Geodesic Webs on a Two-Dimensional Manifold and Euler Equations

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Abstract

We prove that any planar 4-web defines a unique projective structure in the plane in such a way that the leaves of the foliations are geodesics of this projective structure. We also find conditions for the projective structure mentioned above to contain an affine symmetric connection, and conditions for a planar 4-web to be equivalent to a geodesic 4-web on an affine symmetric surface. Similar results are obtained for planar d-webs, $d > 4$, provided that additional $d - 4$ second-order invariants vanish.

1 Introduction

In this paper, which is a continuation of the paper [1], we study geodesic webs, i.e., webs whose leaves are totally geodesic in a torsion-free affine connection.

We study in detail the planar case and prove that any planar 4-web defines a unique projective structure in the plane in such a way that the leaves of the foliations are geodesics of this projective structure. We also find conditions for the projective structure mentioned above to contain an affine symmetric connection, and conditions for a planar 4-web to be equivalent to a geodesic 4-web on an affine symmetric surface.

Similar results are obtained for planar d-webs, $d > 4$, provided that additional $d - 4$ second-order invariants vanish.

We also apply the obtained results to a surface of constant curvature and to the linear webs. This allows us to prove the Gronwall-type theorem (see [7] and [5]) and its natural generalizations in the case of geodesic webs.

For this, first, we find necessary and sufficient conditions for the foliation defined by level sets of a function to be totally geodesic in the torsion-free connection. This brings us to what we call the flex equation. The flex equation possesses the infinite-dimensional pseudogroup of gauge symmetries. Factorization of the flex equation with respect to this pseudogroup leads us to the Euler equation as well as to natural generalizations of it. This reduction gives us a way to solve the flex equation.

Second, we apply these conditions to find a linearity criterion for planar webs mentioned above. We formulate these conditions by means of the flex equation(s) and show how to describe linear webs in terms of the Euler equation.

For all these webs we find conditions which web function of a linear web must satisfy.

2 Linear Connections in Nonholonomic Coordinates

Let M be a smooth manifold of dimension n. Let vector fields $\partial_1, ..., \partial_n$ form a basis in the tangent bundle and let $\omega^1, .., \omega^n$ be the dual basis. Then

$$[\partial_i, \partial_j] = \sum_k c_{ij}^k \partial_k$$

for some functions $c_{ij}^k \in C^\infty(M)$, and

$$d\omega^k + \sum_{i<j} c_{ij}^k \omega^i \wedge \omega^j = 0.$$

Let ∇ be a linear connection in the tangent bundle, and let Γ_{ij}^k be the Christoffel symbols of second type. Then

$$\nabla_i(\partial_j) = \sum_k \Gamma_{ji}^k \partial_k,$$

where $\nabla_i \overset{\text{def}}{=} \nabla_{\partial_i}$, and

$$\nabla_i(\omega^k) = -\sum_j \Gamma_{ij}^k \omega^j.$$

The covariant differential of a vector field

$$d_\nabla : \mathbf{D}(M) \to \mathbf{D}(M) \otimes \Omega^1(M),$$

and the covariant differential of a differential 1-form

$$d_\nabla : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M)$$

take the following form:

$$d_\nabla(\partial_i) = \sum_{k,j} \Gamma_{ij}^k \; \partial_k \otimes \omega^j,$$

and

$$d_\nabla(\omega^k) = -\sum_{i,j} \Gamma_{ij}^k \omega^j \otimes \omega^i.$$

Remark that if the connection ∇ is torsion-free, then

$$\Gamma_{ji}^{k} - \Gamma_{ij}^{k} = c_{ij}^{k}.$$

For the curvature tensor $R = d_{\nabla}^{2} : \mathbf{D}(M) \to \mathbf{D}(M) \otimes \Omega^{2}(M)$ one has

$$R(\partial_i, \partial_j) : \partial_k \longmapsto \sum_{l} R_{kij}^{l} \partial_l,$$

where

$$R_{kij}^{l} = \partial_i \left(\Gamma_{jk}^{l} \right) - \partial_j \left(\Gamma_{ik}^{l} \right) + \sum_{m} \left(\Gamma_{jk}^{m} \Gamma_{im}^{l} - \Gamma_{ik}^{m} \Gamma_{jm}^{l} - c_{ij}^{m} \Gamma_{mk}^{l} \right).$$

3 Geodesic Foliations and Flex Equations

The covariant differential

$$d_{\nabla} : \Omega^{1}(M) \to \Omega^{1}(M) \otimes \Omega^{1}(M)$$

splits into the direct sum $d_{\nabla} = d_{\nabla}^{a} \oplus d_{\nabla}^{s}$ according to the splitting of tensors into the sum of skew-symmetric and symmetric ones:

$$\Omega^{1}(M) \otimes \Omega^{1}(M) = \Omega^{2}(M) \oplus S^{2}(M).$$

Then the connection is a torsion-free if and only if the skew-symmetric component coincides with the de Rham differential:

$$d_{\nabla}^{a} = d : \Omega^{1}(M) \to \Omega^{2}(M).$$

In the nonholonomic coordinates the symmetric component d_{∇}^{s} has the form

$$d_{\nabla}^{s}\left(\omega^{k}\right) = -\sum_{i,j} \Gamma_{ij}^{k} \omega^{i} \cdot \omega^{j}$$

where $\cdot$ means the symmetric product of differential 1-forms.

