Wave operators on quantum algebras via noncanonical quantization

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Abstract

We suggest a method to quantize basic wave operators of Relativistic Quantum Mechanics (Laplace, Maxwell, Dirac ones) without using canonical coordinates. We define two-parameter deformations of the Minkowski space algebra and its 3-dimensional reduction via the so-called Reflection Equation Algebra and its modified version. Wave operators on these algebras are introduced by means of quantized partial derivatives described in two ways. In particular, they are given in so-called pseudospherical form which makes use of a q-deformation of the Lie algebra $sl(2)$ and quantum versions of the Cayley-Hamilton identity.

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1 Introduction

By noncanonical quantization we mean a quantization which does not use canonically conjugated coordinates or fields looking like the Darboux coordinates. A passage from the algebra $\mathrm{Sym}(\mathfrak{g})$ to the algebra $U(\mathfrak{g}_\hbar)$ is an example of such a quantization. Hereafter, the notation $\mathfrak{g}_\hbar$ stands for the Lie algebra with the bracket $\hbar\,[\,,\,]$ where $\hbar\in\mathbb{K}$ is a parameter and $\mathfrak{g}$ is a Lie algebra over the ground field $\mathbb{K}$ with the bracket $[\,,\,]$. In virtue of the PBW theorem there exists a $\mathfrak{g}$-covariant[1] linear map $\alpha:\mathrm{Sym}(\mathfrak{g})\to U(\mathfrak{g}_\hbar)$ which induces isomorphisms on each homogeneous component $\mathrm{Sym}^{(k)}(\mathfrak{g})\to \mathrm{Gr}^{(k)}\,U(\mathfrak{g}_\hbar)$ where $\mathrm{Gr}\,U(\mathfrak{g}_\hbar)$ is the graded algebra associated to the filtered algebra $U(\mathfrak{g}_\hbar)$. Explicitly, this map can be defined as follows: to an element $p(x_1,...,x_n)\in\mathrm{Sym}(\mathfrak{g})$ where $\{x_i\}$ is a basis in the algebra $\mathfrak{g}$ and p is a symmetric polynomial in x_i the map α associates the element $p(\hat{x}_1,...,\hat{x}_n)\in U(\mathfrak{g}_\hbar)$. Hereafter the variables with hat (for example, $\hat{x}_i$) are treated to be elements of the algebra $U(\mathfrak{g}_\hbar)$ whereas those without hat are assumed to be commutative.

This map enables us to introduce a new $\mathfrak{g}$-covariant product in the algebra $\mathrm{Sym}(\mathfrak{g})$

$$f\star_h g := \alpha^{-1}(\alpha(f)\circ\alpha(g)),\qquad \forall f,\,g\in\mathrm{Sym}(\mathfrak{g})$$

where $\circ$ is the product in the algebra $U(\mathfrak{g}_\hbar)$. The Poisson counterpart of the product $\star_h$ is the linear Poisson-Lie bracket $\{\,,\,\}_{PL}$ associated to the Lie bracket $[\,,\,]$.

By assuming a Lie algebra $\mathfrak{g}$ to be $gl(n)$ we can further deform the algebra $U(\mathfrak{g}_\hbar)$ into a two-parameter algebra $\mathcal{L}_{\hbar,q}$ which is called the modified Reflection Equation Algebra (mREA). Its explicit description is given in section 3. The algebra $\mathcal{L}_{\hbar,q}$ as well as its specialization at $\hbar=0$ (denoted $\mathcal{L}_q$ and called the Reflection Equation Algebra (REA)) is

[1] We assume $\mathfrak{g}$ to act on itself by the adjoint action extended onto the algebras in question via the Leibnitz rule.

covariant w.r.t. the Quantum Group (QG) $U_q(sl(n))$. Moreover, the Poisson counterpart of this two-parameter deformation is a Poisson pencil generated by the linear Poisson-Lie bracket and another one which corresponds to the deformation $\mathrm{Sym}(gl(n)) \Rightarrow \mathcal{L}_q$. Observe that this Poisson pencil admits a restriction to the space $\mathrm{Sym}(sl(n))$. The quantum counterpart of the restricted Poisson pencil can be introduced via quotienting the algebra $\mathcal{L}_{\hbar,q}$ over the ideal generated by a central element (the quantum trace). We denote this quotient by $\mathcal{SL}_{\hbar,q}$ and its specialization at $\hbar = 0$ by $\mathcal{SL}_q$.

The passage from the algebra $\mathrm{Sym}(gl(n))$ to that $\mathcal{L}_q$ or $\mathcal{L}_{\hbar,q}$ (resp., an analogous passage from $\mathrm{Sym}(sl(n))$ to $\mathcal{SL}_q$ or $\mathcal{SL}_{\hbar,q}$) is also an example of noncanonical quantization. We call it $q-$(resp., $(\hbar, q)$-)quantization in contrast with the $\hbar$-quantization consisting in the above mentioned passage from $\mathrm{Sym}(\mathfrak{g})$ to $U(\mathfrak{g}_\hbar)$. Note that a map $\alpha : \mathrm{Sym}(gl(n)) \to \mathcal{L}_{\hbar,q}$ (resp., $\alpha : \mathrm{Sym}(sl(n)) \to \mathcal{SL}_{\hbar,q}$), similar to that above, and consequently a product $\star_{\hbar,q}$ can be also defined for a generic q but it is not covariant w.r.t. the action of either classical or quantum group.

In this article we are dealing with the coordinate algebra $\mathbb{K}[\mathbb{R}^4]$ of the Minkowski space $\mathbb{R}^4$, its $(\hbar, q)$-quantization, and its two specializations: at $q = 1$ ($\hbar$-quantization) and at $\hbar = 0$ (q-quantization). All algebras in question are real (i.e. $\mathbb{K} = \mathbb{R}$) but sometimes we need their complexification: in this case we put $\mathbb{K} = \mathbb{C}$.

Now we explicitly describe the quantum algebra arising from the $\hbar$-quantization of the algebra $\mathbb{K}[\mathbb{R}^4]$. To this end we consider a map $\mathbb{R}^4 \to u(2) \cong u(2)^*$ defined as follows

$$(t,\, x,\, y,\, z) \mapsto \begin{pmatrix} t+z & x+\mathbf{i}\,y \\ x-\mathbf{i}\,y & t-z \end{pmatrix}.$$

Hereafter $\mathbf{i} = \sqrt{-1}$. This map enables us to identify the coordinate algebra $\mathbb{K}[\mathbb{R}^4]$ of the Minkowski space with the symmetric algebra $\mathrm{Sym}(u(2))$. This is the reason why we treat the enveloping algebra $U(u(2)_\hbar)$ to be an $\hbar$-quantum counterpart of the algebra $\mathbb{K}[\mathbb{R}^4]$. As for the $q-$ and $(\hbar, q)$-counterparts of this algebra their explicit description is given below. Also, we deal with "sl-reductions" of these algebras which are deformations of the algebra $\mathbb{K}[\mathbb{R}^3]$.

In the present paper we suggest a method of defining analogs of the main wave operators of Relativistic Quantum Mechanics (Laplace, Maxwell, and Dirac ones) on all mentioned algebras without using canonical coordinates.

Note that q-analogs of the Minkowski space algebra and these operators were considered in a number of papers [OSWZ], [AKR], [MM], [M], [K], [P]. In contrast with all these papers we realize a *two-parameter* quantization (deformation) of the initial commutative algebras (even for the $\hbar$-quantization our method seems to be new). Besides, by developing elements of differential calculus on the q-Minkowski space algebra and its sl-reduction we use a new method of defining partial derivatives and other braided vector fields needed for construction of q-deformed operators. the definition of q-deformed operators.

Recall that the classical Laplace operator is associated to a metric[2] on a regular variety. On the Minkowski space its explicit form is

$$\Delta_{\mathbb{K}[\mathbb{R}^4]} = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2.$$

In this case it is also called d'Alembertian.

As for the classical Maxwell operator, it is defined on the space $\Omega^1(\mathbb{R}^4)$ of one-differential forms where it reads

$$\mathrm{Mw}_{\mathbb{K}[\mathbb{R}^4]}(\omega) = \partial\, d\,\omega = \Delta(\omega) - d\,\partial(\omega), \ \text{ where } \ \omega \in \Omega^1(\mathbb{R}^4), \ \ \partial = *^{-1}\, d\, *,$$

d is the de Rham operator, and $*$ is the Hodge one. Note that the original Maxwell equations are defined on the space $\Omega^2(\mathbb{R}^4)$ of two differential forms but they can be readily reduced to the Maxwell operator in the form above.

[2]Note that we employ the term "Laplace operator" in a large sense by admitting that the metric coming in its definition can be indefinite.

Observe that $\Omega^1(\mathbb{R}^4)$ can be presented as a free $\mathbb{K}[\mathbb{R}^4]$-module $\mathbb{K}[\mathbb{R}^4]^{\oplus 4}$ if we identify the differential form $\alpha\, dt + \beta\, dx + \gamma\, dy + \delta\, dz$ with the column $(\alpha\, \beta,\, \gamma,\, \delta)^{\mathsf{T}}$. Hereafter the symbol $^{\mathsf{T}}$ stands for the transposition. Then the Maxwell operator becomes

$$\mathrm{Mw}_{\mathbb{K}[\mathbb{R}^4]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix} = \begin{pmatrix}\Delta_{\mathbb{K}[\mathbb{R}^4]}(\alpha)\\ \Delta_{\mathbb{K}[\mathbb{R}^4]}(\beta)\\ \Delta_{\mathbb{K}[\mathbb{R}^4]}(\gamma)\\ \Delta_{\mathbb{K}[\mathbb{R}^4]}(\delta)\end{pmatrix} - \begin{pmatrix}\partial_t\\ \partial_x\\ \partial_y\\ \partial_z\end{pmatrix}(\partial_t\, -\partial_x,\, -\partial_y,\, -\partial_z)\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}.$$

It is easy to see that the vectors $(\partial_t\varphi,\, \partial_x\varphi,\, \partial_y\varphi,\, \partial_z\varphi)^{\mathsf{T}}$ belong to the kernel of the Maxwell operator. This is an expression of the gauge freedom of the Maxwell equation $\mathrm{Mw}_{\mathbb{K}[\mathbb{R}^4]}(\omega) = 0$.

