The Numerical Calculation of Casimir Forces

A computational challenge

Per Kristen Jakobsen

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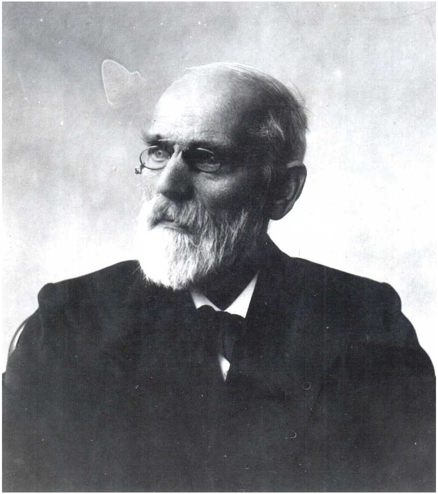
Overview

- Short Timeline
- Main research themes.
- A very short introduction to quantum electrodynamics (QED)
- Calculating Casimir forces by QED mode summation.
- On the numerical calculation of classical optical forces.
- Casimir forces from Matsubara-Green's functions.
- Boundary integral equations for Matsubara-Green's functions.
- Symmetries and reduction.
- Regularization.

Short Timeline

Johannes Diderik Van Der Waals



1873 Finite volume effects and binary molecular interactions were used by J. D. Van Der Waals to derive a generalized equation of state

$$(p + \frac{a}{\rho})(\rho - b) = k_B T$$

where p, ρ, T are the pressure,density and temperature. The coefficients a and b are in general temperature dependent.

Heike Karnerlingh Onnes

1901 The virial expansion for the pressure of a many-particle system in terms of the density was introduced by H. Onnes.

$$\frac{p}{k_B T} = \rho + B_2(T)\rho^2 + B_3(T)\rho^3 + \cdots$$

The virial coefficients are determined by the nature of the interaction forces between the individual molecules in the system.

1913 W. H. Keesom derive the averaged interaction energy beween two molecules of permanent dipolemoments $\mu_1$and μ_2 at temperature T

$$U = \begin{cases} -\frac{2}{3}\frac{\mu_1^2\mu_2^2}{R^6}\ \frac{1}{k_B T} & k_B T >> \frac{\mu_1\mu_2}{R^3} \\ -2\frac{\mu_1\mu_2}{R^3} & k_B T << \frac{\mu_1\mu_2}{R^3} \end{cases}$$

Keesom interpret Van der Waals forces (causing deviations from the perfect gas equation of state) using this *orientation effect*.

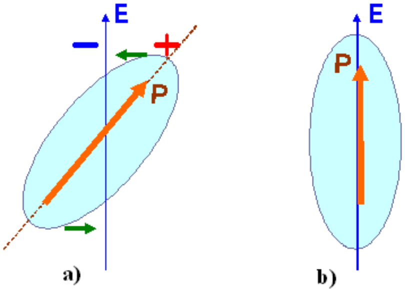


FIG. 2

1920-22 P. Debye and H. Fakckenhagen generalize the Keesom model by including the effect of molecular polarizabilities. For two identical molecules with permanent dipolemoment,μ, and polarizabilities,α, they predict an additional interaction energy

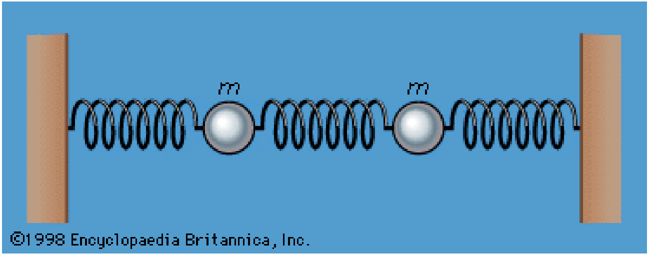
$$U = -\frac{2\alpha\mu^2}{R^6}$$

The Van Der Waals forces is explain the in terms of this *induction* effect.

1936 F. London use quantum mechanical zero point fluctuations of a system of coupled electronic oscillators to derive an interaction energy of the form

$$U = \frac{-3\hbar\omega\alpha^2}{4R^6}$$

beween molecules without permanent dipolemoment.



1937 Hamaker derive the Van Der Waals-London forces between spheres and between a sphere and an infinite wall by summing the dipole interaction energy over the volume of the objects.

1940-46 Verwey and Overbeek, building on Hamakers work, develop a theory of stability of colloidal suspensions.The theory fits the experimental data only if the Van Der Waals-London interaction potential between the colloidal particles falls off faster than R^{-6}.

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Hendrik Casimir

1948 H. Casimir and D. Polder, acting on a suggestion by N. Bohr, compute the interaction energy between two dipoles using QED.

$$U = -23\frac{\hbar c\alpha_1\alpha_2}{4\pi}\left(\frac{1}{R^7}\right)$$

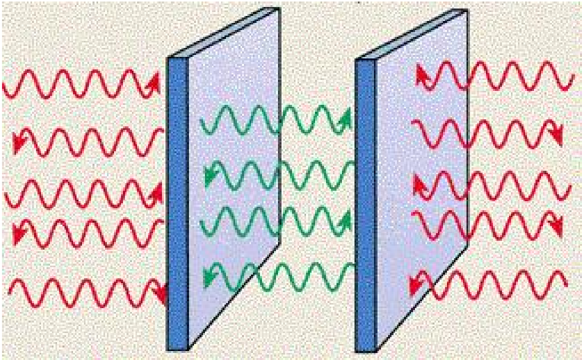
1948 H. Casimir predict, using mode summation in QED, that the interaction energy between two parallel, perfectly conducting, neutral metal plates,separated by a distance R is

$$U = -\frac{\hbar c \pi^2}{720}\left(\frac{1}{R^3}\right)$$

The principle of virtual work predicts an attractive force

$$F = \frac{\partial U}{\partial R} = \frac{\pi^2 \hbar c}{240}\left(\frac{1}{R^4}\right)$$

Casimir interpreted this as the *Zero-point pressure.*



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1956-1960 Experimental measurements of the force between parallel plates by Sparnaay ,Derjaguin,Abrikosova,Kitchener,Proser and several others confirm Casimirs theoretical prediction.

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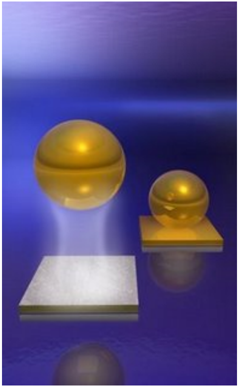
1969 Brown and Maclay compute local Casimir forces using the Quantum Electrodynamical Stress Tensor.

Main Research Themes since 1970

- Physical factors influencing the attractive/repulsive aspect of the Casimir force.(Dispersion,geometry,dimensions etc.)

The Holy Grail



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- Timedependent Casimir forces.(Davies-Unruhe effect)

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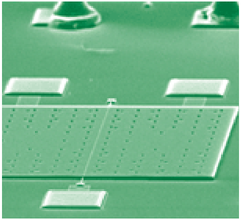
- Casimir forces for other quantum fields.(Hadron bag model, Kaluza-Klein models,The GRID)

- Consequences of the Casimir force for nanoscale engineering.(Stiction,frictionless bearings etc.)



