Dimension of the solutions space of PDEs

Boris Kruglikov & Valentin Lychagin

[1] Inst. of Mathematics and Statistics, University of Tromsø, Tromsø 90-37, Norway
[2] kruglikov@math.uit.no lychagin@math.uit.no

Abstract

We discuss the dimensional characterization of the solutions space of a formally integrable system of partial differential equations and provide certain formulas for calculations of these dimensional quantities.

Keywords: Solutions space, Cartan's test, Cohen-Macaulay module, involutive system, compatibility, formal integrability.

1 Introduction: what is the solutions space?

Let $\mathcal{E}$ be a system of partial differential equations (PDEs[1]). We would like to discuss the dimensional characterization of its solutions space.

However it is not agreed upon what should be called a solution. We can choose between global or local and even formal solutions or jet-solutions to a certain order. Hyperbolic systems hint us about shock waves as multiple-valued solutions and elliptic PDEs suggest generalized functions or sections.

A choice of category, i.e. finitely differentiable C^k, smooth C^∞ or analytic C^ω together with many others, plays a crucial role. For instance there are systems of PDEs that have solutions in one category, but lacks them in another (we can name the famous Lewy's example of a formally integrable PDE without smooth or analytic solutions, [L]).

In this paper we restrict to local or even formal solutions. The reason is lack of reasonable existence and uniqueness theorems (in the case of global solutions even for ODEs). In addition this helps to overcome difficulties with blow-ups and multi-values.

If the category is analytic, then Cartan-Kähler theorem [Ka] guarantees local solutions of formally integrable equations [Go] and even predicts their quantity. We then measure it by certain dimension characteristics.

[1] MSC numbers: 35N10, 58A20, 58H10; 35A30.

If the category is smooth, formal integrability yields existence of solutions only if coupled with certain additional conditions (see for instance [Ho]). Thus it is easier in this case to turn to formal solutions, which in regular situations give the same dimension characteristics. With this vague idea let us call the space of solutions $\mathrm{Sol}(\mathcal{E})$.

With this approach it is easy to impose a topology on the solutions space. However we shall encounter the situations, when the topological structure is non-uniform.

To illustrate the above discussion, let's consider some model ODEs (in which case we possess existence and uniqueness theorem). The space of local solutions for the ODE $y' = y^2$ is clearly one-dimensional, but the space of global solutions (continuous pieces until the blow-up) has two disconnected continuous pieces (solutions $y = (a-x)^{-1}$ for $a < 0$ or for $a > 0$) and a singular point (solution $y = 0$). Another example is the equation $y'^2 + y^2 = 1$, the solutions on$(-\varepsilon, \varepsilon)$ of which form S^1, but the space of global solutions is $\mathbb{R}^1$ (both united with two singular points in $\mathrm{Sol}(\mathcal{E})$).

We would like to observe the "biggest" piece of the space of $\mathrm{Sol}(\mathcal{E})$, so that in our dimensional count we ignore isolated and special solutions or their families and take those of connected components, that have more parameters in.

It will be precisely the number of parameters, on which a general solution depends, that we count as a dimensional characteristic. Let us discuss the general idea how to count it and then give more specified definitions.

Note that in this paper we consider only (over)determined systems of PDEs. Most results will work for underdetermined systems, but we are not concerned with them.

2 Understanding dimension of the solutions space

Let us treat at first the case of linear PDEs systems (the method can be transferred to non-linear case). We consider formal solutions and thus assume the system of PDEs $\mathcal{E}$ is formally integrable. We also assume the system $\mathcal{E} = \mathcal{E}_k$ is of pure order k, which shall be generalized later.

Thus for some vector bundle $\pi : E(\pi) \to M$ we identify $\mathcal{E}$ as a subbundle $\mathcal{E}_k \subset J^k(\pi)$ (see [S, Go, KLV]) and let $\mathcal{E}_l \subset J^l(\pi)$ be its $(l-k)$-th prolongations, $l \geq k$. Then the fibres $\mathcal{E}_x^\infty \subset J_x^\infty(\pi)$ at points $x \in M$ can be viewed as spaces of formal solutions of $\mathcal{E}$ at $x \in M$. To estimate size of $\mathcal{E}_x^\infty$ we consider the spaces of linear functions on $\mathcal{E}_{l,x}$, i.e. the space $\mathcal{E}_{l,x}^*$. The projections $\pi_{l,l-1} : \mathcal{E}_{l,x} \to \mathcal{E}_{l-1,x}$ induce embeddings $\pi_{l,l-1}^* : \mathcal{E}_{l-1,x}^* \hookrightarrow \mathcal{E}_{l,x}^*$, and we have the projective limit

$$\mathcal{E}_x^* = \cup_l \mathcal{E}_{l,x}^*.$$

Remark that $\mathcal{E}^*$ is the module over all scalar valued differential operators on π, while the kernel of the natural projection $J_x^\infty(\pi)^* \to \mathcal{E}_x^*$ can be viewed as the space of scalar valued differential operators on π vanishing on the solutions of the PDEs system $\mathcal{E}$ at the point $x \in M$. Thus elements of $\mathcal{E}_x^*$ are linear functions on the formal solutions $\mathcal{E}_x^\infty$.

We would like to choose "coordinates" among them, which will estimate dimension of the formal solution space. To do this we consider the graded module associated with filtred module $\mathcal{E}_x^*$:

$$g^*(x) = \bigoplus_{l \geq 0} g_l^*(x),$$

where $g_l(x)$ are the symbols of the equation at $x \in M$:

$$g_l(x) = \mathcal{E}_{l,x}/\mathcal{E}_{l-1,x} \subset S^l T_x^* \otimes \pi_x$$

(we let $\mathcal{E}_l = J^l(\pi)$ for $l < k$), and reduce analysis of $\mathcal{E}_x^*$ to investigation of the symbolic module g_x^*.