The Dirac operator in its classical version reads

$$\mathrm{Dir} = \gamma^0\, \partial_\mathrm{t} + \gamma^1\, \partial_\mathrm{x} + \gamma^2\, \partial_\mathrm{y} + \gamma^3\, \partial_\mathrm{z}$$

where $\gamma^i,\ i = 0, 1, 2, 3$ are the Dirac matrices.

By using a map $\alpha : \mathbb{K}[\mathbb{R}^4] \cong \mathrm{Sym}(u(2)) \to U(u(2)_\hbar)$ we can push any operator $\mathrm{Op} : \mathbb{K}[\mathbb{R}^4] \to \mathbb{K}[\mathbb{R}^4]$ forward to the quantum algebra in question by putting

$$\mathrm{Op}_\alpha(\mathrm{x}) =: \alpha\, \mathrm{Op}\, \alpha^{-1}(\mathrm{x}) \qquad \forall \mathrm{x} \in \mathrm{U}(\mathrm{u}(2)_\hbar).$$

In particular, in this way the partial derivatives can be transferred to the algebra $U(u(2)_\hbar)$. By doing so, we get $u(2)$-covariant $\hbar$-counterparts of the above mentioned operators. However, the images of the partial derivatives defined in this way in the algebra $U(u(2)_\hbar)$ are not subject to the Leibnitz rule any more.

As for the q- and $(\hbar, q)$-counterparts of the algebra $\mathbb{K}[\mathbb{R}^4]$ we can also push any operator $\mathrm{Op} : \mathbb{K}[\mathbb{R}^4] \to \mathbb{K}[\mathbb{R}^4]$ forward to these algebras by using an analog of the above map α. However, under this way of proceeding the images of the above wave operators in the quantum algebras lose their covariance property. So, we introduce q-analogs of the wave operators without using any map α but only by analogy with the classical case. As a result, we get "q-wave operators" in question which are invariant (covariant) w.r.t. the QG $U_q(sl(2))$ action.

Observe that in order to define such operators on q-analogs of the algebras $\mathbb{K}[\mathbb{R}^3]$ and $\mathbb{K}[\mathbb{R}^4]$ we first introduce "partial q-derivatives" on them. We do it without using any version of the Leibnitz rule. Instead, we employ a special "q-symmetrized" form of elements of these algebras. Also, we present the q-derivatives in the so-called pseudospherical form which is close to a formula expressing the usual partial derivatives in terms of the spherical coordinates.

Emphasize that this form of partial q-derivatives allows us to get q-analogs of the Laplace and Maxwell operators on the quantum hyperboloid (cf. [DG2]). Besides, we employ a noncommutative version of the Cayley-Hamilton identity valid for a matrix which comes in definition of the algebra in question. In particular, this identity allows us to find a quantum analog of the radius (squared).

Again, emphasize particularities of our approach to quantization of the algebras and operators in question.

- Our quantum algebras arise from a two-parameter deformation of the initial commutative algebras.
- We do not use any form of the Leibnitz rule in defining q-analogs of partial derivatives and other vector fields.
- We realize q-analogs of the operators in question in the so-called pseudospherical form.
- A noncommutative version of the Cayley-Hamilton identity is essentially involved in our q-calculus.

We hope our approach to be useful for quantizing gauge models different from electrodynamics.

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2 $\hbar$-quantization of wave operators

In this section we consider an $\hbar$-quantization of the algebra $\mathbb{K}[\mathbb{R}^4]$ and the wave operators. Let $\alpha : \mathrm{Sym}(u(2)) \cong \mathbb{K}[\mathbb{R}^4] \to U(u(2)_\hbar)$ be the map mentioned in Introduction which identifies elements presented in completely symmetrized form in both algebras. It is known that the image of the subalgebra $\mathrm{Sym}(u(2))^{u(2)}$ (which is formed by elements invariant w.r.t. the action of the Lie algebra $u(2)$) under the map α is not isomorphic to the subalgebra $U(u(2)_\hbar)^{u(2)}$ (in similar notations) which is the center of the algebra $U(u(2)_\hbar)$. However, for a large class of Lie algebras $\mathfrak{g}$ there exist another $\mathfrak{g}$-invariant map of the algebra $\mathrm{Sym}(\mathfrak{g})$ into $U(\mathfrak{g})$ such that its restriction to the subalgebra $\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}$ gives an isomorphism $\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \cong U(\mathfrak{g})^{\mathfrak{g}}$. This was shown by M.Duflo [D] with the use of the character formula.

We suggest another (purely algebraic) method of constructing such a map for the case $\mathfrak{g} = u(2)$. First of all, observe that

$$\mathrm{Sym}(u(2)) \cong \mathrm{Sym}(su(2)) \otimes \mathbb{K}[t] \quad \text{and} \quad U(u(2)_\hbar) \cong U(su(2)_\hbar) \otimes \mathbb{K}[\hat{t}]$$

where $\hat{t}$ is a central element of the Lie algebra $u(2)$. The multiplication table of the algebra $u(2)_h$ is assumed to be

$$[\hat{x}, \hat{y}] = \hbar\, \hat{z}, \quad [\hat{y}, \hat{z}] = \hbar\, \hat{x}, \quad [\hat{z}, \hat{x}] = \hbar\, \hat{y}, \quad [\hat{x}, \hat{t}] = [\hat{y}, \hat{t}] = [\hat{z}, \hat{t}] = 0.$$

Recall that the hat marked letters stand for the elements of NC algebra $U(su(2)_\hbar)$.

So, in order to construct an isomorphism with the desired properties it suffices to do so for the algebras $\mathrm{Sym}(su(2))$ and $U(su(2)_\hbar)$. Namely, given such a map on the algebra $\mathrm{Sym}(su(2))$ we can extend it to the algebra $\mathrm{Sym}(u(2))$ by assuming it to be identical on the factor $\mathbb{K}[t]$ (we identify the generators t in the both algebras). In what follows we also use the notation $\mathbb{K}[\mathbb{R}^3]$ for the algebra $\mathrm{Sym}(su(2))$.

Each of the algebras $\mathbb{K}[\mathbb{R}^3]^{su(2)}$ and $U(su(2)_\hbar)^{su(2)}$ is generated by the only *Casimir element*, namely by

$$\mathrm{Cas}_{\mathbb{K}[\mathbb{R}^3]} = \mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2 \quad \text{and} \quad \mathrm{Cas}_{\mathrm{U(su(2)_\hbar)}} = \hat{\mathrm{x}}^2 + \hat{\mathrm{y}}^2 + \hat{\mathrm{z}}^2$$

respectively. Here $\{x, y, z\}$ are the (commutative) generators of $\mathbb{K}[\mathbb{R}^3]$ corresponding to the generators $\{\hat{x}, \hat{y}, \hat{z}\}$.

Now we introduce an isomorphism $\Upsilon : \mathbb{K}[\mathbb{R}^3]^{su(2)} \to U(su(2)_\hbar)^{su(2)}$ setting by definition

$$\Upsilon(x^2 + y^2 + z^2) = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 - \frac{\hbar^2}{4}\, 1, \tag{2.1}$$

where the symbol 1 at the last summand denotes the $U(su(2)_\hbar)$ unit element. As a consequence of the above definition, $\Upsilon(p(x^2 + y^2 + z^2)) = p(\hat{x}^2 + \hat{y}^2 + \hat{z}^2 - \frac{\hbar^2}{4} 1)$ for any polynomial p in one variable.

The necessity of the shift by $-\hbar^2/4$ in the right hand side of (2.1) can be justified by the following reason. Let us consider an isomorphism of *vector spaces* $\mathbb{K}[\mathbb{R}^3]$ and $su(2)$ given explicitly by the correspondence

$$(x, y, z) \;\mapsto\; L = \begin{pmatrix} -\mathbf{i} z & -\mathbf{i}\, x - y \\ -\mathbf{i}\, x + y & \mathbf{i}\, z \end{pmatrix}.$$

This matrix satisfies the CH identity

$$L^2 + (x^2 + y^2 + z^2)\mathrm{Id} = 0$$

which defines the characteristic equation on the spectrum of the matrix L:

$$\mu^2 + (x^2 + y^2 + z^2) = 0. \tag{2.2}$$

The roots μ_1 and μ_2 of this equation treated as elements of an algebraic extension of the algebra $\mathrm{Sym}(su(2))$ are the eigenvalues of the above matrix L.

Quite similar considerations can also be done for the case of NC algebra $U(su(2)_\hbar)$. Namely, consider a matrix $\hat{L}$ with noncommutative entries

$$\hat{L} = \begin{pmatrix} -\mathbf{i}\hat{z} & -\mathbf{i}\,\hat{x} - \hat{y} \\ -\mathbf{i}\,\hat{x} + \hat{y} & \mathbf{i}\,\hat{z} \end{pmatrix}.$$

As can be easily verified by a straightforward calculation, the matrix $\hat{L}$ satisfies the CH identity of the form

$$\hat{L}^2 - \hbar\,\hat{L} + (\hat{x}^2 + \hat{y}^2 + \hat{z}^2)\mathrm{Id} = 0.$$

This leads to the characteristic equation on the "spectrum" of the matrix $\hat{L}$

$$\hat{\mu}^2 - \hbar\,\hat{\mu} + (\hat{x}^2 + \hat{y}^2 + \hat{z}^2) = 0. \tag{2.3}$$

The roots $\hat{\mu}_1$ and $\hat{\mu}_2$ of this equation are the elements of an algebraic extension of the centrum $Z(U(su(2)_\hbar))$.

