On integrability of the Camassa–Holm equation and its invariants

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To Valentin Lychagin, ad honorem his 60th birthday

Abstract Using geometrical approach exposed in Refs. [12,13], we explore the Camassa–Holm equation (both in its initial scalar form, and in the form of 2×2-system). We describe Hamiltonian and symplectic structures, recursion operators and infinite series of symmetries and conservation laws (local and nonlocal).

1 Introduction

The Camassa–Holm equation was introduced in [4] in the form

$$u_t+\mu u_x-u_{xxt}+3uu_x=2u_xu_{xx}+uu_{xxx},\qquad \mu\in\mathbb{R}, \tag{1}$$

and was intensively explored afterwards (see, for example, Refs. [5–7,17]). Its superizations were also constructed, see [1,19]. Since (1) is not an evolution equation, its integrability properties (existence and even definition of Hamiltonian structures, conservation laws, etc.) are not standard to establish.

One of the ways widely used to overcome this difficulty is to introduce a new unknown $m=u-u_{xx}$ and transform Eq. (1) to the system

$$\begin{cases} m_t=-um_x-(2m+\mu)u_x,\\ m=u-u_{xx}, \end{cases} \tag{2}$$

which has *almost* evolutionary form. We stress this "almost", because the second equation in (2) (that can be considered as a constrain to the first one) disrupts the picture and, at best, necessitates to invert the operator $1-D_x^2$. At worst, dealing with Eq. (2) as with an evolution equation may lead to fallacious results.

In our approach based on the geometrical framework exposed in Ref. [3], we treat the equation at hand as a submanifold in the manifold of infinite jets and consider two natural

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extensions of this equation, cf. with Ref [16]. The first one is called the ℓ-covering and serves the role of the tangent bundle. The second extension, ℓ^*-covering, is the counterpart to the cotangent bundle. The key property of these extensions is that the spaces of their nonlocal (in the sense of [15]) symmetries and cosymmetries contain all essential integrability invariants of the initial equation. The efficiency of the method was tested for a number of problems (see Refs. [12–14]) and we apply it to the Camassa–Holm equation here.

In Section 2 we briefly expose the necessary definition and facts. Section 3 contains computations for the Camassa–Holm equation in its matrix version (computations and results are more compact in this representation), while in Section 4 we reformulate them for the original form (1) and compare later the results obtained for the two alternative presentations. Finally, Section 5 contains discussion of the results obtained. Throughout our exposition we use a very stimulating conceptual parallel between categories of smooth manifolds and differential equations proposed initially by A.M. Vinogradov and in its modern form presented in Table 1.

Table 1 Conceptual parallel between two categories

Manifolds	**Equations**
Smooth manifold	Infinitely prolonged equation
Point	Formal solution
Smooth function	Conservation law
Vector field	Higher symmetry
Differential 1-form	Cosymmetry
de Rham complex	$(n-1)$st line of Vinogradov's $\mathscr{C}$-spectral sequence
Tangent bundle	ℓ-covering
Cotangent bundle	ℓ^*-covering

This table is not just a toy dictionary but a quite helpful tool to formulate important definitions and results. For example, a bivector on a smooth manifold M may be understood as a derivation of the ring $C^\infty(M)$ with values in $C^\infty(T^*M)$. Translating this statement to the language of differential equations we come to the definition of variational bivectors and their description as shadows of symmetries in the ℓ^*-covering (see Theorem 2 below). Another example: any vector field (differential 1-form) on M may be treated as a function on T^*M (on TM). Hence, to any symmetry (cosymmetry) there corresponds a conservation law on the space of the ℓ^*-covering (ℓ-covering). This leads to the notions of nonlocal vectors and forms that, in turn, provide a basis to construct weakly nonlocal structures (see Subsections 3.4 and 3.5). Of course, these parallels are not completely straightforward (in technical aspects, especially), but extremely enlightening and fruitful.

The idea of this paper arose in the discussions one of the authors had with Volodya Roubtsov in 2007. We agreed to write two parallel texts on integrability of the Camassa–Holm equation that reflect our viewpoints. The reader can now compare our results with the ones presented in [20].

2 Underlying theory

We present here a concise exposition of the theoretical background used in the subsequent sections, see Refs. [3, 13, 15].

2.1 Equations, symmetries, etc.

Let $\pi\colon E \to M$ be a fiber bundle and $\pi_\infty\colon J^\infty(\pi) \to M$ be the bundle of its infinite jets. To simplify our exposition we shall assume that π is a vector bundle. In all applications below π is the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$. We consider *infinite prolongations* of differential equations as submanifolds $\mathscr{E} \subset J^\infty(\pi)$ and retain the notation π_∞ for the restriction $\pi_\infty|_{\mathscr{E}}$. Any such a manifold is endowed with the *Cartan distribution* which spans at every point tangent spaces to the graphs of jets. A *symmetry* of $\mathscr{E}$ is a vector field that preserves this distribution. The set of symmetries is a Lie algebra over $\mathbb{R}$ denoted by $\operatorname{sym}\mathscr{E}$.

For any equation $\mathscr{E}$ its *linearization operator* $\ell_{\mathscr{E}}\colon \varkappa \to P$ is defined, where $\varkappa$ is the module[1] of sections of the pullback $\pi_\infty(\pi)$ and P is the module of sections of some vector bundle over $\mathscr{E}$. Then $\operatorname{sym}\mathscr{E}$ can be identified with solutions of the equation

$$\ell_{\mathscr{E}}(\varphi) = 0, \qquad \varphi \in \varkappa. \tag{3}$$

For two symmetries $\varphi_1, \varphi_2 \in \operatorname{sym}\mathscr{E}$ their commutator is denoted by $\{\varphi_1, \varphi_2\}$.

Denote by Λ_h^i the module of horizontal i-forms on $\mathscr{E}$ and introduce the notation

$$\hat{Q} = \operatorname{Hom}_{\mathscr{F}}(Q, \Lambda_h^n), \qquad n = \dim M,$$

for any module Q. The *adjoint* to $\ell_{\mathscr{E}}$ operator

$$\ell_{\mathscr{E}}^*\colon \hat{P} \to \hat{\varkappa} \tag{4}$$

arises and solutions of the equation

$$\ell_{\mathscr{E}}^*(\psi) = 0, \qquad \psi \in \hat{P}, \tag{5}$$

are called *cosymmetries* of $\mathscr{E}$; the space of cosymmetries is denoted by $\operatorname{cosym}\mathscr{E}$.

Let $d_h\colon \Lambda_h^i \to \Lambda_h^{i+1}$ be the *horizontal de Rham differential*. A *conservation law* of the equation $\mathscr{E}$ is a closed form $\omega \in \Lambda_h^{n-1}$. To any conservation law there corresponds its *generating function* $\delta\omega \in \operatorname{cosym}\mathscr{E}$, where $\delta\colon E_1^{0,n-1} \to E_1^{1,n-1}$ is the differential in the E_1 term of Vinogradov's $\mathscr{C}$-spectral sequence, see [21]. In the evolutionary case δ coincides with the Euler–Lagrange operator. A conservation law is *trivial* if its generating function vanishes. In particular, d_h-exact conservation laws are trivial.

A vector field on $\mathscr{E}$ is called a *$\mathscr{C}$-field* if it lies in the Cartan distribution. A differential operator $\Delta\colon P \to Q$, P and Q being $\mathscr{F}$-modules, is called a *$\mathscr{C}$-differential operator* if it is locally expressed in terms of $\mathscr{C}$-fields. For example, $\ell_{\mathscr{E}}$ is a $\mathscr{C}$-differential operator.

A $\mathscr{C}$-differential operator $\mathsf{H}\colon \hat{P} \to \varkappa$ is said to be a *variational bivector* on $\mathscr{E}$ if

$$\ell_{\mathscr{E}} \circ \mathsf{H} = \mathsf{H}^* \circ \ell_{\mathscr{E}}^*. \tag{6}$$

This condition means that H takes cosymmetries of $\mathscr{E}$ to symmetries. If $\mathscr{E}$ is an evolution Eq. (6) implies also that $\mathsf{H}^* = -\mathsf{H}$. A bivector H is a *Hamiltonian structure*[2] on $\mathscr{E}$ if $[\![\mathsf{H},\mathsf{H}]\!]^{\mathrm{s}} = 0$, where $[\![\cdot,\cdot]\!]^{\mathrm{s}}$ is the *variational Schouten bracket* (see also [11]). Two Hamiltonian structures are *compatible* if $[\![\mathsf{H}_1,\mathsf{H}_2]\!]^{\mathrm{s}} = 0$.

A $\mathscr{C}$-differential operator $\mathsf{S}\colon \varkappa \to \hat{P}$ is called a *variational 2-form on $\mathscr{E}$* if

$$\ell_{\mathscr{E}}^* \circ \mathsf{S} = \mathsf{S}^* \circ \ell_{\mathscr{E}}. \tag{7}$$

[1] All the modules below are modules over the ring $\mathscr{F}$ of smooth functions on $\mathscr{E}$.

[2] It is more appropriate to call these objects *Poisson structures*, but we follow the tradition accepted in the theory of integrable systems.

Such operators take symmetries to cosymmetries and in evolutionary case are skew-adjoint. They are elements of the term $E_1^{2,n-1}$ of Vinogradov's $\mathscr{C}$-spectral sequence. A variational form is a *symplectic structure* on the equation $\mathscr{E}$ if it is *variationally closed*, i.e., $\delta\mathsf{S} = 0$, where $\delta\colon E_1^{2,n-1} \to E_1^{3,n-1}$ is the corresponding differential.

We shall also consider *recursion $\mathscr{C}$-differential operators* $\mathsf{R}\colon \varkappa \to \varkappa$ and $\hat{\mathsf{R}}\colon \hat{P} \to \hat{P}$ satisfying the conditions

$$\ell_{\mathscr{E}} \circ \mathsf{R} = \mathsf{R}' \circ \ell_{\mathscr{E}}, \qquad \ell_{\mathscr{E}}^* \circ \hat{\mathsf{R}} = \hat{\mathsf{R}}' \circ \ell_{\mathscr{E}}^* \tag{8}$$

for some $\mathscr{C}$-differential operators $\mathsf{R}'\colon P \to P$ and $\hat{\mathsf{R}}'\colon \hat{\varkappa} \to \hat{\varkappa}$. An operator R satisfies the *Nijenhuis condition* if $[\![\mathsf{R},\mathsf{R}]\!]^{\mathrm{n}} = 0$, where $[\![\cdot,\cdot]\!]^{\mathrm{n}}$ is the *variational Nijenhuis bracket*. A recursion operator R is *compatible* with a Hamiltonian structure H is the composition $\mathsf{R} \circ \mathsf{H}$ is a Hamiltonian structure as well.

