was shown in [Akivis, Goldberg, Lychagin, 2004] that this 4-web is linearizable. The direct calculation shows that the curvature 2-form of this 4-web vanishes: $L\omega_1 \wedge \omega_2 = 0 \longrightarrow$ our W_4 is of maximum rank three.

- Three abelian relations admitted by W_4 are:

$$\log x - \log y - \log f(x,y) = 0,$$

$$\log \frac{1-x}{x} - \log \frac{1-y}{y} + \log g(x,y) = 0,$$

$$\log(1-x) - \log(1-f(x,y))$$

$$+ \log(1-g(x,y)) = 0.$$

- **Theorem 15** (Blaschke–Howe, 1932; Sophus Lie, 1882) For $d = 3, 4$, d-web of maximum rank is algebraizable.

$$\boxed{f_1''(x) + f_2''(x) + f_3''(x) + f_4''(x) = 0}$$

for any straight line intersecting 4 curves

V. 4-WEBS OF RANK TWO

- **Theorem 16** A planar 4-web is of rank two if and only if $\boxed{G_{ij} = 0,\ i,j = 1,2,}$ where

$$
\boxed{
\begin{aligned}
G_{11} = {} & a c_0 (c_{2,2} - c_{2,1}) + a c_2 (c_{0,1} - c_{0,2}) \\
& - a\,(1-a)\, c_1 c_2 \\
& + (2a_2 - a_1 - a a_2)\, c_0 c_2 - K c_0^2, \\
G_{12} = {} & a c_0 (c_{1,2} - c_{1,1}) + a c_1 (c_{0,1} - c_{0,2}) \\
& - a\,(1-a)\, c_1^2 + \left(a_2^2 + a_{12} - a_{22}\right) c_0^2 \\
& + (2a_2 - a_1 - 2 a a_2) c_0 c_1, \\
G_{21} = {} & c_0 (c_{2,1} - a c_{2,2}) + c_2 (a c_{0,2} - c_{0,1}) \\
& - 2 a_2 c_0 c_2 + a\,(1-a)\, c_2^2, \\
G_{22} = {} & c_0 (c_{1,1} - a c_{1,2}) + c_1 (a c_{0,2} - c_{0,1}) \\
& + a\,(1-a)\, c_1 c_2 - a_2 c_0 c_1 \\
& - a_2 (1-a) c_0 c_2 + (a_{22} - K)\, c_0^2.
\end{aligned}
}
$$

- **Example 17** Consider the planar 4-web W_4 formed by the coordinate lines $y = const.,\ x = const.$ and by the level sets of the functions

$$\boxed{f(x,y) = x + y \text{ and } g(x,y) = x^2 + y^2.}$$

For this 4-web, we have $c_0 \neq 0;\ \ c_1 = c_2 = 0;$ and therefore $G_{ij} = 0 \rightarrow W_4$ is of rank two.

- Two abelian relations admitted by the web are

$$\boxed{x + y - f(x,y) = 0;\ x^2 + y^2 - g(x,y) = 0.}$$

- Remark that the conditions of linearizability from [Akivis, Goldberg, Lychagin, 2004] do not hold for this web. This implies

Proposition 18 General 4-webs admitting two abelian equations are not necessarily linearizable.

VI. 4-WEBS OF RANK ONE

- **Theorem 19** A planar 4-web is of rank one if and only if the web is of one of the following types:

1. $\boxed{c_0 = 0,\ J_1 = J_2 = 0,}$ where

$$\begin{aligned} J_1 = {} & a_2 c_1 c_2 (c_1 - c_2) + a c_2^2 (c_{1,2} - c_{1,1}) \\ & + c_1 c_2 (c_{1,1} + a(c_{2,1} - c_{1,2} - c_{2,2})) \\ & + c_1^2 (a c_{2,2} - c_{2,1}), \\ J_2 = {} & c_1^2 (c_1 - c_2)^2 K \\ & + (c_{1,11} - c_{1,12}) c_1 c_2 (c_2 - c_1) \\ & + c_1^2 (c_1 - c_2)(c_{2,11} - c_{2,12}) \\ & - c_2 (2c_1 - c_2) c_{1,1} (c_{1,2} - c_{1,1}) \\ & + c_1^2 c_{2,1} (c_{1,2} - c_{2,2} + c_{2,1}) \\ & + c_1^2 c_{1,1} (c_{2,2} - 2c_{2,1}) \end{aligned}$$

and $c_1 \neq c_2,\ c_1 \neq 0.$

2. $\boxed{c_0 = 0,\ c_1 = c_2 \neq 0,\ J_3 = 0,}$ where

$$\begin{aligned} J_3 = {} & (a_{22} - a_{12})(1 - a) + a_2 (a_2 - a_1) \\ & - (1 - a)^2 K. \end{aligned}$$