Lemma 1 *The foliation defined by a differential 1-form θ is totally geodesic in the connection ∇ if and only if*

$$d_{\nabla}^{a}(\theta) = \alpha \wedge \theta, \quad d_{\nabla}^{s}(\theta) = \beta \cdot \theta$$

for some differential 1-forms α and β.

Proof. See Proposition 2 in [1]. □

Corollary 1 *The foliation defined by a differential 1-form θ is totally geodesic in the torsion-free connection ∇ if and only if*

$$d_{\nabla}^{s}(\theta) = \beta \cdot \theta$$

for a differential 1-form β.

Corollary 2 *The foliation defined by level sets of a function f is totally geodesic in the torsion-free connection ∇ if and only if the quadratic form*

$$d^s_\nabla d(f) \in S^2(M)$$

vanishes on the level sets $f = \mathrm{const}$.

Lemma 2 *Assume that differential 1-forms $\tau_1, .., \tau_n$ are linearly independent on M. A quadratic form*

$$Q = \sum_{ij} Q_{ij} \tau_i \cdot \tau_j$$

vanishes on the distribution $\tau_1 + \cdots + \tau_n = 0$ if and only if we have

$$Q_{ii} + Q_{jj} = 2Q_{ij}$$

for all i, j.

Proof. One has

$$\sum_{ij} Q_{ij} \tau_i \cdot \tau_j = (\tau_1 + \cdots + \tau_n) \cdot (x_1 \tau_1 + \cdots + x_n \tau_n)$$

for some functions $x_1, ..., x_n$.

Then $2Q_{ij} = x_i + x_j$. Taking $i = j$, we get $x_i = Q_{ii}$, and then $Q_{ii} + Q_{jj} = 2Q_{ij}$. □

Now we apply this lemma to the quadratic form $Q = d^s_\nabla d(f)$ and $\tau_i = f_i \omega^i$, where $f_i = \partial_i(f)$. We have

$$df = \sum_i f_i \omega^i = \sum_i \tau_i$$

and

$$\begin{aligned} d^s_\nabla (df) &= d^s_\nabla \left(\sum_k f_k \omega^k \right) = \sum_k f_k d^s_\nabla \left(\omega^k\right) + \sum_i df_i \cdot \omega^i \\ &= -\sum_{i,j,k} f_k \Gamma^k_{ij} \omega^i \cdot \omega^j + \sum_{i,j} \partial_i (\partial_j (f)) \omega^i \cdot \omega^j \\ &= \sum_{i,j} \left(\partial_i (\partial_j (f)) - \sum_k \Gamma^k_{ij} \partial_k (f) \right) \omega^i \cdot \omega^j \\ &= \sum_{i,j} \left(\partial_i (f_j) - \sum_k \Gamma^k_{ij} f_k \right) \frac{\tau_i \cdot \tau_j}{f_i f_j}. \end{aligned}$$

In other words,

$$2Q_{ij} = \frac{\partial_i (f_j) + \partial_j (f_i)}{f_i f_j} - \sum_k (\Gamma^k_{ij} + \Gamma^k_{ji}) \frac{f_k}{f_i f_j}.$$

In what follows we shall assume that the connection ∇ is torsion-free, $\partial_i = \frac{\partial}{\partial xi}$, in some local coordinates, and as a result $\Gamma^i_{jk} = \Gamma^i_{kj}$.

Summarizing, we get the following result.

Theorem 3 *The foliation defined by level sets of a function f is totally geodesic in the torsion-free connection ∇ if and only if the function f satisfies the following system of differential equations:*

$$f_j^2(f_{ii} - \sum_k \Gamma^k_{ii}\, f_k) - 2f_i f_j(f_{ij} - \sum_k \Gamma^k_{ij} f_k) + f_i^2(f_{jj} - \sum_k \Gamma^k_{jj} f_k) = 0 \qquad (1)$$

for all $i < j$, where

$$f_{ij} = (\partial_i(f_j) + \partial_j(f_i))/2.$$

We call such a system a *flex system.*

Consider a d-web W_d, $d \geq n+1$, formed by the level sets of functions $f_\alpha(x_1, ..., x_n), \alpha = 1, \ldots, d$.