2.2 Nonlocal theory

Let $\mathscr{E}$ and $\tilde{\mathscr{E}}$ be equations and $\xi : \tilde{\mathscr{E}} \to \mathscr{E}$ be a fiber bundle. Denote by $\mathscr{C}$ and $\tilde{\mathscr{C}}$ the Cartan distributions on $\mathscr{E}$ and $\tilde{\mathscr{E}}$, resp. We say that ξ is a *covering* if for any $\tilde{\theta} \in \tilde{\mathscr{E}}$ the differential $d_{\tilde{\theta}}\xi$ isomorphically maps $\tilde{\mathscr{C}}_{\tilde{\theta}}$ onto $\mathscr{C}_{\xi(\tilde{\theta})}$. A particular case of coverings (the so-called *Abelian* coverings) is naturally associated with closed horizontal 1-forms[3].

By definition, any $\mathscr{C}$-field X on $\mathscr{E}$ can be uniquely lifted to a $\mathscr{C}$-field $\tilde{X}$ on $\tilde{\mathscr{E}}$ such that $d\xi(\tilde{X}) = X$. Consequently, any $\mathscr{C}$-differential operator $\Delta\colon P \to Q$ is extended to a $\mathscr{C}$-differential operator

$$\tilde{\Delta}\colon \tilde{\mathscr{F}} \otimes_{\mathscr{F}} P \to \tilde{\mathscr{F}} \otimes_{\mathscr{F}} Q,$$

$\tilde{\mathscr{F}}$ being the algebra of smooth functions on $\tilde{\mathscr{E}}$.

A *nonlocal* ξ-(co)symmetry of $\mathscr{E}$ is a (co)symmetry of the covering equation $\tilde{\mathscr{E}}$. They are solutions of the equations $\ell_{\tilde{\mathscr{E}}}\varphi = 0$ and $\ell_{\tilde{\mathscr{E}}}^*\psi = 0$, resp. Along with these two equations one can consider the equations

$$(1)\ \ \tilde{\ell}_{\mathscr{E}}\varphi = 0, \qquad (2)\ \ \tilde{\ell}_{\mathscr{E}}^*\psi = 0. \tag{9}$$

Their solutions are called ξ*-shadows* of symmetries and cosymmetries, resp. A shadow of symmetry is a derivation $\mathscr{F} \to \tilde{\mathscr{F}}$ that preserves the Cartan distributions. For any two shadows of symmetries φ_1 and φ_2 their commutator $\{\varphi_1,\varphi_2\}$ can be defined. This commutator is a shadow in a new covering that is canonically determined by φ_1 and φ_2.

2.3 The ℓ- and ℓ^*-coverings

Let $\mathscr{E} \subset J^\infty(\pi)$ be an equation. Its *ℓ-covering* $\tau\colon \mathscr{L}(\mathscr{E}) \to \mathscr{E}$ is obtained by adding to $\mathscr{E}$ the equation $\ell_{\mathscr{E}}(q) = 0$, where q is a new variable. Dually, the *ℓ^*-covering* $\tau^*\colon \mathscr{L}^*(\mathscr{E}) \to \mathscr{E}$ is constructed by adding the equation $\ell_{\mathscr{E}}^*(p) = 0$ with a new variable p. They are the exact counterparts of the tangent and cotangent bundles in the category of differential equations. By the reasons that will become clear later, we regard both q and p as *odd* variables. The main point of our method is the fundamental relation between integrability invariants of $\mathscr{E}$ and shadows in τ and τ^*. To formulate this relation, let us give an auxiliary definition: for an arbitrary operator equation $A \circ \Delta = \nabla \circ B$ we say that Δ is a *trivial solutions* if Δ is of the

[3] When $\dim M = 2$, Abelian coverings are associated with conservation laws of the equation $\mathscr{E}$.

form $\Delta = \Delta' \circ B$. Classes of solutions modulo trivial ones will be called *nontrivial solutions*. Then the following results hold.

Theorem 1 (shadows in the ℓ-covering) *There is a one-to-one correspondence between nontrivial solutions of the equation*

$$\ell_{\mathscr{E}} \circ \mathsf{R} = \mathsf{R}' \circ \ell_{\mathscr{E}}, \qquad \mathsf{R}\colon \varkappa \to \varkappa,$$

and τ-shadows of symmetries linear w.r.t. the variables q. In a similar way, there is a one-to-one correspondence between nontrivial solutions of the equation

$$\ell^*_{\mathscr{E}} \circ \mathsf{S} = \mathsf{S}' \circ \ell_{\mathscr{E}}, \qquad \mathsf{S}\colon \varkappa \to \hat{P},$$

and τ-shadows of cosymmetries linear w.r.t. the variables q.

Theorem 2 (shadows in the ℓ^*-covering) *There is a one-to-one correspondence between nontrivial solutions of the equation*

$$\ell_{\mathscr{E}} \circ \mathsf{H} = \mathsf{H}' \circ \ell^*_{\mathscr{E}}, \qquad \mathsf{H}\colon \hat{P} \to \varkappa,$$

and τ^*-shadows of symmetries linear w.r.t. the variables p. In a similar way, there is a one-to-one correspondence between nontrivial solutions of the equation

$$\ell^*_{\mathscr{E}} \circ \hat{\mathsf{R}} = \hat{\mathsf{R}}' \circ \ell^*_{\mathscr{E}}, \qquad \hat{\mathsf{R}}\colon \hat{P} \to \hat{P},$$

and τ^*-shadows of cosymmetries linear w.r.t. the variables p.

Theorem 3 *Let R_1 and R_2 be recursion operators for symmetries on $\mathscr{E}$ and Φ_{R} denote the τ-shadow corresponding to R. Then $[\![\mathsf{R}_1, \mathsf{R}_2]\!]^{\mathrm{n}} = 0$ iff $\{\Phi_{\mathsf{R}_1}, \Phi_{\mathsf{R}_1}\} = 0$. Similarly, if H_1 and H_2 are bivectors then $[\![\mathsf{H}_1, \mathsf{H}_2]\!]^{\mathrm{s}} = 0$ iff $\{\Phi^*_{\mathsf{H}_1}, \Phi^*_{\mathsf{H}_1}\} = 0$, where Φ^*_{H} denotes the τ^*-shadow corresponding to H.*

In both cases the curly brackets denote the *super* bracket of shadows that arises due to oddness of the variables q and p. Additional discussion of Theorem 3 the reader will find in Remark 2.

Theorem 4 *To any cosymmetry of $\mathscr{E}$ there canonically corresponds a conservation law of $\mathscr{L}(\mathscr{E})$. Dually, to any symmetry of $\mathscr{E}$ there canonically corresponds a conservation law of $\mathscr{L}^*(\mathscr{E})$.*

2.4 Computational scheme

Let locally the equation $\mathscr{E}$ be given by the system

$$\begin{cases} F^1(x^1, \dots, x^n, \dots, \frac{\partial^{|\sigma|} u^j}{\partial x^\sigma}, \dots) = 0, \\ \dots \\ F^r(x^1, \dots, x^n, \dots, \frac{\partial^{|\sigma|} u^j}{\partial x^\sigma}, \dots) = 0, \end{cases} \tag{10}$$

where $j = 1, \dots, m$ and $|\sigma| \le k$.

Step 1 consists of writing out defining equations for symmetries and cosymmetries of $\mathscr{E}$. Let $\{u^j_\sigma\}^{j\in J}_{\sigma\in S}$ be *internal coordinates* on $\mathscr{E}$, S and J being some sets of (multi)indices and u^j_σ corresponding to $\partial^{|\sigma|}u^j/\partial x^\sigma$. Then any $\mathscr{C}$-field on $\mathscr{E}$ is a linear combination of the *total derivatives*

$$D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{\sigma\in S, j\in J} u^j_{\sigma i}\frac{\partial}{\partial u^j_\sigma}, \qquad i = 1,\dots,n. \tag{11}$$

The linearization of $\mathscr{E}$ is the matrix operator with the entries

$$(\ell_{\mathscr{E}})^l_j = \sum_{\sigma\in S}\frac{\partial F^l}{\partial u^j_\sigma}D_\sigma, \qquad j\in J, \quad l = 1,\dots,r. \tag{12}$$

A symmetry $\varphi = (\varphi^1,\dots,\varphi^m)$ enjoys the equation

$$\sum_{\sigma\in S, j\in J}\frac{\partial F^l}{\partial u^j_\sigma}D_\sigma(\varphi^j) = 0, \qquad l = 1,\dots,r, \tag{13}$$

and the corresponding field is the *evolutionary vector field*

$$\mathbf{\ni}_\varphi = \sum_{\sigma\in S, j\in J} D_\sigma(\varphi^j)\frac{\partial}{\partial u^j_\sigma}, \tag{14}$$

while the bracket of symmetries φ_1, φ_2 is given by

$$\{\varphi_1,\varphi_2\}^j = \mathbf{\ni}_{\varphi_1}(\varphi_2^j) - \mathbf{\ni}_{\varphi_2}(\varphi_1^j), \qquad j = 1,\dots,m. \tag{15}$$

The operator adjoint to (12) is

$$(\ell^*_{\mathscr{E}})^j_l = \sum_{\sigma\in S}(-1)^{|\sigma|}D_\sigma\circ\frac{\partial F^l}{\partial u^j_\sigma}, \qquad j\in J, \quad l = 1,\dots,r, \tag{16}$$

and a cosymmetry $\psi = (\psi^1,\dots,\psi^r)$ satisfies the equation

$$\sum_{\sigma,l}(-1)^{|\sigma|}D_\sigma\Big(\frac{\partial F^l}{\partial u^j_\sigma}\psi^l\Big) = 0, \qquad j\in J. \tag{17}$$

Step 2. Here we look for closed 1-forms and construct Abelian coverings associated to them. A horizontal form $\omega = \sum_i A_i\,dx^i$ is closed if

$$D_{x^\alpha}(A_\beta) = D_{x^\beta}(A_\alpha), \qquad 1\le\alpha<\beta\le n. \tag{18}$$

Such a form gives rise to a nonlocal variable $w = w_\omega$ that satisfies the equations

$$\frac{\partial w}{\partial x^i} = A_i, \qquad i = 1,\dots,n. \tag{19}$$

These equations are compatible on (10) due to (18). Recall that for $n = 2$ closed 1-forms coincide with conservation laws. The total derivatives lifted to the covering equation $\tilde{\mathscr{E}}$ are

$$\tilde{D}_{x^i} = D_{x^i} + A_i\frac{\partial}{\partial w}, \qquad i = 1,\dots,n. \tag{20}$$

Step 3. At this step we compute a number of particular symmetries and cosymmetries (using equations (13) and (17), resp.). They are used to construct canonical nonlocal variables on the ℓ^*-covering (*nonlocal vectors*) and on the ℓ-covering (*nonlocal forms*), resp., at Step 4. We also use them as seed elements in series generated by recursion operators.

