Compatible structures on Lie algebroids and Monge-Ampère operators

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Abstract

We study pairs of structures, such as the Poisson-Nijenhuis structures, on the tangent bundle of a manifold or, more generally, on a Lie algebroid or a Courant algebroid. These composite structures are defined by two of the following, a closed 2-form, a Poisson bivector or a Nijenhuis tensor, with suitable compatibility assumptions. We establish the relationships between PN-, $P\Omega$- and ΩN-structures. We then show that the non-degenerate Monge-Ampère structures on 2-dimensional manifolds satisfying an integrability condition provide numerous examples of such structures, while in the case of 3-dimensional manifolds, such Monge-Ampère operators give rise to generalized complex structures or generalized product structures on the cotangent bundle of the manifold.

Introduction

On the tangent bundle of a manifold or, more generally, on a Lie algebroid, we consider pairs of structures, such as the Poisson-Nijenhuis structures which give rise to hierarchies of Poisson structures (also called Hamiltonian structures) that play a very important role in the theory of completely integrable systems. These structures are defined by closed 2-forms, Poisson bivectors or $(1,1)$-tensors with vanishing Nijenhuis torsion. When suitable compatibility assumptions are introduced, one obtains composite structures called complementary 2-forms, PN-, $P\Omega$- and ΩN-structures. Krasil'shchik contributed to the study of the algebraic nature of Hamiltonian and bi-Hamiltonian structures and was the first to underline the cohomological nature of their compatibility condition (see [24] and references therein). While the first part of this article is a comprehensive survey of the relationships between such composite structures and the related notion of Hitchin pairs, the second part provides numerous examples arising from the theory of Monge-Ampère equations.

Our formulations and proofs make essential use of the *big bracket*, the even graded bracket on the space $\mathcal{F}$ of functions on the cotangent bundle of a Lie algebroid considered as a supermanifold. What we call the big bracket was first introduced by Kostant and Sternberg [23]; its use in the theory of Lie bialgebras is due to Lecomte and Roger [26] and was developed by one of us [15]. Roytenberg extended it to Lie algebroids [33] and Courant algebroids [34]. Recently, it has been used by Antunes [1] in the study of composite structures arising in the theory of sigma-models. In practice, all proofs are reduced to a straightforward use of the graded Jacobi identity, sometimes repeatedly. While many of our results can be found in the literature (see [8], which contains the references to earlier work by Magri, Gelfand and Dorfman, Fokas and Fuchssteiner, see [31] [32] [21], and the more recent articles [37] [38] [10] [1]), we claim that our method unifies results, generalizing the known properties from the case of manifolds to that of Lie algebroids and Courant algebroids. Our main argument is that the big bracket formalism can be applied to problems in the geometric theory of partial differential equations developed in [29][30] [25] and [2] [3]. We also stress that this theory can be considered in the general framework of Lie algebroids, and we wish to introduce a general abstract theory of Monge-Ampère structures on arbitrary Lie algebroids. In particular, the symplectic Monge-Ampère equations defined by n-forms on the cotangent bundle of a smooth, n-dimensional manifold M and, more generally, the Jacobi first-order systems, defined by a set of 2-forms on an $m+2$-dimensional manifold M, can be viewed as "deformations" of the standard Lie algebroid structure on the tangent bundle $T(T^*M)$ of T^*M. We shall indicate some links between our approach and the approach to the geometric structures developed by Hitchin [14] and Gualtieri [12] in their studies of generalized complex and Kähler structures, a new and fast developing field of differential geometry.

In Section 1, we introduce the big bracket, we recall the definition of Lie algebroids and give the explicit expression for the *Dorfman bracket* on the double of a Lie bialgebroid which is a *derived bracket* [16] [18] of the big bracket. The Courant algebroid structure of the double of a Lie bialgebroid is defined by the skew-symmetrized version of the Dorfman bracket, called the *Courant bracket*. Section 2 deals with general facts and formulas involving bivectors, forms and $(1,1)$-tensors that will be used in subsequent sections, and with Grabowski's formula (2.9) that expresses the Nijenhuis torsion of a $(1,1)$-tensor in terms of the big bracket [10]. In Section 3, we show that the adjoint actions of a non-degenerate 2-form and of its inverse bivector induce a representation of $\mathfrak{sl}_2$ on $\mathcal{F}$, we define the *primitive elements* and describe a Hodge-Lepage type decomposition of the elements in $\mathcal{F}$.

Section 4 is a study of the *complementary 2-forms* introduced by Vaisman [37] [38]. We prove that, given a Lie algebroid A, "ω is a complementary 2-form for the Poisson bivector π" is a sufficient condition for the bracket obtained by first dualizing the Lie algebroid structure of A by π and dualizing again by ω to be a Lie algebroid bracket on A, whose expression we easily derive. A remark concerning the corresponding modular class (Section 4.4) will be used in Section 13.3. Sections 5 to 10 contain the detailed analysis of the structures introduced by Magri and Morosi [31] [32] defined by a Poisson bivector and a Nijenhuis tensor, called PN-structures (Section 5), by compatible Poisson tensors (Section 6), by a closed 2-form and a Poisson bivector, called $P\Omega$-structures (Section 7) and by a closed 2-form and a Nijenhuis tensor, called ΩN-structures (Section 8) and *Hitchin pairs* introduced by Crainic [6] (Section 9). A table and a diagram summarize the relationships between these various structures.

Section 11 deals with Nijenhuis tensors on Courant algebroid. We state Grabowski's theorem [10] that characterizes *generalized complex structures* by a simple equation in terms of the big bracket.

In Sections 12-14, we describe the geometry of the symplectic *Monge-Ampère equations* and relate it to the structures discussed in the previous sections, using the formalism of the big bracket. Some of these results are reformulations of results in [25] and [2] [3]. Section 12 introduces Monge-Ampère structures on manifolds and the associated Monge-Ampère operators and equations. We recall the definition of the *effective forms* (also called *primitive forms*) and the one-to-one correspondence between Monge-Ampère operators and effective forms. Section 13 is devoted to the case of Monge-Ampère structures on 2-dimensional manifolds, with an emphasis on the non-degenerate case, when the Pfaffian of the defining 2-form is nowhere vanishing. We show that in the integrable case, *i.e.*, when the Monge-Ampère operator is equivalent to an operator with constant coefficients, the Monge-Ampère structure gives rise to PN- and ΩN-structures and to a deformed Lie algebroid structure on $T(T^*M)$ which is unimodular. More generally, a non-degenerate Monge-Ampère structure of divergence type defines a generalized almost complex structure on T^*M. If the defining 2-form is closed, this structure is integrable and corresponds to a Hitchin pair. The von Karman equation is an example where the integrability condition is not satisfied and the associated composite structures do not satisfy the compatibility condition. We then consider the first-order Jacobi differential systems which generalize the Monge-Ampère equations, and we describe the associated geometric structures on 2-dimensional manifolds. In Section 14, we proceed to study Monge-Ampère operators on 3-dimensional manifolds, recall the classification of the non-degenerate Monge-Ampère operators, and we prove that when the operator is non-degenerate, *i.e.*, when the Hitchin Pfaffian is nowhere-vanishing, and has constant coefficients, there is either an associated generalized complex structure or generalized product structure on T^*M. We conclude with a short discussion of two definitions of the generalized Calabi-Yau manifolds.

1 The big bracket

When $A \to M$ is a vector bundle, let $T^*[2]A[1]$ denote the cotangent bundle of the graded manifold $A[1]$ obtained from A by assigning degree 0 to the coordinates on the base and degree 1 to the coordinates on the fibers. The space $\mathcal{F}$ of smooth functions on $T^*[2]A[1]$ is a bigraded Poisson algebra [33] [20]. (See [15] for the case where M is a point and therefore A is a vector space.) If (x^i, ξ^a), $i = 1, \ldots, \dim M$ and $a = 1, \ldots, \mathrm{rank} A$, are coordinates on $A[1]$, then coordinates on $T^*[2]A[1]$ are $(x^i, \xi^a, p_i, \theta_a)$, with bidegrees $(0,0), (0,1), (1,1), (1,0)$, respectively. If an element u of $\mathcal{F}$ is of bidegree $(p+1, q+1)$, we call $|u| = p+q+2$ its (total) degree and we call (p,q) its *shifted bidegree*, $p \geq -1$, $q \geq -1$. The space $\mathcal{F}^{p,q}$ of elements of $\mathcal{F}$ of shifted bidegree (p,q) contains the space of sections of $\wedge^{p+1} A \otimes \wedge^{q+1} A^*$.

As the cotangent bundle of a graded manifold, $T^*[2]A[1]$ is canonically equipped with an even Poisson structure. We denote the even Poisson bracket on $\mathcal{F}$ by $\{\ ,\ \}$, and we call it the *big bracket*. The big bracket satisfies $\{x^i, p_j\} = \delta^i_j$ and $\{\xi^a, \theta_b\} = \delta^a_b$, so that $\{f, p_j\} = \partial_j f$, where $f \in C^\infty(M)$ and $\partial_j f = \frac{\partial f}{\partial x^j}$. This bracket is of bidegree $(-1,-1)$ and of shifted bidegree $(0,0)$. It is skew-symmetric, $\{u,v\} = -(-1)^{|u|\,|v|}\{v,u\}$, for all u and $v \in \mathcal{F}$, and it satisfies the Jacobi identity,

$$\{u, \{v, w\}\} = \{\{u, v\}, w\} + (-1)^{|u|\,|v|}\{v, \{u, w\}\} \ ,$$

for all u, v and $w \in \mathcal{F}$. We often use the Jacobi identity in the form,

$$\{\{u, v\}, w\} = \{u, \{v, w\}\} + (-1)^{|v|\,|w|}\{\{u, w\}, v\} \ .$$

The big bracket satisfies the Leibniz rule,

$$\{u, v \wedge w\} = \{u, v\} \wedge w + (-1)^{|u|\,|v|} v \wedge \{u, w\} \ ,$$

or

$$\{u \wedge v, w\} = u \wedge \{v, w\} + (-1)^{|v|\,|w|}\{u, w\} \wedge v \ .$$

The space of sections of a vector bundle E is denoted by ΓE. We call a section of $\wedge^\bullet E$ (resp., $\wedge^\bullet E^*$) a multivector (resp., a form) on E. Accordingly, we use the terms vector, bivector, k-form, (p,q)-tensor, etc. All manifolds and maps are assumed to be smooth.

1.1 Lie algebroids

A *Lie algebroid* structure on $A \to M$ is an element μ of $\mathcal{F}$ of shifted bidegree $(0,1)$ such that

$$\{\mu, \mu\} = 0 \ .$$

The *Schouten bracket* of multivectors, *i.e.*, sections of $\wedge^\bullet A$, X and Y, is

$$[X, Y]_\mu = \{\{X, \mu\}, Y\} \ .$$

In particular, this formula defines the Lie bracket of X and $Y \in \Gamma A$ as well as the anchor of A, $\rho : A \to TM$, by

$$\rho(X) f = \{\{X, \mu\}, f\} \ ,$$

for $X \in \Gamma A$ and $f \in C^\infty(M)$.

The *Lie algebroid differential* acting on sections of $\wedge^\bullet A^*$ is denoted by d_μ, thus

$$d_\mu = \{\mu, \cdot\} \ .$$

The Lie derivative of forms by $X \in \Gamma A$ is defined to be the graded commutator, $\mathcal{L}^\mu_X = [i_X, d_\mu]$.

A *Lie bialgebroid* is defined by $\mu \in \mathcal{F}^{0,1}$ and $\gamma \in \mathcal{F}^{1,0}$ such that $\{\mu+\gamma, \mu+\gamma\} = 0$. More generally, a *proto-bialgebroid* is defined by $S = \phi+\mu+\gamma+\psi$, where $\psi \in \Gamma(\wedge^3 A^*)$ and $\phi \in \Gamma(\wedge^3 A)$, such that $\{S, S\} = 0$. *Lie quasi-bialgebroids* correspond to $\psi = 0$, while *quasi-Lie bialgebroids* correspond to $\phi = 0$.

1.2 The Dorfman bracket

If (A, μ, γ) is a Lie bialgebroid, its double is the vector bundle, $A \oplus A^*$, equipped with the *Dorfman bracket* defined by

$$[u, v]_D = \{\{u, \mu + \gamma\}, v\} \ , \tag{1.1}$$

for u and $v \in \Gamma(A \oplus A^*)$. The skew-symmetrized Dorfman bracket is called the *Courant bracket* and $A \oplus A^*$ with the Dorfman bracket is a *Courant algebroid*. Since the Dorfman bracket (see [8] [18] [10]) is a derived bracket, it is a Loday-Leibniz bracket and therefore satisfies the (graded) Jacobi identity in the sense that, for each $u \in \Gamma(A \oplus A^*)$, $[u, \cdot]_D$ is a derivation of the bracket $[\ ,\ ]_D$ (see [16] [18]). More generally, Formula (1.1) defines a Loday-Gerstenhaber bracket on $\Gamma(\wedge^\bullet A \otimes \wedge^\bullet A^*)$.

Explicitly, for $X \in \Gamma A$ and $\alpha \in \Gamma(A^*)$,

$$\begin{aligned}
[X, \alpha]_D &= \{\{X, \mu\}, \alpha\} + \{\{X, \gamma\}, \alpha\} = \{X, \{\mu, \alpha\}\} + \{\mu, \{X, \alpha\}\} - \{\{\gamma, X\}, \alpha\} \\
&= i_X(d_\mu \alpha) + d_\mu(i_X \alpha) - i_\alpha(d_\gamma X) = \mathcal{L}^\mu_X \alpha - i_\alpha(d_\gamma X) \ ,
\end{aligned}$$

while

$$\begin{aligned}
[\alpha, X]_D &= \{\{\alpha, \mu\}, X\} + \{\{\alpha, \gamma\}, X\} = -\{\{\mu, \alpha\}, X\} + \{\alpha, \{\gamma, X\}\} + \{\gamma, \{\alpha, X\}\} \\
&= -i_X(d_\mu \alpha) + i_\alpha(d_\gamma X) + d_\gamma(i_\alpha X) = \mathcal{L}^\gamma_\alpha X - i_X(d_\mu \alpha) \ .
\end{aligned}$$

Therefore, for X and $Y \in \Gamma A$, α and $\beta \in \Gamma(A^*)$,

$$[X + \alpha, Y + \beta]_D = [X, Y]_\mu + \mathcal{L}^\gamma_\alpha Y - i_\beta(d_\gamma X) + [\alpha, \beta]_\gamma + \mathcal{L}^\mu_X \beta - i_Y(d_\mu \alpha) \ . \tag{1.2}$$

In the case of the standard Courant algebroid, $TM \oplus T^*M$, by assumption, $\gamma = 0$ and d_μ is the de Rham differential, d. Thus, for $X \in \Gamma(TM)$ and $\alpha \in \Gamma(T^*M)$,

$$[X, \alpha]_D = \{\{X, \mu\}, \alpha\} = \{X, \{\mu, \alpha\}\} + \{\mu, \{X, \alpha\}\} = i_X(d\alpha) + d(i_X \alpha) = \mathcal{L}_X \alpha \ ,$$

and

$$[\alpha, X]_D = \{\{\alpha, \mu\}, X\} = -\{\{\mu, \alpha\}, X\} = -i_X(d\alpha) \ .$$

In addition, it is clear that, for vector fields X and Y, $[X,Y]_D$ is the Lie bracket, and for 1-forms, α and β, $[\alpha,\beta]_D = 0$, this bracket vanishes on pairs of 1-forms. Therefore

$$[X+\alpha, Y+\beta]_D = [X,Y] + \mathcal{L}_X\beta - i_Y(d\alpha) \ . \tag{1.3}$$

We compute these brackets on $T^*[2]TM[1]$ in local coordinates, $(x^i, \xi^i, p_i, \theta_i)$. Here $\mu = p_i\xi^i$. Let $X = X^i\theta_i$ and $\alpha = \alpha_i\xi^i$. Then

$$[X,\alpha]_D = \{\{X^i\theta_i, p_j\xi^j\}, \alpha_k\xi^k\} = \{X^ip_i - \partial_jX^i\theta_i\xi^j, \alpha_k\xi^k\} = X^i\partial_i\alpha_k\xi^k + \partial_jX^i\alpha_i\xi^j \ ,$$

which is the expression of $\mathcal{L}_X\alpha$ in local coordinates. Similarly,

$$[\alpha,X]_D = \{\{\alpha_i\xi^i, p_j\xi^j\}, X^k\theta_k\} = \{-\partial_j\alpha_i\xi^i\xi^j, X^k\theta_k\} = -X^k\partial_k\alpha_i\xi^i + X^k\partial_j\alpha_k\xi^j \ ,$$

which is the expression of $-i_X(d\alpha)$ in local coordinates.