The web W_d is said to be a *geodesic web* if the leaves of all its foliations are totally geodesic.

Corollary 4 *A web W_d is geodesic if and only if conditions* (1) *are satisfied for all web functions $f_\alpha(x_1, ..., x_n), \alpha = 1, \ldots, d$.*

The conditions for a planar 3-web to be geodesic were found in another form in [3].

In dimension $n = 2$ we get the only one differential equation which we have named the *flex equation* (see [5]) for the case of flat connections:

$$f_2^2\,(f_{11} - \Gamma^1_{11} f_1 - \Gamma^2_{11} f_2) - 2 f_1 f_2\,(f_{12} - \Gamma^1_{12} f_1 - \Gamma^2_{12} f_2) + f_1^2\,(f_{22} - \Gamma^1_{22} f_1 - \Gamma^2_{22} f_2) = 0. \qquad (2)$$

In what follows, we shall often use the following definition.

Definition 1 *The flex of a function $f(x, y)$ is*

$$\text{Flex } f = f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy} = -\det \begin{pmatrix} f_{xx} & f_{xy} & f_x \\ f_{xy} & f_{yy} & f_y \\ f_x & f_y & 0 \end{pmatrix}.$$

4 Factorization of Flex Equation and Euler Equation

The flex equation possesses the infinite-dimensional group of gauge symmetries of the form

$$f \longmapsto \Phi(f)$$

for any diffeomorphism $\Phi : \mathbb{R} \to \mathbb{R}$. Factorization of this equation with respect to this pseudogroup leads us to the classical Euler equation as well as to some generalizations of it.

In order to factorize the flex equation, we define the function

$$w = \frac{f_y}{f_x},$$

which is the first-order differential invariant of the gauge pseudogroup.

Indeed, we have:

$$\frac{\Phi\left(f\right)_y}{\Phi\left(f\right)_x} = \frac{f_y}{f_x},$$

and the flex equation can be rewritten in terms of this invariant as follows:

$$\partial_y w - w\partial_x w = \Pi^2_{11} w^3 - 3\Pi^2_{12} w^2 - 3\Pi^1_{12} w + \Pi^1_{22},$$

where

$$\Pi^1_{22} = \Gamma^1_{22},\ \ \Pi^1_{12} = -\frac{1}{3}(\Gamma^2_{22} - 2\Gamma^1_{12}),\ \ \Pi^2_{12} = -\frac{1}{3}(\Gamma^1_{11} - 2\Gamma^2_{12}),\ \ \Pi^2_{11} = \Gamma^2_{11}$$

are the Thomas parameters (see [9]).

We call the above equation *Euler's equation associated with the connection* ∇.

Given Cauchy data $w(x, 0) = w_0(x)$, one can solve the above Euler equation by the standard method of characteristics, and then find the function $f\left(x, y\right)$ as a first integral of the vector field:

$$\partial_y - w\partial_x.$$

5 Geodesic 4-Webs and Projective Structures

Let us rewrite equation (2) as follows:

$$\Pi^1_{22} f_1^3 - 3\Pi^1_{12} f_1^2 f_2 - 3\Pi^2_{12} f_1 f_2^2 + \Pi^2_{11} f_2^3 = \text{Flex } f. \tag{3}$$

Note that equation (3) appeared also in the paper [6], where the author studied the general theory of systems of linear second-order PDEs.

We shall consider equation (3) as a linear equation for the components Π^i_{jk} of the connection.

Remind that two affine connections, say ∇ and $\widetilde{\nabla}$, are *projectively equivalent* if there is a differential 1-form ρ such that

$$\nabla_X(Y) - \widetilde{\nabla}_X(Y) = \rho(X)Y + \rho(Y)X$$

for all vector fields X and Y (see, for example, [8], p. 17).

A *projective structure* on a manifold may be defined as a class of projectively equivalent affine connections.

The coefficients Π^k_{ij} completely determine the equivalence class of the connection given by (Γ^i_{jk}).

Assume that a geodesic 4-web is given by web functions

$$f_1(x,y), f_2(x,y), f_3(x,y), f_4(x,y).$$

Then equation (3) gives the following linear system with respect to the Thomas parameters (or the so-called projective connection):

$$\begin{aligned}
\Pi^1_{22} f^3_{1,1} - 3\Pi^1_{12} f^2_{1,1} f_{1,2} - 3\Pi^2_{12} f_{1,1} f^2_{1,2} + \Pi^2_{11} f^3_{1,2} &= \text{Flex } f_1,\\
\Pi^1_{22} f^3_{2,1} - 3\Pi^1_{12} f^2_{2,1} f_{2,2} - 3\Pi^2_{12} f_{2,1} f^2_{2,2} + \Pi^2_{11} f^3_{2,2} &= \text{Flex } f_2,\\
\Pi^1_{22} f^3_{3,1} - 3\Pi^1_{12} f^2_{3,1} f_{3,2} - 3\Pi^2_{12} f_{3,1} f^2_{3,2} + \Pi^2_{11} f^3_{3,2} &= \text{Flex } f_3,\\
\Pi^1_{22} f^3_{4,1} - 3\Pi^1_{12} f^2_{4,1} f_{4,2} - 3\Pi^2_{12} f_{4,1} f^2_{4,2} + \Pi^2_{11} f^3_{4,2} &= \text{Flex } f_4.
\end{aligned} \tag{4}$$