This g^* is the module over the symmetric algebra $ST_xM = \oplus S^i(T_xM)$ and its support $\mathrm{Char}_x^{\mathbb{C}}(\mathcal{E}) \subset \mathbb{P}^{\mathbb{C}} T_x^* M$ is a complex projective variety consisting of complex characteristic vectors. The values $\mathcal{K}_p$ of the symbolic module g_x^* at characteristic covectors $p \in {}^{\mathbb{C}}T_x^* \setminus 0$ form a family of vector spaces over $\mathrm{Char}_x^{\mathbb{C}}(\mathcal{E})$, which we call *characteristic sheaf.*

By the Noether normalization lemma ([E]) there is a subspace $U \subset T_xM$ such that the homogeneous coordinate ring $ST_xM/\operatorname{Ann} g^*(x)$ of $\mathrm{Char}_x^{\mathbb{C}}(\mathcal{E})$ is a finitely generated module over SU. It follows that $g^*(x)$ is a finitely generated module over SU too.

If $g^*(x)$ is a Cohen-Macaulay module (see [E], but we recall the definition later in a more general situation, then $g^*(x)$ is a free SU-module (we called the respective PDEs systems $\mathcal{E}$ Cohen-Macaulay in [KL$_2$] and discussed their corresponding reduction).

Let σ be the rank of this module, and $p = \dim U$. By the above discussion these numbers can be naturally called *formal functional rank* and *formal functional dimension* of the solutions space $\mathcal{E}_x^\infty$ at the point $x \in M$, because they describe on how many functions of how many variables a general jet-solution formally depends (we shall omit the word "formally" later), or how many "coordinates" from $\mathcal{E}_x^*$ should be fixed to get a formal solution.

If the symbolic module is not Cohen-Macaulay, the module $g^*(x)$ over SU is not free, but finitely generated and supported on $\mathbb{P}^{\mathbb{C}} U^*$. Let $\mathbb{F}(U)$ be the field of homogeneous functions P/Q, where $P, Q \in SU$, $Q \neq 0$, considered as polynomials on U^*. Thus $\mathbb{F}(U)$ is the field of meromorphic (rational) functions on U^*.

Consider $\mathbb{F}(U) \otimes g^*(x)$ as a vector space over $\mathbb{F}(U)$. Keeping the same definition for σ, let us call the dimension of this vector space p *formal rank* of $\mathcal{E}$ at the point $x \in M$.

It is clear that for Cohen-Macaulay systems the two notions coincide. However since $g^*(x)$ over SU is not free, we would like to give more numbers to characterize the symbolic module.

Let us choose a base $e_1, \dots, e_r$ of $\mathbb{F}(U) \otimes g^*(x)$ such that $e_1, \dots, e_r$ are homogeneous elements of $g^*(x)$ and denote by $\Gamma_1 \subset g^*(x)$ the SU-submodule generated by this base. It is easy to check that Γ_1 is a free SU-module. For the quotient module $M_1 = g^*(x)/\Gamma_1$ we have the following property:

$$\operatorname{Ann} h \neq 0 \text{ in } SU, \text{ for any } h \in M_1.$$

Therefore $\mathrm{Ann}\, M_1 \neq 0$ and the support Ξ_1 of M_1 is a proper projective variety in $\mathbb{P}^{\mathbb{C}}U^*$.

We apply the Noether normalization lemma to Ξ_1, we get a subspace $U_1 \subset U$, such that M_1 is a finitely generated module over SU_1. Its rank will be the next number p_1 and we also get $\sigma_1 = \dim U_1$, which we can call the next formal rank and formal dimension.

Applying this procedure several more times we get a sequence of varieties Ξ_i and numbers (p_i, σ_i), which depends, in general, on the choice of the flag $U \supset U_1 \supset U_2 \supset \dots$ and the submodules Γ_i of SU_{i-1}.

Thus we resolve our symbolic module via the exact 3-sequences

$$0 \to \Gamma_1 \to g^* \to M_1 \to 0 \;\text{ over } SU, \qquad 0 \to \Gamma_2 \to M_1 \to M_2 \to 0 \;\text{ over } SU_1, \qquad \dots$$

(with $\mathrm{Supp}\, M_i = \mathrm{Supp}\, \Gamma_{i+1} \supsetneq \mathrm{Supp}\, M_{i+1}$) etc.

3 Cartan numbers

In Cartan's study of PDEs systems $\mathcal{E}$ (basically viewed as exterior differential systems in this approach) he constructed a sequence of numbers s_i, which are basic for his involutivity test. These numbers depend on the flag of subspaces one chooses for investigation of the system and so have no invariant meaning.

The classical formulation is that a general solution depends on s_p functions of p variables, s_{p-1} functions of $(p-1)$ variables, $\dots$, s_1 functions of 1 variable and s_0 constants (we adopt here the notations from [BCG^3]; in Cartan's notations [C] we should rather write s_p, $s_p + s_{p-1}$, $s_p + s_{p-1} + s_{p-2}$ etc). However as Cartan notices just after the formulation [C], this statement has only a calculational meaning.

Nevertheless two numbers are absolute invariants and play an important role. These are Cartan genre, i.e. the maximal number p such $s_p \neq 0$, but $s_{p+1} = 0$, and Cartan integer $\sigma = s_p$.

As a result of Cartan's test a general solution depends on σ functions of p variables (and some number of functions of lower number of variables, but this number can vary depending on a way we parametrize the solutions). Here general solution is a local analytic solution obtained as a result of application of Cartan-Kähler (or Cauchy-Kovalevskaya) theorem and thus being parametrized by the Cauchy data.