3. $\boxed{c_0 = 0,\ c_1 = 0,\ c_2 \neq 0,\ J_4 = 0,}$

where

$$J_4 = a_{12}a - a_1 a_2 - Ka^2.$$

4. $\boxed{c_0 \neq 0,\ J_{10} = J_{11} = J_{12} = 0,}$ where

$$J_{10} = G_{11}G_{22} - G_{21}G_{12},$$

$$J_{11} = c_0(G_{21,1}G_{22} - G_{22,1}G_{21}) + (a_2c_0 - ac_1)G_{21}^2 + (ac_2 - a_2c_0 + ac_1)G_{21}G_{22} - ac_2G_{22}^2,$$

$$J_{12} = c_0(G_{21,2}G_{22} - G_{22,2}G_{21}) + (a_2c_0 - ac_1)G_{21}^2 + a(c_2 - c_1)G_{21}G_{22} - c_2G_{22}^2.$$

- **Example 20** We consider the planar 4-web formed by the coordinate lines $y = const.,\ x = const.$ and by the level sets of the functions

$$f(x,y) = \frac{xy^2}{(x-y)^2} \text{ and } g(x,y) = \frac{x^2y}{(x-y)^2}.$$

  By direct calculation, we find that for this 4-web we have

$$c_0 = 0,\ c_1, c_2 \neq 0,\ c_1 \neq c_2,\ J_1 = J_2 = 0 \rightarrow$$

  W_4 of type 1 indicated in Theorem 19 and the W_4 is of rank one.
- The only abelian equation admitted by W_4 is

$$\log x - \log y + \log f(x,y) - \log g(x,y) = 0.$$

- The conditions of linearizability from [Akivis, Goldberg, Lychagin, 2004] do not hold for this web $\longrightarrow$

  Proposition 21 General 4-webs admitting one abelian equation are not necessarily linearizable.

- **Theorem 21** A nonparallelizable planar 4-web W_4 with a basic invariant $a = const$ has rank 0.

Proof $c_0, c_1, c_2 \neq 0 \to$ rank$W_4 < 3$

$G_{ij} \neq 0 \to$ rank$W_4 < 2$

$J_{10} = G_{11}G_{22} - G_{12}G_{21} \neq 0 \to$ rank$W_4 < 1 \to$

rank$W_4 = 0$ **Q.E.D.**

VII. 5-WEBS OF MAXIMUM RANK

- We now take $d = 5$. Because the maximum rank $\pi(d)$ of $W(d, 2, 1)$ is $\pi(d) = \frac{1}{2}(d-1)(d-2)$, the maximum rank of $W(5, 2, 1)$ is $\pi(5) = 6$.
- $A_1\omega_1 + A_2\omega_2 + w\omega_3 + u\omega_4 + v\omega_5 = 0$, where

$$\omega_3 = -\omega_1 - \omega_2,$$
$$\omega_4 = -a\omega_1 - \omega_2,$$
$$\omega_5 = -b\omega_1 - \omega_2,$$

and a and b are the basic invariants of the 5-web $W_5 = W(5, 2, 1)$.

- **Theorem 22** A planar 5-web is of maximum rank six if and only if its curvature K and the covariant derivatives K_1, K_2, K_{11}, K_{12} and K_{22} of K are expressed in terms of the 5-web basic invariants a and b and their covariant derivatives up to the 4th order as follows:

$$K = ..., \quad K_1 = ..., \quad K_2 = ...,$$

$$K_{11} = ..., \quad K_{12} = ..., \quad K_{22} = ...$$

- $K = ...$ is equivalent to the 5-web curvature 2-form = the sum of the curvature 2-forms of 10 3-subwebs $[\xi, \eta, \zeta]$.
- $K = ..., \quad K_i = ..., \quad K_{ij} = ...$ do not imply the 5-web linearizability conditions from [(Akivis, Goldberg, Lychagin, 2004] $\longrightarrow$

  Proposition 23 The general 5-web of maximum rank is not linearizable (algebraizable).