Now the map Υ can be prolonged onto the abovementioned algebraical extensions by the following rule

$$\Upsilon(\mu_i) = \hat{\mu}_i - \frac{\hbar}{2}1,$$

where the last summand in this formula is motivated by the relations $\mu_1 + \mu_2 = 0$ and $\hat{\mu}_1 + \hat{\mu}_2 = \hbar 1$. Note, that this map is ambiguously defined at the level of spectral values μ_i since their numbering is arbitrary. But this ambiguity does not affect the action of Υ on the *symmetric* polynomials in the spectral values. Namely, to any symmetric polynomial $p(\mu_1, \mu_2) \in \mathrm{Sym}(su(2))$ the map Υ associates the only element

$$p(\hat{\mu}_1 - \frac{\hbar}{2}1, \hat{\mu}_2 - \frac{\hbar}{2}1) \in Z(U(su(2)_\hbar)).$$

In particular, this leads to formula (2.1).

In the sequel we also need a noncompact counterpart of the algebra $su(2)$, namely that $sl(2)$, and the corresponding CH identity. In the complexification $su(2)_\mathbb{C} \cong sl(2)_\mathbb{C}$ of the Lie algebra $su(2)$ we consider the basis $(\hat{b},\, \hat{h},\, \hat{c})$ where

$$\hat{x} = \frac{\mathbf{i}(\hat{b} + \hat{c})}{2}, \quad \hat{y} = \frac{\hat{c} - \hat{b}}{2}, \quad \hat{z} = \frac{\mathbf{i}\hat{h}}{2}.$$

Then the commutation relations in terms of the new generators read

$$[\hat{h}, \hat{b}] = 2\hbar\,\hat{b}, \quad [\hat{h}, \hat{c}] = -2\hbar\,\hat{c}, \quad [\hat{b}, \hat{c}] = \hbar\,\hat{h}.$$

Note that in the enveloping algebra of $su(2)_\mathbb{C} \cong sl(2)_\mathbb{C}$ we have the following connection of two forms of the Casimir element

$$\mathrm{Cas}_{U(sl_\hbar(2))} = \frac{\hat{h}^2}{4} + \frac{\hat{b}\,\hat{c} + \hat{c}\,\hat{b}}{2} = -(\hat{x}^2 + \hat{y}^2 + \hat{z}^2) = -\mathrm{Cas}_{U(su_\hbar(2))}.$$

In the new basis the matrix $\hat{L}$ takes the form

$$\hat{L} = \begin{pmatrix} \frac{\hat{h}}{2} & \hat{b} \\ \hat{c} & -\frac{\hat{h}}{2} \end{pmatrix}$$

while the CH identity reads

$$\hat{L}^2 - \hbar\,\hat{L} - \left(\frac{\hat{h}^2}{4} + \frac{\hat{b}\,\hat{c} + \hat{c}\,\hat{b}}{2}\right)\mathrm{Id} = 0.$$

Now, we treat $\hat{\mu}_i$ and μ_i to be roots of the equations

$$\hat{\mu}^2 - \hbar\,\hat{\mu} - \left(\frac{\hat{h}^2}{4} + \frac{\hat{b}\,\hat{c} + \hat{c}\,\hat{b}}{2}\right) = 0 \qquad \text{and} \quad \mu^2 - \left(\frac{h^2}{4} + \frac{b\,c + c\,b}{2}\right) = 0$$

respectively. Finally, the map Υ can be expressed in terms of the algebras $\mathbb{K}[\mathbb{R}^3]^{sl(2)}$ and $U(sl(2)_\hbar)^{sl(2)}$.

On the next step, we extend the map Υ to the whole algebra $\mathbb{K}[\mathbb{R}^3]$ by assuming Υ to be linear $sl(2)$-covariant and setting by definition

$$\Upsilon(b^k\, p\left(\frac{h^2}{4}+\frac{b\,c+c\,b}{2}\right)) = \hat{b}^k\, p\left(\frac{\hat{h}^2}{4}+\frac{\hat{b}\,\hat{c}+\hat{c}\,\hat{b}}{2}+\frac{\hbar^2}{4}1\right),$$

where p is a polynomial in one variable and k is any positive integer number. By this condition the map Υ is uniquely defined on the whole algebra $\mathbb{K}[\mathbb{R}^3]$. In order to show this fact, we observe that b^k is the highest weight element of the irreducible $sl(2)$-submodule $V_k \subset \mathbb{K}[\mathbb{R}^3]$ with the integer spin k . Explicitly, this submodule is spanned by the elements b^k, $C(b^k)$, $C^2(b^k), ..., C^{2k}(b^k)$ where B, H, C are standard generators of the algebra $sl(2)$ acting on the algebra $\mathbb{K}[\mathbb{R}^3]$ by the vector fields arising from the adjoint action. By passing to the generators $x,\ y,\ z$ (resp., $\hat{x},\ \hat{y},\ \hat{z}$) in the complexification of the both algebras we get an isomorphism $\Upsilon : \mathbb{K}[\mathbb{R}^3] \to U(su(2)_\hbar)$.

Now, we observe that the partial derivatives in commuting variables x, y, z can be presented as follows

$$\partial_x = \frac{y\,Z - z\,Y}{r^2} + \frac{x}{r}\partial_r = \frac{y\,Z - z\,Y}{\rho} + 2\,x\,\partial_\rho\,,\ c.p. \tag{2.4}$$

where c.p. stands for the cyclic permutation $x \to y \to z \to x$, $\rho = r^2 = (x^2+y^2+z^2)$ and

$$X = z\,\partial_y - y\,\partial_z\,,\ c.p.$$

This looks like a passage to the spherical coordinates (r is the radial variable in the space $\mathbb{R}^3$) but instead of the angle derivatives we use the vector fields $X,\ Y,\ Z$ defined above. Since at each point $(x, y, z) \in \mathbb{R}^3$ these fields are tangent to the sphere of the radius $r = \sqrt{x^2+y^2+z^2}$, they are not independent but subject to the relation

$$x\,X + y\,Y + z\,Z = 0,$$

and, besides, they commute with the derivative ∂_r in the radial variable r. Also, note that

$$\partial_r\,x = \frac{x}{r},\quad \partial_r\,y = \frac{y}{r},\quad \partial_r\,z = \frac{z}{r}, \tag{2.5}$$

or on passing to the derivative in the variable ρ

$$\partial_\rho\,x = \frac{x}{2\rho},\quad \partial_\rho\,y = \frac{y}{2\rho},\quad \partial_\rho\,z = \frac{z}{2\rho}. \tag{2.6}$$

With the use of the vector fields $X,\ Y,\ Z$ and the derivative ∂_ρ we present the Laplace operator on the space $\mathbb{K}[\mathbb{R}^3]$ as follows

$$\Delta_{\mathbb{K}[\mathbb{R}^3]} = \frac{X^2+Y^2+Z^2}{r^2} + \frac{1}{r^2}\partial_r(r^2\,\partial_r) = \frac{X^2+Y^2+Z^2}{\rho} + 6\partial_\rho + 4\rho\partial_\rho^2. \tag{2.7}$$

Since ρ appears in the denominator of this formula we have to enlarge our algebra by adding the negative powers $\rho^m,\ m=-1,-2,...$ to it. More precisely, we should pass to the algebra

$$\mathcal{A} = \mathrm{Sym}(su(2)) \otimes \mathbb{K}[\rho^{-1}]/\langle \rho\,\rho^{-1} - 1\rangle$$

where the notation $\langle \mathfrak{A} \rangle$ stands for the two-sided ideal generated by a given subset $\mathfrak{A}$ of an algebra.

As a vector space, the algebra $\mathcal{A}$ is isomorphic to the direct sum of the $su(2)$-modules

$$\mathcal{A} \cong \bigoplus_{k\geq 0} V_k \otimes \mathbb{K}[\rho, \rho^{-1}]. \tag{2.8}$$

The basis in the algebra $\mathcal{A}$ coming from the above isomorphism is formed by elements

$$\rho^m\, C^l(b^k), \quad k \in \mathbb{Z}_+,\ m \in \mathbb{Z},\ l = 0, 1, ..., 2k$$

and will be called *canonical* in what follows. If an element of the algebra $\mathcal{A}$ is expanded in this basis we say that it is represented in *the canonical form.*

On the other hand, we correspondingly enlarge the algebra $U(su(2)_\hbar)$, adding an inverse element to the central element $\hat\rho = \mathrm{Cas}_{U(su_\hbar(2))} - \frac{\hbar^2}{4}1$ and passing to the quotient

$$\mathcal{A}_\hbar = U(su(2)_\hbar) \otimes \mathbb{K}[\hat\rho^{-1}]/\langle \hat\rho\hat\rho^{-1} - 1\rangle.$$

Note, that as an $su(2)$-module the algebra $\mathcal{A}_\hbar$ is isomorphic to $\mathcal{A}$ and can be expanded into the direct sum of $su(2)$-modules analogous to (2.8).

Then, we extend the map Υ onto the whole algebra $\mathcal{A}$ in the natural way by setting

$$\Upsilon(g\rho^m) = \hat g \hat\rho^m, \qquad m \in \mathbb{Z}, \quad \forall g \in V_k,$$

where $\hat g \in V_k \subset U(su(2)_\hbar)$ is the element corresponding to g as explained above. So, the extended map Υ establishes a bijection between the algebras $\mathcal{A}$ and $\mathcal{A}_\hbar$.