Solving system (4), we get

$$\begin{aligned}
\Pi^1_{22} &= \sum_{i=1}^{4} \frac{\prod\limits_{k\neq i} f_{k,2}}{\prod\limits_{k\neq i} J(f_i, f_k)} \text{ Flex } f_i,\\
\Pi^2_{11} &= \sum_{i=1}^{4} \frac{\prod\limits_{k\neq i} f_{k,1}}{\prod\limits_{k\neq i} J(f_i, f_k)} \text{ Flex } f_i;
\end{aligned} \tag{5}$$

$$\begin{aligned}
-3\Pi^1_{12} &= \sum_{i=1}^{4} \frac{\sum\limits_{k\neq i} f_{k,1} \prod\limits_{l\neq i,k} f_{l,2}}{\prod\limits_{m\neq i} J(f_i, f_m)} \text{ Flex } f_i,\\
-3\Pi^2_{12} &= \sum_{i=1}^{4} \frac{\sum\limits_{k\neq i} f_{k,2} \prod\limits_{l\neq i,k} f_{l,1}}{\prod\limits_{m\neq i} J(f_i, f_m)} \text{ Flex } f_i,
\end{aligned} \tag{6}$$

where

$$J(f_i, f_j) = \det \begin{pmatrix} f_{i,1} & f_{i,2} \\ f_{j,1} & f_{j,2} \end{pmatrix}$$

is the Jacobian of the functions $f_i(x,y)$ and $f_j(x,y)$. Note that in the planar case we have $\Pi^2_{22} = -\Pi^1_{12}$ and $\Pi^1_{11} = -\Pi^2_{12}$. Thus, formulas (5) and (6) give all Thomas parameters Π^i_{jk}.

Remark that system (4) is invariant with respect to the gauge transformations $f_i \to \Phi_i(f_i)$, $i = 1, 2, 3, 4$, and therefore, solutions (5) and (6) do not depend on functions $\{f_i\}$, but they are completely determined by the geodesic 4-web.

Summarizing, we get the following result.

Theorem 5 *Any planar* 4*-web defines a unique projective structure in the plane in such a way that the leaves of the foliations are geodesics of this projective structure.*

This theorem allows us to get the following Gronwall-type theorem (see [7]).

Corollary 6 *Any diffeomorphism sending a planar* 4*-web* W_4 *into a planar* 4*-web* $\widetilde{W}_4$ *is a projective transformation of the corresponding projective structures.*

6 Geodesic Webs and Symmetric Projective Structures

We say that a projective structure is *symmetric* if the class of projectively equivalent affine connections contains an affine symmetric connection.

Let us consider the case when the projective structure determined by a 4-web is symmetric. Then $\nabla R = 0$ for some affine connection from the class.

Denote

$$\Gamma^1_{12} = \sigma, \ \alpha = \Gamma^2_{22} - 2\Gamma^1_{12}, \ \beta = \Gamma^1_{11} - 2\Gamma^2_{12}, \ \Gamma^2_{12} = \tau. \tag{7}$$

Then

$$\Gamma^1_{11} = 2\tau + \beta, \ \Gamma^2_{22} = 2\sigma + \alpha. \tag{8}$$

In order to simplify formulae, we choose coordinates x, y in the plane in such a way that $f_1(x, y) = x, \ f_2(x, y) = y$.

Then (5) gives

$$\Gamma^2_{11} = 0, \ \Gamma^1_{22} = 0, \tag{9}$$

and (6) becomes

$$\alpha = \frac{f_{4,y} \,\mathrm{Flex}\, f_3}{f_{3,x} f_{3,y} \Delta} - \frac{f_{3,y} \,\mathrm{Flex}\, f_4}{f_{4,x} f_{4,y} \Delta}, \ \ \beta = -\frac{f_{4,x} \,\mathrm{Flex}\, f_3}{f_{3,x} f_{3,y} \Delta} + \frac{f_{3,x} \,\mathrm{Flex}\, f_4}{f_{4,x} f_{4,y} \Delta}, \tag{10}$$

where

$$\Delta = f_{3,x} f_{4,y} - f_{3,y} f_{4,x}.$$

By a straightforward computation, one can find the components of the curvature tensor $R(\nabla) = \{R^i_{jkl}\}$ of the connection :

$$R(\nabla) = \begin{Vmatrix} R^1_{112} & R^1_{212} \\ R^2_{112} & R^2_{212} \end{Vmatrix} = \begin{Vmatrix} \sigma_x - 2\tau_y - \beta_y + \sigma\tau & -\sigma_y + \sigma^2 + \sigma\alpha \\ \tau_x - \tau^2 - \tau\beta & 2\sigma_x - \tau_y + \alpha_x - \sigma\tau \end{Vmatrix}.$$