Remark 1.1 When μ is replaced by $\mu + H$, where H is a d_μ-closed 3-form, the equation $\{\mu+H, \mu+H\} = 0$ is satisfied, and one obtains the *Dorfman bracket with background*, $[\ ,\ ]_{D,H}$, on $\Gamma(A\oplus A^*)$,

$$[X+\alpha, Y+\beta]_{D,H} = [X+\alpha, Y+\beta]_D + i_{X\wedge Y}H \ , \tag{1.4}$$

making $A\oplus A^*$ a *twisted Courant algebroid* [35] [33] [12] .

Any 2-form B on A defines a gauge transformation, $\widehat{B} : X+\alpha \mapsto X+\alpha+i_XB$, satisfying

$$[\widehat{B}(X+\alpha), \widehat{B}(Y+\beta)]_{D,H} = \widehat{B}([X+\alpha, Y+\beta]_{D,H-d_\mu B}) \ .$$

If B is d_μ-closed, then $\widehat{B}$ is an automorphism of $(A\oplus A^*, [\ ,\ ]_{D,H})$.

2 Tensors and the big bracket

We shall need various preliminary results concerning tensors on a Lie algebroid.

2.1 Bivectors, forms and $(1,1)$-tensors

Let $\pi^\sharp : A^* \to A$ be the map defined by a bivector π, where $\pi^\sharp\alpha = i_\alpha\pi$, for $\alpha \in \Gamma(A^*)$. Then

$$\pi^\sharp\alpha = \{\alpha, \pi\} \ . \tag{2.1}$$

Let $\omega^\flat : A \to A^*$ be the map defined by a 2-form ω, where $\omega^\flat X = -i_X\omega$, for $X \in \Gamma A$. Then

$$\omega^\flat X = \{\omega, X\} \ . \tag{2.2}$$

Let $N^{\hat{}} : \Gamma A \to \Gamma A$ be the linear map induced by a vector bundle endomorphism of A. Then $N^{\hat{}}$ can be identified with a $(1,1)$-tensor on A, more precisely with a section N of $A^* \otimes A$, by setting

$$N^{\hat{}}(X) = \{X, N\} \; , \tag{2.3}$$

for all $X \in \Gamma A$. In local coordinates, if $N^{\hat{}}$ has components N^a_b, then $N = N^a_b \xi^b \theta_a$. We shall not distinguish between $N^{\hat{}}$ and N, and we shall abbreviate $N^{\hat{}}$ to N.

Lemma 2.1 *The map $N = \pi^\sharp \circ \omega^\flat : A \to A$ considered as a section of $A^* \otimes A$ is*

$$N = \{\pi, \omega\} \; .$$

Proof By the Jacobi identity, since $\{X, \pi\} = 0$, for all $X \in \Gamma A$,

$$\{X, N\} = \{X, \{\pi, \omega\}\} = \{\{\omega, \pi\}, X\} = \{\{\omega, X\}, \pi\} = \{\omega^\flat X, \pi\} = (\pi^\sharp \circ \omega^\flat)(X) \; .$$

This proves the result, in view of (2.3). □

In local coordinates, let $\pi = \frac{1}{2}\pi^{ab}\theta_a\theta_b$ and $\omega = \frac{1}{2}\omega_{ab}\xi^a\xi^b$. For $\alpha = \alpha_a\xi^a \in \Gamma(A^*)$, $\pi^\sharp\alpha = \pi^{ab}\alpha_a\theta_b$. For $X = X^a\theta_a \in \Gamma A$, $\omega^\flat X = \omega_{ab}X^b\xi^a$, whence $(\pi^\sharp \circ \omega^\flat)(X) = \pi^{ab}\omega_{ac}X^c\theta_b$. On the other hand,

$$\{\pi, \omega\} = \frac{1}{4}\pi^{ab}\omega_{cd}\{\theta_a\theta_b, \xi^c\xi^d\} = \pi^{ab}\omega_{bc}\theta_a\xi^c \; ,$$

whence $\{X, \{\pi, \omega\}\} = \pi^{ab}\omega_{ac}X^c\theta_b$.

In particular, if π is non-degenerate, and if π and ω are inverses of one another, by definition, $\pi^\sharp \circ \omega^\flat = \mathrm{Id}_A$ and $\omega^\flat(X) = -i_X\omega$, for all $X \in \Gamma A$. Then the $(1,1)$-tensor $\{\pi, \omega\}$ is the identity of A, Id_A. In Section 3 below, we denote the adjoint action of Id_A, $\{\mathrm{Id}_A, \cdot\}$, by $\mathbf{I}$.

In local coordinates, $\pi^{ab}\omega_{ac} = \delta^b_c$ and $\{\pi, \omega\} = \xi^a\theta_a$, satisfying $\{X, \xi^a\theta_a\} = X$, for all $X \in \Gamma A$.

This relation is a particular case of a general result, proved in [20]: when π and ω are inverses of one another, for $u \in \mathcal{F}^{p,q}$, in particular for $u \in \Gamma(\wedge^{p+1}A \otimes \wedge^{q+1}A^*)$,

$$\{\{\pi, \omega\}, u\} = \{\mathrm{Id}_A, u\} = (q-p)u \; , \tag{2.4}$$

or, in local coordinates,

$$\{\xi^a\theta_a, u\} = (q-p)u \; . \tag{2.5}$$

Therefore, for a bracket μ (resp., cobracket γ) on A,

$$\{\mathrm{Id}_A, \mu\} = \mu \qquad (\text{resp.,} \quad \{\mathrm{Id}_A, \gamma\} = -\gamma)$$

and for a 3-form ψ (resp., 3-tensor ϕ) on A, $\{\mathrm{Id}_A, \psi\} = 3\psi$ (resp., $\{\mathrm{Id}_A, \phi\} = -3\phi$).

2.2 Deformed brackets and torsion

Let (A, μ) be a Lie algebroid. Let $N \in \Gamma(A^* \otimes A)$ be a $(1,1)$-tensor on A, an element of shifted bidegree $(0,0)$. Then the *deformed structure*,

$$\mu_N = \{N, \mu\} \ ,$$

defines an anchor $\rho \circ N$ and a skew-symmetric bracket on A which we shall denote by $[\ ,\ ]^\mu_N$, Explicitly,

$$[X, Y]^\mu_N = \{\{X, \{N, \mu\}\}, Y\} \ , \tag{2.6}$$

for X and $Y \in \Gamma A$.

Lemma 2.2 *The bracket $[\ ,\ ]^\mu_N$ is such that, for X, $Y \in \Gamma A$,*

$$[X, Y]^\mu_N = [NX, Y]_\mu + [X, NY]_\mu - N[X, Y]_\mu \ . \tag{2.7}$$

Proof By definition,

$$\begin{aligned}
[X, Y]^\mu_N = \{\{X, \{N, \mu\}\}, Y\} &= \{\{\{X, N\}, \mu\}, Y\} + \{\{N, \{X, \mu\}\}, Y\} \\
&= [NX, Y]_\mu + \{N, \{\{X, \mu\}, Y\}\} + \{\{N, Y\}, \{X, \mu\}\} \\
&= [NX, Y]_\mu + [X, NY]_\mu - N[X, Y]_\mu \ ,
\end{aligned}$$

where we have used the Jacobi identity and the definition of $[\ ,\ ]_\mu$. □

The bracket $[\ ,\ ]^\mu_N$ is called the *deformed* [21] (or *contracted* [4] [5] [10]) bracket of $[\ ,\ ]_\mu$. By $\mathcal{T}_\mu N$ we denote the *Nijenhuis torsion* of N defined by

$$(\mathcal{T}_\mu N)(X, Y) = [NX, NY]_\mu - N([NX, Y]_\mu + [X, NY]_\mu) + N^2[X, Y]_\mu \ , \tag{2.8}$$

for all X and $Y \in \Gamma A$. It is clear that $(\mathcal{T}_\mu N)(X, Y) = [NX, NY]_\mu - N([X, Y]^\mu_N)$.

Proposition 2.3 *In terms of the big bracket,*

$$\mathcal{T}_\mu N = \frac{1}{2}(\{N, \{N, \mu\}\} - \{N^2, \mu\}) \ , \tag{2.9}$$

and

$$\frac{1}{2}\{\{N, \mu\}, \{N, \mu\}\} = \{\mu, \mathcal{T}_\mu N\} \ . \tag{2.10}$$

Proof See Grabowski [10] for Formula (2.9), which is proved by a simple calculation. Formula (2.10) follows from (2.9) by an application of the Jacobi identity. □

Remark 2.4 Formula (2.9) can also be viewed as a particular case of Formulas (5.22) and (5.16) of [16], taking into account the fact that, for vector-valued forms, the big bracket and the Richardson-Nijenhuis bracket coincide up to sign (see [15]), or as a particular case of Formula (3.14) of [18]. Formula (2.9) also appears in a slightly different form in [8], Section 3.3. It plays an essential role in [1].

2.3 Nijenhuis structures

Let $N \in \Gamma(A^* \otimes A)$ be a $(1,1)$-tensor on A, thus N is an element of shifted bidegree $(0,0)$. Then the *deformed structure* bracket is defined by (2.6), and its explicit expression is Formula (2.7) above. We have denoted the *Nijenhuis torsion* of N by $\mathcal{T}_\mu N$. The following result (see, *e.g.*, [21]) is an immediate corollary of Proposition 2.3.

Theorem 2.5 *A necessary and sufficient condition (resp., a sufficient condition) for the deformed structure $\mu_N = \{N, \mu\}$ to be a Lie algebroid structure on A is*

$$\{\mu, \mathcal{T}_\mu N\} = 0 \ .$$

(resp., $\mathcal{T}_\mu N = 0$).

When $\{\mu, \mathcal{T}_\mu N\} = 0$, we call $\mathcal{T}_\mu N$ a d_μ*-cocycle*.

Remark 2.6 The deformed Lie algebroid structure μ_N is compatible with μ in the sense that $\mu + \mu_N$ is a Lie algebroid structure, *i.e.*, $\{\mu + \mu_N, \mu + \mu_N\} = 0$.

The operator on $\Gamma(\wedge^\bullet A^*)$ associated to μ_N is $d_{\mu_N} = \{\mu_N, \cdot\} = \{\{N, \mu\}, \cdot\}$.

By definition, a $(1,1)$-tensor N is an *almost complex structure* if $N^2 = -\mathrm{Id}_A$, and an almost complex structure N is a *complex structure* if $\mathcal{T}_\mu N = 0$.

Proposition 2.7 *An almost complex structure N is a complex structure if and only if*

$$\{\{N, \mu\}, N\} = \mu \ . \tag{2.11}$$

Proof Equation (2.11) follows from (2.9) and the relation $\{\mathrm{Id_A}, \mu\} = \mu$, a particular case of (2.4). □

2.4 Bivectors and 3-forms

Lemma 2.8 *If π is a bivector and ψ is a 3-form on A, then, for X, $Y \in \Gamma A$,*

$$\{\{X, \{\psi, \pi\}\}, Y\} = \pi^\sharp(i_{X\wedge Y}\psi) \ , \tag{2.12}$$

with the convention $i_{X\wedge Y} = i_X \circ i_Y$.

Proof Since $\{X, \pi\} = \{Y, \pi\} = 0$, by the Jacobi identity,

$$\{\{X, \{\psi, \pi\}\}, Y\} = \{\{\{X, \psi\}, \pi\}, Y\} = \{\{\{X, \psi\}, Y\}, \pi\} \ .$$

Now $\{X, \psi\} = i_X\psi$, $\{\{X, \psi\}, Y\} = i_X i_Y \psi$. Applying (2.1) to the 1-form $\alpha = \{\{X, \psi\}, Y\}$, we obtain (2.12). □

In local coordinates, let $\pi = \frac{1}{2}\pi^{ab}\theta_a\theta_b$ and $\psi = \frac{1}{6}\psi_{abc}\xi^a\xi^b\xi^c$. Then $\{\psi, \pi\} = \frac{1}{2}\pi^{cd}\psi_{abc}\xi^a\xi^b\theta_d$. For $X = X^a\theta_a$, $\{X, \{\psi, \pi\}\} = \pi^{cd}\psi_{abc}X^a\xi^b\theta_d$, and for $Y = Y^a\theta_a$, $\{\{X, \{\psi, \pi\}\}, Y\} = \pi^{dc}\psi_{abc}X^aY^b\theta_d$. On the other hand, $i_X i_Y\psi = \psi_{abc}X^bY^a\xi^c$, and $\pi^\sharp(i_{X\wedge Y}\psi) = \pi^{dc}\psi_{abc}X^aY^b\theta_d$.

