GEODESIC WEBS OF HYPERSURFACES

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Аннотация

In the present paper we study geometric structures associated with webs of hypersurfaces. We prove that with any geodesic $(n+2)$-web on an n-dimensional manifold there is naturally associated a unique projective structure and, provided that one of web foliations is pointed, there is also associated a unique affine structure. The projective structure can be chosen by the claim that the leaves of all web foliations are totally geodesic, and the affine structure by an additional claim that one of web functions is affine.

These structures allow us to determine differential invariants of geodesic webs and give geometrically clear answers to some classical problems of the web theory such as the web linearization and the Gronwall theorem.

1 Introduction

In the present paper we study geometric structures associated with webs of hypersurfaces. We prove that with any geodesic $(n+2)$-web on an n-dimensional manifold there is naturally associated a unique projective structure and, provided that one of web foliations is pointed, there is also associated a unique affine structure. The projective structure can be chosen by the claim that the leaves of all web foliations are totally geodesic, and the affine structure by an additional claim that one of web functions is affine.

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This work is the direct continuation of the papers [1], [2], [3] of the authors on geodesic webs in the plane. In [8] a similar problem of existence of projective structures was studied by a different method which, as indicated by the author of [8], is not invariant. In contrast to [2], in the current work we use the language of differential forms. This method allows us to considerably simplify formulas and give implicit expressions for invariants of a geodesic web.

2 Affine connections

Let $M = M^n$ be a smooth manifold of dimension n, let ∇ be a torsion-free affine connection in the cotangent bundle T^*M, and let d_∇ be the covariant differential:

$$d_\nabla : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M).$$

This differential can be represented in the form

$$d_\nabla = d \oplus d^s_\nabla,$$

where d is the de Rham differential, and

$$d^s_\nabla : \Omega^1(M) \to S^2(\Omega^1)(M)$$

is the symmetric part of the differential d_∇.

3 Geodesic foliations and projective structures

A nonvanishing differential 1-form $\omega \neq 0$ defines a one-codimensional foliation if $\omega \wedge d\omega = 0$. The leaves of this foliation are totally geodesic in the connection ∇ if and only if (see [2])

$$d^s_\nabla \omega = \theta \cdot \omega, \tag{1}$$

where θ is a differential 1-form.

We call a function f *totally geodesic* in the connection ∇ if its level hypersurfaces $f = \mathrm{const}$. are totally geodesic in the connection ∇, and we call a function f *affine* in the connection ∇ if

$$d^s_\nabla \, df = 0. \tag{2}$$

The dimension of the solution space of equation (2) is called the *affine rank* of the connection ∇.

Note that the affine rank of the connection ∇ is equal to the dimension of the manifold M if and only if the connection ∇ is flat.

Two connections ∇ and ∇' are *projectively equivalent* if and only if they have the same geodesics or (see [10]) if

$$d^s_\nabla(\omega) - d^s_{\nabla'}(\omega) = \rho \cdot \omega$$

for some 1-form ρ and all 1-forms ω. Moreover, relation (1) shows that projectively equivalent connections have the same totally geodesic foliations.

4 Webs

A *d-web of codimension one* is a set of d one-codimensional foliations on M if the foliations are given by differential 1-forms $\omega_i, i = 1, \dots, d$, and every n of them are linearly independent. Denote such a web by $< \omega_1, \dots, \omega_d >$.

For a given connection ∇, we say that a d-web is *geodesic* if the leaves of all foliations are totally geodesic in the connection ∇.

Two sets $< \omega_1, \dots, \omega_d >$ and $< s_1 \omega_1, \dots, s_d \omega_d >$ define the same d-web if $s_1 \neq 0, \dots, s_d \neq 0$, where $s_i \in C^\infty(M)$.

Suppose that forms $\omega_1, \dots, \omega_d$ define a d-web. Take n of them, say, $\omega_1, \dots, \omega_n$, as a basis. Then in the basis $\omega_1, \dots, \omega_n$, the forms $\omega_i, i \geq n+1$, can we written as follows:

$$a_{i1}\omega_1 + \cdots + a_{in}\omega_n + \omega_i = 0, \tag{3}$$

where the coordinates a_{ij} do not vanish, and for simplicity in what follows, we denote $a_{i\,n+2}$ just a_i. By choosing the factors $s_i, i = 1, \ldots, n$, we can get the formula

$$\omega_1 + \cdots + \omega_n + \omega_{n+1} = 0. \tag{4}$$

In what follows, we shall also use normalizations (3) and (4). Remark that under normalizations (3) and (4) the forms $\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_d$ and $s_1\,\omega_1, \ldots, s_n\,\omega_n, s_{n+1}\,\omega_{n+1}, \ldots, s_d\,\omega_d$ define the same d-web if and only if $s = s_1 = \ldots s_n = s_{n+1} = \cdots = s_d$, and the points $a^{(i)} = [a_{i1} : \cdots : a_{in}]$ of the projective space $\mathbb{RP}^{n-1}$ are invariants of the web. We shall call them the *basis invariants* (see [1]).