Computing components of ∇R, we get the following system of differential equations for components τ and σ:

$$\begin{aligned}
\sigma_{xx} - 2\,\tau_{xy} - \beta_{xy} &= (2\tau + \beta)\sigma_x - 2\,\sigma\tau_x - (3\,\tau + \beta)(2\tau_y + \beta_y) + 2\,\sigma\tau(2\tau + \beta), \\
\sigma_{xy} - 2\,\tau_{yy} - \beta_{yy} &= (3\,\sigma + \alpha)\sigma_x - (7\,\sigma + 2\alpha)\tau_y - (3\,\sigma + \alpha)\beta_y - 2\,\tau\sigma_y + 2\,\sigma\tau(2\,\sigma + \alpha), \\
\alpha_{xx} + 2\,\sigma_{xx} - \tau_{xy} &= (3\,\tau + \beta)\alpha_x + (7\tau + 2\beta)\,\sigma_x + 2\,\sigma\tau_x - (3\,\tau + \beta)\tau_y - 2\,\sigma\tau(2\,\tau + \beta), \\
\alpha_{xy} + 2\,\sigma_{xy} - \tau_{yy} &= (3\,\sigma + \alpha)\alpha_x + 2(3\,\sigma + \alpha)\sigma_x - (2\,\sigma + \alpha)\tau_y + 2\,\tau\sigma_y - 2\,\sigma\tau(2\,\sigma + \alpha),
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\tau_{xx} &= 2(3\,\tau+\beta)\tau_x+\tau\beta_x-2\,\tau(2\,\tau^2+3\,\tau\beta+\beta^2),\\
\tau_{xy} &= \tau\sigma_x+\tau\alpha_x+(3\,\tau+\beta)\tau_y+2\,\tau\beta_y+2\,\sigma\tau_x-2\,\sigma\tau(2\,\tau+\beta),\\
\sigma_{xy} &= (3\,\sigma+\alpha)\sigma_x+2\,\sigma\alpha_x+\sigma\tau_y+2\,\tau\sigma_y+\sigma\beta_y-2\,\sigma\tau(2\,\sigma+\alpha),\\
\sigma_{yy} &= 3(2\sigma+\alpha)\sigma_y-2\,\sigma\alpha^2+3\alpha\,\sigma_y+\sigma\alpha_y-2\,\sigma^2(2\,\sigma+3\,\alpha).
\end{aligned} \tag{12}$$

Consider a system consisting of the first and the last equations of (11) and (12). Solving this system, we find all second derivatives $\tau_{xx}, \tau_{xy}, \tau_{yy}$ and $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ in terms of $\alpha, \beta, \sigma, \tau$ and first-order derivatives of σ and τ :

$$\begin{aligned}
\sigma_{xx} &= 2\,\sigma\tau_x+(4\,\tau+\beta)\sigma_x+(\tau-\beta)\beta_y+2\,\tau\alpha_x+\beta_{xy}-2\,\sigma\tau(2\,\tau+\beta),\\
\sigma_{xy} &= (3\,\sigma+\alpha)\sigma_x+2\,\sigma\alpha_x+\sigma\tau_y+2\,\tau\sigma_y+\sigma\beta_y-2\,\sigma\tau(2\sigma+\alpha),\\
\sigma_{yy} &= 3(2\sigma+\alpha)\sigma_y+\sigma\alpha_y-2\,\sigma(\alpha^2+2\,\sigma^2+3\,\sigma\alpha),\\
\tau_{xx} &= 3(2\,\tau+\beta)\tau_x+\tau\beta_x-2\,\tau(\beta^2+2\,\tau^2+3\,\tau\beta),\\
\tau_{xy} &= \tau\sigma_x+\tau\alpha_x+(3\,\tau+\beta)\tau_y+2(\tau\beta_y+\,\sigma\tau_x)-2\,\sigma\tau(2\,\tau+\beta),\\
\tau_{yy} &= (\sigma-\alpha)\alpha_x+(4\,\sigma+\alpha)\tau_y+2(\tau\sigma_y+\,\sigma\beta_y)+\alpha_{xy}-2\,\sigma\tau(2\,\sigma+\alpha).
\end{aligned} \tag{13}$$

Substituting these expressions into the second and third equations of (11), we get the following relations for α and β:

$$\begin{aligned}
\alpha_{xx}+2\beta_{xy} &= \beta\alpha_x+2\beta\beta_y,\\
2\alpha_{xy}+\beta_{yy} &= 2\alpha\alpha_x+\alpha\beta_y.
\end{aligned} \tag{14}$$

Remark that for affine symmetric connections we have tr $R=0$, and it is easy to check that this is the only compatibility condition for system (13) under conditions (14) for α and β.