Step 4 consists of construction of the ℓ- and ℓ^*-coverings and introduction of canonical nonlocal variables over them (see Step 3). The ℓ-covering is obtained by adding to (9) the system of equations

$$\sum_{\sigma\in S, j\in J} \frac{\partial F^l}{\partial u^j_\sigma}\frac{\partial q^j}{\partial x^\sigma} = 0, \qquad l = 1,\dots,r, \tag{21}$$

cf. with Eq. (12), while the ℓ^*-covering is given by

$$\sum_{\sigma,l}(-1)^{|\sigma|}\frac{\partial\left(\frac{\partial F^l}{\partial u^j_\sigma}p^l\right)}{\partial x^\sigma} = 0, \qquad j \in J, \tag{22}$$

that comes from (16).

If φ is a symmetry of $\mathscr{E}$ then one can introduce a covering over $\mathscr{L}^*(\mathscr{E})$ described by the system

$$\frac{\partial \bar p}{\partial x^i} = \sum_{\sigma,l}\Delta^l_{\sigma,i}(\varphi)p^l_\sigma, \qquad i = 1,\dots,n, \tag{23}$$

where $\Delta^l_{\sigma,i}$ are $\mathscr{C}$-differential operators (see Theorem 4). In a similar way, to any cosymmetry ψ there corresponds a covering

$$\frac{\partial \bar q}{\partial x^i} = \sum_{\sigma,j}\nabla^j_{\sigma,i}(\psi)q^j_\sigma \qquad i = 1,\dots,n, \tag{24}$$

$\nabla^j_{\sigma,i}$ being $\mathscr{C}$-differential operators as well. We omit here a general description of these operators and refer the reader to the particular case of our interest exposed in Sections 4.4 and 4.5.

Step 5. We now use Theorems 1 and 2 to construct recursion operators and Hamiltonian and symplectic structures. Let $\psi_1,\dots,\psi_s$ cosymmetries of $\mathscr{E}$. Let us consider the covering $\widetilde{\mathscr{L}(\mathscr{E})}$ over $\mathscr{L}(\mathscr{E})$ with the nonlocal variables $\bar q_1,\dots,\bar q_s$ defined by (24) and lift the operators $\ell_{\mathscr{E}}$ and $\ell^*_{\mathscr{E}}$ to this covering. Then the following result specifies Theorem 1:

Theorem 5 *Let $\Phi = (\Phi^1,\dots,\Phi^m)$ be a solution of the equation $\tilde\ell_{\mathscr{E}}(\Phi) = 0$ on $\widetilde{\mathscr{L}(\mathscr{E})}$ linear w.r.t. q^α_σ and $\bar q_\beta$:*

$$\Phi^j = \sum_{\alpha,\sigma} a^{\alpha,j}_\sigma q^\alpha_\sigma + \sum_\beta b^j_\beta \bar q_\beta.$$

Then the operator

$$\mathsf{R} = \sum_\sigma a^{\alpha,j}_\sigma D_\sigma + \sum_\beta b^j_\beta D^{-1}_{x^i}\circ\sum_\sigma \nabla^\alpha_{\sigma,i}(\psi_\beta)D_\sigma,$$

takes shadow of symmetries to shadows of symmetries. In a similar way, to any solution $\Psi = (\Psi^1,\dots,\Psi^r)$,

$$\Psi^j = \sum_{\alpha,\sigma} c^{\alpha,j}_\sigma q^\alpha_\sigma + \sum_\beta d^j_\beta \bar q_\beta$$

there corresponds the operator

$$\mathsf{S} = \sum_\sigma c_\sigma^{\alpha,j} D_\sigma + \sum_\beta d_\beta^j D_{x^i}^{-1} \circ \sum_\sigma \nabla_{\sigma,i}^\alpha(\psi_\beta) D_\sigma$$

that takes shadows of symmetries to shadows of cosymmetries.

In a dual way, consider symmetries $\varphi_1, \dots, \varphi_s$ of the equation $\mathscr{E}$ and the covering $\widetilde{\mathscr{L}^*(\mathscr{E})}$ over $\mathscr{L}^*(\mathscr{E})$ with the nonlocal variables $\bar{p}_1, \dots, \bar{p}_s$ defined by (23). Then, lifting $\ell_\mathscr{E}$ and $\ell_\mathscr{E}^*$, we obtain a similar specification of Theorem 2:

Theorem 6 *Let $\Phi = (\Phi^1, \dots, \Phi^m)$ be a solution of the equation $\tilde{\ell}_\mathscr{E}(\Phi) = 0$ on $\widetilde{\mathscr{L}^*(\mathscr{E})}$ linear w.r.t. p_σ^α and $\bar{p}_\beta$:*

$$\Phi^j = \sum_{\alpha,\sigma} a_\sigma^{\alpha,j} p_\sigma^\alpha + \sum_\beta b_\beta^j \bar{p}_\beta.$$

Then the operator

$$\mathsf{H} = \sum_\sigma a_\sigma^{\alpha,j} D_\sigma + \sum_\beta b_\beta^j D_{x^i}^{-1} \circ \sum_\sigma \Delta_{\sigma,i}^\alpha(\varphi_\beta) D_\sigma,$$

takes shadow of cosymmetries to shadows of symmetries. In a similar way, to any solution $\Psi = (\Psi^1, \dots, \Psi^r)$,

$$\Psi^j = \sum_{\alpha,\sigma} c_\sigma^{\alpha,j} p_\sigma^\alpha + \sum_\beta d_\beta^j \bar{p}_\beta$$

there corresponds the operator

$$\hat{\mathsf{R}} = \sum_\sigma c_\sigma^{\alpha,j} D_\sigma + \sum_\beta d_\beta^j D_{x^i}^{-1} \circ \sum_\sigma \Delta_{\sigma,i}^\alpha(\varphi_\beta) D_\sigma$$

that takes shadows of cosymmetries to shadows of cosymmetries.

After finding the operators R, S, H and $\hat{\mathsf{R}}$ we check conditions (6) and (7) and compute necessary Schouten and Nijenhuis brackets.

Step 6 The last step consists of establishing algebraic relations between the invariants constructed above.

3 The matrix version

We consider Eq. (1) in the form

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{25}$$

i.e., set $\mu = 0$, and, similar to (2), introduce a new variable $w = \alpha u - u_{xx}$, where α is a new real constant. Consequently, the initial equation transforms to the system

$$\begin{cases} w_t &= -2u_x w - uw_x, \\ w &= \alpha u - u_{xx}. \end{cases} \tag{26}$$

We choose the following variables for internal local coordinates on the infinite prolongation of Eq. (26):

$$x,\ t,\ w_l = \frac{\partial^k w}{\partial x^k},\ u_{0,k} = \frac{\partial^k u}{\partial t^k},\ u_{1,k} = \frac{\partial^{k+1} u}{\partial x \partial t^k}, \qquad k = 0, 1, \dots$$

Then the total derivatives in these coordinates will be of the form

$$D_x = \frac{\partial}{\partial x} + \sum_{k\geq 0} w_{k+1}\frac{\partial}{\partial w_k} + \sum_{k\geq 0} u_{1,k}\frac{\partial}{\partial u_{0,k}} + \sum_{k\geq 0} D_t^k(\alpha u - w)\frac{\partial}{\partial u_{1,k}},$$

$$D_t = \frac{\partial}{\partial t} - \sum_{k\geq 0} D_x^k(2u_{1,0}w + uw_1)\frac{\partial}{\partial w_k} + \sum_{k\geq 0} u_{0,k+1}\frac{\partial}{\partial u_{0,k}} + \sum_{k\geq 0} u_{1,k+1}\frac{\partial}{\partial u_{1,k}}.$$

We introduce the following gradings:

$$|x| = -1,\ |t| = -2,\ |u| = 1,\ |w| = 3,\ |\alpha| = 2$$

and extend them in a natural way to all polynomial functions of the internal coordinates. Then all computations can be restricted to homogeneous components.

3.1 Nonlocal variables

In subsequent computations we shall need the following nonlocal variables arising from conservation laws and defined by the equations

$$\begin{aligned}
&(s_2)_x = w,\\
&(s_2)_t = (-u^2\alpha - 2uw + u_1^2)/2;\\
&(s_3)_x = uw,\\
&(s_3)_t = -2u^2w + uu_{1,1} - u_1u_{0,1};\\
&(s_6)_x = w(uw - u_{1,1}),\\
&(s_6)_t = (-u^4\alpha^2 - 4u^3w\alpha + 2u^2u_1^2\alpha + 4u^2u_1w_1 - 4u^2w^2 - 4uu_{0,2}\alpha + 12uu_1^2w + 4uu_{1,1}w\\
&\qquad - u_1^4 + 4u_1u_{1,2} - 4u_1u_{0,1}w)/4;\\
&(s_7)_x = -2u^3w_2 + 60u^2w^2 - 36uu_{1,1}w + 30u_{0,2}w + 27u_1^2u_{1,1},\\
&(s_7)_t = -104u^4w\alpha - 28u^4w_2 - 132u^3u_1w_1 + 36u^3u_{1,1}\alpha - 40u^3w^2 - 18u^2u_{0,2}\alpha - 48u^2u_1^2w\\
&\qquad - 36u^2u_1u_{0,1}\alpha + 144u^2u_{1,1}w + 66u^2u_{0,1}w_1 - 78uu_{0,2}w - 36uu_1^2u_{1,1} + 18uu_1u_{1,2}\\
&\qquad - 36uu_1u_{0,1}w + 30uu_{1,3} + 18uu_{1,1}^2 - 36uu_{0,1}^2\alpha + 9u_{0,2}u_1^2 + 18u_{0,2}u_{1,1} - 30u_{0,3}u_1\\
&\qquad + 36u_1^3u_{0,1} + 36u_1u_{1,1}u_{0,1} - 18u_{1,2}u_{0,1} + 18u_{0,1}^2w.
\end{aligned}$$

The variable s_i is of grading i and computational experiment shows that for every grading $i = 4n - 2 + \varepsilon$, $\varepsilon = 0, 1$, there exist an s_i such that $|s_i| = i$.