Let us denote the images of the partial derivatives ∂_x, ∂_y, ∂_z in the algebra $U(su(2)_\hbar)$ under the map Υ by $\partial_{\hat x}$, $\partial_{\hat y}$, and $\partial_{\hat z}$ respectively. It would be interesting to get their explicit form in intrinsic terms of the algebra $\mathcal{A}_\hbar$. However, the image of the Laplace operator (2.7) can be presented explicitly. We use the notation $\Delta_{U(su(2)_\hbar)}$ for this image.

Proposition 1 *The following relation holds*

$$\Delta_{U(su(2)_\hbar)} f = \left(\frac{X^2 + Y^2 + Z^2}{\hat\rho} + 6\partial_{\hat\rho} + 4\hat\rho\, \partial^2_{\hat\rho}\right) f \tag{2.9}$$

where we assume that $f \in \mathcal{A}_\hbar$ is written in the canonical form and the $su(2)$-generators X, Y, Z are represented in the algebra $U(su(2)_\hbar)$ as the adjoint action operators.

As for the action of the derivative $\partial_{\hat\rho}$ on the elements of $\mathcal{A}_\hbar$ presented in the canonical form, we define it to be

$$\partial_{\hat\rho}(\hat g \hat\rho^m) = \frac{k\hat g}{2\hat\rho}\hat\rho^m + m\hat g\hat\rho^{m-1}$$

for any $\hat g \in V_k \subset U(su(2)_h)$. The first summand of the above expression stems from the classical formulae (2.6).

The formula for $\Delta_{U(su(2)_\hbar)}$ is nothing but the Υ-image of the Laplace operator (2.7). However, we would emphasize once more that (2.9) is *only* applicable to the elements $f \in \mathcal{A}_\hbar$ presented in the *canonical form.* Also, note that the operator $X^2 + Y^2 + Z^2$ is scalar and its eigenvalue on the component V_k is equal to $-\frac{(k+1)(k+3)}{4}$.

Let us pass to the four-dimensional space algebra. First, we add the negative powers of ρ in the algebras we are dealing with, i.e. we consider the quotient algebras

$$\tilde{\mathcal{A}} = \mathbb{K}[\mathbb{R}^4] \otimes \mathbb{K}[\rho^{-1}]/\langle \rho\rho^{-1} - 1\rangle \quad \text{and} \quad \tilde{\mathcal{A}}_\hbar = U(u(2)_\hbar) \otimes \mathbb{K}[\rho^{-1}]/\langle \hat\rho\hat\rho^{-1} - 1\rangle.$$

Then we can define a map $\tilde\Upsilon : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}_\hbar$ which is the extension of Υ considered above. Since the map $\tilde\Upsilon$ identifies the subalgebras $\mathbb{K}[t]$ in the both algebras $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}_\hbar$ the derivative $\partial_{\hat t}$ in $\hat t$ in the algebra $\tilde{\mathcal{A}}_\hbar$ acts in the same way as the classical derivative ∂_t.

Now we are able to explicitly present the $\hbar$-analogs of the Laplace, Maxwell and Dirac operators on the algebra $\tilde{\mathcal{A}}_\hbar$ (we call them $\hbar$-Laplace, $\hbar$-Maxwell, and $\hbar$-Dirac operators respectively). Being the $\tilde\Upsilon$-image of the Laplace operator in the algebra $\tilde{\mathcal{A}}$, the $\hbar$-Laplace operator reads as follows

$$\Delta_{U(u(2)_\hbar)} = \partial^2_{\hat t} - \partial^2_{\hat x} - \partial^2_{\hat y} - \partial^2_{\hat z}.$$

Also, the $\hbar$-Maxwell and $\hbar$-Dirac operators are defined by the same formulae as in the classical case (see Introduction) but the derivatives ∂_t, ∂_x, ∂_y and ∂_z should be respectively replaced by their images $\partial_{\hat{t}}$, $\partial_{\hat{x}}$, $\partial_{\hat{y}}$, and $\partial_{\hat{z}}$ which are to be applied to vectors composed of the elements of the algebra $\mathcal{A}_\hbar$.

Mostly, the properties of these "$\hbar$-operators" are classical. Thus, similarly to the classical Maxwell operator, the kernel of its $\hbar$-counterpart consists of the elements of the form

$$\mathrm{Ker}\,\mathrm{Mw}_{U(u(2)_\hbar)} = \{(\partial_{\hat{t}}\varphi,\, \partial_{\hat{x}}\varphi,\, \partial_{\hat{y}}\varphi,\, \partial_{\hat{z}}\varphi)^{\mathsf{T}} \mid \forall\, \varphi \in \mathcal{A}_\hbar\}.$$

This is true since the $\hbar$-Laplacian commutes with the derivatives $\partial_{\hat{t}}$, $\partial_{\hat{x}}$, $\partial_{\hat{y}}$ and $\partial_{\hat{z}}$ in virtue of their definition.

As for the $\hbar$-Dirac operator $\mathrm{Dir}_{\mathrm{U(u(2)_\hbar)}} = \gamma^0\,\partial_{\hat{\mathrm{t}}} + \gamma^1\,\partial_{\hat{\mathrm{x}}} + \gamma^2\,\partial_{\hat{\mathrm{y}}} + \gamma^3\,\partial_{\hat{\mathrm{z}}}$, its square equals $\Delta_{U(u(2)_\hbar)}\,\mathrm{Id}$ since the operators $\partial_{\hat{t}}$, $\partial_{\hat{x}}$, $\partial_{\hat{y}}$, $\partial_{\hat{z}}$ commute with each other.

3 q-Minkowski space algebra and its truncated version

Now we pass to the "braided counterparts" of the algebras and operators considered in the last section. The term "braided" refers to objects related to a braiding. An operator

$$R : V^{\otimes 2} \to V^{\otimes 2}$$

is called *a braiding* if it satisfies the quantum Yang-Baxter equation

$$(R\otimes \mathrm{Id})(\mathrm{Id}\otimes R)(R\otimes \mathrm{Id}) = (\mathrm{Id}\otimes R)(R\otimes \mathrm{Id})(\mathrm{Id}\otimes R).$$

All factors in this relation are operators acting in the space $V^{\otimes 3}$.

In this paper we mainly deal with a braiding coming from the QG $U_q(sl(2))$. In this case the space V is a two dimensional fundamental $U_q(sl(2))$-module. As is well known, upon fixing a basis $\{x_1,\, x_2\}$ in the space V, the aforementioned braiding is represented by the matrix

$$R = R_q = \begin{pmatrix} q & 0 & 0 & 0\\ 0 & q-q^{-1} & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & q \end{pmatrix}. \tag{3.1}$$

in the corresponding basis $\{x_1\otimes x_1, x_1\otimes x_2, x_2\otimes x_1, x_2\otimes x_2\}$ of the space $V^{\otimes 2}$. Besides, the operator R satisfies the second order relation

$$(q\mathrm{Id} - R)(q^{-1}\mathrm{Id} + R) = 0.$$

Such type braidings are called *the Hecke symmetries*. The parameter q is assumed to be generic. At $q = 1$ (this value is not excluded) this symmetry becomes *involutive*: $R^2_{q=1} = \mathrm{Id}$.

With any braiding R we associate an algebra generated by the unity and elements l_i^j, $1 \le i, j \le n = \dim V$ subject to the relations

$$R(L\otimes \mathrm{Id})R(L\otimes \mathrm{Id}) - (L\otimes \mathrm{Id})R(L\otimes \mathrm{Id})R = 0. \tag{3.2}$$

Here $L = \|l_i^j\|$ is the $n\times n$ matrix with entries l_i^j. This algebra is called *the reflection equation algebra* (REA) associated with the braiding R and will be denoted $\mathcal{L}_q(n)$. Below we constrain ourselves to the two dimensional case with the braiding R defined in (3.1). General properties of REA and its representation theory can be found in [GPS].

Note that the algebra $\mathcal{L}_q(2)$ is a graded quadratic one (i.e. it is determined by the homogeneous quadratic relations on the generators). Besides, for a generic q we have $\dim \mathcal{L}_q^{(k)}(2) = \dim \mathrm{Sym}^{(k)}(gl(2))$ where the index (k) stands for the degree k homogeneous components of these algebras. S. Majid, U. Meyer [MM] and P. Kulish [K] suggested to consider $\mathcal{L}_q(2)$ with the braiding (3.1) as a q-analog of the Minkowski space algebra[3].

[3] Often one introduces an involution in the complexification of this algebra and then considers it as a q-analog of $\mathrm{Sym}(u(2))$. However, this algebra cannot be realized as a real one. This is the reason why we prefer to deal with the q-analogs of the algebras $\mathrm{Sym}(gl(2))$ and $\mathrm{Sym}(sl(2))$.

Now, we consider the structure of relations (3.2) in more detail. Introducing the following notations for the entries of the matrix L

$$L = \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.3}$$

we get that in the basis $\{a,\, b,\, c,\, d\}$ the system (3.2) reduces to six independent relations

$$\begin{array}{ll} qab - q^{-1}ba = 0 & q(bc-cb) - (q-q^{-1})a(d-a) = 0 \\ qca - q^{-1}ac = 0 & q(cd-dc) - (q-q^{-1})ca = 0 \\ ad - da = 0 & q(db-bd) - (q-q^{-1})ab = 0. \end{array} \tag{3.4}$$

It would be more convenient for us to work in another basis of $\mathcal{L}_q$. Let us pass to the set of generators $\{l,\, h,\, b,\, c\}$ where $l = q^{-1}a + qd$, and $h = a - d$. As a consequence of (3.4) we get the following relations for the new generators

$$\begin{array}{ll} q^2hb - bh + (q-q^{-1})lb = 0 & bl - lb = 0 \\ q^2ch - hc + (q-q^{-1})lc = 0 & cl - lc = 0 \\ 2_q\, q\,(bc-cb) + \left(q^2-1\right)h^2 + (q-q^{-1})lh = 0 & hl - lh = 0. \end{array} \tag{3.5}$$

Now, consider a quadratic-linear version of the algebra $\mathcal{L}_q$ defined as follows

$$\begin{array}{ll} q^2hb - bh + (q-q^{-1})lb = 2_q\,\hbar b & bl - lb = 0 \\ q^2ch - hc + (q-q^{-1})lc = 2_q\,\hbar c & cl - lc = 0 \\ q2_q\,(bc-cb) + \left(q^2-1\right)h^2 + (q-q^{-1})lh = 2_q\,\hbar h & hl - lh = 0. \end{array} \tag{3.6}$$

Hereafter, $2_q = q + q^{-1}$. We denote the above agebra $\mathcal{L}_{\hbar,q}(2)$ and call it the *modified REA (mREA)*. Emphasize that at $q = 1$ we have $\mathcal{L}_{q=1}(2) \cong \mathrm{Sym}(gl(2))$ and $\mathcal{L}_{q=1,\hbar} \cong U(gl(2)_\hbar)$.