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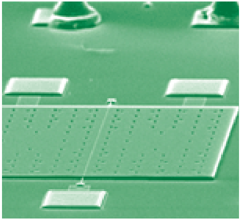


- Improved calculational tools and renormalization techniques.

A very short introduction to Quantum Electrodynamics

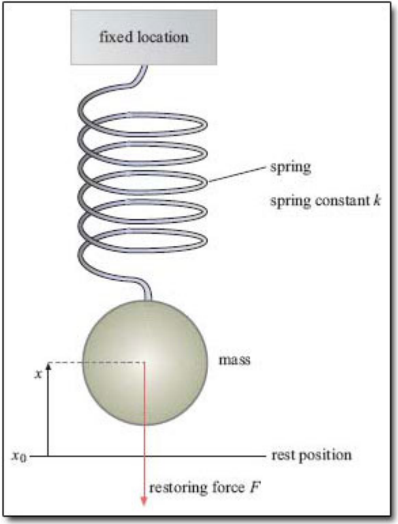
The Quantum Harmonic Oscillator

The Harmonic oscillator is a Hamiltonian dynamical system described by a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

The equations of motion are

$$\begin{aligned} \frac{dq}{dt} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q} = -m\omega^2 q \end{aligned}$$


fixed location
spring
spring constant k
mass
x
x0
rest position
restoring force F

Hamiltonian systems are quantized through the procedure of canonical quantization. The dual pair of variables p and q are replaced by Hermitian operators $\widehat{p}$ and $\widehat{q}$ that satisfy the commutation relation

$$[\widehat{q}, \widehat{p}] \equiv \widehat{q}\widehat{p} - \widehat{p}\widehat{q} = i\hbar$$

The Hamiltonian function become the Hamiltonian operator

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + \frac{1}{2}m\omega^2\widehat{q}^2$$

The equation of motion are the Heisenberg equations

$$\begin{aligned} i\hbar\frac{d\widehat{q}}{dt} &= \left[\widehat{q},\widehat{H}\right] = \frac{\widehat{p}}{m} \\ i\hbar\frac{d\widehat{p}}{dt} &= \left[\widehat{p},\widehat{H}\right] = -m\omega^2\widehat{q} \end{aligned}$$

These equations can be simplified by introducing non-Hermitean operators a and $a^\dagger$ through

$$\begin{aligned} \widehat{q} &= i\sqrt{\frac{\hbar}{2m}}(a-a^\dagger) \\ \widehat{p} &= \sqrt{\frac{m\hbar\omega}{2}}(a+a^\dagger) \end{aligned}$$

These operators satisfy the commutation relation

$$\left[a,a^\dagger\right] = 1$$

The Hamiltonian expressed in terms of a and $a^\dagger$ is

$$\widehat{H} = \hbar\omega(a^\dagger a + \frac{1}{2})$$

The energy levels of the harmonic oscillator is determined by the Hermitian operator $N = a^\dagger a$. Eigenstates and eigenvalues of N are determined by

$$N\,|n\rangle = n\,|n\rangle$$

The eigenvalues n are real and nonnegative since N is a positive hermitian operator. The commutation relation for a and $a^\dagger$ imply

$$\begin{aligned} a\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned}$$

a and $a^\dagger$ are the lowering and raising operators.

Since the eigenvalues n are all nonnegative we must have

$$a\left|0\right\rangle = 0$$

- $\left|0\right\rangle$ is the ground state of the system

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- N is the number operator and $|n\rangle$ are the number eigenstates.
- The integer n counts the number of discrete excitations or "particles" in eigenstate $|n\rangle$.
- The energy levels are discrete

$$E_n = \hbar\omega(n + \frac{1}{2}), \qquad n = 0, 1, 2, \ldots$$

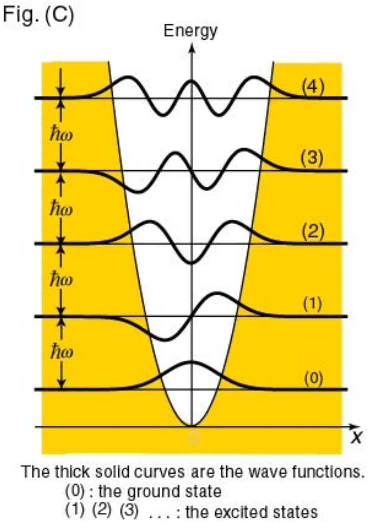
Fig. (C)
Energy
ħω
ħω
ħω
ħω
(4)
(3)
(2)
(1)
(0)
x
The thick solid curves are the wave functions.
(0) : the ground state
(1) (2) (3) . . . : the excited states

Classical Electrodynamics

The Maxwell equations in a source free region are

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{E} &= 0 \end{aligned}$$

Introducing vector and scalar potentials, $\mathbf{A}$ and φ we get

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\ \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi \end{aligned}$$

And

$$\begin{aligned}
\nabla^2 \varphi &= -\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} \\
\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} &= \boldsymbol{\nabla} \left[\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t}\right]
\end{aligned}$$

Using the Coulomb (radiation) gauge, $\nabla \cdot \mathbf{A} = 0$, and assuming, $\varphi = 0$, since there are no sources, we get simply

$$\begin{aligned}
\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} &= 0 \\
\nabla \cdot \mathbf{A} &= 0
\end{aligned}$$

Solutions harmonic in time

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x})e^{-i\omega t} + \mathbf{A}^*(\mathbf{x})e^{i\omega t}$$

satisfy the boundary value problem

$$\begin{aligned} \nabla^2 \mathbf{A}(\mathbf{x}) + k^2 \mathbf{A}(\mathbf{x}) &= 0, \quad k = \frac{\omega}{c} \\ \nabla \cdot \mathbf{A}(\mathbf{x}) &= 0 \end{aligned}$$

The electric and magnetic field corresponding to a solution of the boundary value problem (electromagnetic mode) are

$$\begin{aligned} \mathbf{E}(\mathbf{x},t) &= \frac{1}{c}\left(\frac{d\alpha(t)}{dt}\mathbf{A}(\mathbf{x}) + \frac{d\alpha^*(t)}{dt}\mathbf{A}^*(\mathbf{x})\right) \\ \mathbf{B}(\mathbf{x},t) &= \alpha(t)\nabla \times \mathbf{A}(\mathbf{x}) + \alpha^*(t)\nabla \times \mathbf{A}^*(\mathbf{x}) \end{aligned}$$

where $\alpha(t) = \alpha(0)e^{i\omega t}$.