In addition, we found conservation laws of fractional gradings:

$$\begin{aligned}
&(s_{1/2})_x = w^{1/2},\\
&(s_{1/2})_t = -w^{1/2}u;\\
&(s_{-1/2})_x = w^{-3/2}(6w\alpha + w_2),\\
&(s_{-1/2})_t = w^{-3/2}(4uw\alpha - uw_2 + 14w^2);\\
&(s_{-3/2})_x = w^{-7/2}(12w^2\alpha^2 + 12ww_2\alpha - 2ww_4 + 7w_2^2),\\
&(s_{-3/2})_t = w^{-7/2}(16uw^2\alpha^2 + 16uww_2\alpha + 2uww_4 - 7uw_2^2 - 124w^3\alpha - 32w^2w_2);\\
&(s_{-5/2})_x = w^{-11/2}(216w^3\alpha^3 + 540w^2w_2\alpha^2 - 180w^2w_4\alpha + 20w^2w_6 + 882ww_2^2\alpha
\end{aligned}$$

$$
\begin{aligned}
&\quad - 378ww_2w_4 - 16ww_3^2 + 837w_2^3),\\
(s_{-5/2})_t &= w^{-11/2}(320uw^3\alpha^3 + 1008uw^2w_2\alpha^2 - 360uw^2w_4\alpha - 20uw^2w_6\\
&\quad + 1908uww_2^2\alpha + 378uww_2w_4 + 16uww_3^2 - 837uw_2^3 - 64u_1w^2w_3\alpha - 1400w^4\alpha^2\\
&\quad - 1868w^3w_2\alpha + 580w^3w_4 + 64w^2w_1w_3 - 2322w^2w_2^2),
\end{aligned}
$$

etc.

3.2 Symmetries

A symmetry $\varphi = (\varphi^w, \varphi^u)$ of Eq. (26) must satisfy the linearized equation

$$
\begin{cases}
D_t(\varphi^w) + uD_x(\varphi^w) + 2u_1\varphi^w + 2wD_x(\varphi^u) + w_1\varphi^u = 0,\\
\varphi^w + D_x^2(\varphi^u) - \alpha\varphi^u = 0.
\end{cases}
$$

Direct computations lead to the following results.

(x,t)-independent symmetries. One can observe two types of symmetries that are independent of x and t. The first one consists of symmetries of integer gradings:

$$
\begin{aligned}
\varphi_1^w &= w_1,\\
\varphi_1^u &= u_1;\\
\varphi_2^w &= uw_1 + 2u_1w,\\
\varphi_2^u &= -u_{0,1};\\
\varphi_5^w &= u^2w_1\alpha + 2uww_1 - u_1^2w_1 - 2u_{1,1}w_1 - 4u_{0,1}w\alpha,\\
\varphi_5^u &= u^2u_1\alpha + 2u^2w_1 + 6uu_1w - u_1^3 + 2u_{1,2} - 2u_{0,1}w;\\
\varphi_6^w &= u^3w_1\alpha + 2u^2u_1w\alpha + 8u^2ww_1 - uu_1^2w_1 + 12uu_1w^2 - 2uu_{1,1}w_1\\
&\quad + 2u_{0,2}w_1 - 2u_1^3w + 2u_1u_{0,1}w_1 + 4u_{1,2}w - 4u_{0,1}w^2,\\
\varphi_6^u &= 2u^3w_1 + 6u^2u_1w - 3u^2u_{0,1}\alpha - 6uu_{0,1}w - 2u_{0,3} + 3u_1^2u_{0,1},
\end{aligned}
$$

etc. Symmetries of the second type have semi-integer gradings:

$$
\begin{aligned}
\varphi_{-3/2}^w &= (-4w^2w_1\alpha + 4w^2w_3 - 18ww_1w_2 + 15w_1^3)/(2w^{7/2}),\\
\varphi_{-3/2}^u &= -2w_1/(w^{3/2});\\
\varphi_{-5/2}^w &= (-48w^4w_1\alpha^2 + 80w^4w_3\alpha - 32w^4w_5 - 520w^3w_1w_2\alpha + 320w^3w_1w_4\\
&\quad + 560w^3w_2w_3 + 560w^2w_1^3\alpha - 1820w^2w_1^2w_3 - 2520w^2w_1w_2^2 + 6930ww_1^3w_2\\
&\quad - 3465w_1^5)/(12w^{13/2}),\\
\varphi_{-5/2}^u &= (-12w^2w_1\alpha + 8w^2w_3 - 40ww_1w_2 + 35w_1^3)/(3w^{9/2}),
\end{aligned}
$$

etc. All symmetries are local and $|\varphi_\gamma| = \gamma$.

If one adds to the nonlocal setting the variables s_γ (see above) then an additional series of nonlocal symmetries arises:

$$
\bar{\varphi}_{-1}^w = (s_{1/2}w^{-1/2}(4w^2w_1\alpha - 4w^2w_3 + 18ww_1w_2 - 15w_1^3) + 16tw^3w_1
$$

$$+2w(-4w^2\alpha - 4ww_2 + 5w_1^2))/(16w^3),$$

$$\bar{\varphi}^u_{-1} = (s_{1/2}w^{-1/2}w_1 + 4tu_1w - 2w)/(4w);$$

$$\bar{\varphi}^w_{-2} = (3s_{1/2}w^{-1/2}(-48w^4w_1\alpha^2 + 80w^4w_3\alpha - 32w^4w_5 - 520w^3w_1w_2\alpha + 320w^3w_1w_4 + 560w^3w_2w_3 + 560w^2w_1^3\alpha - 1820w^2w_1^2w_3 - 2520w^2w_1w_2^2 + 6930ww_1^3w_2 - 3465w_1^5) + 2s_{-1/2}w^{-1/2}w^3(-4w^2w_1\alpha + 4w^2w_3 - 18ww_1w_2 + 15w_1^3) + 2w(96w^4\alpha^2 + 384w^3w_2\alpha - 192w^3w_4 - 740w^2w_1^2\alpha + 1364w^2w_1w_3 + 960w^2w_2^2 - 5514ww_1^2w_2 + 3465w_1^4))/(4w^6),$$

$$\bar{\varphi}^u_{-2} = (3s_{1/2}w^{-1/2}(-12w^2w_1\alpha + 8w^2w_3 - 40ww_1w_2 + 35w_1^3) - 2s_{-1/2}w^{-1/2}w^3w_1 + 2w(24w^2\alpha + 24ww_2 - 35w_1^2))/w^4,$$

etc.

(x,t)-dependent symmetries. The first three symmetries that depend on x and t are

$$\varphi^w_0 = t(-uw_1 - 2u_1w) + w,$$

$$\varphi^u_0 = tu_{0,1} + u;$$

$$\varphi^w_3 = t(-u^2w_1\alpha - 2uww_1 + u_1^2w_1 + 2u_{1,1}w_1 + 4u_{0,1}w\alpha) + 2(s_2w_1 + 2uw\alpha + u_1w_1),$$

$$\varphi^u_3 = t(-u^2u_1\alpha - 2u^2w_1 - 6uu_1w + u_1^3 - 2u_{1,2} + 2u_{0,1}w) + 2(s_2u_1 + 2uw - 2u_{1,1});$$

$$\varphi^w_4 = t(-u^3w_1\alpha - 2u^2u_1w\alpha - 8u^2ww_1 + uu_1^2w_1 - 12uu_1w^2 + 2uu_{1,1}w_1 - 2u_{0,2}w_1 + 2u_1^3w - 2u_1u_{0,1}w_1 - 4u_{1,2}w + 4u_{0,1}w^2) + 2(s_2(uw_1 + 2u_1w) + s_3w_1 + 4uw^2 - 4u_{1,1}w - 2u_{0,1}w_1),$$

$$\varphi^u_4 = t(-2u^3w_1 - 6u^2u_1w + 3u^2u_{0,1}\alpha + 6uu_{0,1}w + 2u_{0,3} - 3u_1^2u_{0,1}) + 2(-s_2u_{0,1} + s_3u_1 + u^3\alpha + 3u^2w - uu_1^2 + 3u_{0,2}).$$

All these symmetries, except for the first one, are nonlocal (description of the nonlocal variable is given in Subsection 3.1) and, as above, the subscript denotes the grading.

3.3 Cosymmetries

The defining equation for cosymmetries $\psi = (\psi^w, \psi^u)$ is the adjoint to the linearization of (26):

$$\begin{cases} D_t(\psi^w) + uD_x(\psi^w) - u_1\psi^w - \psi^u = 0, \\ 2wD_x(\psi^w) + w_1\psi^w - D_x^2(\psi^u) + \alpha\psi^u = 0. \end{cases}$$

Similar to symmetries, we consider two types of cosymmetries.

(x,t)-independent cosymmetries. They are local and may be of integer and semi-integer gradings:

$$\psi^w_3 = 1,$$

$$\psi^u_3 = -u_1;$$

$$
\begin{aligned}
\psi_4^w &= u,\\
\psi_4^u &= u_{0,1};\\
\psi_7^w &= u^2\alpha + 2uw - u_1^2 - 2u_{1,1},\\
\psi_7^u &= -u^2u_1\alpha - 2u^2w_1 - 6uu_1w + u_1^3 - 2u_{1,2} + 2u_{0,1}w;\\
\psi_8^w &= u^3\alpha + 4u^2w - uu_1^2 - 2uu_{1,1} + 2u_{0,2} + 2u_1u_{0,1},\\
\psi_8^u &= -2u^3w_1 - 6u^2u_1w + 3u^2u_{0,1}\alpha + 6uu_{0,1}w + 2u_{0,3} - 3u_1^2u_{0,1}
\end{aligned}
$$

etc. and

$$
\begin{aligned}
\psi_{3/2}^w &= w^{-1/2},\\
\psi_{3/2}^u &= 0;\\
\psi_{1/2}^w &= (4w^2\alpha + 4ww_2 - 5w_1^2)/(4w^3w^{1/2}),\\
\psi_{1/2}^u &= 2w_1/(ww^{1/2});\\
\psi_{-1/2}^w &= (48w^4\alpha^2 + 160w^3w_2\alpha - 64w^3w_4 - 280w^2w_1^2\alpha + 448w^2w_1w_3 + 336w^2w_2^2\\
&\quad - 1848ww_1^2w_2 + 1155w_1^4)/(48w^6w^{1/2}),\\
\psi_{-1/2}^u &= (12w^2w_1\alpha - 8w^2w_3 + 40ww_1w_2 - 35w_1^3)/(3w^4w^{1/2}),
\end{aligned}
$$

etc.