Taking into account the evident relation $\mathcal{L}_q(2) = \mathcal{L}_{q,\hbar=0}(2)$ we could, at the first glance, treat $\mathcal{L}_q(2)$ to be a "q-commutative" algebra and $\mathcal{L}_{\hbar,q}(2)$ to be a "q-noncommutative" one. However, unless $q = \pm 1$, the algebras $\mathcal{L}_q(2)$ and $\mathcal{L}_{\hbar,q}(2)$ are isomorphic to each other: this isomorphism is realized by a change of the basis $l_i^j \to l_i^j - \frac{\hbar}{q-q^{-1}}\,\delta_i^j 1$. Thus, we cannot distinguish the "q-commutative" algebra from the "q-noncommutative" one. Below we consider other *nonisomorphic* candidates for the role of q-counterparts of the algebras $\mathrm{Sym}(gl(2))$ and $U(gl(2)_\hbar)$ respectively.

With this purpose, we note that at $q^2 \neq 1$ the central element l enters the defining relations of the algebras $\mathcal{L}_q(2)$ and $\mathcal{L}_{\hbar,q}(2)$ in a specific way (see the left columns of (3.5) and (3.6)). It is just this property of the algebras in question which leads to the isomorphism $\mathcal{L}_q(2) \cong \mathcal{L}_{\hbar,q}(2)$ at $q^2 \neq 1$.

To overcome this obstacle we first pass to the quotient algebras ("sl-reduction")

$$\mathcal{SL}_q(2) = \mathcal{L}_q(2)/\langle l\rangle \quad \text{and} \quad \mathcal{SL}_{\hbar,q}(2) = \mathcal{L}_{\hbar,q}(2)/\langle l\rangle.$$

Emphasize that, having killed l, we get *nonisomorphic* algebras $\mathcal{SL}_q(2)$ and $\mathcal{SL}_{\hbar,q}(2)$ which can be with more evidence called respectively q-commutative and q-noncommutative ones. Besides, $\mathcal{SL}_q(2)$ and $\mathcal{SL}_{\hbar,q}(2)$ are respectively one-parameter and two-parameter deformations of the algebra $\mathbb{K}[\mathbb{R}^3]$.

Thus, the algebra $\mathcal{SL}_{\hbar,q}(2)$ is generated by the elements b, h, c subject to the system of relations

$$q^2hb - bh = 2_q\hbar\, b, \quad 2_q\, q(bc-cb) + (q^2-1)h^2 = 2_q\hbar\, h, \quad q^2ch - hc = 2_q\hbar\, c. \tag{3.7}$$

The corresponding relations on the $\mathcal{SL}_q(2)$ generators can be obtained from the above ones by setting $\hbar = 0$.

Denote by $\mathcal{L}$ and $\mathcal{SL}$ the following $\mathbb{K}$-linear spans of generators :

$$\mathcal{L} = \mathrm{span}(b, h, c, l) \quad \text{and} \quad \mathcal{SL} = \mathrm{span}(b, h, c).$$

Also, consider a *q-skew-symmetric* subspace $I_-^q \subset \mathcal{SL}^{\otimes 2}$ given by the linear span

$$I_-^q = \mathrm{span}(q^2 h\otimes b - b\otimes h,\ 2_q\, q(b\otimes c - c\otimes b) + (q^2-1)h\otimes h,\ q^2 c\otimes h - h\otimes c). \qquad (3.8)$$

These notations allow us to present the algebra $\mathcal{SL}_q(2)$ as a quotient $T(\mathcal{SL})/\langle I_-^q\rangle$ where $T(\mathcal{SL})$ is the free tensor algebra of the space $\mathcal{SL}$.

Note that the space $\mathcal{L}$ can be endowed with an action of the QG $U_q(sl(2))$ in such a way that the subspace $\mathcal{SL}\subset\mathcal{L}$ becomes a $U_q(sl(2))$-submodule (the quantum group trivially acts on the central element l) as well as the subspace I_-^q becomes a $U_q(sl(2))$-submodule of the $U_q(sl(2))$-module $\mathcal{SL}^{\otimes 2}$. Moreover, the algebras $\mathcal{L}_q(2)$ and $\mathcal{SL}_q(2)$ can be endowed with the corresponding action of $U_q(sl(2))$ and their algebraic structure is compatible with this action, that is $\mathcal{L}_q(2)$ and $\mathcal{SL}_q(2)$ have the structure of $U_q(sl(2))$-algebras. The reader is referred to [DG1] for an explicit description of all these actions.

Now we consider new q-analogs of the algebras $\mathrm{Sym}(gl(2))$ and $U(gl(2)_\hbar)$ constructing them as the tensor products $\widetilde{\mathcal{L}}_q(2) = \mathcal{SL}_q(2)\otimes\mathbb{K}[l]$ and $\widetilde{\mathcal{L}}_{\hbar,q}(2) = \mathcal{SL}_{\hbar,q}(2)\otimes\mathbb{K}[l]$. These new algebras differ slightly from $\mathcal{L}_q(2)$ and $\mathcal{L}_{\hbar,q}(2)$: in order to get the defining relations of their generators it suffices to remove the linear in l terms from the relations in the left columns of (3.5) and (3.6). Thus, we have

$$\widetilde{\mathcal{L}}_q(2) = T(\mathcal{L})/\langle\widetilde{I_-^q}\rangle, \qquad \widetilde{I_-^q} = \mathrm{span}(I_-^q,\ l\otimes b - b\otimes l,\ l\otimes h - h\otimes b,\ l\otimes c - c\otimes l). \qquad (3.9)$$

Finally, we get two nonisomorphic algebras: "q-commutative" algebra $\widetilde{\mathcal{L}}_q(2)$ and "q-noncommutative" one $\widetilde{\mathcal{L}}_{\hbar,q}(2)$ which are respectively one-parameter and two-parameter deformations of the commutative algebra $\mathrm{Sym}(gl(2))$. The algebra $\widetilde{\mathcal{L}}_{\hbar,q}(2)$ will be called the *truncated mREA* while the algebra $\widetilde{\mathcal{L}}_q(2)$ will be referred to as the *truncated REA*. (Similar algebras with close properties can be defined for any "skew-invertible" Hecke symmetry, cf. [GPS].) The algebra $\widetilde{\mathcal{L}}_q(2)$ related to the Hecke symmetry (3.1) will be also called the *truncated q-Minkowski space algebra* and denoted $\mathbb{K}_q[\mathbb{R}^4]$.

Below we describe some of the properties of $\mathbb{K}_q[\mathbb{R}^4]$. However, we begin with its quotient $\mathbb{K}_q[\mathbb{R}^3] = \mathbb{K}_q[\mathbb{R}^4]/\langle l\rangle$. Evidently, $\mathbb{K}_q[\mathbb{R}^3] = \mathcal{SL}_q(2)$. Observe that the space $\mathcal{SL}^{\otimes 2}$ is a direct sum of the q-skew-symmetric space I_-^q and the q-symmetric space I_+^q which can be presented as a direct sum $I_+^q = V_2\oplus V_0$ where

$$V_0 = \mathrm{span}(q^{-1}b\otimes c + \frac{1}{2_q}\, h\otimes h + qc\otimes b),$$
$$V_2 = \mathrm{span}(b\otimes b,\ q^2 b\otimes h + h\otimes b,\ q^3 b\otimes c - qh\otimes h + q^{-1}c\otimes b,\ q^2 h\otimes c + c\otimes h,\ c\otimes c). \qquad (3.10)$$

The subspaces V_0 and V_2 are respectively spin 0 and spin 2 $U_q(sl(2))$-modules. Since $I_+^q \bigcap I_-^q = \{0\}$, then the projection of the second degree component $\mathcal{SL}_q^{(2)}(2)$ of the algebra $\mathcal{SL}_q(2)$ onto the space I_+^q is well defined. Therefore, any element $a\in\mathcal{SL}_q^{(2)}(2)$ has a unique image in the space I_+^q. This image we shall call the q-symmetric form of the element a.

The point is that any homogeneous element of the algebra $\mathcal{SL}_q(2)$ of degree > 2 can be also presented in a q-symmetric form. More precisely, for homogeneous elements of the third, the fourth (and so on) degrees the corresponding q-symmetric form is identified with an element from the intersections of the following subspaces

$$\mathcal{SL}\otimes I_+^q \bigcap I_+^q\otimes\mathcal{SL} \subset \mathcal{SL}^{\otimes 3}$$
$$\mathcal{SL}^{\otimes 2}\otimes I_+^q \bigcap \mathcal{SL}\otimes I_+^q\otimes\mathcal{SL} \bigcap I_+^q\otimes\mathcal{SL}^{\otimes 2} \subset \mathcal{SL}^{\otimes 4} \qquad (3.11)$$
$$\text{(and so on).}$$

On expanding any given element of the algebra $\mathcal{SL}_q(2)$ into a sum of homogeneous ones we can present this element in a q-symmetric form. The homogeneous elements of the degree 0 and 1 are considered to be in the q-symmetric form by definition.