3 A representation of $\mathfrak{sl}_2$

For $u \in \mathcal{F}^{p,q}$, we shall call $w(u) = q - p$ the *weight* of u. Let π be a non-degenerate bivector and ω a 2-form on A which are inverses of one another. Then $\{\pi, \omega\} = \mathrm{Id}_A$. Set $\mathrm{ad}_\omega = \{\omega, \cdot\}$ and $\mathrm{ad}_\pi = \{\pi, \cdot\}$. Then $\mathbf{I} = \{\{\pi, \omega, \}, \cdot\}$ acts on $\mathcal{F}$ by

$$\mathbf{I}(u) = w(u)u \ ,$$

for $u \in \mathcal{F}^{p,q}$ (see Formula (2.4)). Let $\mathrm{ad}'_\pi = \{., \pi\} = -\mathrm{ad}_\pi$ be the right adjoint action of π. Then

$$[\mathbf{I}, \mathrm{ad}_\omega] = 2\mathrm{ad}_\omega \ , \qquad [\mathbf{I}, \mathrm{ad}'_\pi] = -2\mathrm{ad}'_\pi \qquad [\mathrm{ad}_\omega, \mathrm{ad}'_\pi] = \mathbf{I} \ ,$$

where $[\ ,\ ]$ denotes the commutator of operators. Therefore the operators $(\mathrm{ad}_\omega, \mathrm{ad}'_\pi, \mathbf{I})$ define a representation of $\mathfrak{sl}_2$ on the linear space $\mathcal{F}$ which restricts to the linear space of all tensors, analogous to the representation on forms in [25]. Then

$$\mathrm{ad}_\omega(\mathcal{F}^{p,q}) \subset \mathcal{F}^{p-1,q+1}, \ \mathrm{ad}'_\pi(\mathcal{F}^{p,q}) \subset \mathcal{F}^{p+1,q-1} \ ,$$

so that $w(\mathrm{ad}_\omega u) = w(u) + 2$ and $w(\mathrm{ad}'_\pi u) = w(u) - 2$.

Definition 3.1 *An element $u \in \mathcal{F}^{p,q}$ is called* primitive *if $u \in \ker(\mathrm{ad}'_\pi)$, i.e., $\{u, \pi\} =$ 0.*

The next statement follows from the definitions.

Lemma 3.2 *For π and ω inverses of one another,* $\ker(\mathrm{ad}'_\pi) \cap \ker(\mathrm{ad}_\omega) \subset \oplus_{p \geq -1}\mathcal{F}^{p,p}$.

The inverse inclusion is not valid since counter-examples are furnished by $(1,1)-$ tensors N of shifted bidegree $(0,0)$ such that the 2-form $\{\mathrm{ad}_\omega, N\}$ or the bivector $\{\mathrm{ad}'_\pi, N\}$ does not vanish, *e.g.*, when N is a multiple of the identity.

The following theorem is an analogue of the Hodge-Lepage decompositions in Kähler [41] and symplectic [27] [25] geometry. We first prove a lemma.

Lemma 3.3 *Let $u \in \mathcal{F}$ be of weight $w(u)$. Then, for any $k \geq 0$,*

$$\mathbf{I}(\mathrm{ad}^k_\omega u) = (w(u) + 2k)\mathrm{ad}^k_\omega u \ .$$

If u is primiitve, then

$$\mathrm{ad}'_\pi(\mathrm{ad}^k_\omega u) = -k(w(u) + k - 1)\mathrm{ad}^{k-1}_\omega u \ . \qquad \square$$

Proof The first formula follows from $w(\mathrm{ad}^k_\omega u) = w(u) + 2k$. The second is proved by recursion on k. $\square$

From the complete reducibility of finite-dimensional representations of semi-simple Lie algebras, and from Lemma 3.3 we obtain the following result [29] [25].

Theorem 3.4 *Any element $u \in \mathcal{F}^{p,q}$ admits the decomposition,*

$$u = u_0 + \mathrm{ad}_\omega u_1 + \mathrm{ad}^2_\omega u_2 + \ldots + \mathrm{ad}^k_\omega u_k + \ldots \ , \tag{3.1}$$

where each $u_k, k \geq 0$, is a uniquely defined primitive element of $\mathcal{F}^{p+k,q-k}$ of weight $w(u) - 2k$.

4 Complementary 2-forms for Poisson structures

The complementary 2-forms with respect to a Poisson structure on a Lie algebroid were defined and studied by Vaisman [37] [38]. We shall describe the complementary 2-forms on a Lie algebroid A and their properties by means of the big bracket on $\mathcal{F}$. The method of proof using the big bracket gives a clear view of their nature and properties.

4.1 Poisson bivectors

We recall several well known facts concerning the Poisson structures on Lie algebroids [33] [19].

Lemma 4.1 *Let (A, μ) be a Lie algebroid. If $\pi \in \Gamma(\wedge^2 A)$, then*

$$\gamma_\pi = \{\pi, \mu\}$$

is of shifted bidegree $(1, 0)$, and γ_π is a Lie algebroid structure on A^* if and only if

$$\{\gamma_\pi, \gamma_\pi\} = 0 \ . \tag{4.1}$$

The next lemma gives conditions for the construction, from a bivector on a Lie algebroid, of a Lie algebroid structure on the dual vector bundle. Since $\{\mu, \mu\} = 0$,

$$\{\gamma_\pi, \gamma_\pi\} = \{\{\pi, \mu\}, \{\pi, \mu\}\} = \{\{\{\pi, \mu\}, \pi\}, \mu\} = \{[\pi, \pi]_\mu, \mu\} \ .$$

Therefore

Lemma 4.2 *A necessary and sufficient condition for γ_π to be a Lie algebroid structure on A^* is*

$$\{\mu, [\pi, \pi]_\mu\} = 0 \ , \tag{4.2}$$

while a sufficient condition is

$$[\pi, \pi]_\mu = 0 \ , \tag{4.3}$$

i.e., π is a Poisson bivector.

The bracket defined by $\gamma_\pi = \{\pi, \mu\}$ on $\Gamma(\wedge^\bullet A^*)$ is usually denoted simply by $[\ ,\ ]_\pi$. Thus, by definition,

$$\{\{\alpha, \{\pi, \mu\}\}, \beta\} = [\alpha, \beta]_\pi \ ,$$

for all α and $\beta \in \Gamma(\wedge^\bullet A^*)$. The following lemma is proved in [33] [19].

Lemma 4.3 *The bracket defined by $\gamma_\pi = \{\pi, \mu\}$ on $\Gamma(\wedge^\bullet A^*)$ is the Koszul bracket of forms. In particular, for all $f \in C^\infty(M)$, $\alpha, \beta \in \Gamma(A^*)$,*

$$\{\{\alpha, \{\pi, \mu\}\}, f\} = ((\rho \circ \pi^\sharp)\alpha) \cdot f \ ,$$

$$\{\{\alpha, \{\pi, \mu\}\}, \beta\} = \mathcal{L}^\mu_{\pi^\sharp\alpha}\beta - \mathcal{L}^\mu_{\pi^\sharp\beta}\alpha - d_\mu(\pi(\alpha, \beta)) \ .$$

Remark 4.4 A bivector π is Poisson if and only if $\gamma_\pi = \{\pi, \mu\}$ is primitive in the sense of Definition 3.1. Assume that π is a non-degenerate bivector, with inverse ω. We consider the decomposition (3.1) of the structure $\mu \in \mathcal{F}^{0,1}$,

$$\mu = \mu_0 + \mathrm{ad}_\omega \mu_1 \ ,$$

where $\mu_0 \in \mathcal{F}^{0,1}$ and $\mu_1 \in \mathcal{F}^{1,0}$ are primitive, $\mathrm{ad}'_\pi \mu_i = 0$, $i = 0, 1$, and of weight 1 and -1, respectively. Then, using Lemma 3.3, we obtain

$$\gamma_\pi = \{\pi, \mu\} = \{\pi, \mu_0 + \mathrm{ad}_\omega \mu_1\} = -\mathrm{ad}'_\pi \mu_0 - \mathrm{ad}'_\pi \mathrm{ad}_\omega \mu_1 = \mu_1 \ .$$

Thus $\mu = \mu_0 + \mathrm{ad}_\omega \gamma_\pi$, where μ_0 and γ_π are primitive.

4.2 Dualization and composition

We now dualize the construction of Section 4.1. Let (A^*, γ) be a Lie algebroid. If $\omega \in \Gamma(\wedge^2 A^*)$, then $\widetilde{\mu} = \{\gamma, \omega\}$ is of shifted bidegree $(0, 1)$ and $\widetilde{\mu}$ is a Lie algebroid structure on A if and only if

$$\{[\omega, \omega]_\gamma, \gamma\} = 0 \ , \tag{4.4}$$

while a sufficient condition is

$$[\omega, \omega]_\gamma = 0 \ . \tag{4.5}$$

We shall now combine the two preceding constructions and consider the following scheme,

$$\boxed{(A, \mu) \overset{(\pi)}{\rightsquigarrow} (A^*, \gamma_\pi = \{\pi, \mu\}) \overset{(\omega)}{\rightsquigarrow} (A, \widetilde{\mu} = \{\gamma_\pi, \omega\} = \{\{\pi, \mu\}, \omega\}) \ .}$$

The following definition is due to Vaisman [37].

Definition 4.5 *A 2-form satisfying* (4.5) *when $\gamma = \gamma_\pi = \{\pi, \mu\}$ is called a* complementary 2-form *for π.*

Since, in this case, $\gamma = \{\pi, \mu\}$, by Lemma 4.3, $[\omega, \omega]_\gamma = \{\{\omega, \gamma\}, \omega\}$ is equal to $[\omega, \omega]_\pi$, where $[\ ,\ ]_\pi$ is the Koszul bracket .

Let π be an arbitrary bivector and ω an arbitrary 2-form. Let us determine sufficient conditions for $\widetilde{\mu} = \{\gamma_\pi, \omega\}$ to be a Lie algebroid structure on A, *i.e.*, to satisfy

$$\{\widetilde{\mu}, \widetilde{\mu}\} = 0 \ . \tag{4.6}$$

Proposition 4.6 *(i) Let π be a bivector on (A,μ) such that $\gamma_\pi = \{\pi,\mu\}$ satisfies $\{\gamma_\pi,\gamma_\pi\} = 0$. A necessary and sufficient condition for $\widetilde{\mu} = \{\gamma_\pi,\omega\} = \{\{\pi,\mu\},\omega\}$ to be a Lie algebroid structure on A is $\{[\omega,\omega]_\pi,\gamma_\pi\} = 0$.*

(ii) Let π be a bivector on (A,μ). A sufficient condition for $\widetilde{\mu} = \{\gamma_\pi,\omega\} = \{\{\pi,\mu\},\omega\}$ to be a Lie algebroid structure on A is

$$\begin{cases} [\pi,\pi]_\mu = 0 & (\pi \textit{ is Poisson}) \ , \\ [\omega,\omega]_\pi = 0 & (\omega \textit{ is a complementary 2-form for } \pi) \ . \end{cases}$$

Proof Using the Jacobi identity we compute

$$\begin{aligned} \{\widetilde{\mu},\widetilde{\mu}\} = \{\{\gamma_\pi,\omega\},\{\gamma_\pi,\omega\}\} &= \{\gamma_\pi,\{\omega,\{\gamma_\pi,\omega\}\}\} + \{\{\gamma_\pi,\{\gamma_\pi,\omega\}\},\omega\} \\ &= \{\gamma_\pi,\{\omega,\{\gamma_\pi,\omega\}\}\} + \frac{1}{2}\{\{\{\gamma_\pi,\gamma_\pi\},\omega\},\omega\}. \end{aligned}$$

Let us assume that $\gamma_\pi = \{\pi,\mu\}$, and that π satisfies $\{\gamma_\pi,\gamma_\pi\} = 0$, which is equivalent to (4.2). Condition (4.6) becomes

$$\{\gamma_\pi,\{\{\omega,\gamma_\pi\},\omega\}\} = 0 \ ,$$

i.e.,

$$\{\gamma_\pi,[\omega,\omega]_\pi\} = 0 \ . \tag{4.7}$$

This proves part (i), and part (ii) follows immediately. □

4.3 Lie algebroid structure defined by a complementary 2-form

Let us determine an explicit expression for the anchor and bracket of $(A,\widetilde{\mu})$. By the Jacobi identity,

$$\widetilde{\mu} = \{\{\pi,\mu\},\omega\} = \mu_1 + \mu_2 \ , \tag{4.8}$$

where we have set

$$\mu_1 = \{\{\pi,\omega\},\mu\} \qquad \text{and} \qquad \mu_2 = \{\pi,\{\mu,\omega\}\} \ .$$

We set

$$N = \{\pi,\omega\} \ ,$$

then $\mu_1 = \{N,\mu\}$ and we write $\mu_2 = \{\pi,\{\mu,\omega\}\} = \{\psi,\pi\}$, where ψ is the 3-form $-\{\mu,\omega\} = -d_\mu\omega$.

By definition, the anchor of $(A,\widetilde{\mu})$ is $\widetilde{\rho}$ such that $\widetilde{\rho}(X)f = \{\{X,\widetilde{\mu}\},f\}$, for all $X \in \Gamma A$ and $f \in C^\infty(M)$. Then $\widetilde{\rho} = \rho \circ N$, where ρ is the anchor of A. In fact, for $X \in \Gamma A$ and $f \in C^\infty(M)$,

$$\{\{X,\mu_1\},f\} = \{\{X,\{N,\mu\}\},f\} = \rho(NX)\cdot f$$

and

$$\{\{X,\mu_2\},f\} = \{\{X,\{\psi,\pi\}\},f\} = 0 \ .$$

Let us consider the bracket defined, for X, $Y \in \Gamma A$, by

$$[X,Y]_{\widetilde{\mu}} = \{\{X,\widetilde{\mu}\},Y\} \ .$$

By Lemma 2.2, the bracket $[\ ,\ ]_{\mu_1}$ is the bracket $[\ ,\ ]^{\mu}_N$ recalled in (2.7). The theorem below follows from Lemmas 2.1, 2.2 and 2.8.

Theorem 4.7 *Let π be a bivector and ω a 2-form on (A,μ). Then π and ω satisfy* (4.6) *if and only if the bracket of sections of A defined by*

$$[X,Y]_{\widetilde{\mu}} = [X,Y]^{\mu}_N - \pi^{\sharp}(i_{X\wedge Y}d_{\mu}\omega) \ , \tag{4.9}$$

for all X, $Y \in \Gamma A$, where $N = \pi^{\sharp}\circ\omega^{\flat}$, is a Lie algebroid bracket with anchor $\rho\circ N$.

In order to compare (4.9) with Formula (3.3) in [37], we remark that $B = -N$, so that $[\ ,\ ]'_E$ is the opposite of $[\ ,\ ]_{\widetilde{\mu}}$. As a corollary of Proposition 4.6 (ii) and Theorem 4.7, we obtain the following results which were proved in [37].

Corollary 4.8 *If π is a Poisson bivector and ω is a complementary 2-form for π, then*

(i) Formula (4.9) *defines a Lie bracket on the space of sections of A, and*

(ii) if, in addition, $d_{\mu}\omega = 0$, then bracket $[\ ,\ ]^{\mu}_N$, where $N = \pi^{\sharp}\circ\omega^{\flat}$, is a Lie bracket.

In part (ii) of this corollary the assumption that ω be d_{μ}-closed can be replaced by the weaker assumption that, for all X and $Y \in \Gamma A$, $i_{X\wedge Y}d_{\mu}\omega \in \ker(\pi^{\sharp})$.