Hence we can think of p as of *functional dimension* and of σ as of *functional rank* of the solutions space $\mathrm{Sol}(\mathcal{E})$. In fact, we adopt this terminology further on in the paper, because as was shown in the previous section it correctly reflects the situation.

These numbers can be computed via the characteristic variety. If the characteristic sheaf over $\mathrm{Char}^{\mathbb{C}}(\mathcal{E})$ has fibers of dimension k, then

$$p = \dim \mathrm{Char}^{\mathbb{C}}(\mathcal{E}) + 1, \;\; \sigma = k \cdot \deg \mathrm{Char}^{\mathbb{C}}(\mathcal{E}).$$

The first formula is a part of Hilbert-Serre theorem ([H]), while the second is more complicated. Actually Cartan integer σ was calculated in [BCG^3] in general situation and the formula is as follows.

Let $\mathrm{Char}^{\mathbb{C}}(g) = \cup_\epsilon \Sigma_\epsilon$ be the decomposition of the characteristic variety into irreducible components and $d_\epsilon = \dim \mathcal{K}_x$ for a generic point $x \in \Sigma_\epsilon$. Then

$$\sigma = \sum d_\epsilon \cdot \deg \Sigma_\epsilon.$$

It is important that these numbers coincide with the functional dimension and rank of the previous section. Moreover the sequence of Cartan numbers s_i is related to the sequence (p_i, σ_i) of the previous section.

This can be seen from the general approach of the next and following sections, which treat the case of systems $\mathcal{E}$ of PDEs of different orders (we though make presentation for the symbolic systems, with interpretation for general systems being well-known [S, KLV, KL$_2$]).

4 Symbolic systems

Consider a vector space T of dimension n (tangent space to the set of independent variables, substitute to T_xM) and a vector space N of dimension m (tangent space to the set of dependent variables, substitute to $\pi_x = \pi^{-1}(x)$).

Spencer δ-complex is de Rham complex of polynomial N-valued differential forms on T:

$$0 \to S^kT^* \otimes N \xrightarrow{\delta} S^{k-1}T^* \otimes N \otimes T^* \xrightarrow{\delta} \cdots \xrightarrow{\delta} S^{k-n}T^* \otimes N \otimes \Lambda^n T^* \to 0,$$

where $S^iT^* = 0$ for $i < 0$. Denote by

$$\delta_v = i_v \circ \delta : S^{k+1}T^* \otimes N \to S^kT^* \otimes N$$

the differentiation along the vector $v \in T$.

The l-th prolongation of a subspace $h \subset S^kT^* \otimes N$ is

$$h^{(l)} = \{p \in S^{k+l}T^* \otimes N \ : \ \delta_{v_1} \dots \delta_{v_l} p \in h \ \forall v_1, \dots, v_l\} = S^lT^* \otimes h \cap S^{k+l}T^* \otimes N.$$

Definition. *A sequence of subspaces $g_k \subset S^kT^* \otimes N$, $k \geq 0$, with $g_0 = N$ and $g_k \subset g_{k-1}^{(1)}$, is called a symbolic system.*

If a system of PDEs $\mathcal{E}$ is given as $F_1 = 0, \dots, F_r = 0$, where F_i are scalar PDEs on M, then $T = TM, N \simeq \mathbb{R}^m$ and the system $g \subset ST^* \otimes N$ is given as $f_1 = 0, \dots, f_r = 0$, where $f_i = \sigma(F_i)$ are symbols of the differential operators at the considered point (or jet for non-linear PDEs).

With every such a system we associate its Spencer δ-complex of order k:

$$0 \to g_k \xrightarrow{\delta} g_{k-1} \otimes T^* \xrightarrow{\delta} g_{k-2} \otimes \Lambda^2 T^* \to \dots$$
$$\to g_i \otimes \Lambda^{k-i} T^* \xrightarrow{\delta} \cdots \xrightarrow{\delta} g_{k-n} \otimes \Lambda^n T^* \to 0.$$

Definition. *The cohomology group at the term $g_i \otimes \Lambda^j T^*$ is denoted by $H^{i,j}(g)$ and is called the Spencer δ-cohomology of g.*

Note that $g_k = S^k T^* \otimes N$ for $0 \le k < r$ and the first number $r = r_{\min}(g)$, where the equality is violated is called the minimal order of the system. Actually the system has several orders:

$$\operatorname{ord}(g) = \{k \in \mathbb{Z}_+ \mid g_k \neq g_{k-1}^{(1)}\}.$$

Multiplicity of an order r is:

$$m(r) = \dim g_{r-1}^{(1)}/g_r = \dim H^{r-1,1}(g).$$

Hilbert basis theorem implies finiteness of the set of orders (counted with multiplicities):

$$\operatorname{codim}(g) := \dim H^{*,1}(g) = \sum m(r) < \infty.$$

Starting from the maximal order of the system $k = r_{\max}$ we have: $g_{k+l} = g_k^{(l)}$.

If we dualize the above construction over $\mathbb{R}$, then Spencer δ-differential transforms to a homomorphism over the algebra of polynomials ST and $g^* = \oplus_i g_i^*$ becomes an ST-module. This module is called a *symbolic module* and it plays an important role in understanding PDEs.