Moreover, adding the condition tr $R=0$, or

$$\alpha_x-\beta_y+3(\sigma_x-\tau_y)=0, \tag{15}$$

does not produce any new compatibility condition.

In other words, assuming conditions (14) for α and β, the PDE system of equations (13) and (15) with respect to σ and τ is a formally integrable system of finite type. It is easy to see that the solution space of the system has dimension 5.

Summarizing, we arrive at the following result.

Theorem 7 (i) *The projective structure defined by a planar* 4*-web is symmetric if and only if conditions* (14) *hold.*

(ii) *A planar* 4*-web given by functions* $\{x, y, f_3(x,y), f_4(x,y)\}$ *is locally equivalent to a geodesic* 4*-web on an affine symmetric surface if and only if conditions* (14) *hold for the functions* α *and* β *defined by equation* (10).

Remark 1 *If a planar* 4*-web satisfies conditions of the above theorem, then there exists a* 5*-dimensional family* $\nabla_t,\ t \in \mathbb{R}^5$*, of affine symmetric connections such that all leaves of the web are geodesics for* ∇_t*. Take one of them, say* ∇_{t_0}*. Then for cases* $R(\nabla_{t_0}) \neq 0$*, if* $\det R(\nabla_{t_0}) > 0$*, then the* 4*-web is equivalent to a geodesic* 4*-web either on the standard* 2*-sphere or to a geodesic* 4*-web on the Lobachevskii plane. If* $\det R(\nabla_{t_0}) < 0$*, then the* 4*-web is equivalent to a geodesic* 4*-web on the de Sitter plane. If* $\det R(\nabla_{t_0}) = 0$*, then the* 4*-web is equivalent to a geodesic* 4*-web on the affine torus or on the Klein bottle* (*see, for example,* [4]).

7 Planar Geodesic *d*-Webs and Projective Structures

Consider now a planar d-web W_d defined by d web functions $f_\alpha(x,y),\ \alpha = 1,...,d$. Such a web has $\binom{d}{4}$ 4-subwebs $[\alpha,\beta,\gamma,\varepsilon]$ defined by the foliations X_α, X_β, X_γ, and $X_\varepsilon,\ \alpha,\beta,\gamma,\varepsilon = 1,...,d$.

If a d-web W_d is geodesic, then each of $\binom{d}{4}$ its 4-subwebs $[\alpha,\beta,\gamma,\varepsilon]$ is also geodesic, and by Theorem 8, each of them determines its own unique projective structure. These $d-4$ projective structures coincide with a projective structure defined by one of them, let's say, by the 4-subweb $[1,2,3,4]$, if $d-4$ second-order invariants given by the flex equations vanish.

Thus we have proved the following result.

Theorem 8 (i) *A planar d-web,* $d \geq 5$*, defined by web functions* $< f_1,\dots,f_d >$ *is geodesic if and only if the functions* $f_5,\dots,f_d$ *satisfy flex equations* (3)*, in which the components* $\Pi^1_{22},\Pi^2_{11},\Pi^1_{12}$ *and* Π^2_{12} *are given by formulae* (5) *and* (6)*.*

(ii) *A planar d-web,* $d \geq 5$*, defined by web functions* $< x,y,f_3,\dots,f_d >$ *is locally equivalent to a geodesic d-web with respect to a symmetric projective structure if and only if the* $d-4$ *conditions for geodesicity mentioned above are satisfied and, in addition, conditions* (14) *hold.*

Remark 2 *Theorem 4 shows that one has* $d-4$ *second-order conditions on web functions and two more fourth-order conditions to have a geodesic d-web in a symmetric projective structure.*

8 Planar Linear Webs

In what follows, we shall use coordinates x,y on the plane in which the Christoffel symbols Γ^i_{jk} vanish. We shall assume that a d-web W_d is formed by the level sets of web functions $f_1(x,y),\ f_2(x,y),.....,\ f_d(x,y)$.

The following theorem, which immediately follows from formula (3), gives a criterion for W_d to be linear in the coordinates x,y.