Remark 4.9 If π is a non-degenerate Poisson bivector, its inverse ω is a complementary 2-form for π. In fact [21], $[\omega,\cdot]_{\pi} = d_{\mu}$ and therefore $[\omega,\omega]_{\pi} = d_{\mu}\omega = 0$. In this case $N = \mathrm{Id}_A$ and $\mu_2 = 0$, therefore $\widetilde{\mu} = \mu$.

4.4 The modular class of $(A,\widetilde{\mu})$

Consider a Poisson bivector π on A and a complementary 2-form ω with respect to π. Assume that A is orientable and let λ be a nowhere-vanishing section of $\wedge^{\mathrm{top}}(A^*)$ that defines an isomorphism, $*_{\lambda}$, from multivectors to forms. Let $d^{\pi}_{\omega} = -[\omega,.]_{\pi}$ be the Lie algebroid cohomology operator of A with structure $\widetilde{\mu} = \{\{\pi,\mu\},\omega\}$. Each of the operators on the sections of $\wedge^{\bullet}A$,

$$\partial^{\pi}_{\omega} = [d_{\pi}, i_{\omega}] \ ,$$

and

$$\partial^{\pi}_{\omega,\lambda} = -(*_{\lambda})^{-1}d^{\pi}_{\omega}*_{\lambda}$$

generates $[\ ,\ ]_{\widetilde{\mu}}$ and also has square 0 since ω is a complementary 2-form with respect to π. The 1-form $\xi_{\pi,\omega,\lambda}$ on A defined by

$$\partial^{\pi}_{\omega,\lambda} - \partial^{\pi}_{\omega} = i_{\xi_{\pi,\omega,\lambda}}$$

is a $d_{\widetilde{\mu}}$-cocycle. Its class is the modular class of the Lie algebroid $(A, \widetilde{\mu})$ [40].

In the following sections, A denotes a vector bundle over a manifold M, and we let (A, μ) be a Lie algebroid, so that, by assumption, $\{\mu, \mu\} = 0$. We will sometimes abbreviate (A, μ) by A.

5 What is a PN-structure on a Lie algebroid?

We have reviewed the Nijenhuis structures in Section 2.3. We now consider Nijenhuis structures on Lie algebroids equipped with a Poisson structure.

5.1 Compatibility

Given a bivector π and a $(1,1)$-tensor N on (A, μ), we consider both

$$\mu_N = \{N, \mu\} \ ,$$

which defines an anchor $\rho \circ N$ and a bracket $[\ ,\ ]^{\mu}_{N}$ on A, and

$$\gamma_\pi = \{\pi, \mu\} \ ,$$

which defines an anchor $\rho \circ \pi^\sharp$ and a bracket on A^* that we have denoted by $[\ ,\ ]_\pi$. We assume that

$$N \circ \pi^\sharp = \pi^\sharp \circ N^* \ , \tag{5.1}$$

where N^* denotes the transpose of N satisfying $< N(X), \alpha > = < X, N^*(\alpha) >$, for all $X \in \Gamma A$ and $\alpha \in \Gamma(A^*)$, so that $N \circ \pi^\sharp$ defines a bivector π_N by $\pi^\sharp_N = N \circ \pi^\sharp$. Then,

$$\pi_N = \frac{1}{2}\{\pi, N\} \ .$$

We introduce a *compatibility condition* for π and N by requiring that the bracket $[\ ,\ ]^{\mu}_{N}$ twisted by π, which is $\{\pi, \{N, \mu\}\}$, be equal to the bracket $[\ ,\ ]_\pi$ deformed by N^*, which is $\{\{\pi, \mu\}, N\}$. Thus we set

$$C_\mu(\pi, N) = \{\pi, \{N, \mu\}\} + \{N, \{\pi, \mu\}\} \ , \tag{5.2}$$

which is a section of $\wedge^2 A \otimes A^*$.

Definition 5.1 *A bivector π and a $(1,1)$-tensor N on (A, μ) are called* compatible *if they satisfy* (5.1) *and*

$$C_\mu(\pi, N) = 0 \ .$$

A PN-structure *on (A, μ) is defined by a Poisson bivector and a Nijenhuis tensor on (A, μ) which are compatible.*

5.2 PN structures and Lie bialgebroid structures

A necessary and sufficient condition for (μ_N, γ_π) to define a Lie bialgebroid structure on (A, A^*) is $\{\mu_N + \gamma_\pi, \mu_N + \gamma_\pi\} = 0$. When N is a Nijenhuis $(1,1)$-tensor, $\{\mu_N, \mu_N\} = 0$, and when π is a Poisson bivector, $\{\gamma_\pi, \gamma_\pi\} = 0$. Therefore in this case the condition $\{\mu_N + \gamma_\pi, \mu_N + \gamma_\pi\} = 0$ is equivalent to $\{\mu_N, \gamma_\pi\} = 0$.

Lemma 5.2 *Let $C'_\mu(\pi, N) = 2\{\mu_N, \gamma_\pi\}$. Then $C'_\mu(\pi, N) = \{\mu, C_\mu(\pi, N)\}$.*

Proof By the Jacobi identity,

$$C'_\mu(\pi, N) = 2\{\{N, \mu\}, \{\pi, \mu\}\} = \{\{N, \mu\}, \{\pi, \mu\}\} + \{\{\pi, \mu\}, \{N, \mu\}\}$$

$$= \{\{\{N, \mu\}, \pi\}, \mu\} + \{\{\{\pi, \mu\}, N\}, \mu\} = \{\mu, C_\mu(\pi, N)\}\ . \qquad \square$$

Theorem 5.3 *Let N be a Nijenhuis $(1,1)$-tensor and π a Poisson bivector on (A, μ).*

(i) The vanishing of $\{\mu, C_\mu(\pi, N)\}$ is a necessary and sufficient condition for (μ_N, γ_π) to define a Lie bialgebroid structure on (A, A^*). In particular, if π and N are compatible, then (μ_N, γ_π) is a Lie bialgebroid structure.

(ii) If the d_μ-exact 1-forms generate $\Gamma(A^*)$ locally as a $C^\infty(M)$-module, then a Poisson bivector π and a Nijenhuis tensor N define a PN-structure on (A, μ) if and only if the pair (μ_N, γ_π) defines a Lie bialgebroid structure on (A, A^*).

Proof Only (ii) needs to be proved. From $C'_\mu(\pi, N) = \{\mu, C_\mu(\pi, N)\}$ we obtain

$$\{C_\mu(\pi, N), \{\mu, f\}\} = \{\{C_\mu(\pi, N), \mu\}, f\} = -\{C'_\mu(\pi, N), f\}\ ,$$

for all $f \in C^\infty(M)$. Thus $C'_\mu(\pi, N) = 0$ implies $C_\mu(\pi, N)(d_\mu f, \cdot) = 0$, for all $f \in C^\infty(M)$. Under the assumptions of part (ii) of the theorem, $C_\mu(\pi, N)$ vanishes identically since it is $C^\infty(M)$-linear. $\square$

The equivalence stated in the theorem was proved in [17] for the case when A is the tangent bundle of a manifold. This equivalence may fail for Lie algebroids which are not tangent bundles, a fact observed by Grabowski and Urbanski [11].

Remark 5.4 The compatibility condition $C_\mu(\pi, N) = 0$ implies that the brackets $[\ ,\ ]_{\pi_N}$, $([\ ,\ ]^\mu_N)_\pi$ and $([\ ,\ ]_\pi)_{N^*}$ coincide. In fact, from $\{\pi, \{N, \mu\}\} = \{\{\pi, \mu\}, N\}$, we obtain $\{\mu, \{N, \pi\}\} = \{\{\mu, N\}, \pi\} + \{N, \{\mu, \pi\}\} = 2\{\{\mu, N\}, \pi\} = 2\{N, \{\mu, \pi\}\}$ or $\{\mu, \pi_N\} = \{\mu_N, \pi\} = \{\{\pi, \mu\}, N\}$.

Remark 5.5 The condition $C'_\mu(\pi, N) = 0$ is equivalent to each of the following:

- The operator $d_N = \{\{N, \mu\}, \cdot\} = [i_N, d_\mu]$ is a derivation of $[\ ,\ ]_\pi$,
- $(\Gamma(\wedge^\bullet A^*), [\ ,\ ]_\pi, d_N)$ is a differential Gerstenhaber algebra,
- The operator $d_\pi = \{\{\pi, \mu\}, \cdot\} = [\pi, \cdot]_\mu$ is a derivation of $[\ ,\ ]^\mu_N$,
- $(\Gamma(\wedge^\bullet A), [\ ,\ ]^\mu_N, d_\pi)$ is a differential Gerstenhaber algebra.

6 On compatible Poisson structures

For some of the results in this section, see [8] and earlier articles cited there.

Two Poisson bivectors on (A,μ) are said to be *compatible*, or to form a *Hamiltonian pair* or to define a *bi-Hamiltonian structure*, if their sum is a Poisson bivector. Thus Poisson bivectors π and π_1 are compatible if and only if $\{\{\pi,\mu\},\pi_1\}=0$.

Let π be a bivector on (A,μ) and N a $(1,1)$-tensor. Assume that $N\circ\pi^\sharp$ is a bivector, *i.e.*, $N\circ\pi^\sharp=\pi^\sharp\circ N^*$. We have set $\pi_N^\sharp=N\circ\pi^\sharp$. Then

$$\pi_N=\frac{1}{2}\{\pi,N\}\ .$$

In particular, if a bivector π is non-degenerate and has inverse ω, then $N=\{\pi_N,\omega\}$.

Proposition 6.1 *Assume that π is a non-degenerate Poisson bivector on (A,μ) with inverse ω, and N is a $(1,1)$-tensor such that π and π_N satisfy $\{\{\pi,\mu\},\pi_N\}=0$. Then*

$$\{\{\mu,\pi\},N\}+\{\{\mu,N\},\pi\}=0\ , \tag{6.1}$$

and

$$\{\{N,\mu\},\omega\}=0\ . \tag{6.2}$$

If, in addition, π_N is a Poisson bivector, then

$$\{\{\mu,\pi_N\},N\}+\{\{\mu,N\},\pi_N\}=0\ . \tag{6.3}$$

Proof Since ω is the inverse of π, $\{\pi,\omega\}$ is the identity of A and, by (2.4), for $u\in\mathcal{F}^{p,q}$,

$$\{u,\{\pi,\omega\}\}=(p-q)u\ . \tag{6.4}$$

By assumption, $\{\{\pi,\mu\},\pi\}=0$ and $\{\{\pi,\mu\},\pi_N\}=0$.

(i) From the compatibility of π and π_N and the Jacobi identity, we derive

$$0=\{\{\{\pi_N,\mu\},\pi\},\omega\}=\{\{\{\pi_N,\mu\},\omega\},\pi\}+\{\{\pi_N,\mu\},\{\pi,\omega\}\}\ .$$

Because π is a Poisson bivector, $\{\mu,\omega\}=0$. Therefore

$$0=\{\{\{\pi_N,\omega\},\mu\},\pi\}+\{\{\pi_N,\mu\},\{\pi,\omega\}\}\ .$$

Using the relations $\pi_N=\frac{1}{2}\{\pi,N\}$, $N=\{\pi_N,\omega\}$, and (6.4), we obtain

$$0=\{\{N,\mu\},\pi\}+\frac{1}{2}\{\{\pi,N\},\mu\}=\{\{N,\mu\},\pi\}+\frac{1}{2}\{\pi,\{N,\mu\}\}+\frac{1}{2}\{\{\pi,\mu\},N\}$$

$$=\frac{1}{2}\{\{N,\mu\},\pi\}+\frac{1}{2}\{\{\pi,\mu\},N\}\ .$$

(ii) To prove (6.2), we compute

$$\{\{\pi,\{N,\mu\}\},\pi\} = \{\{\{\pi,N\},\mu\},\pi\} + \{\{N,\{\pi,\mu\}\},\pi\} = 4\{\pi_N,\{\mu,\pi\}\} = 0 \ ,$$

and we know that the vanishing of $\{\{\pi,\{N,\mu\}\},\pi\}$ is equivalent to $\{\{N,\mu\},\omega\} = 0$.

(iii) To prove (6.3), we use the assumption $\{\{\pi_N,\mu\},\pi_N\} = 0$ to obtain

$$\begin{aligned} 0 &= \{\{\{\pi_N,\mu\},\pi_N\},\omega\} = \{\{\pi_N,\mu\},N\} + \{\{\{\pi_N,\mu\},\omega\},\pi_N\} \\ &= \{\{\pi_N,\mu\},N\} + \{\{N,\mu\},\pi_N\} \ , \end{aligned}$$

thus proving (6.3). □

From (6.1), it follows that

$$\{\pi_N,\mu\} = \frac{1}{2}\{\{\pi,N\},\mu\} = \{\{\pi,\mu\},N\} = \{\{\mu,N\},\pi\} \ , \tag{6.5}$$

and from (6.3), it follows that

$$\frac{1}{2}\{\{\pi_N,N\},\mu\} = -\{\{\mu,\pi_N\},N\} = \{\{\mu,N\},\pi_N\} \ . \tag{6.6}$$

Lemma 6.2 *When $N = \{\sigma,\tau\}$, where σ is a bivector and τ a 2-form, then*

$$N^2 = -\frac{1}{2}\{\{N,\sigma\},\tau\} \ .$$

Proof This formula is proved by a simple calculation. □

The following essential result in the theory of bi-hamiltonian systems was proved by Magri and Morosi in [31] and also by Gelfand and Dorfman [8] in the algebric framework of Hamiltonian pairs and by Fuchssteiner and Fokas [9] in their study of Hamiltonian structures for evolution equations. See [21] for the case of Lie algebroids.

Theorem 6.3 *Let π and π_1 be compatible Poisson structures, with π non-degenerate. Set $N = \pi_1^\sharp \circ (\pi^\sharp)^{-1}$. Then*

(i) the Nijenhuis torsion of the $(1,1)$-tensor N vanishes.

(ii) the pair (π,N) is a PN-structure,

(iii) the pair (π_1,N) is a PN-structure.

Proof (i) Let ω be the inverse of π. Then $N = \{\pi_1,\omega\}$. Applying Lemma 6.2 to $\sigma = \pi_1$ and $\tau = \omega$, we obtain from Formula (2.9):

$$\begin{aligned} 2\,\mathcal{T}_\mu N &= \{N,\{N,\mu\}\} - \{N^2,\mu\} \\ &= \{N,\{N,\mu\}\} + \frac{1}{2}\{\{\{N,\pi_1\},\omega\},\mu\} \ . \end{aligned}$$

Using (6.2) and (6.3) yields

$$\begin{aligned} 2\,\mathcal{T}_\mu N &= \{N,\{N,\mu\}\} + \{\{\{N,\mu\}.\pi_1\},\omega\} \\ &= \{N,\{N,\mu\}\} + \{\{N,\mu\},N\} = 0\ . \end{aligned}$$

(ii) Equation (6.1) expresses the vanishing of $C_\mu(\pi, N)$.