The electromagnetic energy in the mode is

$$
\begin{aligned}
H &= \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\mathbf{E}^2(\mathbf{x})+\mathbf{B}^2(\mathbf{x})\right)dV \\
&= \frac{k^2}{2\pi}\left|\alpha(t)\right|^2
\end{aligned}
$$

where the mode is assumed to be normalized

$$
\int_{\mathbb{R}^3}\left|\mathbf{A}(\mathbf{x})\right|^2 dV = 1
$$

The complex equation of motion for the mode amplitude $\alpha(t)$ is $\alpha''(t) = -\omega^2\alpha(t)$. Introducing real quantities

$$
\begin{aligned}
q &= \frac{i}{c\sqrt{4\pi}}\left(\alpha-\alpha^*\right) \\
p &= \frac{k}{\sqrt{2\pi}}\left(\alpha+\alpha^*\right)
\end{aligned}
$$

the equation of motion become Hamiltonian with dual variables p and q and with Hamiltonian equal to the electromagnetic energy

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$$

The amplitude of a electromagnetic field mode of frequency ω

$$\Updownarrow$$

Harmonic oscillator with mass $m = 1$ and frequency ω

Quantization of an Electromagnetic field mode

We quantize a electromagnetic field mode by replacing the Harmonic oscillator for the amplitude with the corresponding Quantum Harmonic oscillator. This is equivalent to the replacement

$$\begin{aligned}\alpha(t) &\rightarrow \sqrt{\frac{2\pi\hbar c^2}{\omega}}a(t)\\ \alpha^*(t) &\rightarrow \sqrt{\frac{2\pi\hbar c^2}{\omega}}a^\dagger(t)\end{aligned}$$

The vector potential become the operator

$$\mathbf{A}(\mathbf{x},t) = \sqrt{\frac{2\pi\hbar c^2}{\omega}}\left(a(t)\mathbf{A}(\mathbf{x}) + a^\dagger(t)\mathbf{A}^*(\mathbf{x})\right)$$

The electric and magnetic field become operators

$$\begin{aligned}\mathbf{E}(\mathbf{x},t) &= i\sqrt{2\pi\hbar\omega}\left(a(t)\mathbf{A}(\mathbf{x}) - a^\dagger(t)\mathbf{A}^*(\mathbf{x})\right)\\ \mathbf{B}(\mathbf{x},t) &= \sqrt{\frac{2\pi\hbar c^2}{\omega}}\left(a(t)\nabla\times\mathbf{A}(\mathbf{x}) + a^\dagger(t)\nabla\times\mathbf{A}^*(\mathbf{x})\right)\end{aligned}$$

The Hamiltonian, or the energy operator, for the field mode become

$$H = \hbar\omega(a^\dagger a + \frac{1}{2})$$

- The energy eigenmodes are determined by the eigenmodes for the number operator $N = a^\dagger a$.

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- The state $|0\rangle$ has no photons in the field but nevertheless has finite energy $H = \frac{1}{2}\hbar\omega$.
- This is the *zero-point* state of the field mode.

The **E** and **B** field operators does not commute with the number operator

$\Downarrow$

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For a energy eigenstate $|n\rangle$ we have

$$\begin{aligned}
\langle \mathbf{E}(\mathbf{x},\mathbf{t})\rangle &= \langle n|\,\mathbf{E}(\mathbf{x},\mathbf{t})\,|n\rangle \\
&= i\sqrt{2\pi\hbar\omega}\left(\langle n|\,a(t)\,|n\rangle\,\mathbf{A}(\mathbf{x}) - \langle n|\,a^{\dagger}(t)\,|n\rangle\,\mathbf{A}^{*}(\mathbf{x})\right) \\
&= 0 \\
\langle \mathbf{B}(\mathbf{x},t)\rangle &= \langle n|\,\mathbf{B}(\mathbf{x},\mathbf{t})\,|n\rangle \\
&= \sqrt{\frac{2\pi\hbar c^{2}}{\omega}}\left(\langle n|\,a(t)\,|n\rangle\,\nabla\times\mathbf{A}(\mathbf{x}) + \langle n|\,a^{\dagger}(t)\,|n\rangle\,\nabla\times\mathbf{A}^{*}(\mathbf{x})\right) \\
&= 0
\end{aligned}$$

using

$$\begin{aligned}
\langle n|\,a(t)\,|n\rangle &= \sqrt{n}\,\langle n|n-1\rangle = 0 \\
\langle n|\,a^{\dagger}(t)\,|n\rangle &= \sqrt{n+1}\,\langle n|n+1\rangle = 0
\end{aligned}$$

For the electric field intensity we have

$$\begin{aligned}
\left\langle \mathbf{E}^2(\mathbf{x},t)\right\rangle &= \langle n|\, \mathbf{E}^2(\mathbf{x},\mathbf{t})\,|n\rangle \\
&= -(2\pi\hbar\omega)\{\langle a(t)a(t)\rangle\,(\mathbf{A}(\mathbf{x}))^2 \\
&\quad -\left\langle a(t)a^\dagger(t)+a^\dagger(t)a(t)\right\rangle |\mathbf{A}(\mathbf{x})|^2 \\
&\quad +\left\langle a^\dagger(t)a^\dagger(t)\right\rangle (\mathbf{A}^*(\mathbf{x}))^2\}
\end{aligned}$$

and

$$\begin{aligned}
\langle a(t)a(t)\rangle &= \left\langle a^\dagger(t)a^\dagger(t)\right\rangle = 0 \\
\left\langle a(t)a^\dagger(t)+a^\dagger(t)a(t)\right\rangle &= \left\langle 2a^\dagger(t)a(t)+1\right\rangle = 2n+1
\end{aligned}$$

Thus we get

$$\left\langle \mathbf{E}^2(\mathbf{x},t)\right\rangle = 4\pi\hbar\omega\,|\mathbf{A}(\mathbf{x})|^2\,n + \left\langle \mathbf{E}^2(\mathbf{x},t)\right\rangle_0$$

where $\left\langle \mathbf{E}^2(\mathbf{x},t)\right\rangle_0$ is the intensity of the zero point field mode

$$\left\langle \mathbf{E}^2(\mathbf{x},t)\right\rangle_0 = \langle 0|\,\mathbf{E}^2(\mathbf{x},\mathbf{t})\,|0\rangle = 2\pi\hbar\omega\,|\mathbf{A}(\mathbf{x})|^2$$

- The *shape* of the intensity pattern is determined by the *classical* mode.
- The *visibility* of the intensity pattern is determined by the number of *photons* in the mode.

Quantization of the Electromagnetic field

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- Each mode is a classical Harmonic oscillator and is replaced by a Quantum Harmonic oscillator.
- The lowering and raising operator corresponding to different modes commute, $[a_\mu, a_{\mu'}] = 0, [a_\mu, a^\dagger_{\mu'}] = \delta_{\mu,\mu'}$

$$\begin{aligned}
\mathbf{A}(\mathbf{x}, t) &= \sum_\mu \sqrt{\frac{2\pi\hbar c^2}{\omega}} \left(a_\mu(t)\mathbf{A}_\mu(\mathbf{x}) + a^\dagger_\mu(t)\mathbf{A}^*_\mu(\mathbf{x}) \right) \\
\mathbf{E}(\mathbf{x}, t) &= \sum_\mu i\sqrt{2\pi\hbar\omega} \left(a_\mu(t)\mathbf{A}_\mu(\mathbf{x}) - a^\dagger_\mu(t)\mathbf{A}^*_\mu(\mathbf{x}) \right) \\
\mathbf{B}(\mathbf{x}, t) &= \sum_\mu \sqrt{\frac{2\pi\hbar c^2}{\omega}} \left(a_\mu(t)\nabla \times \mathbf{A}_\mu(\mathbf{x}) + a^\dagger_\mu(t)\nabla \times \mathbf{A}^*_\mu(\mathbf{x}) \right)
\end{aligned}$$

The Hamiltonian for the field is

$$H = \sum_{\mu} \hbar\omega_{\mu}(a_{\mu}^{\dagger}a_{\mu} + \frac{1}{2})$$

and the *vacuum* is determined by the condition

$$a_{\mu}|0>= 0$$

The vacuum state, or *zero-point field*, belong to an infinite dimensional Hilbert space called the *Fock space*.
The energy of the zero-point field is

$$U = \langle 0| \mathbf{H} |0\rangle = \frac{1}{2}\hbar\sum_{\mu}\omega_{\mu}$$

In what sense does electromagnetic zero-point field exist?