Similar to the case of symmetries, when one adds nonlocal variables s_γ an additional series of nonlocal cosymmetries arises:

$$
\begin{aligned}
\bar\psi_1^w &= (3s_{1/2}w^{-1/2}(-4w^2\alpha - 4ww_2 + 5w_1^2) - 2s_{-1/2}w^{-1/2}w^3 + 96tw^3 - 10ww_1)/(96w^3),\\
\bar\psi_1^u &= (-s_{1/2}w^{-1/2}w_1 - 4tu_1w + 2w)/(4w);\\
\bar\psi_0^w &= (3s_{1/2}w^{-1/2}(48w^4\alpha^2 + 160w^3w_2\alpha - 64w^3w_4 - 280w^2w_1^2\alpha + 448w^2w_1w_3\\
&\quad + 336w^2w_2^2 - 1848ww_1^2w_2 + 1155w_1^4) + 4s_{-1/2}w^{-1/2}w^3(4w^2\alpha + 4ww_2 - 5w_1^2)\\
&\quad + 12s_{-3/2}w^{-1/2}w^6 + 2w(400w^2w_1\alpha - 276w^2w_3 + 1366ww_1w_2 - 1155w_1^3))/(16w^6),\\
\bar\psi_0^u &= (3s_{1/2}w^{-1/2}(12w^2w_1\alpha - 8w^2w_3 + 40ww_1w_2 - 35w_1^3) + 2s_{-1/2}w^{-1/2}w^3w_1\\
&\quad + 2w(-24w^2\alpha - 24ww_2 + 35w_1^2))/w^4;\\
\bar\psi_{-1}^w &= (27s_{1/2}w^{-1/2}(320w^6\alpha^3 + 2240w^5w_2\alpha^2 - 1792w^5w_4\alpha + 512w^5w_6 - 5040w^4w_1^2\alpha^2\\
&\quad + 16128w^4w_1w_3\alpha - 6912w^4w_1w_5 + 12096w^4w_2^2\alpha - 14592w^4w_2w_4 - 8832w^4w_3^2\\
&\quad - 81312w^3w_1^2w_2\alpha + 52800w^3w_1^2w_4 + 181632w^3w_1w_2w_3 + 42944w^3w_2^3\\
&\quad + 60060w^2w_1^4\alpha - 274560w^2w_1^3w_3 - 569712w^2w_1^2w_2^2 + 1021020ww_1^4w_2 - 425425w_1^6)\\
&\quad + 18s_{-1/2}w^{-1/2}w^3(48w^4\alpha^2 + 160w^3w_2\alpha - 64w^3w_4 - 280w^2w_1^2\alpha + 448w^2w_1w_3\\
&\quad + 108s_{-3/2}w^{-1/2}w^6(4w^2\alpha + 4ww_2 - 5w_1^2) + 40s_{-5/2}w^{-1/2}w^9 + 2w(69120w^4w_1\alpha^2\\
&\quad + 336w^2w_2^2 - 1848ww_1^2w_2 + 1155w_1^4) + 34160w^4w_5 + 624816w^3w_1w_2\alpha\\
&\quad - 349880w^3w_1w_4 - 605776w^3w_2w_3 - 665280w^2w_1^3\alpha + 1991880w^2w_1^2w_3\\
&\quad + 2772036w^2w_1w_2^2 - 7636860ww_1^3w_2 + 3828825w_1^5))/(72w^9),\\
\bar\psi_{-1}^u &= (3s_{1/2}w^{-1/2}(240w^4w_1\alpha^2 - 320w^4w_3\alpha + 128w^4w_5 + 2240w^3w_1w_2\alpha
\end{aligned}
$$

$$- 1344w^3w_1w_4 - 2240w^3w_2w_3 - 2520w^2w_1^3\alpha + 7728w^2w_1^2w_3 + 10416w^2w_1w_2^2 - 29568ww_1^3w_2 + 15015w_1^5) + 4s_{-1/2}w^{-1/2}w^3(12w^2w_1\alpha - 8w^2w_3 + 40ww_1w_2 - 35w_1^3) + 12s_{-3/2}w^{-1/2}w^6w_1 + 2w(-384w^4\alpha^2 - 1536w^3w_2\alpha + 768w^3w_4 + 3360w^2w_1^2\alpha - 5732w^2w_1w_3 - 3840w^2w_2^2 + 23422ww_1^2w_2 - 15015w_1^4))/w^7,$$

etc.

(x,t)-dependent cosymmetries. All them are nonlocal:

$$\begin{aligned}
\psi_5^w &= t(-u^2\alpha - 2uw + u_1^2 + 2u_{1,1}) + 2(s_2 + u_1),\\
\psi_5^u &= t(u^2u_1\alpha + 2u^2w_1 + 6uu_1w - u_1^3 + 2u_{1,2} - 2u_{0,1}w) + 2(-s_2u_1 - 2uw + 2u_{1,1});\\
\psi_6^w &= t(-u^3\alpha - 4u^2w + uu_1^2 + 2uu_{1,1} - 2u_{0,2} - 2u_1u_{0,1}) + 2(s_2u + s_3 - 2u_{0,1}),\\
\psi_6^u &= t(2u^3w_1 + 6u^2u_1w - 3u^2u_{0,1}\alpha - 6uu_{0,1}w - 2u_{0,3} + 3u_1^2u_{0,1}) + 2(s_2u_{0,1} - s_3u_1\\
&\quad - u^3\alpha - 3u^2w + uu_1^2 - 3u_{0,2}),
\end{aligned}$$

etc.

3.4 Nonlocal forms

Recall that nonlocal forms are nonlocal variables of a special type on the ℓ-covering. The ℓ-covering itself is obtained from Eq. (26) by adding two additional equations

$$\begin{cases} q_t^w &= -uq_x^w - 2u_xq^w - 2wq_x^u - w_xq^u,\\ q_{xx}^u &= \alpha q^u - q^w, \end{cases}$$

where q^w and q^u are new odd variables. The total derivatives on the ℓ-covering are

$$\begin{aligned}
\tilde{D}_x &= D_x + \sum_{k\geq 0} q_{k+1}^w \frac{\partial}{\partial q_k^w} + \sum_{k\geq 0} q_{1,k}^u \frac{\partial}{\partial q_{0,k}^u} + \sum_{k\geq 0} \tilde{D}_t^k(\alpha q^u - q^w)\frac{\partial}{\partial q_{1,k}^u},\\
\tilde{D}_t &= D_t - \sum_{k\geq 0} \tilde{D}_x^k(uq_1^w + 2u_1q^w + 2wq_1^u + w_1q^u)\frac{\partial}{\partial q_k^w} + \sum_{k\geq 0} q_{0,k+1}^u \frac{\partial}{\partial q_{0,k}^u} + \sum_{k\geq 0} q_{1,k+1}^u \frac{\partial}{\partial q_{1,k}^u}.
\end{aligned}$$

The nonlocal form Q_i associated to a cosymmetry $\psi_i = (\psi_i^w, \psi_i^u)$ (see Subsection 3.3) is defined by the equations

$$\begin{aligned}
\tilde{D}_x(Q^i) &= \psi_i^w q^w,\\
\tilde{D}_t(Q^i) &= -u\psi_i^w q^w + (D_x(\psi_i^u) - 2w\psi_i^w)q^u - \psi_i^u q_1^u.
\end{aligned}$$

3.5 Nonlocal vectors

Dually to nonlocal forms, nonlocal vectors arise as special nonlocal variables on the ℓ^*-covering associated to symmetries of the initial equation. The ℓ^*-covering is the extension of Eq. (26) by two new equations

$$\begin{cases} p^w_t = -up^w_x + u_x p^w + p^u, \\ p^u_{xx} = 2wp^w_x + w_x p^w + \alpha p^u, \end{cases}$$

where $p = (p^w, p^u)$ is a new odd variable. The total deriavatives are given by

$$\tilde{D}_x = D_x + \sum_{k\geq 0} p^w_{k+1}\frac{\partial}{\partial p^w_k} + \sum_{k\geq 0} p^u_{1,k}\frac{\partial}{\partial p^u_{0,k}} + \sum_{k\geq 0} \tilde{D}^k_t(2wp^w_1 + w_1 p^w + \alpha p^u)\frac{\partial}{\partial p^u_{1,k}},$$
$$\tilde{D}_t = D_t + \sum_{k\geq 0} \tilde{D}^k_x(-up^w_1 + u_1 p^w + p^u)\frac{\partial}{\partial p^w_k} + \sum_{k\geq 0} p^u_{0,k+1}\frac{\partial}{\partial p^u_{0,k}} + \sum_{k\geq 0} p^u_{1,k+1}\frac{\partial}{\partial p^u_{1,k}}.$$

The nonlocal vector P_i associated to a symmetry $\varphi = (\varphi^w, \varphi^u)$ (see Subsection 3.2) is defined by the equations

$$\tilde{D}_x(P^i) = \varphi^w_i p^w,$$
$$\tilde{D}_t(P^i) = -(u\varphi^w_i + 2w\varphi^u_i)p^w - D_x(\varphi^u_i)p^u + \varphi^u_i p^u_1.$$

3.6 Recursion operators for symmetries

The defining equations for these operators are

$$\begin{cases} \tilde{D}_t(\mathsf{R}^w) + u\tilde{D}_x(\mathsf{R}^w) + 2u_1\mathsf{R}^w + 2w\tilde{D}_x(\mathsf{R}^u) + w_1\mathsf{R}^u = 0, \\ \mathsf{R}^w + \tilde{D}^2_x(\mathsf{R}^u) - \alpha\mathsf{R}^u = 0 \end{cases}$$

(see Theorem 5), where the total derivatives are those described in Subsection 3.4. The following two solutions are essential:

$$\begin{aligned} \mathsf{R}^w_{-1} &= (Q^{3/2}w^{-1/2}(-4w^2w_1\alpha + 4w^2w_3 - 18ww_1w_2 + 15w_1^3) \\ &\quad - 4q^w_2 w^2 + 10q^w_1 ww_1 + q^w(4w^2\alpha + 8ww_2 - 15w_1^2))/w^3, \\ \mathsf{R}^u_{-1} &= 4(-Q^{3/2}w^{-1/2}w_1 + q^w)/w \end{aligned}$$

and

$$\begin{aligned} \mathsf{R}^w_3 &= Q^3 w_1 + q^u_1 w_1 + 2q^u w\alpha, \\ \mathsf{R}^u_3 &= Q^3 u_1 + q^w u - q^u_{1,1} + q^u w. \end{aligned}$$

The corresponding operators are of the form

$$\mathsf{R}_{-1} = \frac{1}{8w^3}\begin{pmatrix} h_{12}w^{-1/2}D_x^{-1}\circ w^{-1/2}/2 - 4w^2D_x^2 + 10ww_1D_x + h_8 & 0 \\ -4w^2w_1w^{-1/2}D_x^{-1}\circ w^{-1/2}/2 + 4w^2 & 0 \end{pmatrix}$$

and

$$\mathsf{R}_3 = \begin{pmatrix} w_1 D_x^{-1} & w_1 D_x + 2w\alpha \\ u_1 D_x^{-1} + u & -D_{xt} + w \end{pmatrix},$$

where $h_{12} = -4w^2 w_1 \alpha + 4w^2 w_3 - 18ww_1 w_2 + 15w_1^3$, $h_8 = 4w^2\alpha + 8ww_2 - 15w_1^2$. All other solutions obtained in our computations corresponded to operators that are generated by the two above.