(iii) Equation (6.3) expresses the vanishing of $C_\mu(\pi_1, N)$. □

In [21] and in the references cited above, it is proved more generally that each π_k defined by $\pi_k^\sharp = N^k \circ \pi^\sharp$, $k \in \mathbb{N}$, is a Poisson bivector, and the bivectors π_k are pairwise compatible. The preceding theorem admits a converse.

Theorem 6.4 *If π and π_1 are non-degenerate Poisson bivectors, and if $N = \pi_1^\sharp \circ (\pi^\sharp)^{-1}$ has vanishing Nijenhuis torsion, then π and π_1 are compatible.*

Proof Assume that π is non-degenerate and let ω be its inverse. Since π is a Poisson bivector, $\{\mu,\omega\} = 0$. From this fact, the fact that π_1 is Poisson and the formula $N = \{\pi_1, \omega\}$, we obtain (6.3) which implies

$$\{\{N,\pi_1\},\mu\} = 2\{\{N,\mu\},\pi_1\}\ . \tag{6.7}$$

From Lemma 6.2 applied to $N = \{\pi_1,\omega\}$, we obtain

$$2\mathcal{T}_\mu N = \{N,\{N,\mu\}\} + \frac{1}{2}\{\{\{N,\pi_1\},\omega\},\mu\}\ .$$

Using $\{\mu,\omega\} = 0$ and (6.7), we obtain

$$2\mathcal{T}_\mu N = \{N,\{N,\mu\}\} + \{\{\{N,\mu\},\pi_1\},\omega\}\ ,$$

whence

$$\mathcal{T}_\mu N = \frac{1}{2}\{\{\{N,\mu\},\omega\},\pi_1\}\ . \tag{6.8}$$

We now assume that π_1 also is non-degenerate, and we denote its inverse by ω_1. Then $2\{\mathcal{T}_\mu N, \omega_1\} = \{\{N,\mu\},\omega\}$ and the vanishing of $\mathcal{T}_\mu N$ implies the vanishing of $\{\{N,\mu\},\omega\}$. We then remark that the vanishing of $\{\{N,\mu\},\omega\}$ is equivalent to the vanishing of $\{\{\pi,\{N,\mu\}\},\pi\}$. Since $\{\{\pi_1,\mu\},\pi\} = \frac{1}{4}\{\{\pi,\{N,\mu\}\},\pi\}$, the vanishing of $\mathcal{T}_\mu N$ implies that π and π_1 are compatible. □

Remark 6.5 Let π be a non-degenerate Poisson bivector with inverse ω and let N be a $(1,1)$-tensor. Assume that π_N defined by $(\pi_N)^\sharp = N \circ \pi^\sharp$ is a Poisson bivector. In view of Lemma 2.8 and Lemma 7.1 below, Formula (6.8) means that

$$(\mathcal{T}_\mu N)(X,Y) = \frac{1}{2}(\pi_N)^\sharp(i_{X\wedge Y}(d_N\omega))\ ,$$

for all X and $Y \in \Gamma A$.

In the next sections we shall review and compare the $P\Omega$- and ΩN-structures of Magri and Morosi [31] and the Hitchin pairs of Crainic [6].

7 What is a $P\Omega$-structure on a Lie algebroid?

In [31], Magri and Morosi defined the $P\Omega$- and ΩN-structures on manifolds and, more recently, in his study of generalized complex structures, Crainic defined Hitchin pairs on manifolds [6]. These notions admit straightforward generalizations to the case of Lie algebroids which we now define and study. When the Lie algebroid is TM with its standard Lie algebroid structure, these definitions recover the classical case. Most of the results in this section are particular cases of the general theorem of Antunes[1] on Poisson quasi-Nijenhuis structures with background, see [1], Theorem 4.1.

If N is a $(1,1)$-tensor on a Lie algebroid (A,μ), let $\mu_N = \{N,\mu\}$ be the deformed bracket satisfying (2.7). Define the operator on forms $i_N = \{N,\cdot\}$ and let d_N be the operator considered in Remark 5.5,

$$d_N = [i_N, d_\mu] \ ,$$

where $[\ ,\ ]$ is the graded commutator. In particular, if a form α is d_μ-closed, then $d_\mu(\alpha_N) = -d_N\alpha$, where $\alpha_N = i_N\alpha$. The following simple lemma was proved in [21]. We present an alternative proof.

Lemma 7.1 *Let N be a $(1,1)$-tensor on (A,μ). The operators on forms $d_N = [i_N, d_\mu]$ and $d_{\mu_N} = \{\mu_N,\cdot\}$ coincide.*

Proof For any form α, $d_{\mu_N}\alpha = \{\mu_N,\alpha\} = \{\{N,\mu\},\alpha\} = \{N,\{\mu,\alpha\}\} - \{\mu,\{N,\alpha\}\}$, while $d_N\alpha = [i_N,d_\mu](\alpha) = i_N\{\mu,\alpha\} - \{\mu, i_N\alpha\} = \{N,\{\mu,\alpha\}\} - \{\mu,\{N,\alpha\}\}$. ☐

Let N be a $(1,1)$-tensor and ω a 2-form on (A,μ) such that

$$\omega^\flat \circ N = N^* \circ \omega^\flat \ . \tag{7.1}$$

Then ω_N defined by $\omega_N^\flat = \omega^\flat \circ N$ is a 2-form and

$$\omega_N = \frac{1}{2} i_N\omega = \frac{1}{2}\{N,\omega\} \ .$$

Let π be a bivector and let ω be a 2-form on the Lie algebroid (A,μ). Set $N = \pi^\sharp \circ \omega^\flat$. Then (7.1) is satisfied and

$$N = \{\pi,\omega\} \ .$$

We shall now prove identities relating π, ω and N when $N = \{\pi,\omega\}$.

Lemma 7.2 *Let π be a bivector and ω a 2-form on (A,μ), and let N be the $(1,1)$-tensor $N = \pi^\sharp \circ \omega^\flat$. The 2-form ω satisfies*

$$\frac{1}{2}[\omega,\omega]_\pi + d_N\omega = -\frac{1}{2} d_\mu(\omega_N) \ . \tag{7.2}$$

[1] The convention for the definition of $\omega^\flat$ in [1] is the opposite of ours.

Proof In fact, by the Jacobi identity,

$$\begin{aligned}[\omega,\omega]_\pi &= \{\{\omega,\{\pi,\mu\}\},\omega\} = \{\{\{\omega,\pi\},\mu\},\omega\} + \{\{\pi,\{\omega,\mu\}\},\omega\} \\ &= \{\{\omega,\pi\},\{\mu,\omega\}\} + \{\{\{\omega,\pi\},\omega\},\mu\} + \{\{\pi,\omega\},\{\omega,\mu\}\} \\ &= 2\{\{\omega,\pi\},\{\mu,\omega\}\} - \{\{\{\pi,\omega\},\omega\},\mu\}\ .\end{aligned}$$

Since $N = \{\pi,\omega\}$ and $d_\mu\omega = \{\mu,\omega\}$, we obtain $[\omega,\omega]_\pi = -2\{N,\{\mu,\omega\}\}+\{\mu,\{N,\omega\}\}$, hence (7.2). □

Remark 7.3 When (π, N) is a PN-structure, $(\Gamma(\wedge^\bullet(A^*)), [\ ,\ ]_\pi, d_N)$ is a differential graded Lie algebra, and this formula expresses the fact that 2-forms such that $i_N\omega$ is d_μ-closed are Maurer-Cartan elements in this DGLA.

We shall prove two additional identities relating π, ω and N. We recall that $[\pi,\pi]_\mu = \{\{\pi,\mu\},\pi\}$ and that $C_\mu(\pi,N)$ is defined by Formula (5.2).

Lemma 7.4 *A bivector π, a 2-form ω and a $(1,1)$-tensor N related by $N = \pi^\sharp \circ \omega^\flat$ satisfy the relations*

$$\{[\pi,\pi]_\mu,\omega\} = \{\{\pi,d_\mu\omega\},\pi\} + C_\mu(\pi,N)\ , \tag{7.3}$$

and

$$\{\{[\pi,\pi]_\mu,\omega\},\omega\} = \{\{\pi,d_\mu\omega\},\pi\},\omega\} - \{\{\pi,N\},d_\mu\omega\} + 2\{\pi,d_N\omega\} + 4\mathcal{T}_\mu N\ . \tag{7.4}$$

Proof Applying the Jacobi identity, we obtain

$$\begin{aligned}\{\{\{\pi,\mu\},\pi\},\omega\} &= \{\{\{\pi,\mu\},\omega\},\pi\} + \{\{\pi,\mu\},\{\pi,\omega\}\} \\ &= \{\{\pi,\{\mu,\omega\}\},\pi\} + \{\{\{\pi,\omega\},\mu\},\pi\} + \{\{\pi,\mu\},\{\pi,\omega\}\}\ .\end{aligned}$$

Therefore, for $N = \{\pi,\omega\}$, we obtain (7.3). Furthermore,

$$\begin{aligned}&\{C_\mu(\pi,N),\omega\} = \{\omega,\{\{N,\mu\},\pi\}\} + \{\omega,\{\{\pi,\mu\},N\}\} \\ &= \{\{N,\mu\},\{\omega,\pi\}\} + \{\pi,\{\{N,\mu\},\omega\}\} + \{\omega,\{\pi,\{\mu,N\}\}\} + \{\omega,\{\{\pi,N\},\mu\}\} \\ &= -\{\{N,\mu\},N\} + \{\pi,d_N\omega\} + \{\{\omega,\pi\},\{\mu,N\}\}\} \\ &\quad + \{\{\{\mu,N\},\omega\},\pi\} + \{\{\omega,\{\pi,N\}\},\mu\} + \{\{\pi,N\},\{\omega,\mu\}\} \\ &= 2\{N,\{N,\mu\}\} + 2\{\pi,d_N\omega\} - \{\mu,\{\{N,\pi\},\omega\}\} + \{\{N,\pi\},d_\mu\omega\}\ .\end{aligned}$$

We shall now make use of Formula (2.9) and Lemma 6.2 which imply

$$4\mathcal{T}_\mu N = 2\{N,\{N,\mu\}\} - \{\{\{N,\pi\},\omega\},\mu\}\ .$$

Thus

$$\{C_\mu(\pi,N),\omega\} = \{\{N,\pi\},d_\mu\omega\} + 2\{\pi,d_N\omega\} + 4\mathcal{T}_\mu N\ .$$

Whence the relation (7.4). □

Definition 7.5 *A bivector π and a 2-form ω on a Lie algebroid (A,μ) define a $P\Omega$-structure if π is a Poisson bivector, ω is d_μ-closed, and $d_\mu(\omega_N)=0$, where $N=\pi^\sharp\circ\omega^\flat$ and ω_N is the 2-form such that $\omega_N^\flat=\omega^\flat\circ N$.*

Since $d_\mu\omega=0$, the condition $d_\mu(\omega_N)=0$ in the definition of a $P\Omega$-structure is equivalent to the condition $d_N\omega=0$, so that $P\Omega$-structures can be characterized as follows.

Proposition 7.6 *A Poisson bivector π and a d_μ-closed 2-form ω on (A,μ) define a $P\Omega$-structure if and only if $d_N\omega=0$, where $N=\pi^\sharp\circ\omega^\flat$.*

Since $N=\{\pi,\omega\}$ and $\omega_N=\frac{1}{2}i_N\omega=\frac{1}{2}\{N,\omega\}$, in terms of the big bracket, the conditions in the definition of $P\Omega$ structure are

$$\{\{\pi,\mu\},\pi\}=0,\qquad \{\mu,\omega\}=0,\qquad \text{and}\qquad \{\mu,\{N,\omega\}\}=0\ .$$

We can relate the notion of $P\Omega$ structure to that of complementary 2-form.

Theorem 7.7 *A Poisson bivector π and a 2-form ω on (A,μ) define a $P\Omega$-structure if and only if ω is a d_μ-closed complementary 2-form for π.*

Proof In fact, by the Jacobi identity,

$$[\omega,\omega]_\pi=\{\{\omega,\{\pi,\mu\}\},\omega\}=\{\{\{\omega,\pi\},\mu\},\omega\}+\{\{\{\omega,\mu\},\pi\},\omega\}$$

$$=-\{\{N,\mu\},\omega\}+\{\{\pi,d_\mu\omega\},\omega\}=-d_N\omega+\{N,d_\mu\omega\}\ .$$

Thus, if ω is both d_μ- and d_N-closed, it is a complementary 2-form. Conversely is ω is a d_μ-closed complementary 2-form, then $d_N\omega=0$. $\square$

The following theorem generalizes a result of [31].

Theorem 7.8 *If a Poisson bivector π and a d_μ-closed 2-form ω on (A,μ) define a $P\Omega$-structure, then the pair (π,N), where $N=\pi^\sharp\circ\omega^\flat$, is a PN-structure. Conversely, if (π,N) is a PN-structure and π is non-degenerate, then (π,ω), where $\omega^\flat=(\pi^\sharp)^{-1}\circ N$ is a $P\Omega$-structure.*

Proof If (π,ω) is a $P\Omega$-structure, Equation (5.1) is obviously satisfied. It follows from (7.3) that, when π is a Poisson bivector and ω is a d_μ-closed 2-form, π and $N=\{\pi,\omega\}$ are compatible. It follows from (7.4) that, if in addition $d_N\omega=0$, then $\mathcal{T}_\mu N=0$. Therefore (π,N) is a PN-structure.