- The entire universe, including any detector, is immersed in the zero-point field. Therefore only deviations from the zero-point field can be detected directly.
  - The field Hamiltonian can be replaced by a physical Hamiltonian

$$
\begin{aligned}
H_{ph} &= H - \langle 0| H |0\rangle \\
&= \hbar\omega(a^\dagger a + \frac{1}{2}) - \frac{1}{2}\hbar\omega \\
&= \hbar\omega a^\dagger a
\end{aligned}
$$

  - H and H_{ph} give the same Heisenberg equation of motion for a and $a^\dagger$, thus the same physics.
- The zero-point field induce a small shift in the energy levels of hydrogen atoms interacting with the field. This *Lamb-shift* has been detected experimentally(1947).

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    - The zero-point field reappear in the form of the homogenous solution.
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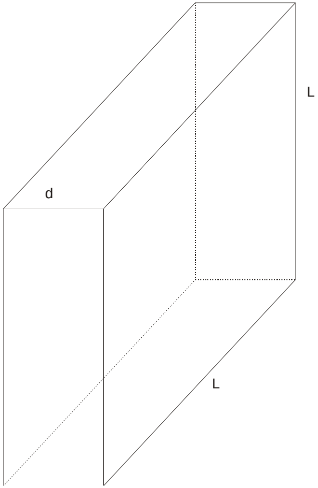
- In General Relativity the zero of energy is not arbitrary. The zero-point field should give a space-time independent contribution to the Stress Energy tensor and thereby contribute to the curvature of space.
- The introduction of physical objects and/or nontrivial topology through boundary conditions will change the zero-point field:
  - Changes in the zero-point field will lead to forces between objects (Casimir forces).
  - Casimir forces has been detected and agree with theory at the 5% level(1997).

Calculating Casimir forces by QED mode summation.

Casimir force between parallel plates



- The modes of the cavity are determined by the equations

$$\begin{aligned} \nabla^2 \mathbf{A}(\mathbf{x}) + k^2 \mathbf{A}(\mathbf{x}) &= 0 \\ \nabla \cdot \mathbf{A} &= 0 \end{aligned}$$

and the following boundary conditions at the walls of the cavity

$$\begin{aligned} \mathbf{n} \times \mathbf{A} &= 0 \\ \mathbf{n} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) &= 0 \end{aligned}$$

- The modes of the cavity are determined by the equations

$$\begin{aligned}\nabla^2\mathbf{A}(\mathbf{x})+k^2\mathbf{A}(\mathbf{x}) &= 0\\ \nabla\cdot\mathbf{A} &= 0\end{aligned}$$

and the following boundary conditions at the walls of the cavity

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- The boundary conditions and the Helmholtz equation give

$$\begin{aligned}A_x &= a_x\sqrt{\frac{8}{V}}\cos(k_x x)\sin(k_y y)\sin(k_z z)\\ A_y &= a_y\sqrt{\frac{8}{V}}\sin(k_x x)\cos(k_y y)\sin(k_z z)\\ A_z &= a_z\sqrt{\frac{8}{V}}\sin(k_x x)\sin(k_y y)\cos(k_z z)\end{aligned}$$

- where
  - $V = L^2 L_z, a_x^2 + a_y^2 + a_z^2 = 1$
  - $k_x = \frac{l\pi}{L}, k_y = \frac{m\pi}{L}, k_z = \frac{n\pi}{d}$ where l, m, n are positive integers and zero.

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- The divergence condition give

$$\frac{\pi}{L}\left(la_x + ma_y\right) + \frac{\pi}{d} na_z = 0$$

  - Two independent modes when all integers are nonzero.
  - One independent mode if one of the integers is zero.

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  - Two independent modes when all integers are nonzero.
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- The frequency of a mode determined by integers (l, m, n) is

$$\omega_{lmn} = \pi c \sqrt{\left(\frac{l\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{d}\right)^2}$$

The zero point energy is computed by assigning an energy $\frac{1}{2}\hbar\omega$ to each mode

$$U(d) = \sum_{l,m,n}^{\sim} \hbar\omega(k_x, k_y, n)$$

$$\approx \frac{\hbar c L^2}{\pi^2} \int_0^\infty \int_0^\infty dk_x dk_y \sum_n^{\sim} \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{d}\right)^2}$$

Subtracting the energy when the plates are far apart we get

$$\begin{aligned} E(d) &= U(d) - U(\lambda d) \\ &= \frac{\hbar c L^2}{\pi^2} \int_0^\infty \int_0^\infty dk_x dk_y (\tilde{\sum_n} \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{d}\right)^2} \\ &\quad - \frac{\lambda d}{\pi} \int_0^\infty dk_z \sqrt{k_x^2 + k_y^2 + \lambda^2 k_z^2}) \\ &= \left(\frac{\pi^2 \hbar c}{4d^3}\right) L^2 \left(\frac{1}{2} F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^\infty d\kappa F(\kappa)\right) \end{aligned}$$

where

$$F(x) = \int_0^\infty dk \sqrt{x + k^2} f\left(\left(\frac{\pi}{d}\right)\sqrt{x + k^2}\right)$$

and where $f(y)$ is a function with

$$\begin{aligned} f(y) &= 1, \quad y << k_m \\ f(y) &= 0, \quad y << k_m \end{aligned}$$

where k_m is some cutoff wavenumber. Using the Euler-Maclaurin summation formula

$$\sum_{n=1}^{\infty} F(n) - \int_0^{\infty} F(k) = -\frac{F(0)}{2} - \frac{F'(0)}{12} + \frac{F'''(0)}{720} + \cdots$$

we find that only the $F'''(0)$ term contribute and

$$U(d) = -\left(\frac{\pi^2 \hbar c}{720 d^3}\right) L^2$$

The force per unit area is thus

$$F(d) = \frac{\pi^2 \hbar c}{240 d^4}$$

Casimir force on a perfect metal shell

Where $\lambda \approx R >> 1$. Define renormalized energy by

$$U(S, R, g_a) = \frac{1}{2}\hbar \sum_{\mu} \omega_\mu g_a(\omega_\mu) < \infty$$

where $g_a(\omega)$ is a cutoff function

$$\lim_{a->\infty} g_a(\omega) = 1$$

The Casimir energy is defined by

$$E(S) = \lim_{a\to\infty} \lim_{R\to\infty} (U(S, R, g_a) - U(\lambda S, R, g_a))$$

and the corresponding Casimir force by

$$F(S) = \frac{\partial E}{\partial S}$$

Where $\frac{\partial}{\partial S}$ represents differentiation with respect to the parameters describing the surface S.