The above procedure of "q-symmetrization" can be explicitly done via projectors related to the Birman-Wenzl-Murakami algebra (cf. [OP] for the definition). Here we want only to remark that this method is valid for all sl-reduced REA associated with Hecke symmetries of the Temperley-Lieb type[4].

A remarkable property of the spaces I_+^q and I_-^q is that they are orthogonal to each other w.r.t. a $U_q(sl(2))$-covariant pairing $\mathcal{SL}^{\otimes 2}\otimes\mathcal{SL}^{\otimes 2}\to\mathbb{K}$. We start the construction from the space $\mathcal{SL}$ and define a pairing $\mathcal{SL}\otimes\mathcal{SL}\to\mathbb{K}$ on the basis elements by the rules

$$\langle b,c\rangle = q^{-1},\quad \langle h,h\rangle = 2_q,\quad \langle c,b\rangle = q \tag{3.12}$$

(all other terms are trivial). This pairing is $U_q(sl(2))$-covariant and is unique up to a common factor. Thus, the space $\mathcal{SL}$ is autodual: its dual space is isomorphic to the space itself and this isomorphism is $U_q(sl(2))$-covariant.

Now we extend this paring to the map $\mathcal{SL}^{\otimes 2}\otimes\mathcal{SL}^{\otimes 2}\to\mathbb{K}$ as follows

$$\langle u\otimes v, w\otimes z\rangle := \langle u,z\rangle\,\langle v,w\rangle \qquad \forall u\otimes v, w\otimes z\in\mathcal{L}^{\otimes 2}. \tag{3.13}$$

Note that in a similar way, i.e. without any transposing, the pairing can be extended onto the spaces $\mathcal{SL}^{\otimes k}\otimes\mathcal{SL}^{\otimes k}$ for any $k\geq 3$ and this extended pairing is $U_q(sl(2))$-covariant.

Proposition 2 *The spaces I_-^q and I_+^q are orthogonal to each other with respect to the pairing (3.12) - (3.13).*

Proof. By direct computations it is easy to check that the generator $q^{-1}b\otimes c+\frac{1}{2_q}h\otimes h+qc\otimes b$ of the space V_0 is orthogonal to the space I_-^q. It is also evident that $\langle b\otimes b,z\rangle=0$ for all $z\in I_-^q$. Observe that $b\otimes b$ is the highest weight element of the $U_q(sl(2))$-module V_2. Applying the descending element $Y\in U_q(sl(2))$ to the relation $\langle b\otimes b,z\rangle=0$ and taking into account the $U_q(sl(2))$-covariance of the pairing we deduce that $\langle u,z\rangle=0$ for any $u\in V_2$ and any $z\in I_-^q$. Therefore, $I_+^q=V_0\oplus V_2$ is orthogonal to I_-^q. ■

A decomposition of the space $\mathcal{L}^{\otimes 2}$ similar to that above can be done as follows. We present this space as a direct sum of two complimentary subspaces

$$\mathcal{L}^{\otimes 2}=\widetilde{I_-^q}\oplus\widetilde{I_+^q}$$

where $\widetilde{I_-^q}$ is defined in (3.9) and $\widetilde{I_+^q}$ is as follows

$$\widetilde{I_+^q}=I_+^q\oplus\operatorname{span}(l\otimes l,\ l\otimes b+b\otimes l,\ l\otimes h+h\otimes l,\ l\otimes c+c\otimes l).$$

Then we extend the pairing (3.12) to the space $\mathcal{L}$ demanding $\langle l,l\rangle\neq 0$ to be an arbitrary nonzero number and l to be orthogonal to other generators. Then the spaces $\widetilde{I_+^q}$ and $\widetilde{I_-^q}$ will be orthogonal to each othe with respect to the extended pairing. Below we explain the role of this property in computing relations between the q-derivatives.

4 Partial q-derivatives

In this section we discuss the problem of introducing q-analogs of the partial derivatives on the algebras in question. Remark that in general, there is no way to introduce a reasonable analog of the partial derivative on a NC algebra. Nevertheless, if such an algebra is related to a braiding, one often defines a q-analog of derivative by using one or another form of transposing "functions" and derivatives and the Leibnitz rule. In this section we introduce q-derivatives in another way.

[4]If the initial Hecke symmetry is of a more general form, an analog of the space I_+^q and the procedure of q-symmetrization in the related REA is suggested in [GPS]. But a rigorous construction of the corresponding projectors is only done for the homogeneous components of the degree ≤ 3. By passing to the sl-reduced algebra, i.e. by killing the central element l, we can conjecturally get the q-symmetrization procedure for elements of the sl-reduced REA.

First, consider the algebra $\mathcal{SL}_q(2)$ and define an action of the q-derivatives ∂_b^q, ∂_h^q and ∂_c^q on the generators b, h and c in the classical manner:

$$\partial_b^q\, b = \partial_h^q\, h = \partial_c^q\, c = 1, \quad \partial_b^q\, h = \partial_b^q\, c = \partial_h^q\, b = \partial_h^q\, c = \partial_c^q\, b = \partial_c^q\, h = 0. \tag{4.1}$$

Besides, we naturally assume that these q-derivatives kill elements of the ground field.

Second, in order to define the action of the derivatives on higher degree elements we present these elements in the q-symmetric form discussed in the previous section. It suffices to consider only homogeneous elements of degree $m \geq 2$. Let $f_m \in \mathcal{SL}_q(2)$ be an arbitrary element of the degree m and f_m^{sym} be the corresponding symmetric form. Then we put by definition

$$\partial_a^q f_m = m\,(\partial_a^q \otimes \mathrm{Id}^{\otimes(m-1)}) f_m^{sym}, \quad a = b, h, c. \tag{4.2}$$

The above foormula means that we apply the partial q-derivative only to the first factor of the element f_m^{sym}. In so doing we need only formulae (4.1). Thus, neither any transposition of elements from $\mathcal{SL}_q(2)$ and the q-derivatives nor the Leibnitz rule are involved in our definition.

Passing to the algebra $\widetilde{\mathcal{L}}_q(2)$ we should also define the derivative ∂_l^q. It can be done in the classical manner. By contrary, in the algebra $\mathcal{L}_q(2)$ defining such a q-derivative is a more complicated problem since l enter the first column of the relations (3.5) (cf. [DG2]).

Our immediate aim is to find relations between the q-derivatives. In order to make these relations more clear we consider another basis in the space spanned by the q-derivatives:

$$D_c := q\partial_b^q, \quad D_h := 2_q\partial_h^q, \quad D_b := q^{-1}\partial_c^q. \tag{4.3}$$

Remark 1 The sense of this basis becomes clear when comparing the definition of the partial q-derivatives (4.1) with pairing (3.12). Thus, we see that on the linear component of $\mathcal{SL}_q(2)$ the partial q-derivative ∂_b^q up to a factor q^{-1} coincides with the action of the pairing with the generator c: $\partial_b^q a = q^{-1}\langle c, a\rangle$. Therefore, the operators D_c, D_h, D_b in (4.3) act on the linear component as a pairing with the corresponding generator: $D_a z = \langle a, z\rangle$, $a = b, h, c$.

Proposition 3 *On the algebra $\mathcal{SL}_q(2)$ the partial q-derivatives defined in (4.3) satisfy the relations*

$$\begin{aligned} q^2 D_h\, D_b - D_b\, D_h &= 0 \\ 2_q\, q(D_b\, D_c - D_c\, D_b) + (q^2-1) D_h\, D_h &= 0 \\ q^2 D_c\, D_h - D_h\, D_c &= 0. \end{aligned} \tag{4.4}$$

Note, that these relations coincide with the defining relations of the algebra $\mathcal{SL}_q(2)$ (see (3.7) with $\hbar = 0$) if we replace its generators by the corresponding q-derivative: $b \to D_b$ and so on.

Proof. In virtue of definition of q-derivative (4.2) we have to prove the proposition for the q-symmetric elements f^{sym} of $\mathcal{SL}_q(2)$. Therefore, we first prove that the left hand sides of relations (4.4) are zero operators on the subspace I_+^q (see (3.10)). But this is a direct consequence of the Proposition 2 since the left hand side of (4.4) are formally coincide with the basis of I_-^q defined in (3.8) while the action of q-derivatives (4.3) coincide with the pairing (3.12) (see Remark 1).

The general case now follows from the specific form of f^{sym} which belongs to the intersections of the form (3.11). ∎

For the case of the algebra $\widetilde{\mathcal{L}}_q(2)$ we should add q-derivative ∂_l^q in the classical manner and this q-derivative will commute with the other ones. However, finding relations between partial q-derivatives on the algebra $\mathcal{L}_q(2)$ is a more complicated problem.

Remark 2 The above q-derivatives are related in the usual way to the de Rham operator associated with the algebra $\mathcal{L}_q = Sym_q(\mathcal{L})$. Here $\mathcal{L} = \mathrm{span}(l_i^j)$, $Sym_q(\mathcal{L}) = \mathcal{L}_q(n)$, $n \geq 2$.

Let us introduce this operator by using the method of the paper [G]. In this paper there was considered the complex

$$\bigwedge^k(V)\otimes \mathrm{Sym}^m(V)\to \bigwedge^{k+1}(V)\otimes \mathrm{Sym}^{m-1}(V),$$

where V is a space endowed with a Hecke symmetry (say, that coming from $U_q(sl(n))$) and all elements of the spaces $\mathrm{Sym}^m(V)$ (resp., $\bigwedge^k(V)$) are presented in the q-symmetric (resp. q-skew-symmetric) form. The differential d acting in this complex is given by

$$d\,((y_{i_1}\otimes ...\otimes y_{i_k})\otimes (x_{j_1}\otimes ...\otimes x_{j_m})) = mP_+(y_{i_1}\otimes ...\otimes y_{i_k}\otimes x_{j_1})\otimes (x_{j_2}\otimes ...\otimes x_{j_m}),$$

where P_+ is the operator of q-symmetrization. (In contrast with [G] we have introduced in this operator the factor m.)