Conversely, it is clear that (5.1) implies that ω is a 2-form. Assume that π is non-degenerate and let τ be the 2-form inverse of π. Then $\{\{\pi,\tau\},\cdot\}=\mathbf{I}$. Applying $\{\tau,\cdot\}$ to both sides of equation (7.3) yields

$$\{\{[\pi,\pi]_\mu,\omega\},\tau\}=\{\{\{\pi,d_\mu\omega\},\pi\},\tau\}+\{C_\mu(\pi,N),\tau\}\ .$$

Now, by (2.4),

$$\{\{\{\pi, d_\mu\omega\}, \pi\}, \tau\} = \{\{\pi, d_\mu\omega\}, \{\pi, \tau\}\} + \{\{\{\pi, d_\mu\omega\}, \tau\}, \pi\}\} = 4\{d_\mu\omega, \pi\} .$$

Applying $\{\tau, \cdot\}$ once more yields $d_\mu\omega = 0$. Therefore, if π is a non-degenerate Poisson bivector and π and N are compatible, then ω is d_μ-closed. Applying $\{\tau, \cdot\}$ to both sides of equation (7.4) then yields $d_N\omega = 0$ when $[\pi, \pi]_\mu$, $C_\mu(\pi, N)$ and $\mathcal{T}_\mu N$ all vanish. □

Corollary 7.9 *If π is a Poisson bivector and ω is a closed complementary 2-form for π, then the Nijenhuis torsion $\mathcal{T}_\mu(N)$ of $N = \pi^\sharp \circ \omega^\flat$ vanishes.*

Proof This corollary follows from Theorems 7.7 and 7.8. □

8 What is an ΩN-structure on a Lie algebroid?

Definition 8.1 *A 2-form ω and a $(1,1)$-tensor N on a Lie algebroid (A, μ) define an ΩN-structure if $\omega^\flat \circ N = N^* \circ \omega^\flat$, ω is d_μ-closed, N is a Nijenhuis tensor, and $d_\mu(\omega_N) = 0$, where $\omega_N^\flat = \omega^\flat \circ N$.*

Since $\omega_N = \frac{1}{2}\{N, \omega\}$, the conditions in the definition of an ΩN structure, in addition to $\omega^\flat \circ N = N^* \circ \omega^\flat$, are, in terms of the big bracket,

$$\{\mu, \omega\} = 0, \qquad \{N, \{N, \mu\}\} - \{N^2, \mu\} = 0, \qquad \text{and} \qquad \{\mu, \{N, \omega\}\} = 0 .$$

Theorem 8.2 *If (ω, N) is an ΩN-structure and ω is non-degenerate, then (π, N), where $\pi^\sharp = N \circ (\omega^\flat)^{-1}$, is a PN-structure. Conversely, if (π, N) is a PN-structure and π is non-degenerate, then (ω, N), where $\omega^\flat = (\pi^\sharp)^{-1} \circ N$, is an ΩN-structure.*

Proof If (ω, N) is an ΩN-structure, we conclude from (7.4) that $\{[\pi, \pi]_\mu, \omega\}, \omega\} = 0$. If ω is non-degenerate with inverse σ, applying $\{\sigma, \cdot\}$ twice yields $[\pi, \pi]_\mu = 0$. From (7.3) we then see that π and N are compatible, therefore (π, N) is a PN-structure.

Conversely, if (π, N) is a PN-structure, from (7.3) we obtain $\{\{\pi, d_\mu\omega\}, \pi\} = 0$. If π is non-degenerate with inverse τ, applying $\{\tau, \cdot\}$ twice yields $d_\mu\omega = 0$. Then (7.4) yields $\{\pi, d_N\omega\} = 0$. Applying $\{\tau, \cdot\}$ yields $d_N\omega = 0$, whence also $d(\omega_N) = 0$. □

Theorem 8.3 *If (π, ω) is a $P\Omega$-structure, then (ω, N), where $N = \pi^\sharp \circ \omega^\flat$, is an ΩN-structure. Conversely, if (ω, N) is an ΩN-structure and ω is non-degenerate, then (π, ω), where $\pi^\sharp = N \circ (\omega^\flat)^{-1}$, is a $P\Omega$-structure.*

Proof If (π, ω) is a $P\Omega$-structure, we conclude from (7.4) that N is a Nijenhuis tensor and therefore (ω, N) is an ΩN-structure.

Conversely, if (ω, N) is an ΩN-structure, from (7.3), we obtain $\{[\pi, \pi]_\mu, \omega\}, \omega\} = 0$. If ω is non-degenerate with inverse σ, applying $\{\sigma, \cdot\}$ twice yields $[\pi, \pi]_\mu = 0$, and therefore (π, ω) is a $P\Omega$-structure. □

We can relate the notion of ΩN-structure to that of a complementary 2-form. The following proposition is a consequence of Theorems 7.7 and 8.3.

Proposition 8.4 *If ω is a d_μ-closed complementary form for a Poisson bivector π, then (ω, N), where $N = \pi^\sharp \circ \omega^\flat$, is an ΩN-structure. Conversely, if (ω, N) is an ΩN-structure and ω is non-degenerate, then π such that $\pi^\sharp = N \circ (\omega^\flat)^{-1}$ is a Poisson bivector and ω is a d_μ-closed complementary 2-form for π.*

Remark 8.5 Magri and Morosi [31] introduced the ΩN-structures as follows: an ΩN-structure on a manifold M is a pair (ω, N), where ω is a closed 2-form and N is a Nijenhuis tensor such that $\omega^\flat \circ N = N^* \circ \omega^\flat$ and $S(\omega, N) = 0$, where $S(\omega, N)$ is the section of $\wedge^2(T^*M) \otimes T^*M$ such that, for X and $Y \in \Gamma(TM)$, $S(\omega, N)(X, Y) = (\mathcal{L}_{NY}\omega)^\flat(X) - (\mathcal{L}_{NX}\omega)^\flat(Y) - (\omega^\flat \circ N)[X, Y] + d < (\omega^\flat \circ N)(Y), X >$. They then proved that for any 2-form ω and $(1,1)$-tensor N on M, for all vector fields X, Y, Z,

$$S(\omega, N)(X, Y, Z) = d\omega(NX, Y, Z) - d\omega(X, NY, Z) + d(\omega_N)(X, Y.Z) \ ,$$

with $\omega_N^\flat = \omega^\flat \circ N$. Using this formula, they proved that if (π, ω) is a $P\Omega$-structure on M, then (ω, N), where $N = \pi^\sharp \circ \omega^\flat$, is an ΩN-structure. The preceding formula remains valid in the case of a Lie algebroid when d is replaced by d_μ. Therefore Definition 8.1 agrees with the original definition of Magri and Morosi.

9 What is a Hitchin pair on a Lie algebroid?

Recall that a d_μ-closed non-degenerate 2-form is called *symplectic*. The following definition is due to Crainic [6].

Definition 9.1 *A symplectic form ω and a $(1,1)$-tensor N define a Hitchin pair on a Lie algebroid (A, μ) if $\omega^\flat \circ N = N^* \circ \omega^\flat$ and $d_\mu(\omega_N) = 0$, where $\omega_N^\flat = \omega^\flat \circ N$.*

It follows that an ΩN-structure (ω, N) when ω is non-degenerate is a Hitchin pair. Conversely a Hitchin pair (ω, N) is an ΩN-structure if and only the Nijenhuis torsion of N vanishes.

The 2-form $\lambda = -(\omega + (\omega_N)_N)$, where $(\omega_N)_N = \frac{1}{4} i_N(i_N\omega) = \frac{1}{4}\{N, \{N, \omega\}\}$, is called the *twist* of the Hitchin pair.

Lemma 9.2 *Let (ω, N) be a Hitchin pair. The twist 2-form λ satisfies the relation,*

$$\mathcal{T}_\mu N = \{\sigma, d_\mu\lambda\} \ , \tag{9.1}$$

where σ is the inverse of ω.

Proof Since $N = \sigma^\sharp \circ (\omega_N)^\flat$, we can apply Formula (7.4) to σ and ω_N. Since $d_\mu\omega = 0$ and therefore $[\sigma, \sigma]_\mu = 0$, and since $d_\mu(\omega_N) = 0$, Formula (7.4) reduces to

$$\mathcal{T}_\mu(N) = \frac{1}{2}\{d_N\omega_N, \sigma\} \ .$$

Since $d_\mu(\omega_N) = 0$, $d_N(\omega_N) = -d_\mu(i_N\omega_N) = -2d_\mu((\omega_N)_N) = -\frac{1}{2}d_\mu(i_N(i_N\omega))$, and we obtain (9.1) without further computations. $\square$

In view of Lemma 2.8, this result agrees with the computation in [6] (but there is a misprint in the definition of the twist in the unpublished preprint [6]).

When (ω, N) is a Hitchin pair on a Lie algebroid A satisfying the algebraic conditions, $N^2 - \sigma^\sharp \circ \lambda^\flat = -\mathrm{Id}_A$, where λ is the twist 2-form, then $\mathcal{N} = \sigma + N + \lambda$ is a generalized complex structure (see Section 11) on $A \oplus A^*$, *i.e.*, on M if $A = TM$. In matrix form,

$$\mathcal{N} = \begin{pmatrix} N & \sigma \\ \lambda & -N^* \end{pmatrix} .$$

Then $(\sigma, N, -d_\mu\lambda)$ is a Poisson quasi-Nijenhuis structure [36] [1]. In this case, we obtain an alternate proof of (9.1)

The table in the next section summarizes the main definitions and implications of the preceding sections.

In the diagram that summarizes the relationships between PN- , $P\Omega$- and ΩN-structures, the arrows denote implications, and the dotted arrows denote implications under a non-degeneracy assumption.

10 Summary

10.1 Definitions

PN $(NP = PN^*)$

$$\boxed{\{\{\pi,\mu\},\pi\} = 0, \quad \{\{\pi,\mu\},N\} + \{\{N,\mu\},\pi\} = 0, \quad \{N,\{N,\mu\}\} - \{N^2,\mu\} = 0}$$

$P\Omega$

$$\boxed{\{\{\pi,\mu\},\pi\} = 0, \quad \{\mu,\omega\} = 0, \quad \{\{\{\pi,\omega\},\mu\},\omega\} = 0}$$

ΩN $(\omega^\flat \circ N = N^* \circ \omega^\flat)$

$$\boxed{\{\mu,\omega\} = 0, \quad \{N,\{N,\mu\}\} - \{N^2,\mu\} = 0, \quad \{\mu,\{N,\omega\}\} = 0}$$

Hitchin pair $(\omega^\flat \circ N = N^* \circ \omega^\flat)$

$$\boxed{\{\mu,\omega\} = 0, \quad \{\mu,\{N,\omega\}\} = 0}$$

Complementary 2-form

$$\boxed{\{\{\pi,\mu\},\pi\} = 0, \quad \{\{\omega,\{\pi,\mu\}\},\omega\} = 0}$$

10.2 Relationships

$$\boxed{P\Omega \Longrightarrow PN \quad (N = \pi \circ \omega)}$$

$$\boxed{PN \quad and \quad \pi \quad non-degenerate \Longrightarrow P\Omega \quad (\omega = \pi^{-1} \circ N)}$$

$$\boxed{\Omega N \quad and \quad \omega \quad non-degenerate \Longrightarrow PN \quad (\pi = N \circ \omega^{-1})}$$

$$\boxed{PN \quad and \quad \pi \quad non-degenerate \Longrightarrow \Omega N \quad (\omega = \pi^{-1} \circ N)}$$

$$\boxed{P\Omega \Longrightarrow \Omega N \quad (N = \pi \circ \omega)}$$

$$\boxed{\Omega N \quad and \quad \omega \quad non-degenerate \Longrightarrow P\Omega \quad (\pi = N \circ \omega^{-1})}$$

$$\boxed{Hitchin \quad pair \quad and \quad N \quad Nijenhuis \Longleftrightarrow \Omega N \quad and \quad \omega \quad non-degenerate}$$

$$\boxed{\omega \quad closed \quad complementary \quad 2-form \quad for \quad \pi \Longleftrightarrow P\Omega}$$

$$\begin{tikzcd}
 & P\Omega \arrow[dl] \arrow[dr] & \\
PN \arrow[ur, dotted, shift left] \arrow[rr, dotted, shift left] & & \Omega N \arrow[ul, dotted, shift right] \arrow[ll, dotted, shift left]
\end{tikzcd}$$

11 Nijenhuis tensors on Courant algebroids

11.1 Courant algebroids

The double $E = A \oplus A^*$ of any proto-bialgebroid (A, A^*) is a Courant algebroid when equipped with the Dorfman bracket (see Section 1.2) which is the derived bracket defined by

$$[u, v]_S = \{\{u, S\}, v\} \ ,$$

for $u, v \in \Gamma(A \oplus A^*)$. Here S denotes the structure $\phi + \mu + \gamma + \psi$. When (A, A^*) is a Lie bialgebroid, then $S = \mu + \gamma$ and $[\ ,\ ]_S$ is the bracket that we have denoted by $[\ ,\ ]_D$ in Section 1.2. The usual case is $A = TM$ with $\gamma = 0$. (See [12].)

The skew-symmetrization of the Dorfman bracket is the Courant bracket, $[u, v]'_S = \frac{1}{2}([u, v]_S - [v, u]_S)$.

11.2 Nijenhuis tensors and generalized complex structures

For an endomorphism $\mathcal{N}$ of E, the Dorfman torsion $\mathcal{T}_S\mathcal{N}$ (resp., the Courant torsion $\mathcal{T}'_S\mathcal{N}$) of $\mathcal{N}$ is defined by

$$(\mathcal{T}_S\mathcal{N})(u, v) = [\mathcal{N}u, \mathcal{N}v]_S - \mathcal{N}([\mathcal{N}u, v]_S + [u, \mathcal{N}v]_S) + \mathcal{N}^2[u, v]_S$$

(resp., $(\mathcal{T}'_S\mathcal{N})(u,v) = [\mathcal{N}u, \mathcal{N}v]'_S - \mathcal{N}([\mathcal{N}u, v]'_S + [u, \mathcal{N}v]'_S) + \mathcal{N}^2[u,v]'_S)$.

A *generalized almost complex structure* is an endomorphism $\mathcal{N}$ of E which is orthogonal with respect to the symmetric bilinear form, and such that $\mathcal{N}^2 = -\mathrm{Id}_E$. Note that if $\mathcal{N}^2 = -\lambda \mathrm{Id}_E$, with $\lambda \neq 0$, then $\mathcal{N}\mathcal{N}^* = \mathrm{Id}_E$ is equivalent to $\mathcal{N} + \lambda\mathcal{N}^* = 0$. In particular, the orthogonality condition for a generalized almost complex structure is equivalent to $\mathcal{N} + \mathcal{N}^* = 0$.

A generalized almost complex structure $\mathcal{N}$ is a *generalized complex structure* if $\mathcal{T}_S\mathcal{N} = 0$.

Lemma 11.1 *For a generalized almost complex structure $\mathcal{N}$, $\mathcal{T}_S\mathcal{N} = \mathcal{T}'_S\mathcal{N}$.*

Proof Since $\mathcal{N}$ is orthogonal, the equality follows from the relation $[u,v]_S + [v,u]_S = \mathcal{D}(< u, v >)$, where $(\mathcal{D}u)(f) =< \rho u, df >$. □

Thus the integrability condition for an almost complex structure can be expressed either in terms of non-skew symmetric brackets or in terms of skew-symmetric brackets.

As observed by Grabowski in [10], Formula (2.9) is valid when μ denotes the cubic homological function defining the Courant algebroid structure, denoted by S above. Therefore the following theorem is proved.

Theorem 11.2 *The generalized complex structures are the generalized almost complex structures such that*

$$\{\{\mathcal{N}, S\}, \mathcal{N}\} = S \ .$$

The preceding remarks also apply to almost product and almost subtangent structures. They are characterized by $\{\{\mathcal{N}, S\}, \mathcal{N}\} = -S$ and $\{\{\mathcal{N}, S\}, \mathcal{N}\} = 0$, respectively.