- The Casimir force is tiny by macroscopic standards. For two plates of area $1cm^2$ at a distance of $1\mu m$ the force is

$$F \approx 1.3\ 10^{-7}\ N$$

equivalent to the weight of a water droplet half a millimeter in diameter.

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- However as the distance between object decrease down towards nanometer scales the force become highly significant.
- The Casimir force is already a factor to be taken into account in the production and operation of MEMS.
- The force can be both attractive and repulsive and will play an ever increasing role as engineering moves into the nanoscale regime.

Extensions

- Lossless dispersive slabs (The Lifschitz theory at zero temperature rederived by Van Kampen using mode summation in 1961).

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- The Casimir energy must be independent of the cutoff.
- Can not easily be extended to lossy materials.
- Hard to include temperature effects.

Some references for QED and Casimir forces

1. "Photons and Atoms. Introduction to Quantum Electrodynamics", C. Cohen-Tannoudjii,J. Dupond-Roc,G. Grynberg.
2. "The Quantum Vacuum. An introduction to Quantum Electrodynamics",P. W. Milonni.
3. "Physical manifestations of zero-point energy. The Casimir Effect", K. A. Milton

On the numerical calculation of classical optical forces

The stationary Maxwell equations for dispersive dielectric materials are

$$\begin{aligned} \nabla \times \mathbf{E} - i\omega \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} + i\omega \varepsilon_0 \mu_0 n^2 \mathbf{E} &= \mathbf{j} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot (\varepsilon_0 n^2 \mathbf{E}) &= \rho \end{aligned}$$

where $n = n(\mathbf{x}, t)$ is the refractive index of the material. In regions of constant refractive index we get stationary wave equations for $\mathbf{E}$ and $\mathbf{B}$

$$\begin{aligned} -\nabla \times (\nabla \times \mathbf{E}) + \left(\frac{\omega}{c}\right)^2 n^2 \mathbf{E} &= \mathbf{j}_E \\ -\nabla \times (\nabla \times \mathbf{B}) + \left(\frac{\omega}{c}\right)^2 n^2 \mathbf{B} &= \mathbf{j}_B \end{aligned}$$

The optical force on an object with surface S in a vacuum is computed according to

$$\mathbf{F} = \int_S \mathbf{T}(\mathbf{x}) \cdot \mathbf{n} dS$$

where $\mathbf{T}(\mathbf{x})$ is the electromagnetic Stress Tensor

$$\mathbf{T} = \varepsilon_0 \mathbf{E}\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\mathbf{B} - \frac{1}{2}\mathbf{I}\left(\varepsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right)$$

Boundary Integral formulation of the stationary Maxwell equations

Any solution of the stationary Maxwell equations in a region V of constant refractive index, satisfy certain integral identities. These integral identities are derived using

1. A vector-matrix version of Green's second identity

$$\int_V \{(\nabla \times (\nabla \times \boldsymbol{\varphi})) \cdot A - \boldsymbol{\varphi} \cdot (\nabla \times (\nabla \times A))\} \, dV$$
$$= \int_{\partial V} \mathbf{n} \cdot \{(\nabla \times \boldsymbol{\varphi}) \times A + \boldsymbol{\varphi} \times (\nabla \times A)\} \, dS$$

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2. A Green's function for the wave operator corresponding to outgoing waves at infinity

$$G(\mathbf{x},\boldsymbol{\xi}) = g(\mathbf{x},\boldsymbol{\xi})I+\left(\frac{1}{k_0^2n^2}\right)\nabla\nabla g(\mathbf{x},\boldsymbol{\xi})$$
$$g(\mathbf{x},\boldsymbol{\xi}) = -\frac{e^{ik_0n||\mathbf{x}-\boldsymbol{\xi}||}}{4\pi||\mathbf{x}-\boldsymbol{\xi}||}$$

The integral identities are

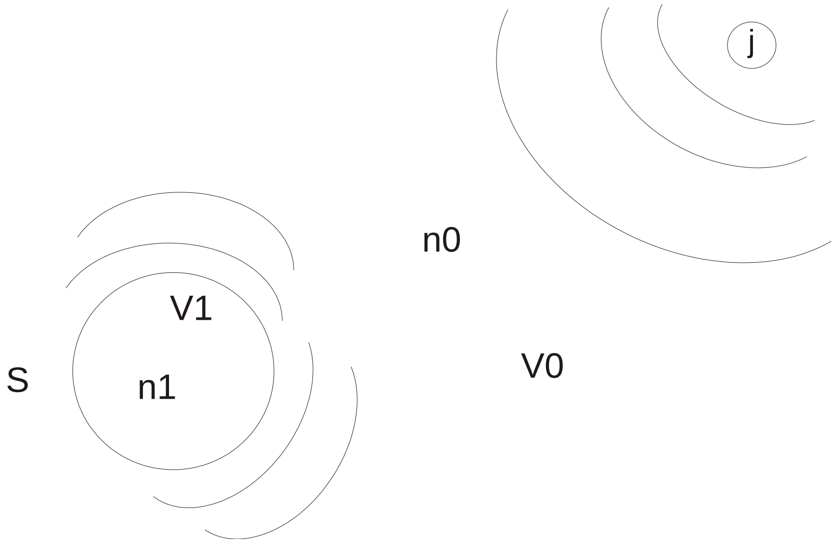
$$\begin{aligned}\mathbf{E}(\xi) &= \mathbf{E}^i(\xi) - \int_{\partial V} \{ik_0 c(\mathbf{B}\times\mathbf{n})\cdot G + (\mathbf{E}\times\mathbf{n})\cdot(\nabla\times G)\}\, dS \\ \mathbf{B}(\xi) &= \mathbf{B}^i(\xi) - \int_{\partial V} \left\{-i\frac{kn^2}{c}(\mathbf{E}\times\mathbf{n})\cdot G + (\mathbf{B}\times\mathbf{n})\cdot(\nabla\times G)\right\} dS\end{aligned}$$

where

$$\begin{aligned}\mathbf{E}^i(\xi) &= \int_V G\cdot\mathbf{j}_E\, dV \\ \mathbf{B}^i(\xi) &= \int_V G\cdot\mathbf{j}_B\, dV\end{aligned}$$

are the field generated by the sources in an infinite homogenous medium of the same index as in V.