Theorem 9 *The d-web W_d is a linear if and only if the web functions are solutions of the differential equation*

$$\text{Flex } f = 0. \tag{16}$$

Note that in algebraic geometry the linearity condition Flex $f = 0$ is also the necessary and sufficient condition for a point (x, y) to be a flex of the curve defined by the equation $f(x, y) = 0$. Here, in (16), Flex $f = 0$ is the equation for finding the function $f(x, y)$ (it should be satisfied for all points (x, y)). This is a reason that we call equation (16) the *flex equation*.

We shall show how to integrate flex equation (16). The main idea of integration is that the factorization of the flex-equation Flex $f = 0$ with respect to the diffeomorphism group produces an Euler equation.

Namely, as we have seen, passing to the differential invariant $w = \frac{f_x}{f_y}$ allows us to reduce the order of the equation. Let us rewrite the flex equation in the form

$$\partial_x \left(\frac{f_x}{f_y} \right) - \left(\frac{f_x}{f_y} \right) \partial_y \left(\frac{f_x}{f_y} \right) = 0. \tag{17}$$

Then integration of the flex equation is equivalent to solution of the following system:

$$\begin{cases} \partial_x w - w \partial_y w & = 0, \\ \partial_x f - w \partial_y f & = 0. \end{cases}$$

The first equation of the system

$$\partial_x w - w \partial_y w = 0$$

is the classical Euler equation in gas-dynamics.

Solutions of this equation are well-known. Namely, if $w_0(y) = w|_{x=0}$ is the Cauchy data, then the solution $w(x, y)$ can be found from the system

$$\begin{cases} y + w_0(\lambda) x - \lambda & = & 0, \\ w(x, y) - w_0(\lambda) & = & 0 \end{cases} \tag{18}$$

by elimination of the parameter λ.

Further, if w is a solution of the Euler equation, then the functions w and f are both first integrals of the vector field $\partial_x - w\partial_y$, and therefore, $f = \Phi(w)$ for some smooth function Φ.

Summarizing, we get the following description of web functions of linear webs.

Proposition 2 *The web functions $f_1(x, y), f_2(x, y), ..., f_d(x, y)$ of a linear d-web have the form*

$$f_1(x, y) = \Phi_1(w_1(x, y)),\ f_2(x, y) = \Phi_2(w_2(x, y)), ...,\ f_d(x, y) = \Phi_d(w_d(x, y)),$$

where $w_1(x, y), w_2(x, y), ..., w_d(x, y)$ are distinct solutions of the Euler equation, and $\Phi_1, \Phi_2, ..., \Phi_d$ are smooth functions.

In particular, using the gauge transformations, we can take

$$f_1(x,y) = w_1(x,y),\ f_2(x,y) = w_2(x,y), ..., \ f_d(x,y) = w_d(x,y).$$

Therefore, the above proposition yields the following description of web functions for linear d-webs.

Theorem 10 *Web functions of linear d-webs can be chosen as d distinct solutions of the Euler equation.*

Example 11 *Assume that for a linear 3-web we have $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = f(x,y)$. Taking $w_0(y) = -2\sqrt{-y}$, we get from (18) that $w = -2(x + \sqrt{x^2 - y})$ and $f = x + \sqrt{x^2 - y}$. The leaves of the third foliation are the tangents to the parabola $y = x^2$.*

Example 12 *Assume that for a linear 5-web we have $f_1(x,y) = x$ and $f_2(x,y) = y$. Taking $(w_3)_0(y) = -2\sqrt{-y}$, $(w_4)_0(y) = y$ and $(w_5)_0(y) = 2y$, we get the linear 5-web with remaining three web functions*

$$f_3 = x + \sqrt{x^2 - y},\ f_4 = \frac{y+1}{1-x},\ \ f_5 = \frac{y}{1-2x}.$$

The last three foliations of this 5-web are the tangents to the parabola $y = x^2$ (see Example 11) and the straight lines of the pencils with the centers $(1,-1)$ and $(\frac{1}{2}, 0)$.

9 Geodesic Webs on Surfaces of Constant Curvature

Theorem 13 *Let (M,g) be a surface of constant curvature with the metric tensor*

$$g = \frac{dx^2 + dy^2}{(1 + \kappa(x^2 + y^2))^2}, \tag{19}$$

where κ is a constant. Then the level sets of a function $f(x,y)$ are geodesics of the metric if and only if the function f satisfies the flex equation

$$\text{Flex } f = \frac{2\kappa\,(xf_x + yf_y)\left(f_x^2 + f_y^2\right)}{1 + \kappa\,(x^2 + y^2)}. \tag{20}$$

Proof. It is easy to see that the Christoffel symbols Γ^i_{jk} are the following

$$\Gamma^1_{12} = \Gamma^2_{22} = \Gamma^1_{21} = -\Gamma^2_{11} = -2\kappa x_2 b,$$
$$\Gamma^1_{11} = \Gamma^2_{21} = \Gamma^2_{12} = -\Gamma^1_{22} = -2\kappa x_1 b,$$

where

$$b = \frac{1}{1 + \kappa\,(x^2 + y^2)}.$$

Substituting these values into the right-hand side of formula (1), we get formula (20). □

Remark that functions of the form $f(x, y) = \Phi\left(\frac{y}{x}\right)$ are solutions of the flex equation, and therefore the level sets of these functions are geodesic.