When the torsion of $\mathcal{N}$ vanishes, $\mathcal{N}$ defines a *deformed structure* $S_{\mathcal{N}} = \{\mathcal{N}, S\}$ on A, and $\mathcal{N}$ maps bracket $[\ ,\ ]_{S_{\mathcal{N}}}$ to bracket $[\ ,\ ]_S$. A necessary and sufficient condition for $S_{\mathcal{N}} = \{\mathcal{N}, S\}$ to be a Dorfman bracket is $\{S, \{\{\mathcal{N}, S\}, \mathcal{N}\}\} = 0$. In fact, Formula (2.10) extends to the Courant algebroid case,

$$\frac{1}{2}\{S_{\mathcal{N}}, S_{\mathcal{N}}\} = \frac{1}{2}\{\{\mathcal{N}, S\}, \{\mathcal{N}, S\}\} = \{S, \mathcal{T}_S\mathcal{N}\} \ .$$

The vanishing of $\mathcal{T}_S(\mathcal{N})$ expresses the fact that $\mathcal{N} : (E, S_{\mathcal{N}}) \to (E, S)$ preserves the brackets.

In the next sections we shall study Monge-Ampère structures as examples of compatible structures and generalized geometries.

12 Monge-Ampère structures on manifolds

The notion of Monge-Ampère structure has its origin in the theory of symplectic Monge-Ampère operators and equations. See [25] for a detailed analysis of symplectic and contact Monge-Ampère operators and equations, together with many examples. Let M be a smooth manifold of dimension n and let T^*M be its cotangent bundle. We shall denote the space of k-forms on T^*M by $\Omega^k(T^*M)$ and the space of vector fields by $\mathcal{X}(T^*M)$. Let $\mathcal{F}(T^*M)$ be the space of functions on the supermanifold $T^*[2](T(T^*M))[1]$.

We shall denote the canonical symplectic 2-form on T^*M by Ω, and its inverse, the canonical bivector, by π_Ω. More generally, we shall denote by π_τ the bivector on T^*M that is the inverse of a non-degenerate 2-form τ on T^*M.

Definition 12.1 *The pair (Ω, ω) is a* Monge-Ampère structure *on M if ω is an n-form on T^*M satisfying the condition $\omega \wedge \Omega = 0$.*

According to the original ideas of Lychagin [29] [25], any symplectic Monge-Ampère operator on M can be defined by an *effective form* on T^*M of degree k, $2 \le k \le n$, *i.e.*, a k-form ω such that $i_{\pi_\Omega}\omega = 0$. When $k = n$, this condition is equivalent to the condition $\omega \wedge \Omega = 0$, so (Ω, ω) is a Monge-Ampère structure if and only if ω is an effective n-form.

A correspondence between forms on T^*M and form-valued differential operators on M is defined as follows. Let k be a positive integer, $k \le n$. For any k-form ω on T^*M, define the *Monge-Ampère operator*, $\Delta_\omega : C^\infty(M) \to \Omega^k(T^*M)$, by

$$\Delta_\omega(f) = (df)^*(\omega) \ , \tag{12.1}$$

for any $f \in C^\infty(M)$. We understand the differential df to be a map, $df : M \to T^*M$, the section of the cotangent bundle defined by the smooth function f. The equation $\Delta_\omega(f) = 0$ is called a *Monge-Ampère equation.*

13 Monge-Ampère structures in dimension 2 and compatible structures

13.1 Monge-Ampère structures in dimension 2

We first consider the simplest geometric examples, those of Monge-Ampère structures in dimension $n = 2$. In this case, $\dim(T^*M) = 4$, and a Monge-Ampère structure is defined by a pair of 2-forms (Ω, ω) on T^*M, where Ω is the canonical 2-form and ω satisfies the effectivity condition, $\omega \wedge \Omega = 0$, or equivalently, $i_{\pi_\Omega}\omega = 0$. We consider the $(1,1)$-tensor on T^*M, $A_\omega = (\pi_\Omega)^\sharp \circ \omega^\flat$, which satisfies, for all $X, Y \in \mathcal{X}(T^*M)$,

$$\omega(X, Y) = \Omega(A_\omega X, Y) \ .$$

It is easy to prove that A_ω satisfies the equation $A_\omega^2 + \mathrm{Pf}(\omega)\mathrm{Id} = 0$, where the Pfaffian, $\mathrm{Pf}(\omega)$, of the 2-form ω is defined by

$$\mathrm{Pf}(\omega)\Omega \wedge \Omega = \omega \wedge \omega \ ,$$

and Id is the identity of $T(T^*M)$.

13.2 Non-degenerate Monge-Ampère structures in dimension 2

By definition, a Monge-Ampère structure (Ω, ω) on $T^*(\mathbb{R}^2)$ is called *non-degenerate* if its Pfaffian $\mathrm{Pf}(\omega)$ is nowhere-vanishing.

When (Ω, ω) is a non-degenerate Monge-Ampère structure, we consider the normalized 2-form $\widetilde{\omega} = \frac{\omega}{\sqrt{|\mathrm{Pf}(\omega)|}}$, with inverse bivector $\pi_{\widetilde{\omega}} = \sqrt{|\mathrm{Pf}(\omega)|}\pi_\omega$. The normalized $(1,1)$-tensor J_ω is defined by $J_\omega = (\pi_\Omega)^\sharp \circ \widetilde{\omega}^\flat$, and it satisfies

$$J_\omega = \frac{A_\omega}{\sqrt{|\mathrm{Pf}(\omega)|}} \ .$$

Then $J_\omega^2 = -\mathrm{Id}$ or $J_\omega^2 = \mathrm{Id}$. The sign of $\mathrm{Pf}(\omega)$ determines whether the corresponding Monge-Ampère operator is elliptic (when $\mathrm{Pf}(\omega) > 0$ and therefore $J_\omega^2 = -\mathrm{Id}$) or hyperbolic (when $\mathrm{Pf}(\omega) < 0$ and therefore $J_\omega^2 = \mathrm{Id}$).

It is proved in [25] that the integrability of J_ω, *i.e.* the condition $\mathcal{T}_\mu(J_\omega) = 0$, where $\mu \in \mathcal{F}^{0,1}(T^*M)$ defines the standard Lie algebroid structure of $T(T^*M)$, is equivalent to the condition that the normalized 2-form $\widetilde{\omega}$ be closed. This integrability condition is also equivalent to the existence of a symplectomorphism mapping the 2-form ω to a form with constant coefficients. The corresponding Monge-Ampère operator Δ_ω is then equivalent to an operator with constant coefficients.

13.3 Properties of non-degenerate Monge-Ampère structures in dimension 2

We show that in the case of dimension 2, non-degenerate Monge-Ampère structures give rise to composite structures.

- By Theorem 6.4 of Section 6, if (Ω, ω) is a non-degenerate Monge-Ampère structure on M satisfying the condition $d\widetilde{\omega} = 0$, then the pairs (π_Ω, J_ω) and $(\pi_{\widetilde{\omega}}, J_\omega)$ are PN-structures on T^*M, *i.e.*, on the Lie algebroid $T(T^*M)$.
- Theorem 7.8 in Section 7 implies that, when $d\widetilde{\omega} = 0$, the pair $(\pi_\Omega, \widetilde{\omega})$ is a $P\Omega$-structure and the pair $(\widetilde{\omega}, J_\omega)$ is an ΩN-structure on T^*M.
- Let μ_{J_ω} be the element of $\mathcal{F}^{0,1}(T^*M)$ defined by

$$\mu_{J_\omega} = \{J_\omega, \mu\} \ ,$$

where as above μ is the Lie algebroid structure of $T(T^*M)$. When J_ω is integrable, μ_{J_ω} defines a new Lie algebroid structure on $T(T^*M)$ deformed by J_ω. By Remark 2.6 of Section 2.3, this structure is compatible with the standard structure,

$$\{\mu + \mu_{J_\omega}, \mu + \mu_{J_\omega}\} = 0 \ .$$

The observation in Section 4.4 can be applied to the modular class of this deformed Lie algebroid structure. Assume that $\widetilde{\omega}$ is closed. Then the deformed structure μ_{J_ω} is equal to $\{\{\pi_\Omega, \mu\}, \widetilde{\omega}\}$. In fact, by the Jacobi identity, since $\{\mu, \widetilde{\omega}\} = 0$,

$$\mu_{J_\omega} = \{J_\omega, \mu\} = \{\{\pi_\Omega, \widetilde{\omega}\}, \mu\} = \{\{\pi_\Omega, \mu\}, \widetilde{\omega}\} \ .$$

Hence, the 1-form on T^*M, $\xi_{\pi_\Omega, \widetilde{\omega}, \lambda}$, defined by the Liouville volume form $\lambda = \frac{1}{2}\Omega \wedge \Omega$, satisfying

$$\partial^{\pi_\Omega}_{\widetilde{\omega},\lambda} - \partial^{\pi_\Omega}_{\widetilde{\omega}} = i_{\xi_{\pi_\Omega, \widetilde{\omega}, \lambda}}$$

is a d_{J_ω}-cocycle whose cohomology class is the modular class of the Lie algebroid $(T(T^*M), \mu_{J_\omega})$ defined by the Nijenhuis operator J_ω.

Proposition 13.1 *The deformed Lie algebroid $(T(T^*M), \mu_{J_\omega})$ which is obtained from a Monge-Ampère structure such that $d\widetilde{\omega} = 0$ is unimodular.*

Proof Since J_ω is a Nijenhuis tensor, the modular class in the d_{J_ω}-cohomology of the Lie algebroid $(T(T^*M), \mu_{J_\omega})$ is the class of the 1-form $d(\mathrm{Tr} J_\omega)$ (see [22] [7]). Since the form ω is effective, the $(1,1)$-tensor A_ω, and hence J_ω, are traceless. □

• Any non-degenerate Monge-Ampère structure (Ω, ω) on M such that ω is closed defines a Hitchin pair (Ω, A_ω) in the sense of Crainic [6] on T^*M. If, in particular, this structure is defined by a non-degenerate Monge-Ampère operator with constant coefficients, the $(1,1)$-tensor A_ω is integrable and (Ω, A_ω) is an ΩN-structure on T^*M.

• Monge-Ampère structures of divergence type were defined in [25]. A pair (Ω, ω), where ω is a 2-form, is called a structure of divergence type if there exists a function ϕ on T^*M such that $\omega + \phi\Omega$ is closed. Following [2], we observe that a non-degenerate structure (Ω, ω) of divergence type, where ω is not necessarily effective, defines a generalized almost complex structure $\mathcal{J}_\omega = \begin{pmatrix} A_\omega & \pi^\sharp_\Omega \\ -\Omega^\flat(\mathrm{Id} + A^2_\omega) & -A^*_\omega \end{pmatrix}$ on T^*M. The pair (Ω, A_ω) is a Hitchin pair if and only if $\mathcal{J}_\omega$ is integrable if and only if ω is closed.

If, in addition, the Monge-Ampère structure (Ω, ω) satisfies the condition $d\widetilde{\omega} = 0$, we obtain another generalized complex structures on T^*M, $\mathbb{J}_\omega = \begin{pmatrix} J_\omega & \pi^\sharp_\Omega \\ 0 & -J^*_\omega \end{pmatrix}$ if Δ_ω is equivalent to an elliptic Monge-Ampère operator with constant coefficients, and $\mathbb{J}'_\omega = \begin{pmatrix} J_\omega & \pi^\sharp_\Omega \\ -2\Omega^\flat & -J^*_\omega \end{pmatrix}$ which corresponds to a hyperbolic Monge-Ampère operator.

The tensor $\mathbb{J}_\omega$ can be written as $\mathbb{J}_\omega = \pi_\Omega + J_\omega$ since

$$\mathbb{J}_\omega(u) = \{u, \pi_\Omega + J_\omega\} \ ,$$

for all $u \in \mathcal{F}(T^*(M)$, and similarly for $\mathcal{J}_\omega$ and $\mathbb{J}'_\omega$.

• The deformed Lie bialgebroid structure on $(T(T^*M), T^*(T^*M))$ defined by $\mathbb{J}_\omega$ induces a new Courant algebroid structure on $T(T^*M) \oplus T^*(T^*M)$ which we shall call a *Monge-Ampère Courant algebroid.* This structure is defined by the function $S_{\mathbb{J}_\omega} = \{\mathbb{J}_\omega, S\} \in \mathcal{F}(T^*M)$, where $S = \mu$ is the standard Courant algebroid structure of $T(T^*M) \oplus T^*(T^*M)$.

The integrability condition of Theorem 11.2 is

$$\{\{\mathbb{J}_\omega, S\}, \mathbb{J}_\omega\} = S \ .$$

When the integrability condition is satisfied, $S_{\mathbb{J}_\omega}$ satisfies $\{S_{\mathbb{J}_\omega}, S_{\mathbb{J}_\omega}\} = 0$ and $S_{\mathbb{J}_\omega}$ maps the Dorfman bracket defined by $S_{\mathbb{J}_\omega}$ to the bracket defined by S.

13.4 The von Karman equation

The conditions $d\omega = 0$ and $d(\frac{\omega}{\sqrt{|\mathrm{Pf}(\omega)|}}) = 0$ are very different. In the former case, there is a pair of symplectic forms on T^*M. The latter condition is the necessary and sufficient condition for the Monge-Ampère structure to be equivalent to a structure with constant coefficients. If the Monge-Ampère structure (Ω, ω) is such that $d\widetilde{\omega} \neq 0$, then the torsion of J_ω does not vanish, the integrability condition is not satisfied. The following example shows that the condition $d\omega = 0$ is not sufficient to define a PN- or an ΩN-structure.

Let (q_1, q_2, p_1, p_2) be the canonical coordinates on $T^*(\mathbb{R}^2) = \mathbb{R}^4$. Let (Ω, ω) be the Monge-Ampère structure on $\mathbb{R}^2$ defined by the 2-form on $T^*(\mathbb{R}^2)$,

$$\omega = p_1 dp_1 \wedge dq_2 - dp_2 \wedge dq_1 \ .$$

The corresponding partial differential equation is the *von Karman equation*,

$$f_{q_1} f_{q_1 q_1} - f_{q_2 q_2} = 0 \ .$$

It is easy to show that $\omega \wedge \Omega = 0$ and $d\omega = 0$. This structure is non-degenerate in the complement of the hyperplane $p_1 = 0$, since the Pfaffian $\mathrm{Pf}(\omega)$ is equal to p_1. In the half-space $p_1 > 0$ (resp., $p_1 < 0$) the von Karman equation is an elliptic (resp., hyperbolic) Monge-Ampère equation. Now $d\widetilde{\omega} \neq 0$ since

$$d\widetilde{\omega} = d(\frac{\omega}{\sqrt{|\mathrm{Pf}(\omega)|}}) = \frac{1}{2}|p_1|^{-3/2} dp_1 \wedge dp_2 \wedge dq_1 \ . \tag{13.1}$$

Therefore, the Monge-Ampère structure (Ω, ω) is not equivalent to a Monge-Ampère structure with constant coefficients. It does not define a PN- nor an ΩN-structure on $\mathbb{R}^4$, nor a Monge-Ampère Courant structure because the equation $\{S_{\mathbb{J}_\omega}, S_{\mathbb{J}_\omega}\} = 0$ is not satisfied. The Poisson tensor inverse to ω is

$$\pi_\omega = \frac{1}{|p_1|} \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_2} - \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_1} \ .$$

The computation of $\{\{\pi_\Omega, \mu\}, \pi_\omega\}$ shows that the Schouten bracket of π_ω and the canonical Poisson tensor π_Ω is the 3-vector,

$$[\pi_\Omega, \pi_\omega]_\mu = -\frac{1}{(p_1)^2}\frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_1} \ .$$

13.5 Generalized Monge-Ampère structures

More generally, we can consider *generalized Monge-Ampère structures* (ω_1, ω_2), where both 2-forms on T^*M, ω_1 and ω_2, are non-degenerate but not necessarily closed. The corresponding equations are systems of non-linear first-order partial differential equations whose non-linearity has a specific form. Such systems, called *Jacobi systems*, are studied in [25]. A Jacobi system is called *non-degenerate* if $\omega_1 \wedge \omega_2 = 0$ and there exists a nowhere vanishing function on T^*M, ϵ, such that $\omega_1 \wedge \omega_1 = \epsilon\, \omega_2 \wedge \omega_2$. The Jacobi systems are of the form, for a pair of functions (u, v) on $M = \mathbb{R}^2$ with coordinates (x, y),

$$\begin{cases} a + bu_x + cu_y + dv_x + ev_y + f\mathcal{J}_{u,v} = 0 \ , \\ A + Bu_x + Cu_y + Dv_x + Ev_y + F\mathcal{J}_{u,v} = 0 \ , \end{cases} \tag{13.2}$$

where $\mathcal{J}_{u,v}$ is the Jacobian determinant of (u, v).