In particular, *characteristic variety* $\operatorname{Char}^{\mathbb{C}}(g) \subset \mathbb{P}^{\mathbb{C}} T^*$ is defined as the support of this module $\operatorname{Supp}(g^*) = \{[p] : (g^*)_p \neq 0\}$ and the *characteristic sheaf* $\mathcal{K}$ over it is the family of vector spaces, which at the point $p \in \operatorname{Char}^{\mathbb{C}}(g)$ equals the value of the module at this point $\mathcal{K}_p = g^*/p \cdot g^*$. For more geometric description see [S, KLV, KL_2].

5 Commutative algebra approach

We will study only local solutions of a system of PDEs $\mathcal{E}$, which we consider in such a neighborhood that type of the symbolic system does not change from point to point (on equation) in the sense that dimensions of g_k, of the characteristic variety $\operatorname{Char}^{\mathbb{C}}(g)$ and of the fibers of $\mathcal{K}$ are the same.

It should be noted that if a system $\mathcal{E}$ is not formally integrable and $\mathcal{E}'$ is obtained from it by the prolongation-projection method [K, M2, KL_2], then the numbers p, σ change in this process, i.e. either the functional dimension or the functional rank decrease. Thus from now on we suppose the system $\mathcal{E}$ is formally integrable.

The numbers p, σ can be described using the methods of commutative algebra. Recall ([AM]) that by Hilbert-Serre theorem the sum

$$f(k) = \sum_{i \le k} \dim g_i^*$$

behaves as a polynomial in k for sufficiently large k. This polynomial is called the *Hilbert polynomial* of the symbolic module g^* corresponding to $\mathcal{E}$ and we denote it by $P_{\mathcal{E}}(z)$. If $p = \deg P_{\mathcal{E}}(z)$ and $\sigma = P_{\mathcal{E}}^{(p)}(z)$, then the highest term of this polynomial is

$$P_{\mathcal{E}}(z) = \sigma z^p + \dots$$

(see [H] for the related statements in algebraic geometry, the interpretation for PDEs is straightforward).

A powerful method to calculate the Hilbert polynomial is resolution of a module. In our case a resolution of the symbolic module g^* exists and it can be expressed via the Spencer δ-cohomology. Indeed, the Spencer cohomology of the symbolic system g is $\mathbb{R}$-dual to the Koszul homology of the module g^* and for algebraic situation this resolution was found in [Gr]. It has the form:

$$0 \to \oplus_q H^{q-n,n}(g) \otimes S^{[-q]} \xrightarrow{\varphi_n} \oplus_q H^{q-n+1,n-1}(g) \otimes S^{[-q]} \xrightarrow{\varphi_{n-1}} \dots$$
$$\to \oplus_q H^{q-1,1}(g) \otimes S^{[-q]} \xrightarrow{\varphi_1} \oplus_q H^{q,0}(g) \otimes S^{[-q]} \xrightarrow{\varphi_0} g^* \to 0,$$

where $S^{[-q]}$ is the polynomial algebra on T_x^*M with grading shifted by q, i.e. $S_i^{[-q]} = S^{i-q}T_xM$, and the maps φ_j have degree 0.

Thus denoting $h^{i,j} = \dim H^{i,j}(g)$ and $\tau_\alpha = \dim S^\alpha TM = \binom{\alpha+n-1}{\alpha}$ we have:

$$\dim g_i = \sum_q \bigl(h^{q,0}\tau_{i-q} - h^{q,1}\tau_{i-q-1} + h^{q,2}\tau_{i-q-2} - \dots + (-1)^n h^{q,n}\tau_{i-q-n}\bigr).$$

Let also $j_\beta = \sum_{\alpha\le\beta}\tau_\alpha = \dim J_v^\beta M = \binom{\beta+n}{n}$ be the dimension of the fiber of the vertical jets $J_v^\beta M$, i.e. the fiber of the jet space $J^\beta M$ over M. Thus we calculate

$$\sum_{i\le k}\dim g_i = \sum_q \bigl(h^{q,0} j_{k-q} - h^{q,1} j_{k-q-1} + h^{q,2} j_{k-q-2} - \dots \pm h^{q,n} j_{k-q-n}\bigr).$$

Finally we deduce the formula for Hilbert polynomial of the symbolic module g^*

$$P_{\mathcal{E}}(z) = \sum_q \Bigl(h^{q,0}\tbinom{z-q+n}{n} - h^{q,1}\tbinom{z-q+n-1}{n} + + h^{q,2}\tbinom{z-q+n-2}{n} - \dots + (-1)^n h^{q,n}\tbinom{z-q}{n}\Bigr).$$

Here

$$\tbinom{z+k}{k} = \frac{1}{k!}(z+1)\cdot(z+2)\cdots(z+k).$$

Denote $S_j(k_1,\dots,k_n) = \sum\limits_{i_1<\dots<i_j} k_{i_1}\cdots k_{i_j}$ the j-th symmetric polynomial and let also

$$s_i^n = \frac{(n-i)!}{n!}\,S_i(1,\dots,n)$$

Thus

$$s_0^n = 1, \quad s_1^n = \frac{n+1}{2}, \quad s_2^n = \frac{(n+1)(3n+2)}{4\cdot 3!}, \quad s_3^n = \frac{n(n+1)^2}{2\cdot 4!},$$
$$s_4^n = \frac{(n+1)(15n^3+15n^2-10n-8)}{48\cdot 5!} \quad \text{etc.}$$

If we decompose

$$\binom{z+n}{n} = \sum_{i=0}^{n} s_i^n \frac{z^{n-i}}{(n-i)!},$$

then we get the expression for the Hilbert polynomial

$$P_{\mathcal{E}}(z) = \sum_{i,j,q} (-1)^i h^{q,i} s_j^n \frac{(z-q-i)^{n-j}}{(n-j)!} = \sum_{k=0}^{n} b_k \frac{z^{n-k}}{(n-k)!},$$

where

$$b_k = \sum_{j=0}^{k} \sum_{q,i} (-1)^{i+j+k} h^{q,i} s_j^n \frac{(q+i)^{k-j}}{(k-j)!}.$$

6 Calculations for the Solutions space

We are going to compute the dimensional characteristics of two important classes of PDEs.