As we have seen (see Theorem 4), a geodesic d-web uniquely defines a projective structure provided that $d - 4$ additional second-order invariants vanish. This leads us to the following Gronwall–type theorem (cf. Corollary 4):

Theorem 14 *Suppose that W_d, $d \geq 4$, is a geodesic d-web given on a surface (M, g) of constant curvature for which $d - 4$ additional second-order invariants vanish. Then any mapping of W_d on a geodesic web $\widetilde{W}_d$ is a projective transformation.*

Proof. We give an alternative proof. By Beltrami's theorem (see [2]), the surface (M, g) can be mapped by a transformation ϕ onto a plane, and the mapping ϕ sends the geodesics of (M, g) into straight lines. Thus the mapping ϕ linearizes the webs W_d and $\widetilde{W}_d$. In [5] (see Remark on p. 99) it was proved that for $d \geq 4$ the Gronwall conjecture is valid, i.e., there exists a unique transformation sending W_d and $\widetilde{W}_d$ into linear d-webs, and this transformation is projective. The mapping of W_d onto $\widetilde{W}_d$ induces a transformation of the corresponding linear d-webs. The latter transformation is projective. As a result, the mapping of W_d onto $\widetilde{W}_d$ is also projective. □

10 Geodesic Webs on Surfaces in $\mathbb{R}^3$

Proposition 3 *Let $(M, g) \subset \mathbb{R}^3$ be a surface defined by an equation $z = z(x, y)$ with the induced metric g and the Levi-Civita connection ∇. Then the flex equation takes the form*

$$\text{Flex } f = \frac{z_x f_x + z_y f_y}{1 + z_x^2 + z_y^2}(f_y^2 z_{xx} - 2 f_x f_y z_{xy} + f_x^2 z_{yy}). \tag{21}$$

Proof. To prove formula (21), note that the metric induced on a surface $z = z(x, y)$ is

$$g = ds^2 = (1 + z_x^2)dx^2 + (1 + z_y^2)dy^2 + 2 z_x z_y dx dy.$$

Computing the Christoffel symbols, we get that

$$\Gamma^1_{11} = \frac{z_x z_{xx}}{1 + z_x^2 + z_y^2}, \ \Gamma^1_{12} = \Gamma^1_{21} = \frac{z_x z_{xy}}{1 + z_x^2 + z_y^2}, \ \Gamma^1_{22} = \frac{z_x z_{yy}}{1 + z_x^2 + z_y^2},$$
$$\Gamma^2_{11} = \frac{z_y z_{xx}}{1 + z_x^2 + z_y^2}, \ \Gamma^2_{12} = \Gamma^2_{21} = \frac{z_y z_{xy}}{1 + z_x^2 + z_y^2}, \ \Gamma^2_{22} = \frac{z_x z_{yy}}{1 + z_x^2 + z_y^2}.$$

Applying these formulas to the right-hand side of (2), we get formula (21). □

For example, functions of the form $f(x, y) = \Phi\left(\frac{x}{y}\right)$ are solutions of the flex equation if and only if the function $z\,(x, y)$ satisfies one of the following equations:

$$yz_x - xz_y = 0,$$

or

$$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = 0.$$

The general solution of the first equation have the form $z = \Omega(x^2 + y^2)$, and of the second one $z(x, y) = x\Psi(\frac{y}{x}) + \Theta(\frac{y}{x})$, where $\Omega(\alpha), \Psi(\alpha)$ and $\Theta(\alpha)$ are arbitrary smooth functions.

If we assume that the foliations $\{x = \text{const}\,.\}$ and $\{y = \text{const}\,.\}$ are geodesic on the surface $z = z(x, y)$, then the flex equation gives $z_{xx} = z_{yy} = 0$, and therefore $z = axy + bx + cy + d$ for some constants a, b, c, and d, and formula (10) takes the form

$$\text{Flex } f = -\frac{2af_x f_y((ay + b)f_x + (ax + c)f_y)}{1 + (ay + b)^2 + (ax + c)^2}.$$

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[1]In the bibliography we will use the following abbreviations for the review journals: JFM for Jahrbuch für die Fortschritte der Mathematik, MR for Mathematical Reviews, and Zbl for Zentralblatt für Mathematik.

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