3.7 Symplectic structures

Symplectic structures, as it follows from Theorem 5, are defined by the equations

$$\begin{cases} -\tilde{D}_t(\mathsf{S}^w) - u\tilde{D}_x(\mathsf{S}^w) + u_1\mathsf{S}^w + \mathsf{S}^u = 0, \\ -2w\tilde{D}_x(\mathsf{S}^w) - w_1\mathsf{S}^w + \tilde{D}_x^2(\mathsf{S}^u) - \alpha\mathsf{S}^u = 0, \end{cases}$$

where the total derivatives were defined in Subsection 3.4. Here are the simplest nontrivial solutions:

$$\begin{aligned} \mathsf{S}_2^w &= Q^{3/2} w^{-1/2}, \\ \mathsf{S}_2^u &= -q^u; \\ \mathsf{S}_5^w &= Q^3 + q_1^u, \\ \mathsf{S}_5^u &= -Q^3 u_1 - q^w u + q_{1,1}^u - q^u w; \\ \mathsf{S}_6^w &= Q^4 + Q^3 u - q_{0,1}^u + q_1^u u - q^u u_1, \\ \mathsf{S}_6^u &= -Q^4 u_1 + Q^3 u_{0,1} - q^w u^2 - q_{0,2}^u + q^u(-u^2\alpha - 2uw + u_1^2) \end{aligned}$$

with the corresponding symplectic operators

$$\begin{aligned} \mathsf{S}_2 &= \begin{pmatrix} w^{-1/2} D_x^{-1} \circ w^{-1/2}/2 & 0 \\ 0 & -1 \end{pmatrix}, \\ \mathsf{S}_5 &= \begin{pmatrix} D_x^{-1} & D_x \\ -u_1 D_x^{-1} - u & D_{xt} - w \end{pmatrix}, \\ \mathsf{S}_6 &= \begin{pmatrix} D_x^{-1} \circ u + u D_x^{-1} & -D_t + uD_x - u_1 \\ -u_1 D_x^{-1} \circ u + u_{0,1} D_x^{-1} - u^2 & -D_t^2 - u^2\alpha - 2uw + u_1^2 \end{pmatrix}. \end{aligned}$$

3.8 Hamiltonian structures

The equations that should be satisfied by a Hamiltonian operator (see Theorem 6) are

$$\begin{cases} \tilde{D}_t(\mathsf{H}^w) + u\tilde{D}_x(\mathsf{H}^w) + 2u_1\mathsf{H}^w + 2w\tilde{D}_x(\mathsf{H}^u) + w_1\mathsf{H}^u = 0, \\ \mathsf{H}^w + \tilde{D}_x^2(\mathsf{H}^u) - \alpha\mathsf{H}^u = 0, \end{cases}$$

where the total derivatives are from Subsection 3.5. In particular, we found the following solutions:

$$\mathsf{H}_{-3}^w = -p_3^w + p_1^w \alpha,$$

$$\begin{aligned}
\mathsf{H}^u_{-3} &= p^w_1;\\
\mathsf{H}^w_{-2} &= 2p^w_1 w + p^w w_1,\\
\mathsf{H}^u_{-2} &= -p^u;\\
\mathsf{H}^w_1 &= P^1 w_1 - 2p^w w w_1 + p^u_1 w_1 + 2p^u w\alpha,\\
\mathsf{H}^u_1 &= P^1 u_1 - 2p^w_1 uw - p^w(uw_1 + 2u_1 w) - p^u_{1,1} + p^u w.
\end{aligned}$$

The corresponding Hamiltonian operators are

$$\begin{aligned}
\mathsf{H}_{-3} &= \begin{pmatrix} -D_x^3 + \alpha D_x & 0 \\ D_x & 0 \end{pmatrix},\\
\mathsf{H}_{-2} &= \begin{pmatrix} 2wD_x + w_1 & 0 \\ 0 & -1 \end{pmatrix},\\
\mathsf{H}_1 &= \begin{pmatrix} w_1 D_x^{-1} \circ w_1 - 2ww_1 & w_1 D_x + 2w\alpha \\ u_1 D_x^{-1} \circ w_1 - 2uwD_x - (uw_1 + 2u_1 w) & -D_{xt} + w \end{pmatrix}.
\end{aligned}$$

3.9 Recursion operators for cosymmetries

By Theorem 6, the equation to find recursion operators for cosymmetries are

$$\begin{cases} -\tilde{D}_t(\hat{\mathsf{R}}^w) - u\tilde{D}_x(\hat{\mathsf{R}}^w) + u_1 \hat{\mathsf{R}}^w + \hat{\mathsf{R}}^u = 0,\\ -2w\tilde{D}_x(\hat{\mathsf{R}}^w) - w_1 \hat{\mathsf{R}}^w + \tilde{D}_x^2(\hat{\mathsf{R}}^u) - \alpha \hat{\mathsf{R}}^u = 0 \end{cases}$$

with the total derivatives given in Subsection 3.5. One of solutions is presented below:

$$\begin{aligned}
\hat{\mathsf{R}}^w_3 &= P^1 - 2p^w w + p^u_1,\\
\hat{\mathsf{R}}^u_3 &= -P^1 u_1 + 2p^w_1 uw + p^w(uw_1 + 2u_1 w) + p^u_{1,1} - p^u w
\end{aligned}$$

The corresponding recursion operator is

$$\hat{\mathsf{R}}_3 = \begin{pmatrix} D_x^{-1} \circ w_1 - 2w & D_x \\ -u_1 D_x^{-1} \circ w_1 + 2uwD_x + uw_1 + 2u_1 w & D_{xt} - w \end{pmatrix}.$$

3.10 Interrelation

We expose here basic facts on structural relations between the above described invariants. The main one is the following

Theorem 7 *Recursion operator* R_3 *and Hamiltonian operator* H_{-3} *constitute a Poisson–Nijenhuis structure on the Camassa–Holm equation. Consequently, all the operators* $\mathsf{R}_3^n \circ \mathsf{H}_{-3}$ *are Hamiltonian and pair-wise compatible. In particular,* $\mathsf{R}_3 \circ \mathsf{H}_{-3} = \alpha \mathsf{H}_{-2}$.

Proof The proof consists of direct computations using the results and techniques of Ref. [8][4]. ∎

Table 2 Distribution of symmetries over gradings

Gradings	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	1	2	3	4	5	6	7	8
Local positive							•	•			•	•		
Local negative	•		•											
Nonlocal positive						•			•	•			•	•
Nonlocal negative		•		•										

Table 3 Distribution of cosymmetries over gradings

Gradings	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3	4	5	6	7	8
Local positive									•	•			•	•
Local negative	•		•		•		•							
Nonlocal positive								•			•	•		
Nonlocal negative		•		•		•								

A visual presentation of how symmetries are distributed over gradings is given in Table 2. How to prove locality of the first two series of symmetries will be discussed in Section 5. Similar presentation for cosymmetries see in Table 3.

The action of Hamiltonian and recursion operators for symmetries (up to a constant multiplier) is given in Diagram (27):

$$\begin{tikzcd}
\dots \arrow[r, "\mathsf{R}_3"'] & \varphi_{-5/2} \arrow[l, bend right, "\mathsf{R}_{-1}"'] \arrow[r, "\mathsf{R}_3"'] & \varphi_{-3/2} \arrow[l, bend right, "\mathsf{R}_{-1}"'] \arrow[r, "\mathsf{R}_3"'] & 0 & \varphi_1 \arrow[l, bend right, "\mathsf{R}_{-1}"'] \arrow[r, "\mathsf{R}_3"'] & \varphi_2 \arrow[l, bend right, "\mathsf{R}_{-1}"'] \arrow[r, "\mathsf{R}_3"'] & \dots \arrow[l, bend right, "\mathsf{R}_{-1}"'] \\
\dots & \psi_{1/2} \arrow[u, "\mathsf{H}_{-3}"] \arrow[ur, "\mathsf{H}_{-2}"] & \psi_{3/2} \arrow[u, "\mathsf{H}_{-3}"] \arrow[ur, "\mathsf{H}_{-2}"] & \psi_3 \arrow[u, "\mathsf{H}_{-3}"] \arrow[ur, "\mathsf{H}_{-2}"] & \psi_4 \arrow[u, "\mathsf{H}_{-3}"] \arrow[ur, "\mathsf{H}_{-2}"] & \psi_7 \arrow[u, "\mathsf{H}_{-3}"] \arrow[ur, "\mathsf{H}_{-2}"] & \dots
\end{tikzcd} \tag{27}$$

The action of recursion operators for cosymmetries and simplectic structures is similar.

We shall now prove commutativity of the local hierarchies.

Lemma 1 *The symmetry $\bar{\varphi}_3$ is a positive hereditary symmetry, i.e., its action on local symmetries, $\varphi \mapsto \{\bar{\varphi}_3, \varphi\}$, coincides, up to a multiplier, with the one of the recursion operator R_3. The only symmetries that vanish under this action are $\varphi_{-3/2}$ and φ_1. In a similar way, the symmetry φ_{-1} is a negative hereditary symmetry and the only symmetry that is taken to zero under its action is φ_1.*

A direct corollary of this result is

Theorem 8 *Local positive and negative symmetries form commutative hierarchies.*

4 The scalar version

Let us consider now the Camassa–Holm equation in its initial form (1) with $\mu = 0$ and, similar to the matrix case introduce a new real parameter α:

$$\alpha u_t - u_{txx} + 3\alpha u u_x = 2u_x u_{xx} + u u_{xxx}. \tag{28}$$

[4] We are pretty sure that the pair $(\mathsf{H}_{-3}, \mathsf{R}_{-1})$ is also a Poisson–Nijenhuis structure and generates an infinite negative hierarchy of Hamiltonian operators, but could not prove this fact because of computer capacity limitations.