The Jacobi systems can be defined invariantly as follows. Let $\mathcal{M} = M \times \mathbb{R}^2$, where M is a 2-dimensional manifold and let ω_i, $i = 1, 2$, be 2-forms on $\mathcal{M}$. We define the differential operators, $\Delta_{\omega_i} : C^\infty(M, \mathbb{R}^2) \to \Omega^2(T^*M)$, by

$$\Delta_{\omega_i}(f) = \omega_i|_{L_f} \ , \quad i = 1, 2 \ , \tag{13.3}$$

where L_f is the graph of the $\mathbb{R}^2$-valued function f on M, a 2-dimensional surface in $\mathcal{M}$. The system (13.2) is then written

$$\Delta_{\omega_i} f = 0, \quad i = 1, 2, \quad f = (u, v).$$

If the restrictions of ω_1 and ω_2 to the surface $L_f \subset \mathcal{M}$ vanish, we shall say that L_f is a *generalized solution* of (13.2). Geometrically, we can assign to each point $m \in \mathcal{M}$ the plane in $\wedge^2(T^*_m\mathcal{M})$, called the *Jacobi plane*, generated by $\omega_1|_m$ and $\omega_2|_m$, thus defining a smooth distribution on $\mathcal{M}$ which corresponds to the system (13.3). The submanifold L_f is an integral manifold for this distribution.

We define a $(1, 1)$-tensor $A \in \Gamma(T\mathcal{M} \otimes T^*\mathcal{M})$ by

$$\omega_2(X, Y) = \omega_1(AX, Y) \ ,$$

for all X and $Y \in \Gamma(T\mathcal{M})$. If the Jacobi system (13.3) is non-degenerate and if, in addition, $\epsilon = 1$ or $\epsilon = -1$, then $A^2 = \epsilon$ and we can associate to such systems an almost complex or almost product structure on $T\mathcal{M}$ (see [25]). Let $\pi_{\omega_i} \in \wedge^2(T\mathcal{M})$,

$i = 1, 2$, be the bivectors which are inverse to the non-degenerate 2-forms ω_i. Suppose that these bivectors satisfy the following conditions,

$$\begin{cases} [\pi_{\omega_1}, \pi_{\omega_1}] = [\pi_{\omega_2}, \pi_{\omega_2}] \ , \\ [\pi_{\omega_1}, \pi_{\omega_2}] = 0 \ . \end{cases} \tag{13.4}$$

A pair of bivectors satisfying the conditions (13.4) is called a *Hitchin pair of bivectors* in [2]. Theorem 6.3 of Section 6 implies that the Jacobi systems on $\mathbb{R}^2$ associated with Hitchin pairs of Poisson bivectors define PN-structures on $\mathbb{R}^4$.

14 Monge-Ampère structures in dimension 3 and generalized geometry

14.1 Classification

Dimension 3 plays an exceptional role in the geometry of Monge-Ampère operators. The classification problem for Monge-Ampère operators and equations on 3-dimensional manifolds can be reduced to a classical problem in geometric invariant theory, the determination of the normal forms of the effective 3-forms in a 6-dimensional real symplectic vector space V, in other words, of the orbits of the symplectic group Sp(6) on the space of effective 3-forms on V. This problem was solved in [30] (see also [25]). There are three types of generic orbits, each with a non-trivial stabilizer, each corresponding to a non-degenerate Monge-Ampère structure with a non-degenerate non-linear Monge-Ampère operator. Let $(q_1, q_2, q_3, p_1, p_2, p_3)$ be the canonical coordinates on $T^*(\mathbb{R}^3) = \mathbb{R}^6$, and let u be a function on $T^*\mathbb{R}^3$. The three types of generic orbits are those of the following 3-forms with constant coefficients, with corresponding Monge-Ampère equations:

$$\omega = dp_1 \wedge dp_2 \wedge dp_3 - dq_1 \wedge dq_2 \wedge dq_3 \ , \qquad \Delta_\omega = \mathrm{hess}(u) - 1 \ , \tag{14.1}$$

$$\omega = dp_1 \wedge dp_2 \wedge dp_3 - dp_1 \wedge dq_2 \wedge dq_3 - dq_1 \wedge dp_2 \wedge dq_3 - dq_1 \wedge dq_2 \wedge dp_3 \ , \Delta_\omega = \mathrm{hess}(u) - \Delta(u) \ , \tag{14.2}$$

$$\omega = dp_1 \wedge dp_2 \wedge dp_3 - dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 - dp_3 \wedge dq_1 \wedge dq_2 \ , \Delta_\omega = \mathrm{hess}(u) - \Box(u) \ , \tag{14.3}$$

where $\Delta = \frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2} + \frac{\partial^2}{\partial q_3^2}$ is the Laplacian, $\Box = \frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2} - \frac{\partial^2}{\partial q_3^2}$ is the D'Alembertian of signature $(2, 1)$, and $\mathrm{hess}(u) = \det(u_{q_i q_j}), 1 \le i, j \le 3$, is the Hessian of the function u, *i.e.*, the determinant of the matrix of second-order partial derivatives of u with respect to q_1, q_2, q_3.

We shall show that, in full analogy to the 2-dimensional case where almost complex (resp., almost product) structures[2] correspond to elliptic (resp., hyperbolic) Monge-

[2]In [3], Banos called these structures "generalized Calabi-Yau structures", but this terminology conflicts with Hitchin's in [14]. Below we shall clarify the difference between these two generalizations of the Calabi-Yau structures.

Ampère operators, there exist three generalized structures in the sense of Grabowski [10] corresponding to the three types of Monge-Ampère structures in dimension 3.

14.2 Hitchin endomorphism and Hitchin Pfaffian

To each Monge-Ampère structure $(\Omega, \omega) \in \Omega^2(T^*M) \times \Omega^3(T^*M)$ on a 3-dimensional manifold M, where ω is effective, are associated the following [13] [25]:

- the Liouville volume form, vol, associated with Ω,

$$\mathrm{vol} = -\frac{1}{6}\Omega \wedge \Omega \wedge \Omega \in \Omega^6(T^*M) \ ,$$

- the Hitchin endomorphism, $H_\omega : \mathcal{X}(T^*M) \to \mathcal{X}(T^*M)$, defined by

$$H_\omega(X) = i_X\omega \wedge \omega \in \Omega^5(T^*M) \simeq \mathcal{X}(T^*M) \ ,$$

for all $X \in \mathcal{X}(T^*M)$, where $\Omega^5(T^*M)$ is identified with $\mathcal{X}(T^*M)$ by means of the Liouville form,

- the Hitchin Pfaffian, λ_ω, defined by

$$\lambda_\omega = \frac{1}{6}\mathrm{Tr}(H_\omega^2) \ ,$$

- the symmetric bilinear form, q_ω, defined by

$$q_\omega(X, Y) = \Omega(H_\omega X, Y) \ ,$$

for all X and $Y \in \mathcal{X}(T^*M)$.

The Hitchin endomorphism and the Hitchin Pfaffian are related by

$$H_\omega^2 = \lambda_\omega \mathrm{Id}.$$

By definition, a Monge-Ampère structure (Ω, ω) on $T^*(\mathbb{R}^3)$ is called *non-degenerate* if its Hitchin Pfaffian λ_ω is nowhere-vanishing.

An essential part of the proof of the above-mentioned classification is the proof that the forms in the orbit of the form ω of (14.1) have negative Hitchin Pfaffian, while those in the orbits of the forms of (14.2) and of (14.3) have positive Hitchin Pfaffian and quadratic forms q_ω of different signatures.

For a 2-form τ, we define the modified Pfaffian $\mathcal{P}f(\tau)$ by

$$\tau \wedge \tau \wedge \Omega = -\frac{1}{3}\mathcal{P}f(\tau)\Omega \wedge \Omega \wedge \Omega \ .$$

The following statement is the result of a straightforward computation.

Proposition 14.1 *The modified Pfaffian $\mathcal{P}f$, the Hitchin endomorphism H_ω and the bilinear form q_ω satisfy the relations*

$$\mathcal{P}f(i_X\omega) = \Omega(H_\omega X, X) = q_\omega(X, X) \equiv -\frac{1}{4}\imath_{\pi_\Omega}\imath_{\pi_\Omega}(i_X\omega \wedge i_X\omega) \ ,$$

for all $X \in \mathcal{X}(T^*M)$.

14.3 Properties of non-degenerate Monge-Ampère structures in dimension 3

We can now draw conclusions analogous to those of the 2-dimensional case of Section 13.

• Any Monge-Ampère structure (Ω, ω) satisfies the conditions of Lemma 2.8 of Section 2.4 and therefore,

$$\{\{X, \{\omega, \pi_\Omega\}\}, Y\} = \pi_\Omega^\sharp(i_{X\wedge Y}\omega) \ ,$$

for all X and $Y \in \mathcal{X}(T^*M)$.

• In the notations of Section 13 we consider the function $S \in \mathcal{F}(T^*M)$,

$$S = \mu + \omega \ ,$$

where ω is a closed effective 3-form on T^*M. Then $\{S, S\} = 0$ and S defines a Courant algebroid structure on $T(T^*M) \oplus T^*(T^*M)$.

• We consider the Hitchin endomorphism H_ω and the Poisson bivector π_Ω inverse of Ω. Then $H_\omega \circ \pi_\Omega^\sharp = \pi_\Omega^\sharp \circ H_\omega^*$. We obtain the following result, the second part of which can be viewed as a corollary of Theorem 2.5 of [1].

Theorem 14.2 *Let ω be the 3-form on T^*M defined by Formula* (14.1) *or* (14.2) *or* (14.3)*, and let $\mathbb{J}_\omega$ be the endomorphism of $T(T^*M) \oplus T^*(T^*M)$ defined by*

$$\mathbb{J}_\omega = \begin{pmatrix} H_\omega & \pi_\Omega^\sharp \\ 0 & -H_\omega^* \end{pmatrix} \ ,$$

where H_ω is the Hitchin endomorphism. If $\lambda_\omega = -1$ (resp., $\lambda_\omega = 1$), the endomorphism $\mathbb{J}_\omega$ is a generalized complex structure (resp., generalized product structure) on $(T(T^*M) \oplus T^*(T^*M), \mu + \omega)$. The triple $(\pi_\Omega, H_\omega, \omega)$ is a Poisson Nijenhuis structure with background on the manifold T^*M.

Proof When $H_\omega^2 = -\mathrm{Id}$ (resp., $+\mathrm{Id}$), the endomorphism $\mathbb{J}_\omega$ is a generalized almost complex (resp., generalized almost product) structure. Because ω has constant coefficients, these structures are integrable [3]. Expressing the vanishing of the torsion of $\mathbb{J}_\omega$ and expanding the terms of $\{\{\pi_\Omega + H_\omega, \mu + \omega\}, \pi_\Omega + H_\omega\} + \{H_\omega^2, \mu + \omega\}$, we obtain, as in [1],

$$\mathrm{ad}^2_{\pi_\Omega}(\mu) = 0, \quad \{\mathrm{ad}_{\pi_\Omega}(\mu), H_\omega\} - \mathrm{ad}_{\pi_\Omega}(\{H_\omega, \mu\}) = \mathrm{ad}^2_{\pi_\Omega}(\omega) \ ,$$

$$\{\{H_\omega, \mu\}, H_\omega\} + \{\mathrm{ad}_{\pi_\Omega}(\omega), H_\omega\} - \mathrm{ad}_{\pi_\Omega}(\{H_\omega, \omega\}) + \{H_\omega^2, \mu\} = 0 \ ,$$

$$\{\{H_\omega, \omega\}, H_\omega\} + \{H_\omega^2, \omega\} = 0 \ .$$

Replacing H_ω^2 par Id or $-$Id, we find that the quadruple $(\pi_\Omega, H_\omega, 0, \omega)$ is a Poisson quasi-Nijenhuis structure with background, *i.e.*, the triple $(\pi_\Omega, H_\omega, \omega)$ is a Poisson-Nijenhuis structure with background in the sense of [1]. □

14.4 Generalized Calabi-Yau structures

Theorem 14.2 answers a natural question: what is the relation between the generalized Calabi-Yau structures in the sense of Hitchin [14] or Gualtieri [12] and the generalized Calabi-Yau structures introduced by Banos [3] in his study of Monge-Ampère structures?

The generalized Calabi-Yau structures in the sense of Hitchin are special generalized complex structures. According to the definition of M. Gualtieri [12] (which is slightly different from Hitchin's [14]), a generalized Calabi-Yau manifold is a manifold with a generalized complex structure and trivial canonical class. Theorem 8.2 shows that the Calabi-Yau Monge-Ampère structures in the sense of Banos are generalized c.p.s. (complex, product or subtangent) structures in the sense of Grabowski [10] and Vaisman [39]. Equations (14.1), (14.2), (14.3) define generalized Calabi-Yau structures on T^*M in the sense of Banos. The Monge-Ampère structures (14.2) and (14.3) (called *special Lagrangian* and *pseudo-special Lagrangian*, respectively) define generalized Calabi-Yau structures in the sense of Gualtieri, while that of (14.1), where $H_\omega^2 = \mathrm{Id}$, does not since it corresponds to a generalized product structure. In the case of (14.2), we obtain the canonical Calabi-Yau structure on $T^*(\mathbb{R}^3) = \mathbb{C}^3$ with the complex structure H_ω, satisfying $H_\omega^2 = -\mathrm{Id}_{\mathbb{R}^6}$.

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