Upon fixing $k=0$ and applying the differential d to an element $f=\sum_j x_j\otimes g_j\in \mathrm{Sym}^m(V)$ we get

$$d\,f = d\,(\sum_j x_j\otimes g_j) = m\,\sum_j d\,x_j\otimes g_j,$$

where $d\,x_j$ is merely another form of the fact that the terms x_j belong to the factor $\bigwedge^1(V)\cong V$. It is natural to define the q-derivatives in the algebra $\mathrm{Sym}_q(V)$ by $\partial^q_{x_j} f = g_j$.

This method can be applied for the definition of the de Rham complex associated with the algebras $\mathcal{L}_q(n)$ and $\mathcal{SL}_q(n)$, once the elements of these algebras can be presented in the q-symmetric form. It is easy to see that the partial q-derivatives on the algebra $\mathcal{SL}_q(2)$ arising from this approach coincide with those defined above (see (4.1)).

Now, we want to present the q-derivatives in a form similar to (2.4) or rather that related to the Lie algebra $sl(2)$ (cf. [DG2]).

However, first, we enlarge the algebra $\mathcal{SL}_q(2)$ by adding the central generator ρ_q^{-1} where

$$\rho_q = \mathrm{Cas}_{\mathbb{K}_q[\mathbb{R}^3]} = \frac{1}{2_q}\left(q^{-1}\,b\,c + \frac{h^2}{2_q} + q\,c\,b\right)$$

is a q-analog of the $sl(2)$-Casimir element. So we introduce algebras

$$\mathcal{A}_q = \mathcal{SL}_q\otimes \mathbb{K}[\rho_q^{-1}]/\langle \rho_q\rho_q^{-1}-1\rangle \quad \text{and} \quad \tilde{\mathcal{A}}_q = \widetilde{\mathcal{L}}_q\otimes \mathbb{K}[\rho_q^{-1}]/\langle \rho_q\rho_q^{-1}-1\rangle.$$

Our immediate aim is to define the q-derivative $\partial^q_{\rho_q}$ in the algebra $\mathcal{A}_q$. By imitating formula (2.6) (more precisely, its $sl(2)$-analog) we set

$$\partial^q_{\rho_q} b = \frac{b}{2\,\rho_q},\quad \partial^q_{\rho_q} h = \frac{h}{2\,\rho_q},\quad \partial^q_{\rho_q} c = \frac{c}{2\,\rho_q}.$$

Also, we assume that this derivative is subject to the usual Leibnitz rule. It is easy to see that this way of introducing the derivative $\partial^q_{\rho_q}$ is compatible with the defining relations of the algebra $\mathcal{L}_q(2)$.

In order to get a q-analog of formulae (2.4) we need q-analogs of the infinitesimal rotations X, Y and Z or, more precisely, q-analogs of the infinitesimal hyperbolical rotations arising from the adjoint action of the algebra $sl(2)$. They can be found from a q-analog of the $sl(2)$ Lie bracket defined as follows. There exists a unique (up to a nontrivial factor) $U_q(sl(2))$-morphism

$$[\,,\,]:\mathcal{SL}\otimes\mathcal{SL}\to\mathcal{SL}.$$

In the explicit form it reads

$$\begin{gathered}[b,b]=0,\quad [b,h]=-w\,b,\quad [b,c]=w\,\tfrac{q}{2_q}\,h,\quad [h,b]=w\,q^2\,b,\\ [h,h]=w\,(q^2-1)h,\quad [h,c]=-w\,c,\quad [c,b]=-w\,\tfrac{q}{2_q}\,h,\quad [c,h]=w\,q^2\,c,\quad [c,c]=0\end{gathered}\tag{4.5}$$

where $w\in\mathbb{K}$ is an arbitrary non-trivial factor.

By definition, the adjoint action corresponding to this bracket reads: $\mathrm{ad}\,(x) \triangleright y = [x, y]$ for any $x, y \in \mathcal{SL}$. Thus, we have three operators $B_q = \mathrm{ad}\,(b)$, $H_q = \mathrm{ad}\,(h)$, $C_q = \mathrm{ad}\,(c)$. In the basis $\{b, h, c\}$ they are represented by the following matrices

$$B_q = w \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & \frac{q}{2_q} \\ 0 & 0 & 0 \end{pmatrix} \quad H_q = w \begin{pmatrix} q^2 & 0 & 0 \\ 0 & q^2-1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad C_q = w \begin{pmatrix} 0 & 0 & 0 \\ -\frac{q}{2_q} & 0 & 0 \\ 0 & q^2 & 0 \end{pmatrix} \tag{4.6}$$

Proposition 4 *The operators B_q, H_q and C_q satisfy the relation*

$$q^{-1} b\, C_q + \frac{h\, H_q}{2_q} + qc\, B_q = 0. \tag{4.7}$$

Note that at $q = 1$ these operators turn into the classical ones if we put $w = 2$. Also, note that we can choose the factor w in such a way that the operators (4.6) would realize a representation of the algebra $\mathcal{SL}_{\hbar,q}(2)$ defined in formulae (3.7).

One can extend the action of operators B_q, H_q and C_q on the higher components of the algebra $\mathcal{L}_q(2)$ such that the relation (4.7) will be valid. This extension can be constructed via the same method as was used for the q-derivatives. Presenting a homogeneous element of the degree m $f_m \in \mathcal{L}_{\hbar,q}(2)$ in the q-symmetric form f_m^{sym} we apply the operators (4.6) to the first factor of the f_m^{sym} with subsequent multiplication by m (see formula (4.2)). Observe, that at any homogeneous component of the degree m of the algebra $\mathcal{L}_q(2)$ the extended operators B_q, H_q and C_q (we shall keep the same notation for them) will satisfy the commutation relations looking like (3.7) but the coefficients in the r.h.s. of these relations will depend on m.

The operators B_q, H_q and C_q as well as all their combinations $\alpha B_q + \beta H_q + \gamma C_q$, α, β, $\gamma \in \mathcal{SL}_q(2)$ are called *the tangent braided vector fields*. Thus, all such fields form a left $\mathcal{SL}_q(2)$-module M which is the quotient of the free $\mathcal{L}_q(2)$-module $\mathcal{L}_q(2)^{\oplus 3}$ over the submodule

$$\bar{M} = \{\varphi(q^{-1} b\, C_q + \frac{h\, H_q}{2_q} + qc\, B_q), \quad \forall\, \varphi \in \mathcal{L}_q(2)\}.$$

Consider the following tangent braided vector fields

$$\begin{aligned} \mathcal{B}_q &= w^{-1}(q^2 h B_q - b H_q), \\ \mathcal{H}_q &= w^{-1}(q 2_q (b C_q - c B_q) + (q^2-1) h H_q), \\ \mathcal{C}_q &= w^{-1}(q^2 c H_q - h C_q). \end{aligned}$$

Here w is the factor entering the definition of bracket (4.5). The following proposition is proved in [DG2].

Proposition 5 *The following operator equalities are valid on the space $\mathcal{SL}$*

$$D_b = \frac{q^{-2}}{2_q \rho_q}\, \mathcal{B}_q + \frac{2b}{2_q}\, \partial_{\rho_q}, \quad D_h = \frac{q^{-2}}{2_q \rho_q}\, \mathcal{H}_q + \frac{2h}{2_q}\, \partial_{\rho_q}, \quad D_c = \frac{q^{-2}}{2_q \rho_q}\, \mathcal{C}_q + \frac{2c}{2_q}\, \partial_{\rho_q}. \tag{4.8}$$

Consequently, the same relation is valid on the whole algebra $\mathcal{SL}_q(2)$ since in our definition of extended q-derivatives and tangent braided vector fields we use the action of the corresponding operator only on the first factor of a q-symmetric element. As for the derivative $\partial^q_{\rho_q}$, a similar way for its action is valid as well. Thus, we get that formulae (4.8) are valid on the whole algebra $\mathcal{SL}_q(2)$.

The form (4.8) of the q-derivatives is called *pseudospherical*.

5 q- and $(\hbar, q)$-quantization of the wave operators

We first consider a q-quantization of the wave operators. In order to define a q-analog of the Laplace operator on the space $\mathbb{K}_q[\mathbb{R}^3]$ we replace each generator in the Casimir element $\mathrm{Cas}_{\mathbb{K}_q[\mathbb{R}^3]}$ by the corresponding partial derivative $b \to D_b$ and so on. Thus, we get the operator

$$\Delta_{\mathbb{K}_q[\mathbb{R}^3]} = \frac{1}{2_q}\left(q^{-1}\, D_b\, D_c + \frac{(D_h)^2}{2_q} + q\, D_c\, D_b\right). \tag{5.1}$$

As for the Laplace operator on the algebra $\mathbb{K}_q[\mathbb{R}^4]$ we have a one parametric freedom in its definition. Namely, we put

$$\Delta_{\mathbb{K}_q[\mathbb{R}^4]} = \epsilon D_l^2 + \frac{1}{2_q}\left(q^{-1}\, D_b\, D_c + \frac{(D_h)^2}{2_q} + q\, D_c\, D_b\right),$$

where $\epsilon = \pm 1$. Note that at the classical point $q = 1$ the signature of the corresponding quadratic form is $(2,2)$ or $(3,1)$ in dependence of ϵ. Here the derivative D_l acts nontrivially only on the central element l: $D_l l = \mathrm{const} \neq 0$ (this reflects the fact that we have a freedom in the normalization of the pairing $\langle l, l\rangle = \mathrm{const} \neq 0$). We leave to the reader checking that these q-Laplace operators are $U_q(sl(2))$-invariant ones.