Involutive systems. These are such symbolic systems $g = \{g_k\}$ that all subspaces g_k are involutive in the sense of Cartan [C, BCG^3] (this definition for the symbolic systems of different orders was introduced in [KL$_5$]). Thanks to Serre's contribution [GS] we can reformulate this via Spencer cohomology as follows.

Denote by $g^{|k\rangle}$ the symbolic system generated by all differential corollaries of the system deduced from the order k:

$$g_i^{|k\rangle} = \begin{cases} S^i T^* \otimes N, & \text{for } i < k; \\ g_k^{(i-k)}, & \text{for } i \geq k. \end{cases}$$

Then the system g is involutive iff $H^{i,j}(g^{|k\rangle}) = 0$ for all $i \geq k$ (this condition has to be checked for $k \in \operatorname{ord}(g)$ only), see [KL$_5$].

In particular, $H^{i,j}(g) = 0$ for $i \notin \operatorname{ord}(g) - 1$, $(i,j) \neq (0,0)$, and the resolution for the symbolic module g^* as well as the formula for the Hilbert polynomial of $\mathcal{E}$ become easier.

Let us restrict for simplicity to the case of systems of PDEs $\mathcal{E}$ of pure first order. Then

$$P_{\mathcal{E}}(z) = h^{0,0}\binom{z+n}{n} - h^{0,1}\binom{z+n+1}{n+1} + h^{0,2}\binom{z+n+2}{n+2} - \dots$$
$$= b_1 \frac{z^{n-1}}{(n-1)!} + b_2 \frac{z^{n-2}}{(n-2)!} + \dots + b_0.$$

Vanishing of the first coefficient $b_0 = 0$ is equivalent to vanishing of Euler characteristic for the Spencer δ-complex, $\chi = \sum_i (-1)^i h^{0,i} = 0$, and this is equivalent to the claim that not all the covectors from ${}^{\mathbb{C}}T^* \setminus 0$ are characteristic for the system g.

The other numbers b_i are given by the general formulas from the previous section, but they simplify in our case. For instance

$$b_1 = \tfrac{n+1}{2}\, b_0 - \sum (-1)^i h^{0,i} i = \sum (-1)^{i+1} i \cdot h^{0,i}.$$

If $\operatorname{codim} \operatorname{Char}^{\mathbb{C}}(\mathcal{E}) = n - p > 1$, then $b_1 = 0$ and in fact then $b_i = 0$ for $i < n - p$, but $b_{n-p} = \sigma$.

Theorem. *If* $\operatorname{codim} \operatorname{Char}^{\mathbb{C}}(\mathcal{E}) = n - p$*, then the functional rank of the system equals*

$$\sigma = \sum_i (-1)^i h^{0,i} \frac{(-i)^{n-p}}{(n-p)!}.$$

Proof. Indeed one successively calculate the coefficients using the formula

$$b_k = \sum_i \sum_{\alpha=0}^{k} (-1)^{i+\alpha} h^{0,i} s^n_{k-\alpha} \frac{i^\alpha}{\alpha!}$$

and notes that b_k equals to the displayed expression plus a linear combination of $b_{k-1}, \dots, b_0$. The claim follows. □

One can extend the above formula for general involutive system and thus compute the functional dimension and functional rank of the solutions space (some interesting calculations can be found in classical works [J, C]).

Cohen-Macaulay systems. A symbolic system g (and the respective PDEs system $\mathcal{E}$) is called Cohen-Macaulay ([KL$_2$]) if the corresponding symbolic module g^* is Cohen-Macaulay, i.e. (see [M1, E] for details)

$$\dim g^* = \operatorname{depth} g^*.$$

Consider an important partial case (we formulate the definition only for symbolic systems; PDEs are treated in [KL$_4$]):

Definition. *A symbolic system* $g \subset ST^* \otimes N$ *(*$n = \dim T$, $m = \dim N$*) of* $\operatorname{codim}(g) = r$ *is called a* generalized complete intersection *if*

- $m \le r < n + m$;
- $\operatorname{codim}_{\mathbb{C}} \operatorname{Char}^{\mathbb{C}}(g) = r - m + 1$;
- $\dim \mathcal{K}_x = 1\ \forall x \in \operatorname{Char}^{\mathbb{C}}(g) \subset P^{\mathbb{C}}T^*$.

Formal integrability of such systems are given by the compatibility conditions expressed via brackets (for scalar systems [KL_1, KL_3]) or multi-brackets (for vector systems [KL_4]). In this case we can calculate Cartan genre and integer directly.

Theorem. *Let $\mathcal{E}$ be a system of generalized complete intersection type and suppose it is formally integrable. Then the functional dimension of* $\mathrm{Sol}(\mathcal{E})$ *is*

$$p = m + n - r - 1$$

and the functional rank is

$$\sigma = S_{r-m+1}(k_1, \dots, k_r) = \sum_{i_1 < \dots < i_{r-m+1}} k_{i_1} \cdots k_{i_{r-m+1}},$$

the l-th symmetric polynomial of the orders $k_1, \dots, k_r$ of the system.

Note that if the last requirement in the definition of generalized complete intersection is changed to $\dim \mathcal{K}_x = d$ everywhere on the characteristic variety, then the functional rank will be multiplied by d:

$$\sigma = d \cdot S_{r-m+1}(k_1, \dots, k_r).$$

However the formal integrability criterion for generalized complete intersections is proved in [KL_4] under assumption that $d = 1$.