Scattering problem



Using the integral relations for the inside and the outside of all object we find the following set of *Müller scattering equations.*

$$n_j^2\mathbf{E}_j^-(\xi) + n_0^2\mathbf{E}_0^+(\xi) = 2n_0^2\mathbf{E}^i(\xi)$$

$$-2\int_{S_j}\left\{ik_0c(\mathbf{B}_j^+\times\mathbf{n})\cdot(n_j^2G_j - n_0^2G_0) + (\mathbf{E}_j^+\times\mathbf{n})\cdot(\nabla\times(n_j^2G_j - n_0^2G_0))\right\} d$$

$$+2\sum_{l\neq j}\int_{S_l}\{ik_0c(\mathbf{B}_l^+\times\mathbf{n})\cdot(n_0^2G_0) + (\mathbf{E}_l^+\times\mathbf{n})\cdot(\nabla\times(n_0^2G_0))\}\,dS$$

$$\mathbf{B}_j^-(\xi) + \mathbf{B}_j^+(\xi) = 2\mathbf{B}^i(\xi)$$

$$-2\int_{S_j}\left\{-i\frac{k_0n_j^2}{c}(\mathbf{E}_l^+\times\mathbf{n})\cdot(n_j^2G_j - n_0^2G_0) + (\mathbf{B}_l^+\times\mathbf{n})\cdot(\nabla\times(G_j - G_0))\right\} d$$

$$+2\sum_{l\neq j}\int_{S_l}\left\{-i\frac{k_0}{c}(\mathbf{B}_l^+\times\mathbf{n})\cdot(n_0^2G_0) + (\mathbf{E}_l^+\times\mathbf{n})\cdot(\nabla\times G_0)\right\} dS$$

Boundary conditions at interfaces

- The magnetic field and the tangential component of the electric field are continuous.
- The normal component of the electric field satisfy

$$(n_0^2 \mathbf{E}_j^+ - n_j^2 \mathbf{E}_j^-) \cdot \mathbf{n} = 0,$$

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- $\mathbf{E}_j^-$ and $\mathbf{B}_j^-$ can be eliminated and we have a closed system of equations for the surface field values $\mathbf{E}_j^+$ and $\mathbf{B}_j^+$.
- The fields can be decomposed into tangential and normal components

$$\boldsymbol{\varphi}_i = (\mathbf{E} \times \mathbf{n})|_{S_i} \qquad \boldsymbol{\varphi}_i^i = \frac{2n_0^2}{n_j^2 + n_0^2}(\mathbf{E}^i \times \mathbf{n})|_{S_i}$$

$$\boldsymbol{\psi}_i = (\mathbf{B} \times \mathbf{n})|_{S_i} \qquad \boldsymbol{\psi}_i^i = (\mathbf{B}^i \times \mathbf{n})|_{S_i}$$

$$u_i = (\mathbf{E} \cdot \mathbf{n})|_{S_i} \qquad u_i^i = (\mathbf{E}^i \cdot \mathbf{n})|_{S_i}$$

$$v_i = (\mathbf{E} \cdot \mathbf{n})|_{S_i} \qquad v_i^i = (\mathbf{E}^i \cdot \mathbf{n})|_{S_i}$$

The resulting system of integral equations is

$$\begin{aligned}
\varphi_i(\xi) &= \varphi_i^i(\xi) + \sum_k \int_{S_k} \{\varphi_k(\mathbf{x}) \cdot K_{ik}(\xi, \mathbf{x}) + \psi_k(\mathbf{x}) \cdot L_{ik}(\xi, \mathbf{x})\}\, dS \\
\psi_i(\xi) &= \psi_i^i(\xi) + \sum_k \int_{S_k} \{\varphi_k(\mathbf{x}) \cdot P_{ik}(\xi, \mathbf{x}) + \psi_k(\mathbf{x}) \cdot Q_{ik}(\xi, \mathbf{x})\}\, dS \\
u_i(\xi) &= u_i^i(\xi) + \sum_k \int_{S_k} \{\varphi_k(\mathbf{x}) \cdot M_{ik}(\xi, \mathbf{x}) + \psi_k(\mathbf{x}) \cdot N_{ik}(\xi, \mathbf{x})\}\, dS \\
v_i(\xi) &= v_i^i(\xi) + \sum_k \int_{S_k} \{\varphi_k(\mathbf{x}) \cdot R_{ik}(\xi, \mathbf{x}) + \psi_k(\mathbf{x}) \cdot S_{ik}(\xi, \mathbf{x})\}\, dS
\end{aligned}$$

where

$$L_{ik}(\xi, \mathbf{x}) = -\frac{2ik_0 c}{n_i^2 + n_0^2}\left(n_k^2 G_k(\xi, \mathbf{x}) - n_0^2 G_0(\xi, \mathbf{x})\right)$$

Computational strategy

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3. Reconstruct the electric and magnetic surface fields from the identities

$$\mathbf{E}_i(\boldsymbol{\xi}) = u_i(\boldsymbol{\xi})\mathbf{n}_i(\boldsymbol{\xi}) + \mathbf{n}_i(\boldsymbol{\xi}) \times \boldsymbol{\varphi}_i(\boldsymbol{\xi})$$
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4. Construct the Stress Tensor

$$\mathbf{T}_i = \varepsilon_0 \mathbf{E}_i \mathbf{E}_i + \frac{1}{\mu_0}\mathbf{B}_i \mathbf{B}_i - \frac{1}{2}\left(\varepsilon_0 \mathbf{E}_i \cdot \mathbf{E}_i + \frac{1}{\mu_0}\mathbf{B}_i \cdot \mathbf{B}_i\right)$$

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5. Compute the surface integral

$$\mathbf{F}_i = \int_{S_i} \mathbf{T}_i(\mathbf{x}) \cdot \mathbf{n}_i dS$$

Numerical accuracy

- Each surface is split into a large number of disjoint parts

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$$\boldsymbol{\varphi}_{ij} = \boldsymbol{\varphi}^i_{ij} + \sum_{kl} \int_{S_{kl}} \left\{ \boldsymbol{\varphi}_k(\mathbf{x}) \cdot K_{ik}(\boldsymbol{\xi}_{ij}, \mathbf{x}) + \boldsymbol{\psi}_k(\mathbf{x}) \cdot L_{ik}(\boldsymbol{\xi}_{ij}, \mathbf{x}) \right\} dS$$

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- A simple approximation scheme for the terms in the sums is

$$\int_{S_{kl}} \boldsymbol{\psi}_k(\mathbf{x}) \cdot L_{ik}(\boldsymbol{\xi}_{ij}, \mathbf{x}) dS \approx \boldsymbol{\psi}_{kl} \cdot L_{ijkl}$$

where the tensor L_{ijkl} is defined by

$$L_{ijkl} = \int_{S_{kl}} L_{ik}(\boldsymbol{\xi}_{ij}, \mathbf{x}) dS$$

- The same approximation applied to the other terms in the scattering equations give us a set of linear algebraic equations.

$$\begin{aligned}\varphi_{ij} &= \varphi^{i}_{ij} + \sum_{kl}\{\varphi_{kl}\cdot K_{ijkl} + \psi_{kl}\cdot L_{ijkl}\}\\ \psi_{ij} &= \psi^{i}_{ij} + \sum_{kl}\{\varphi_{kl}\cdot P_{ijkl} + \psi_{kl}\cdot Q_{ijkl}\}\end{aligned}$$

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- For any realistic application this system must be solved by iteration.
- A large fraction of the work consist of computing the matrix elements and this makes the algoritm well suited for running on large parallel clusters.