Now, we pass to the definition of the q-Maxwell operator on the algebras $\mathbb{K}_q[\mathbb{R}^3]$ and $\mathbb{K}_q[\mathbb{R}^4]$. Respectively, they are given by the following formulae

$$\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^3]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\Delta_{\mathbb{K}_q[\mathbb{R}^3]}(\alpha)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^3]}(\beta)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^3]}(\gamma)\end{pmatrix} - \frac{1}{2_q}\begin{pmatrix}D_c\\ D_h\\ D_b\end{pmatrix}(q^{-1}D_b,\, \frac{D_h}{2_q},\, qD_c)\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

$$\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^4]}\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix} = \begin{pmatrix}\Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\alpha)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\beta)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\gamma)\\ \Delta_{\mathbb{K}_q[\mathbb{R}^4]}(\delta)\end{pmatrix} - \frac{1}{2_q}\begin{pmatrix}D_c\\ D_h\\ D_b\\ D_l\end{pmatrix}(q^{-1}D_b,\, \frac{D_h}{2_q},\, qD_c,\, 2_q\varepsilon D_l)\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}.$$

Here $\alpha,\, \beta,\, \gamma \in \mathbb{K}_q[\mathbb{R}^3]$ (resp., $\alpha,\, \beta,\, \gamma, \delta \in \mathbb{K}_q[\mathbb{R}^4]$).

It is easy to see that a column $(D_c\varphi,\, D_h\varphi,\, D_b\varphi)^{\mathsf{T}}$ (resp., $(D_c\varphi,\, D_h\varphi,\, D_b\varphi,\, D_l\varphi)^{\mathsf{T}}$) belongs to the kernel of the operator $\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^3]}$ (resp., $\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^4]}$).

Remark 3 Note, that making a linear transformation of the basis of the one-form space we pass to an equivalent Maxwell operator $A\mathrm{Mw}_{\mathbb{K}_q[\mathbb{R}^3]}A^{-1}$, where A is an invertible 3×3 numerical matrix. With an appropriate choice of this matrix we can get the kernel of the transformed q-Maxwell operator consisting of the one-forms $(\partial^q_b\varphi,\, \partial^q_h\varphi,\, \partial^q_c\varphi)^{\mathsf{T}}$. Therefore, we can treat the operator $\varphi \to (\partial^q_b\varphi,\, \partial^q_h\varphi,\, \partial^q_c\varphi)^{\mathsf{T}}$ as a q-analog of the gradient, whereas the corresponding operator ∂ becomes

$$(\alpha,\, \beta,\, \gamma)^{\mathsf{T}} \to \frac{1}{2_q}\,(q^{-1}\partial^q_c\,\alpha + 2_q\partial^q_h\beta + q\partial^q_b\gamma).$$

In order to describe a q-analog of the Dirac operator we need a q-version of the Cayley-Hamilton identity for the matrix L (3.3) composed of the REA generators. More precisely we are interested in the sl-reduced version of this matrix. We get this sl-reduced matrix from (3.3) by passing to the basis $\{b, h, c, l\}$ and then killing l. Thus, we have

$$L = \begin{pmatrix}\frac{q\,h}{2_q} & b\\ c & \frac{-h}{2_q q}\end{pmatrix} = b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + h\begin{pmatrix}\frac{q}{2_q} & 0\\ 0 & \frac{-1}{2_q q}\end{pmatrix} + c\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{5.2}$$

The following CH identity can be checked by straightforward computations

$$L^2 - \frac{1}{2_q}\left(q^{-1}\, b\, c + \frac{h^2}{2_q} + q\, c\, b\right)\mathrm{Id} = L^2 - \rho_q \mathrm{Id} = 0. \tag{5.3}$$

Here the elements b, h and c are the $\mathcal{SL}_q(2)$ generators.

Consider the 4×4 matrix with entries belonging to the algebra $\widetilde{\mathcal{L}}_q(2)$

$$\varepsilon l \begin{pmatrix} 0_2 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0_2 \end{pmatrix} + b \begin{pmatrix} \sigma_b & 0_2 \\ 0_2 & -\sigma_b \end{pmatrix} + h \begin{pmatrix} \sigma_h & 0_2 \\ 0_2 & -\sigma_h \end{pmatrix} + c \begin{pmatrix} \sigma_c & 0_2 \\ 0_2 & -\sigma_c \end{pmatrix} \tag{5.4}$$

where 0_2 and I_2 are respectively the trivial and unity 2×2 matrices, and the matrices σ_b, σ_h, σ_c are the multipliers of b, h, c in formula (5.2).

Assuming l, b, h, c to be the $\widetilde{\mathcal{L}}_q(2)$ generators we get that the square of matrix (5.4) equals $(l^2+\rho_q)\mathrm{I}_4$. Now, replacing the generators l, b, h, c by the corresponding partial derivatives we get the q-Dirac operator:

$$\varepsilon D_l \begin{pmatrix} 0_2 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0_2 \end{pmatrix} + D_b \begin{pmatrix} \sigma_b & 0_2 \\ 0_2 & -\sigma_b \end{pmatrix} + D_h \begin{pmatrix} \sigma_h & 0_2 \\ 0_2 & -\sigma_h \end{pmatrix} + D_c \begin{pmatrix} \sigma_c & 0_2 \\ 0_2 & -\sigma_c \end{pmatrix}. \tag{5.5}$$

Note that at $q=1$ we get an operator which differs from the classical Dirac operator because we deal with a deformation of the Lie algebra $sl(2)$ instead of $su(2)$ (for the reason indicated above in the footnote 3).

Of course in order to apply the q-derivatives given in the pseudospherical form (4.8) we have to replace the algebra $\widetilde{\mathcal{L}}_q(2)$ by the quotient $\tilde{\mathcal{A}}_q = \widetilde{\mathcal{L}}_q(2)\otimes\mathbb{K}[\rho_q^{-1}]/\langle\rho_q\rho_q^{-1}-1\rangle$. In the sequel we also need its $(\hbar,q)$-counterpart $\tilde{\mathcal{A}}_{\hbar,q} = \widetilde{\mathcal{L}}_{\hbar,q}(2)\otimes\mathbb{K}[\rho_q^{-1}]/\langle\rho_q\rho_q^{-1}-1\rangle$.

Now we pass to $(\hbar,q)$-quantization of the wave operators. Here we use the same scheme as for the $\hbar$-quantization. First, we define a $U_q(sl(2))$-covariant map $\Upsilon : \tilde{\mathcal{A}}_q \to \tilde{\mathcal{A}}_{\hbar,q}$ similar to that considered in section 2. For this we need the CH identity for the matrix $\hat{L}$, which has the same form as L in (5.2) but composed of the $\mathcal{SL}_{\hbar,q}(2)$ generators (3.7). In order to differ these generators from $\mathcal{SL}_q(2)$ ones we again use the hat notations. So, with the use of (3.7) one can verify the following CH identity

$$\hat{L}^2 - q^{-1}\hbar\hat{L} - \frac{1}{2_q}\left(q^{-1}\,\hat{b}\,\hat{c} + \frac{\hat{h}^2}{2_q} + q\,\hat{c}\,\hat{b}\right)\mathrm{Id} = 0. \tag{5.6}$$

The eigenvalues of the matrices L and $\hat{L}$ are treated to be the roots of the equations induced by identities (5.3) and (5.6) respectively

$$\mu^2 - \frac{1}{2_q}\left(q^{-1}\,b\,c + \frac{h^2}{2_q} + q\,c\,b\right) = 0, \qquad \hat{\mu}^2 - q^{-1}\hbar\hat{\mu} - \frac{1}{2_q}\left(q^{-1}\,\hat{b}\,\hat{c} + \frac{\hat{h}^2}{2_q} + q\,\hat{c}\,\hat{b}\right) = 0.$$

Also, for a generic q the finite spin $U_q(sl(2))$-modules V_k^q analogous to V_k above are well defined in the both algebras. The module V_k^q is spanned by the elements $b^k, Y(b^k),\ Y^2(b^k),\ldots,Y^{2k}(b^k)$ where X, H, Y are now the standard generators of the the QG $U_q(sl(2))$.

For the case of algebras $\mathbb{K}_q[\mathbb{R}^3]$ and $\mathcal{SL}_{\hbar,q}(2)$ we put

$$\Upsilon(b^k p(\mu_1,\mu_2)) = \hat{b}^k p(\hat{\mu}_1 - \frac{q^{-1}\hbar}{2}1, \hat{\mu}_2 - \frac{q^{-1}\hbar}{2}1),$$

where p is a symmetric polynomial in two variables. In particular, we have

$$\Upsilon\left(\frac{1}{2_q}(q^{-1}\,b\,c + \frac{h^2}{2_q} + q\,c\,b)\right) = \frac{1}{2_q}\left(q^{-1}\,\hat{b}\,\hat{c} + \frac{\hat{h}^2}{2_q} + q\,\hat{c}\,\hat{b}\right) - \frac{(q^{-1}\hbar)^2}{4}1.$$

The extension of this map up to a $U_q(sl(2))$-covariant map $\tilde{\mathcal{A}}_q \to \tilde{\mathcal{A}}_{\hbar,q}$ is evident.

With the use of map we can push the q-derivatives forward to the algebra $\tilde{\mathcal{A}}_{\hbar,q}$: $\hat{\partial}_a^q = \Upsilon\partial_a^q\Upsilon^{-1}$, $a=b,h,c$. Then, replacing the q-derivatives $\partial_b^q\ldots$ by their q-noncommutative analogs in formulae for all q-operators in question we get their $(\hbar,q)$-counterparts. Thus, the passage from q-counterparts of wave operators to $(\hbar,q)$-ones is completely analogous to the $\hbar$-quantization.

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