Proof. We shall consider the case of a system g of a pure order: $k_1 = \dots = k_r = k$, $k_i \in \mathrm{ord}(g)$. The case of different orders is similar and will appear elsewhere.

The formula for functional dimension p follows directly from the definition of generalized complete intersection. Let's calculate σ.

We can use interpretation of the Cartan integer σ from §3. Recall that characteristic variety $\mathrm{Char}^{\mathbb{C}}(g)$ is the locus of the characteristic ideal $I(g) = \mathrm{Ann}(g)$, which the the annihilator of g^* in ST.

Since the module is represented by the matrix with polynomial entries (each differential operator Δ_i giving a PDEs system $\mathcal{E}$ is a column Δ_{ij}, $1 \le i \le r$, $1 \le j \le m$; so that their union is a $m \times r$ matrix $M(\Delta)$), its annihilator is given by the zero Fitting ideal (in fact, here we use the condition on grade of the ideal: $\operatorname{depth} \mathrm{Ann}(g) = r - m + 1$, which follows from the conditions of the above definition).

This ideal $\mathrm{Fitt}_0(g)$ is generated by all determinants of $m \times m$ minors of the corresponding to $M(\Delta)$ matrix of symbols $M(\sigma_\Delta)$. These minors are determined by a choice of m from r columns, so that there are $\binom{r}{m}$ determinants and each is a polynomial of degree k^{r-m+1}.

However not all the minors are required to determine $\mathrm{Char}^{\mathbb{C}}(g)$ and this is manifested by the fact, that we sum $\binom{r}{m-1}$ degrees k^{r-m+1} to get the functional rank σ. The easiest way to explain this is via the Hilbert polynomial of the symbolic module g^*.

This can be calculated since under the assumption of generalized complete intersection g^* possesses a resolution in the form of Buchsbaum-Rim complex (see [KL$_4$]):

$$0 \to S^{r-m-1}V^\star \otimes \Lambda^r U \to S^{r-m-2}V^\star \otimes \Lambda^{r-1}U \to \\ \cdots \to \Lambda^{m+1}U \to U \to V \to g^* \to 0,$$

where $V \simeq ST \otimes N^*$ (recall that $\dim N = m$ and $g \subset ST^* \otimes N$) and $U = \underbrace{ST \oplus \cdots \oplus ST}_{r \text{ terms}}$. Star $\star$ means dualization over ST and the tensor products are over ST as well.

Now the claim follows from the detailed investigation of degrees of the homomorphisms in the above exact sequence. To see this we suppose at first that $r = m+n-1$ and use the following assertion.

Lemma. *The following combinatorial formula holds:*

$$m\binom{n+k(n+m-1)}{n} - (n+m-1)\binom{n+k(n+m-2)}{n} \\ + \sum_{j=1}^{n-1}(-1)^{j-1}\binom{j+m-2}{m-1}\binom{n+m-1}{j+m}\binom{(k+1)n-k(1+j)}{n} = \binom{n+m-1}{n}k^n.$$

We would like to comment and interpret the sum on the left hand side of this formula. In our case the system is of finite type (g^* has finite dimension as a vector space) and $\sigma = \sum \dim g_i$ (the sum is finite).

Stabilization of the symbol occurs at the order $i = \sum k_i - 1 = k(n+m-1) - 1$: $g_i = 0$. So we prolong $\mathcal{E}$ to the jets of order $k(n+m-1)$ and the first term is just $\dim J_v^{k(n+m-1)}(M,N)$.

The next term is due to the fact that $\mathcal{E} \subset J^k(M,N)$ is proper. It is given by $r = n+m-1$ equations of order k, we which we differentiate up to $k(n+m-2)$ times along all coordinate directions (prolongation).

There are relations between these derivatives. These are compatibility conditions (1-syzygy of g^*), which appear in the form of multi-brackets [KL$_4$], in our case this bracket uses $(m+1)$-tuples of Δ_i.

There are in turn relations among relations (2-syzygy of g^*), which are identities between multi-brackets (these we call generalized Plücker identities, to appear soon), in our case these latter use $(m+2)$-tuples of the defining operators Δ_i etc.

Due to exact form of the relations (higher syzygies) we get factors $\binom{j+m-2}{m-1}$ in the summations formula of the lemma.

In the case $r < n+m-1$ we should perform a reduction, which is possible by Theorem A [KL$_2$]. Then the functional dimension p grows, but the functional rank remains the same and the previous calculation works. $\square$

7 Examples

Here we show some examples demonstrating the above results.

1. *Intermediate integral* of a system $\mathcal{E} \subset J^k\pi$ is such a system $\tilde{\mathcal{E}} \subset J^{\tilde{k}}\pi$ that $\tilde{k} < k$ and $\mathcal{E} \subset \tilde{\mathcal{E}}^{(k-\tilde{k})}$ (where $\mathcal{E}^{(i)}$ is the i-th prolongation of the system). Since every solution to the system $\mathcal{E}$ is a solution to $\tilde{\mathcal{E}}^{(k-\tilde{k})}$ we conclude: Whenever the functional dimension $p > 0$, we have $\tilde{p} = p$ and $\tilde{\sigma} = \sigma$.

Indeed the solutions of $\tilde{\mathcal{E}}^{(k-\tilde{k})}$ form a finite-dimensional parametric family, such that solutions of $\tilde{\mathcal{E}}$ appear for some fixed values of parameters (because we differentiate with respect to all variables to obtain the prolongation). Thus the number of functions of $p > 0$ variables, on which a general solution depends, will not be altered.