For the internal coordinates we choose the functions

$$u_l = \frac{\partial^l u}{\partial x^l}, \quad u_{l,k} = \frac{\partial^{k+l} u}{\partial x^l \partial t^k}, \qquad l = 0,1,2,\ k \geq 1.$$

The total derivatives in these coordinates are of the form

$$D_x = \frac{\partial}{\partial x} + \sum_{l=0}^{2} u_{l+1}\frac{\partial}{\partial u_l} + \sum_{k\geq 1}\left(u_{1,k}\frac{\partial}{\partial u_{0,k}} + u_{2,k}\frac{\partial}{\partial u_{1,k}} + D_t^k(u_3)\frac{\partial}{\partial u_{2,k}}\right),$$

$$D_t = \frac{\partial}{\partial t} + \sum_{l=0}^{2} u_{l,1}\frac{\partial}{\partial u_l} + \sum_{l=0}^{2}\sum_{k\geq 1} u_{l,k+1}\frac{\partial}{\partial u_{l,k}},$$

where $u_3 = (\alpha u_{0,1} - u_{2,1} + 3\alpha u u_1 - 2u_1 u_2)/u$. The equation becomes homogeneous if assign the following gradings:

$$|x| = -1,\ |t| = -2,\ |u| = 1,\ |\alpha| = 2.$$

4.1 Nonlocal variables

We introduce nonlocal variable associated to conservation laws of the equation at hand:

$$\begin{aligned}
(s_2)_x &= u\alpha - u_2,\\
(s_2)_t &= (-3u^2\alpha + 2uu_2 + u_1^2)/2;\\
(s_3)_x &= u(u\alpha - u_2),\\
(s_3)_t &= -2u^3\alpha + 2u^2u_2 + uu_{1,1} - u_{0,1}u_1;\\
(s_6)_x &= u^3\alpha^2 - 2u^2u_2\alpha - uu_{1,1}\alpha + uu_2^2 + u_{1,1}u_2,\\
(s_6)_t &= (-9u^4\alpha^2 + 12u^3u_2\alpha + 6u^2u_1^2\alpha + 4u^2u_{1,1}\alpha - 4u^2u_2^2 - 8uu_{0,1}u_1\alpha\\
&\quad - 4uu_{0,2}\alpha - 4uu_1^2u_2 + 4uu_1u_{2,1} - 4uu_{1,1}u_2 + 4u_{0,1}u_1u_2 - u_1^4 + 4u_1u_{1,2})/4;\\
(s_7)_x &= 60u^4\alpha^2 - 116u^3u_2\alpha - 12u^2u_1^2\alpha - 40u^2u_{1,1}\alpha + 56u^2u_2^2 - 6uu_{0,1}u_1\alpha\\
&\quad + 28uu_{0,2}\alpha + 12uu_1^2u_2 + 10uu_1u_{2,1} + 40uu_{1,1}u_2 + 2uu_{2,2} + 2u_{0,1}^2\alpha - 4u_{0,1}u_1u_2\\
&\quad - 2u_{0,1}u_{2,1} - 30u_{0,2}u_2 + 27u_1^2u_{1,1},\\
(s_7)_t &= -144u^5\alpha^2 + 240u^4u_2\alpha + 48u^3u_1^2\alpha + 124u^3u_{1,1}\alpha - 96u^3u_2^2\\
&\quad - 156u^2u_{0,1}u_1\alpha - 124u^2u_{0,2}\alpha - 48u^2u_1^2u_2 + 8u^2u_1u_{2,1} - 88u^2u_{1,1}u_2 + 28u^2u_{2,2}\\
&\quad - 56uu_{0,1}^2\alpha + 112uu_{0,1}u_1u_2 + 38uu_{0,1}u_{2,1} + 78uu_{0,2}u_2 - 36uu_1^2u_{1,1} + 18uu_1u_{1,2}\\
&\quad + 18uu_{1,1}^2 + 30uu_{1,3} - 18u_{0,1}^2u_2 + 36u_{0,1}u_1^3 + 36u_{0,1}u_1u_{1,1} - 18u_{0,1}u_{1,2} + 9u_{0,2}u_1^2\\
&\quad + 18u_{0,2}u_{1,1} - 30u_{0,3}u_1.
\end{aligned}$$

4.2 Symmetries

A symmetry φ must satisfy the linearized equation

$$\alpha D_t(\varphi) - D_x^2 D_t(\varphi) - uD_x^3(\varphi) - 2u_1 D_x^2(\varphi) + (3\alpha u - 2u_2)D_x(\varphi) + (3\alpha u_1 - u_3)\varphi = 0.$$

We computed two types of symmetries. Everywhere below the subscript was chosen in a way to correspond the enumeration taken for the matrix case.

(x,t)-independent symmetries. We present the first four of them:

$$\begin{aligned}
&\varphi_1 = u_1,\\
&\varphi_2 = -u_{0,1},\\
&\varphi_5 = 3u^2u_1\alpha - 4uu_{0,1}\alpha - 2uu_1u_2 + 2uu_{2,1} + 2u_{0,1}u_2 - u_1^3 + 2u_{1,2},\\
&\varphi_6 = 2u^3u_1\alpha - 11u^2u_{0,1}\alpha - 2u^2u_1u_2 + 2u^2u_{2,1} + 6uu_{0,1}u_2 + 3u_{0,1}u_1^2 - 2u_{0,3}.
\end{aligned}$$

All these symmetries are local.

(x,t)-dependent symmetries. The first three (x,t)-dependent symmetries are

$$\begin{aligned}
&\varphi_0 = tu_{0,1} + u,\\
&\varphi_3 = 2s_2u_1 + t(-3u^2u_1\alpha + 4uu_{0,1}\alpha + 2uu_1u_2 - 2uu_{2,1} - 2u_{0,1}u_2 + u_1^3 - 2u_{1,2})\\
&\quad + 4(u^2\alpha - uu_2 - u_{1,1}),\\
&\varphi_4 = -2s_2u_{0,1} + 2s_3u_1 + t(-2u^3u_1\alpha + 11u^2u_{0,1}\alpha + 2u^2u_1u_2 - 2u^2u_{2,1}\\
&\quad - 6uu_{0,1}u_2 - 3u_{0,1}u_1^2 + 2u_{0,3}) + 2(4u^3\alpha - 3u^2u_2 - uu_1^2 + 3u_{0,2})
\end{aligned}$$

The only local symmetry in this series is φ_1.

4.3 Cosymmetries

The defining equation for cosymmetries is

$$-\alpha D_t(\psi) + D_x^2D_t(\psi) + uD_x^3(\psi) + u_1D_x^2(\psi) + (u_2 - 3\alpha u)D_x(\psi) = 0.$$

(x,t)-independent cosymmetries. These cosymmetries are local. The first four of them are

$$\begin{aligned}
&\psi_3 = 1,\\
&\psi_4 = u,\\
&\psi_7 = 3u^2\alpha - 2uu_2 - u_1^2 - 2u_{1,1},\\
&\psi_8 = 5u^3\alpha - 4u^2u_2 - uu_1^2 - 2uu_{1,1} + 2u_{0,1}u_1 + 2u_{0,2}.
\end{aligned}$$

(x,t)-dependent cosymmetries. These cosymmetries are nonlocal, except for the first one:

$$\begin{aligned}
&\bar\psi_2 = -tu,\\
&\bar\psi_5 = 2s_2 + t(-3u^2\alpha + 2uu_2 + u_1^2 + 2u_{1,1}) + 2u_1,\\
&\bar\psi_6 = 2s_2u + 2s_3 + t(-5u^3\alpha + 4u^2u_2 + uu_1^2 + 2uu_{1,1} - 2u_{0,1}u_1 - 2u_{0,2}) - 4u_{0,1},
\end{aligned}$$

etc.

4.4 Nonlocal forms

Nonlocal forms arise in the ℓ-covering which is given by the equation

$$\alpha q_t - q_{xxt} - uq_{xxx} - 2u_x q_{xx} + (3\alpha u - 2u_{xx})q_x + (3\alpha u_x - u_{xxx})q = 0$$

with the total derivatives

$$\begin{aligned}
\tilde{D}_x &= D_x + \sum_{l=0}^{2} q_{l+1}\frac{\partial}{\partial q_l} + \sum_{k\geq 1}\left(q_{1,k}\frac{\partial}{\partial q_{0,k}} + q_{2,k}\frac{\partial}{\partial q_{1,k}} + D_t^k(q_3)\frac{\partial}{\partial q_{2,k}}\right),\\
\tilde{D}_t &= D_t + \sum_{l=0}^{2} q_{l,1}\frac{\partial}{\partial q_l} + \sum_{l=0}^{2}\sum_{k\geq 1} q_{l,k+1}\frac{\partial}{\partial q_{l,k}},
\end{aligned}$$

where $q_3 = (\alpha q_{0,1} - q_{2,1} + (3u_1\alpha - u_3)q + (3u\alpha - 2u_2)q_1 - 2u_1 q_2)/u$.

To any cosymmetry ψ_i we associate a nonlocal form Q^i defined by the equations

$$\begin{aligned}
&\tilde{D}_x(Q^i) = (\alpha\psi_i - D_x^2(\psi_i))q,\\
&\tilde{D}_t(Q^i) = ((u_2 - 3u\alpha)\psi_i + uD_x^2(\psi_i))q^u + (u_1\psi_i - uD_x(\psi_i))q_1 + u\psi_i q_2\\
&\qquad - D_x(\psi_i)q_{0,1} + \psi_i q_{1,1}.
\end{aligned}$$

4.5 Nonlocal vectors

Nonlocal vectors arise in the ℓ^*-covering. The latter is the extension of the initial equation by the equation

$$-\alpha p_t + p_{xxt} + up_{xxx} + u_x p_{xx} + (u_{xx} - 3\alpha u)p_x = 0$$

with the total derivatives

$$\begin{aligned}
\tilde{D}_x &= D_x + \sum_{l=0}^{2} p_{l+1}\frac{\partial}{\partial p_l} + \sum_{k\geq 1}\left(p_{1,k}\frac{\partial}{\partial p_{0,k}} + p_{2,k}\frac{\partial}{\partial p_{1,k}} + D_t^k(p_3)\frac{\partial}{\partial p_{2,k}}\right),\\
\tilde{D}_t &= D_t + \sum_{l=0}^{2} p_{l,1}\frac{\partial}{\partial p_l} + \sum_{l=0}^{2}\sum_{k\geq 1} p_{l,k+1}\frac{\partial}{\partial p_{l,k}},
\end{aligned}$$

where $p_3 = ((3u\alpha - u_2)p_1 - u_1 p_2 + \alpha p_{0,1} - p_{2,1})/u$.