- **Example 23** We consider the Bol 5-web formed by the coordinate lines $y = const., \; x = const.$ and by the level sets of the functions

$$f(x,y) = \frac{x}{y}, \quad g_4(x,y) = \frac{1-y}{1-x}, \quad g_5(x,y) = \frac{x-xy}{y-xy}$$

We have $K = 0 \to K_i = K_{ij} = 0$. The RHS of the formulas in Theorem **21** are also $0 \longrightarrow$ the Bol 5-web is of maximum rank 6.

$\mu_{[1,2,3,4]} - \mu_{[1,2,3,5]} = \frac{x(x+y-2)}{y(x-1)(y-1)} \neq 0,$ i.e., the Bol 5-web is not linearizable (algebraizable).

- Six abelian relations for the Bol 5-web are:

$$\ln u_1 - \ln u_2 - \ln u_3 = 0,$$
$$\ln u_3 + \ln u_4 - \ln u_5 = 0,$$
$$\ln(1 - u_1) - \ln(1 - u_2) + \ln u_4 = 0,$$
$$\ln(1 - u_1) - \ln(1 - u_3) + \ln(1 - u_5) = 0,$$
$$\ln \frac{1 - u_1}{u_1} - \ln \frac{1 - u_3}{u_3} + \ln(1 - u_4) = 0,$$
$$\mathbf{D}_2(u_1) - \mathbf{D}_2(u_2) - \mathbf{D}_2(u_3)$$
$$-\mathbf{D}_2(u_4) + \mathbf{D}_2(u_5) = 0,$$

where $u_1 = x,\ u_2 = y,\ u_3 = f(x, y),$ $u_4 = g_4(x, y),\ u_5 = g_5(x, y)$ and

$$\mathbf{D}_2(u) = -\int_0^u \left(\frac{\ln|1-u|}{u} + \frac{\ln|u|}{1-u}\right) du$$
$$+\frac{1}{2} \ln u \ln(1 - u) - \frac{\pi^2}{6},\quad 0 < u < 1,$$

is the version of the original Rogers dilogarithm normalized so that RHS of the last abelian relation is 0.

- **Example 24** We consider the 5-web formed by the coordinate lines $y = const.,\ x = const.$ and by the level sets of the functions

$$f(x,y) = \frac{x}{y}, \quad g_4(x,y) = \frac{x}{(x-1)(y-1)}, \\ g_5(x,y) = \frac{y}{(x-1)(y-1)}.$$

We have $K = 0 \longrightarrow K_i = K_{ij} = 0$. The RHS of the formulas in Theorem **21** are also $0 \longrightarrow$ our 5-web is of maximum rank 6.

$\mu_{[1,2,3,4]} - \mu_{[1,2,3,5]} = 0$, and the 4-subweb $[1, 2, 3, 4]$ is linearizable $\longrightarrow$ our 5-web is linearizable (algebraizable.)

- Six abelian relations for this 5-web are:

$$\ln u_1 - \ln u_2 - \ln u_3 = 0,$$
$$\ln u_3 - \ln u_4 + \ln u_5 = 0,$$
$$\ln \frac{u_1}{1-u_1} - \ln(1-u_2) - \ln u_4 = 0,$$
$$-\ln(1-u_1) + \ln \frac{u_2}{1-u_2} - \ln u_5 = 0,$$
$$\frac{u_1-1}{u_1} + u_2 - u_3 - \frac{1}{u_4} = 0,$$
$$-u_1 + \frac{1-u_2}{u_2} - u_3 - \frac{1}{u_5} = 0,$$

where $u_1 = x,\ u_2 = y,\ u_3 = f(x,y),$
$u_4 = g_4(x,y),\ u_5 = g_5(x,y).$

- Nonlinearizable 5-webs of maximum rank were called exceptional webs. As we saw, the general 5-web of maximum rank is not linearizable (algebraizable). $\longrightarrow$ They are not exceptional webs.

  Among 5-webs of maximum rank there is a class of algebraizable webs.

- On the other hand, among nonalgebraizable d-webs, there are **polylogarithmic d-webs** .

  "Are all webs of maximal rank which are not algebraizable of this type? We do no attempt to formulate this question precisely—intuitively, we are asking whether or not for each k such that there is a "new" $n(k)$-web of maximal rank one of whose abelian relations is a (the?) functional equation with $n(k)$ terms for the k polylogarithm $\mathbf{Li_k}$? [Griffiths, 2001]

  In our opinion, it is natural to call **exceptional** such polylogarithmic webs. [Pirio, Robert, Trépereau]

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