Numerical tests

- The implementation is tested on simple dielectric spheres and layered spheres.
- For these systems the exact solution of the scattering problem is known. These are the Mie-solutions.

Scaling behaviour of the boundary integral method

- We consider N randomly places dielectric spheres.

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N	Running times(sec)	#cpu
1	45	2
2	65	5
3	68	10
4	70	17
5	71	26
6	73	37
7	90	50
9	105	82
12	119	145

Casimir forces from Matsubara-Green's functions

- The basic elements in the description of Casimir forces using Matsubara-Green's functions was developed by Dzyalonshinskii,Lifshitz and Pitaevskii in 1961.

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The force on an nonmagnetic object V with surface S and linear electric response defined by $\varepsilon = \varepsilon(\mathbf{x},\omega)$ is given by

$$\mathbf{F} = \int_0^\infty \left(\int_S \mathcal{T}(\mathbf{x},\mathbf{x},w) \cdot \mathbf{n} \; dS \right) dw$$

$\mathcal{T}$ is the stress tensor evaluated at imaginary frequencies

$$\begin{aligned}\mathcal{T}(\mathbf{x},\boldsymbol{\xi},w) &= \varepsilon(\mathbf{x},iw)G^E(\mathbf{x},\boldsymbol{\xi},w)+\frac{1}{\mu_0}G^B(\mathbf{x},\boldsymbol{\xi},w)\\ &\quad -\frac{1}{2}Tr\left(\varepsilon(\mathbf{x},iw)G^E(\mathbf{x},\boldsymbol{\xi},w)+\frac{1}{\mu_0}G^E(\mathbf{x},\boldsymbol{\xi},w)\right)I\end{aligned}$$

and G^E and G^B are the zero temperature Matsubara-Green's functions.

The Matsubara-Green's functions are calculated from the following equations

$$
\begin{aligned}
\nabla \times \nabla \times \mathcal{E}(\mathbf{x},\boldsymbol{\xi},w) + \left(\frac{w}{c}\right)^2 n^2(\mathbf{x},iw)\mathcal{E}(\mathbf{x},\boldsymbol{\xi},w) &= \delta(\mathbf{x}-\boldsymbol{\xi})I \\
\nabla \times \nabla \times \mathcal{B}(\mathbf{x},\boldsymbol{\xi},w) + \left(\frac{w}{c}\right)^2 n^2(\mathbf{x},iw)\mathcal{B}(\mathbf{x},\boldsymbol{\xi},w) &= \nabla \times (\delta(\mathbf{x}-\boldsymbol{\xi})I) \times \nabla \\
G^E(\mathbf{x},\boldsymbol{\xi},w) &= \frac{\hbar^2 w^2}{\pi}\mathcal{E}(\mathbf{x},\boldsymbol{\xi},w) \\
G^B(\mathbf{x},\boldsymbol{\xi},w) &= \frac{\hbar}{\pi c^2}\mathcal{B}(\mathbf{x},\boldsymbol{\xi},w)
\end{aligned}
$$

The quantities $\mathcal{B}(\mathbf{x},\boldsymbol{\xi},w)$,$\mathbf{n}(\mathbf{x})\times\mathcal{E}(\mathbf{x},\boldsymbol{\xi},w)$ and $\mathbf{n}(\mathbf{x}) \cdot (n^2(\mathbf{x},iw)\mathcal{E}(\mathbf{x},\boldsymbol{\xi},w))$ are continous across interfaces.

Extensions

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2. Temperature effects are included by making $\varepsilon(\mathbf{x},\omega, T)$ a function of the temperature. The formula for $\mathcal{T}$ is the same. The force formula become

$$\mathbf{F} = \frac{T}{4\pi} \sum_{n=0}^{\infty} \left(\int_S \mathcal{T}(\mathbf{x}, \mathbf{x}, w_n) \cdot \mathbf{n} \ dS \right)$$

where $w_n = 2\pi n T$ are the Matsubara frequencies.

Remarks

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- Such identities are realizations of the *fluctuation-dissipation theorem.*

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4. Possibilities for dimensional reduction is severely limited, the problem is almost always 3D.

Boundary Integral equations for Matsubara-Green's functions.

The key element in deriving the boundary integral formulation for the Matsubara-Green's function is the following Green's identity for matrix field A and B.

$$\begin{aligned} & {}^{t}(\nabla \times (\nabla \times A))\, B - {}^{t} A\, (\nabla \times (\nabla \times B)) \\ = \;& {}^{t}\left({}^{t}(\nabla \times A) \times B\right) - {}^{t}\left({}^{t}A \times (\nabla \times B)\right) \end{aligned}$$

where the transpose acts on the left pair of indices and where for matrices we define

$$\begin{aligned} (\nabla \times A)_{ij} &= \varepsilon_{ikl} \partial_{x_k} a_{lj} \\ (C \times D)_{ijk} &= \varepsilon_{jnm} c_{in} d_{mk} \end{aligned}$$

Using integral for of the identity we get the following boundary integral equations for the Matsubara-Green's functions

$$\begin{aligned} n_1^2\mathcal{E}_1(\boldsymbol{\eta},\boldsymbol{\xi}) &+ n_0^2\mathcal{E}_0(\boldsymbol{\eta},\boldsymbol{\xi}) = 2n_j^2\mathcal{E}^i(\boldsymbol{\eta}-\boldsymbol{\xi}) \\ &- 2PV_\eta\int_S\{(n_1^2G_1-n_0^2G_0)(\mathbf{n}\times\mathcal{A}(\mathbf{x},\boldsymbol{\xi})) \\ &+ {}^t(\nabla\times(n_1^2G_1-n_0^2G_0))(\mathbf{n}\times\mathcal{E}(\mathbf{x},\boldsymbol{\xi}))\}ds \end{aligned}$$

$$\begin{aligned} \mathcal{A}_1(\boldsymbol{\eta},\boldsymbol{\xi}) &+ \mathcal{A}_0(\boldsymbol{\eta},\boldsymbol{\xi}) = 2\mathcal{A}^i(\boldsymbol{\eta}-\boldsymbol{\xi}) \\ &- 2PV_\eta\int_S\{-w^2(n_1^2G_1-n_0^2G_0)(\mathbf{n}\times\mathcal{E}(\mathbf{x},\boldsymbol{\xi})) \\ &+ {}^t(\nabla\times(G_1-G_0))(\mathbf{n}\times\mathcal{A}(\mathbf{x},\boldsymbol{\xi}))\}ds \end{aligned}$$

Where $\boldsymbol{\eta}\in S$, and $j=0,1$ if $\boldsymbol{\xi}\in V_j$.

The matrix function $\mathcal{E}^i_j(\boldsymbol{\eta}-\boldsymbol{\xi})$ and $\mathcal{A}^i_j(\boldsymbol{\eta}-\boldsymbol{\xi})$ are the Matsubara-Green's functions generated by the delta function source in the absence of the scattering object

$$\begin{aligned} \mathcal{E}^i(\boldsymbol{\eta}-\boldsymbol{\xi}) &= G_j(\eta-\boldsymbol{\xi}) \\ \mathcal{A}^i(\boldsymbol{\eta}-\boldsymbol{\xi}) &= \boldsymbol{\nabla}\times G_j(\eta-\boldsymbol{\xi}) \end{aligned}$$

for $\boldsymbol{\xi} \in V_j$.