2. If the PDEs system $\mathcal{E}$ is underdetermined, then $p = n$ and $\sigma \geq 1$. Indeed, σ is precisely the under-determinacy degree, i.e. the minimal number of unknown functions that should be arbitrarily fixed to get a determined system. We assume we can do it to get a formally integrable system. If underdetermined system is not formally integrable, compatibility conditions can turn it into determined or over-determined and then decrease p and change σ.

A nice illustration is the Hilbert-Cartan system

$$z'(x) = (y''(x))^2.$$

It has $p = 1$, $\sigma = 1$. But even though a general solution depends on one function of one variable, it cannot be represented in terms of a function and its derivatives only (Hilbert's theorem).

3. As we noticed earlier the similar situation happens to overdetermined system: If $\mathcal{E}$ is not formally integrable, and $\tilde{\mathcal{E}}$ is obtained from $\mathcal{E}$ by prolongation-projection technique (sometimes it is said that $\tilde{\mathcal{E}}$ is the involutive form of $\mathcal{E}$, but this is not true, only a certain prolongation of $\tilde{\mathcal{E}}$ is), then $\tilde{p} < p$ or $[\tilde{p} = p$ and $\tilde{\sigma} < \sigma]$. Indeed, supplement of additional equations shrinks the solution space.

For instance if we consider two second-order scalar differential equations on the plane

$$F\bigl(x, y, u(x,y), Du(x,y), D^2u(x,y)\bigr) = 0, \quad G\bigl(x, y, u(x,y), Du(x,y), D^2u(x,y)\bigr) = 0,$$

such that F and G have no common complex characteristics, then the compatibility condition of this system $\mathcal{E}$ can be expressed via the Mayer bracket ([KL$_1$]): $H = [F, G]_{\mathcal{E}}$. If $H = 0$, then $p = 0$, $\sigma = 4$. If $H \neq 0$, then $p = 0$ and $\sigma \leq 3$, the equality being given by the Frobenius condition for the system $\tilde{\mathcal{E}} = \{F = 0, G = 0, H = 0\}$.

If the system has one common characteristic and is compatible, we have: $p = 1$, $\sigma = 1$. Pairs of such systems are basic examples of Darboux integrability.

4. Evolutionary equations $u_t = L[u]$ provide interesting examples, which usually "contradict" the theory. Consider for instance the heat equation

$$u_t = u_{xx}.$$

It is formally integrable and analytic. We can try to specify the initial condition $u|_{t=0} = \varphi(x)$ and then solve the Cauchy problem, so that we get $p = 1$, $\sigma = 1$. On the other hand we can let $u|_{x=0} = \psi_0(t), u_x|_{x=0} = \psi_1(t)$ and then get $p = 1$, $\sigma = 2$.

If we calculate the numbers using our definitions of functional dimension and functional rank (for instance, via Hilbert polynomial), it turns out that the second approach is correct. Indeed with the first idea we come into trouble with certain Cauchy data: Let, for instance, $\varphi(x) = (1-x)^{-1}$, which is an analytic function around the origin. Then the analytic solution should have the series

$$u(t,x) \doteqdot \frac{1}{1-x} + \frac{2}{1}\frac{t}{(1-x)^3} + \frac{4!}{2!}\frac{t^2}{(1-x)^5} + \dots + \frac{(2n)!}{n!}\frac{t^n}{(1-x)^{2n+1}} + \dots$$

which diverges everywhere outside $t = 0$. The reason why the second approach provides no problem is because the line $\{x = 0\}$ is non-characteristic and we can solve our first order PDE by the classical method of Cauchy characteristics.

Remark however that in the standard courses of mathematical physics the heat equation is solved with the first approach (by Fourier method). How is it possible?

Explanation is that we solve the heat equation then only for positive time $t \geq 0$. Doing the same method in negative direction blows up the solutions immediately (heat goes rapidly to equilibrium, but we cannot predict even closest past)! We here are interested in the solutions, which exist in an open neighborhood of the origin (like in Cauchy-Kovalevskaya theorem), and this contradicts the first approach.

5. Similar problems arise with Cauchy problems in other PDEs systems: one usually applies reduction or fixes gauge, but this can change dimensional characteristics.

For instance, consider the Cauchy problem for the Einstein vacuum equations, which is a system of 10 PDEs of 10 unknown functions. The system is over-underdetermined (i.e. it has compatibility conditions). In wave gauge [CB] its solution depends on several functions on a 3-dimensional space, which are subject to constraint equations, so that $p = 2$. On the other hand, the original Einstein system is invariants under diffeomorphisms and this yields $p = 4$.

One should also be careful with Cauchy data in higher order, since then the definition of characteristics becomes more subtle, see [KL_5].

6. Consider a system $\mathcal{E}$, which describes automorphisms of a given geometric structure. The corresponding symbolic system is $g \subset ST^*\otimes T$. The automorphism group has maximal dimension iff the system is formally integrable. Consider the examples, when the geometric structure is symplectic, complex or Riemannian (all these structures are of the first order).

Let at first g be generated by $g_1 = \mathrm{sp}(n) \subset T^* \otimes T$. Our tangent space $T = T_xM$ is equipped with a symplectic structure ω, and we can identify $T^* \overset{\omega}{\simeq} T$ and we get $g_1 = S^2T^* \subset T^* \otimes T^*$. The prolongations are $g_i = S^{i+1}T^* \subset S^iT^* \otimes T$.