To any symmetry φ_i there corresponds a nonlocal vector P^i defined by the relations

$$\begin{aligned}
&(P^i)_x = (\varphi_i\alpha - D_x^2(\varphi_i))p,\\
&(P^i)_t = ((u_2 - 3u\alpha)\varphi_i + u_1 D_x(\varphi_i) + uD_x^2(\varphi_i))p - uD_x(\varphi_i)p_1 + u\varphi_i p_2\\
&\qquad - D_x(\varphi_i)p_{0,1} + \varphi_i p_{1,1}.
\end{aligned}$$

4.6 Recursion operators for symmetries

The defining equation for these operators is

$$\alpha\tilde{D}_t(\mathsf{R})-\tilde{D}_x^2\tilde{D}_t(\mathsf{R})-u\tilde{D}_x^3(\mathsf{R})-2u_1\tilde{D}_x^2(\mathsf{R})+(3\alpha u-2u_2)\tilde{D}_x(\mathsf{R})+(3\alpha u_1-u_3)\mathsf{R}=0$$

where the total derivatives are those presented in Subsection 4.4. We consider here two nontrivial solutions:

$$\begin{aligned}\mathsf{R}_{-1}=(&4Q^{3/2}(u\alpha-u_2)^{-5/2}u(2u^2u_1\alpha^2+uu_{0,1}\alpha^2-4uu_1u_2\alpha-uu_{2,1}\alpha\\&-u_{0,1}u_2\alpha+2u_1u_2^2+u_2u_{2,1})-4q_2u^2(u\alpha-u_2)^{-1}+4q_1(u\alpha-u_2)^{-3}u(-2u^2u_1\alpha^2\\&-uu_{0,1}\alpha^2+4uu_1u_2\alpha+uu_{2,1}\alpha+u_{0,1}u_2\alpha-2u_1u_2^2-u_2u_{2,1})\\&+2q((u\alpha-u_2)^{-3}(4u^2u_1^2\alpha^2+4uu_{0,1}u_1\alpha^2-8uu_1^2u_2\alpha-4uu_1u_{2,1}\alpha\\&+u_{0,1}^2\alpha^2-4u_{0,1}u_1u_2\alpha-2u_{0,1}u_{2,1}\alpha+4u_1^2u_2^2+4u_1u_2u_{2,1}+u_{2,1}^2)\\&+2u^2\alpha(u\alpha-u_2)^{-1}))/u^2\end{aligned}$$

and

$$\mathsf{R}_3=Q^3u_1-q_{1,1}-q_2u-q_1u_1+q(2u\alpha-u_2).$$

The operator corresponding to the second one is

$$\mathsf{R}_3=\alpha u_1D_x^{-1}-D_{xt}-uD_x^2-u_1D_x+2u\alpha-u_2.$$

The first operator is too complicated to present it here.

4.7 Symplectic structures

The equation defining symplectic structures is

$$-\alpha\tilde{D}_t(\mathsf{S})+\tilde{D}_x^2D_t(\mathsf{S})+u\tilde{D}_x^3(\mathsf{S})+u_1\tilde{D}_x^2(\mathsf{S})+(u_2-3\alpha u)\tilde{D}_x(\mathsf{S})=0,$$

where the total derivatives are from Subsection 4.4. The simplest solutions are

$$\begin{aligned}&\mathsf{S}_5=Q^3,\\&\mathsf{S}_6=Q^4+Q^3u-q_{0,1}-q_1u\end{aligned}$$

while the corresponding operators have the form

$$\begin{aligned}&\mathsf{S}_5=\alpha D_x^{-1},\\&\mathsf{S}_6=D_x^{-1}\circ(\alpha u-u_2)+\alpha uD_x^{-1}-D_t-uD_x.\end{aligned}$$

4.8 Hamiltonian structures

Hamiltonian structures are defined by the equation

$$\alpha\tilde{D}_t(\mathsf{H}) - \tilde{D}_x^2\tilde{D}_t(\mathsf{H}) - u\tilde{D}_x^3(\mathsf{H}) - 2u_1\tilde{D}_x^2(\mathsf{H}) + (3\alpha u - 2u_2)\tilde{D}_x(\mathsf{H}) + (3\alpha u_1 - u_3)\mathsf{H} = 0,$$

where the total derivatives are given in Subsection 4.5. We present here three solutions. Two of them are local and the third one is nonlocal:

$$\begin{aligned}
\mathsf{H}_{-3} &= p_1,\\
\mathsf{H}_{-2} &= -p_{0,1} - p_1 u + p u_1,\\
\mathsf{H}_1 &= P^1 u_1 - p_{1,2} - p_{2,1}u + p_{0,1}u\alpha - p_2 u_{0,1} + p_1 u(-u\alpha + u_2)\\
&\quad + p(-uu_1\alpha + u_{0,1}\alpha + u_1 u_2)
\end{aligned}$$

The corresponding operators are

$$\begin{aligned}
\mathsf{H}_{-3} &= D_x,\\
\mathsf{H}_{-2} &= -D_t - uD_x + u_1,\\
\mathsf{H}_1 &= u_1 D_x^{-1} \circ ((-2uu_1\alpha - u_{0,1}\alpha + 2u_1u_2 + u_{2,1})/u) - uD_x^2 D_t - D_x D_t^2 + u\alpha D_t\\
&\quad - u_{0,1}D_x^2 + u(-u\alpha + u_2)D_x - 4uu_1\alpha + uu_3 + 3u_1u_2 + u_{2,1}.
\end{aligned}$$

4.9 Recursion operators for cosymmetries

These recursion operators are defined by the equation

$$-\alpha\tilde{D}_t(\hat{\mathsf{R}}) + \tilde{D}_x^2 D_t(\hat{\mathsf{R}}) + u\tilde{D}_x^3(\hat{\mathsf{R}}) + u_1\tilde{D}_x^2(\hat{\mathsf{R}}) + (u_2 - 3\alpha u)\tilde{D}_x(\hat{\mathsf{R}}) = 0,$$

where the total derivatives are given in Subsection 4.5. One of the solutions is

$$\hat{\mathsf{R}}_3 = P^1 + p_{1,1} + p_2 u - 2u\alpha + u_2$$

to which the operator

$$\hat{\mathsf{R}}_3 = D_x^{-1} \circ ((-2uu_1\alpha - u_{0,1}\alpha + 2u_1u_2 + u_{2,1})/u) + D_{xt} + uD_x^2 - 2u\alpha + u_2$$

corresponds.

5 Conclusion

We finish this paper with a number of remarks.

Remark 1 First of all, let us stress again what has been done above. We treated the Camassa–Holm equation directly, without artificial assumptions about its evolution or pseudo-evolution nature. The pass from the original form (1) to system (26) (which is *not* in evolution form) was done by technical reasons only. On this way, we found an infinite family of pairwise compatible Hamiltonian structures, recursion operators for symmetries and cosymmetries, and symplectic operators. These structures lead to existence of two commutative series of local symmetries and conservation laws and thus the equation is integrable.

Remark 2 Several comments are worth to be made in relation to Theorem 3. Two cases must be distinguished: local and nonlocal. In both cases the problem reduces to *reconstruction* of nonlocal shadows up to symmetries and subsequent computation of the bracket.

*Local case.*This case is simple and the result is as follows: any shadow in the ℓ-covering corresponding to a recursion operator can be canonically lifted up to a symmetry of the covering equation and the commutator of these symmetries corresponds to the Nijenhuis bracket of the operators. A similar result is valid for the shadows in the ℓ^*-covering corresponding to local bivectors (in this case the commutator of symmetries is related to the Schouten bracket).

*Nonlocal case.*The case of nonlocal operators is much more complicated and rests on the problem of how to commute nonlocal shadows. Seemingly there exists no *natural* definition of such a commutator, but in Ref. [9] we proposed a procedure both to reconstruct and commute shadows. A weak point of this procedure is that it is based on rather cumbersome and intuitively not obvious notion of shadow equivalence. Dealing with equivalence classes of shadows necessitates elimination of certain kind of nonlocal variables that we call *pseudo-constants* and that are intrinsically related to the covering at hand. This not always simple to do and we plan to simplify and clarify the procedure.

Remark 3 Of course, all invariants of the Camassa–Holm equation in its initial form can be obtained from those computed for the system by simple transformations. E.g., to obtain symmetries of Eq. (28) from those of (26) one should substitute in the u-component the variables w_l by $D_x^l(u\alpha - u_2)$. On the other way, any cosymmetry in the scalar case can be constructed directly from the corresponding cosymmetry of Eq. (26) by changing the variables w_l in the w-component by $D_x^l(u\alpha - u_2)$. Similar transformations are applicable to other structures.

Remark 4 Computation of symmetries and cosymmetries in the matrix representation can be simplified using the following observation:

Proposition 1 *The following correspondence between the components of symmetries and cosymmetries of matrix equation* (26) *is valid*:

$$\psi_k^w \xmapsto{D_t + uD_x - u_1} \psi_k^u \xmapsto{\cdot(-1)} \varphi_{k-2}^u \xmapsto{\alpha - D_x^2} \varphi_{k-2}^w.$$

Remark 5 Now we shall show how to prove locality of symmetry hierarchies. Let us introduce a new variable $v = w^2$. Then Eq. (26) transforms to

$$\begin{cases} v_t &= -u_x v - u w_x, \\ v^2 &= \alpha u - u_{xx}, \end{cases} \tag{29}$$

i.e., the first equation acquires potential form[5]. This means that for any symmetry $\varphi = (\varphi^v, \varphi^u)$ the form $\omega_\varphi = \varphi^v\, dx - (u\varphi^v + v\varphi^u)\, dt$ is a conservation law of Eq. (29). Comparing gradings, it is easily checked that all conservation laws of the form ω_φ are trivial and consequently φ^v lies in the image of D_x for any symmetry φ. From this fact it follows that the action of recursion operators is local.

Remark 6 Finally, we indicate relations between the Camassa–Holm equation and the equation describing short capillary-gravity waves (the Neveu–Manna equation, see [2, 18])

$$u_{xy} = u - uu_{xx} - \frac{1}{2}u_x^2 + \frac{\lambda}{2}u_{xx}u_x^2.$$

We strongly believe that there exists a deformation connecting these two equations and their integrable structures are closely related to each other. We intend to discuss this relation elsewhere.

[5] This fact is related to existence of the nonlocal variable $s_{1/2}$, see above.

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