Observations

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- The force formula require us to compute the integral of $\mathcal{E}(\boldsymbol{\eta}, \boldsymbol{\eta})$, exactly at the nonintegrable singularity.
- In order to compute the force we must solve the boundary integral equations for all positions, $\boldsymbol{\xi}$, of the source on the surface.

Symmetries

Let G be the group of affine isometries of space. Thus $g \in G$ implies that

$$g(\mathbf{x})_i = a_{ij}x_j + b_i$$

where $a^t = a$ is an orthogonal matrix. Observe that the source functions $\mathcal{E}^i$ and $\mathcal{A}^i$ satisfy the relations

$$\begin{aligned} a_{pk}a_{qj}\mathcal{E}^i_{pq}(\mathbf{x}) &= \mathcal{E}^i_{kj}(g^{-1}(\mathbf{x})) \\ |a|a_{pk}a_{qj}\mathcal{A}^i_{pq}(\mathbf{x}) &= \mathcal{A}^i_{kj}(g^{-1}(\mathbf{x})) \end{aligned}$$

These relations means that the sources are *invariant* under the action of the group G. The Matsubara-Green's functions are in general not invariant under G. However if the surface S is invariant under a subgroup $H \subset G$ then the Matsubara-Green's functions are also invariant under H. Thus for all $g \in H$ we have

$$\begin{aligned} a_{pk}a_{qj}\mathcal{E}_{pq}(\mathbf{x},\mathbf{y}) &= \mathcal{E}_{kj}(g^{-1}(\mathbf{x}),g^{-1}(\mathbf{y})) \\ |a|a_{pk}a_{qj}\mathcal{A}_{pq}(\mathbf{x},\mathbf{y}) &= \mathcal{A}_{kj}(g^{-1}(\mathbf{x}),g^{-1}(\mathbf{y})) \end{aligned}$$

Casimir force on symmetric objects

- For $g \in H$ we have from the change of variable formula and the invariance of the stress tensor, $\mathcal{T}$, that the Casimir force $\mathbf{f} = (f_i)$ on the object enclosed by the surface S is

$$f_i = \int_S a_{pi} a_{qj} \mathcal{T}_{pq}(\mathbf{x}) a_{lj} n_l(\mathbf{x}) dS$$

- If the surface is invariant under reflection through the origin we have

$$a_{ij} = -\delta_{ij}$$

The force formula then immediately gives

$$\mathbf{f} = 0$$

Thus there is no Casimir *force* on a sphere or a spherical shell, there is however a Casimir *stress*.

Reduction

1. Let $S_0 \subset S$ be a subset of points of S such that S_0 generate the whole of S under the action of the symmetry group H.

$$H(S_0) = S$$

Then we only need to solve the equations for source locations in S_0. This can, depending on the nature of the subgroup H, lead to a substancial reduction in computational effort.

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2. Let H_ξ be the subgroup of H that fix a source location ξ. If H_ξ is a continous subgroup with $H_\xi(\gamma) = S$ where $\gamma \subset S$ is a curve on S, the problem can be reduced to a 1D integral equation on γ.

Regularization

The basic idea for regularizing the Matsubara-Green's functions is to subtract the (infinite) contribution from the sources and consider the difference

$$\triangle\mathcal{E}(\boldsymbol{\eta},\boldsymbol{\xi}) = \mathcal{E}(\boldsymbol{\eta},\boldsymbol{\xi}) - \mathcal{E}^{i}(\boldsymbol{\eta}-\boldsymbol{\xi})$$

as the basic physical quantity.

$$
\begin{aligned}
n_1^2 \triangle \mathcal{E}_1(\boldsymbol{\eta}, \boldsymbol{\xi}) \quad &+ \quad n_0^2 \triangle \mathcal{E}_0(\boldsymbol{\eta}, \boldsymbol{\xi}) = (n_1^2 - n_0^2) G_1(\boldsymbol{\eta} - \boldsymbol{\xi}) + \triangle \mathcal{E}^i(\boldsymbol{\eta} - \boldsymbol{\xi}) \\
&- \quad 2PV_\eta \int_S \{(n_1^2 G_1 - n_0^2 G_0)(\mathbf{n} \times \triangle \mathcal{A}(\mathbf{x}, \boldsymbol{\xi})) \\
&+ \quad {}^t(\nabla \times (n_1^2 G_1 - n_0^2 G_0))(\mathbf{n} \times \triangle \mathcal{E}(\mathbf{x}, \boldsymbol{\xi}))\} ds
\end{aligned}
$$

$$
\begin{aligned}
\triangle \mathcal{A}_1(\boldsymbol{\eta}, \boldsymbol{\xi}) \quad &+ \quad \triangle \mathcal{A}_0(\boldsymbol{\eta}, \boldsymbol{\xi}) = \triangle \mathcal{A}^i(\boldsymbol{\eta} - \boldsymbol{\xi}) \\
&- \quad 2PV_\eta \int_S \{-w^2 (n_1^2 G_1 - n_0^2 G_0)(\mathbf{n} \times \triangle \mathcal{E}(\mathbf{x}, \boldsymbol{\xi})) \\
&+ \quad {}^t(\nabla \times (G_1 - G_0))(\mathbf{n} \times \triangle \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}))\} ds
\end{aligned}
$$

where the new sources are

$$
\begin{aligned}
\triangle\mathcal{E}^i(\boldsymbol{\eta}-\boldsymbol{\xi}) &= -2PV_{\eta}\int_S\{(n_1^2G_1-n_0^2G_0)(\mathbf{x}-\boldsymbol{\eta})(\mathbf{n}\times\mathcal{A}^i(\mathbf{x}-\boldsymbol{\xi}))\\
&\quad +{}^t(\nabla\times(n_1^2G_1-n_0^2G_0))(\mathbf{x}-\boldsymbol{\eta})(\mathbf{n}\times\mathcal{E}^i(\mathbf{x}-\boldsymbol{\xi}))\}ds\\
\triangle\mathcal{A}^i(\boldsymbol{\eta}-\boldsymbol{\xi}) &= -2PV_{\eta}\int_S\{-w^2(n_1^2G_1-n_0^2G_0)(\mathbf{x}-\boldsymbol{\eta})(\mathbf{n}\times\mathcal{E}^i(\mathbf{x}-\boldsymbol{\xi}))\\
&\quad +{}^t(\nabla\times(G_1-G_0))(\mathbf{x}-\boldsymbol{\eta})(\mathbf{n}\times\mathcal{A}^i(\mathbf{x}-\boldsymbol{\xi}))\}ds
\end{aligned}
$$

Remarks

1. For the case when S consists of a finite number of parallel plane surfaces the boundary integral equations are regular in the limit $\xi \longrightarrow \eta$.
2. For general surfaces the integrand has a nonintegrable singularity. Further regularization is needed.