The system is easily checked to be involutive and the only non-vanishing Spencer δ-cohomology groups are

$$H^{0,i}(g) = \Lambda^{i+1}T^*.$$

Then one checks that the Euler characteristic is $\chi = 1 \neq 0$ and so $b_0 \neq 0$. Thus the functional dimension is $p = n$. Indeed the characteristic variety is $\mathbb{P}^{\mathbb{C}}T^*$ because each non-zero covector p is characteristic: $p^2 \in g_1 \simeq S^2T^*$. Next by a theorem from §6 one calculates the functional rank

$$\sigma = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i+1} \frac{(-i)^0}{0!} = \chi = 1.$$

This result is easy to verify: an infinitesimal symplectic transformation has a generating function (Hamiltonian) and so it is determined by one function of n variables.

If we turn to (almost) complex structures J on M, then $g_1 = \mathrm{gl}(\frac{n}{2}, \mathbb{C}) = T^* \otimes_{\mathbb{C}} T$ (space of $\mathbb{C}$-linear endomorphisms of T) and the prolongations are $g_i = S^i_{\mathbb{C}} T^* \otimes_{\mathbb{C}} T$.

The characteristic variety is proper and one calculates that $p = \frac{n}{2}$, $\sigma = n$. The system is again involutive. The second Spencer cohomology is

$$H^{0,2}(g) = \Lambda^2_{\mathbb{C}} T^* \otimes_{\bar{\mathbb{C}}} T,$$

which is the space of $\mathbb{C}$-antilinear skew-symmetric $(2,1)$ tensors (Nijenhuis tensors).

The last example is the algebra of Riemannian isometries (i.e. T is equipped with a Riemannian structure) of a Riemannian metric q on M. The symbol is $g_1 = o(n)$ and the prolongations are zero $g_2 = g_3 = \cdots = 0$.

This system is not involutive. For instance,

$$H^{1,2}(g) = \mathrm{Ker}\big(S^2\Lambda^2T^* \to \Lambda^4T^*\big)$$

(the space of Riemannian curvatures) is non-zero (for $n = \dim T > 1$). Since the system is of finite type, the characteristic variety is empty and $p = 0$. The general solution (isometry) depends on $\sigma = \frac{(n+1)n}{2}$ constants.

We recall, that the above dimensional conclusions are correct if the system $\mathcal{E}$ is integrable, otherwise the space $\mathrm{Sol}(\mathcal{E})$ shrinks. In the above examples this means: the form ω is closed (with just non-degeneracy we have almost-symplectic manifold); the structure J is integrable (Nijenhuis tensor N_J vanishes); the manifold (M, q) has constant sectional curvature (so it is a spacial form).

Literatur

[AM] M. F. Atiyah, I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley (1969).

[C] E. Cartan, *Les systèmes différentiels extérieurs et leurs applications géométriques* (French), Actualités Sci. Ind. **994**, Hermann, Paris (1945).

[CB] Y. Choquet-Bruhat (Foures-Bruhat), *Theoreme d'existence pour certains systemes d'equations aux derivees partielles non lineaires* (French), Acta Math. **88**, 141–225 (1952).

[BCG^3] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths, *Exterior differential systems*, MSRI Publications **18**, Springer-Verlag (1991).

[E] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag (1995).

[Go] H. Goldschmidt, *Integrability criteria for systems of nonlinear partial differential equations*, J. Diff. Geom., **1**(3) (1967), 269–307.

[Gr] M L. Green, *Koszul cohomology and geometry*, in: Lectures on Riemann surfaces", World Sci. Publ., 177–200 (1989).

[GS] V. Guillemin, S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. A.M.S., **70** (1964), 16–47.

[H] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag (1977).

[Ho] L. Hörmander, *Linear partial differential operators*, Springer Verlag (1976).

[J] M. Janet, *Leçons sur les systèms d'équations*, Gauthier-Villers, Paris (1929).

[Ka] E. Kähler, *Einfürhung in die Theorie der Systeme von Differentialgleichungen*, B.G. Teubner, Leipzig (1934).

[KLV] I. S. Krasilschik, V. V. Lychagin, A. M. Vinogradov, *Geometry of jet spaces and differential equations*, Gordon and Breach (1986).

[KL$_1$] B. S. Kruglikov, V. V. Lychagin, *Mayer brackets and solvability of PDEs – I*, Diff. Geom. and its Appl. **17** (2002), 251–272.

[KL$_2$] B. S. Kruglikov, V. V. Lychagin, *Mayer brackets and solvability of PDEs – II*, Trans. Amer. Math. Soc. **358**, no.3 (2005), 1077–1103.

[KL$_3$] B. S. Kruglikov, V. V. Lychagin, *A compatibility criterion for systems of PDEs and generalized Lagrange-Charpit method*, A.I.P. Conference Proceedings, "Global Analysis and Applied Mathematics: International Workshop on Global Analysis", **729**, no. 1, 39-53 (2004).

[KL$_4$] B. S. Kruglikov, V. V. Lychagin, *Multi-brackets of differential operators and compatibility of PDE systems*, Comptes Rendus Math. **342**, no. 8 (2006), 557–561.

[KL$_5$] B. S. Kruglikov, V. V. Lychagin, *Spencer delta-cohomology, restrictions, characteristics and involutive symbolic PDEs*, ArXive: math.DG/0503124.

[K] M. Kuranishi, *On E. Cartan's prolongation theorem of exterior differential systems*, Amer. J. Math. **79** (1957), 1–47.

[L] H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. (2) **66** (1957), 155–158.

[M1] H. Matsumura, *Commutative algebra*, New York: Benjamin (1970).

[M2] Y. Matsushima, *On a theorem concerning the prolongation of a differential system*, Nagoya Math. J. **6**, 1–16 (1953).

[S] D. C. Spencer, *Overdetermined systems of linear partial differential equations*, Bull. Amer. Math. Soc., **75** (1969), 179–239.