5 Geodesic webs

Formula (1) implies the following result:

Теорема 1. *A d-web $< \omega_1, \ldots, \omega_d >$ is geodesic if and only if*

$$d^s_\nabla \omega_i = \theta_i \cdot \omega_i, \quad i = 1, \ldots, d,$$

for some 1-forms θ_i.

Take a basis $\partial_1, \ldots, \partial_n$ of vector fields which is dual to the cobasis $\omega_1, \ldots, \omega_n$: $\omega_i(\partial_j) = \delta_{ij}$. Then

$$[\partial_i, \partial_j] = \sum_k c^k_{ij}\partial_k$$

for some functions $c^k_{ij} \in C^\infty(M)$ and

$$\nabla_{\partial_i}(\partial_j) = \sum_k \Gamma^k_{ji}\partial_k, \quad 1 \le i, j \le n,$$

where Γ^k_{ij} are the Christoffel symbols of the second type of the connection ∇.

The symmetric part d^s_∇ takes the form

$$d^s_\nabla(\omega^k) = -\sum_{i,j} \Gamma^k_{ij}\omega^j \cdot \omega^i. \tag{5}$$

It follows that

$$\Gamma^k_{ji} - \Gamma^k_{ij} = c^k_{ij}. \tag{6}$$

Let us investigate the conditions of total geodesicity of the first $n+1$ foliations of the web. For the foliations defined by the forms $\omega_1, \dots, \omega_n$ these conditions are

$$d^s_\nabla \omega_i = \theta_i \cdot \omega_i, \quad i = 1, \dots, n,$$

where

$$\theta_i = \sum_{j=1}^n \theta_{ij}\omega_j. \tag{7}$$

Comparing relations (5) and (7), we find that

$$\begin{aligned} &\Gamma^k_{ik} + \Gamma^k_{ki} + \theta_{ki} = 0, \quad i, k = 1, \dots, n, \\ &\Gamma^k_{ij} + \Gamma^k_{ji} = 0 \;\text{ if }\; i \neq k \;\text{ and }\; j \neq k. \end{aligned}$$

This and relation (6) imply the following relations among Γ^k_{ij} and c^k_{ij}, θ_{ki}:

$$\Gamma^k_{ik} = \frac{c^k_{ki} - \theta_{ki}}{2}, \tag{8}$$

$$\Gamma^k_{ij} = \frac{c^k_{ji}}{2} \;\text{ if }\; i \neq k \;\text{ and }\; j \neq k. \tag{9}$$

Denote by σ_{ij} and α_{ij} the symmetric and skew-symmetric parts of the matrix (θ_{ij}), i.e.,

$$\sigma_{ij} = \frac{\theta_{ij} + \theta_{ji}}{2}, \quad \alpha_{ij} = \frac{\theta_{ij} - \theta_{ji}}{2},$$

and assume that $t_i = \theta_{ii}$. Then the conditions of total geodesicity of the $(n+1)$st foliation determine completely the symmetric part of θ_{ij},

$$\sigma_{ij} = \frac{t_i + t_j}{2},$$

and give also the following representation of the differential form θ_{n+1}:

$$\theta_{n+1} = \sum_{i=1}^{n} t_i \theta_i.$$

The conditions of total geodesicity of the $(n+2)$nd foliation allow us to find the skew-symmetric part α_{ij}:

$$\alpha_{ij} = \frac{t_j - t_i}{2} + s_{ij},$$

where

$$s_{ij} = s_{ij}^a = \frac{1}{a_i - a_j}\Big(a_i \partial_j - a_j \partial_i\Big) \log \frac{a_j}{a_i},$$

$$\theta_{n+2} = \theta_{n+1} + \sum_{i=1}^{n} \frac{a_{i,i}}{a_i} \omega_i,$$

and $a_{i,i}$ is the covariant derivative of a_i along ∂_i.

Finally, the differential forms θ_i, θ_{n+1}, and θ_{n+2} can be written as follows:

$$\begin{cases} \theta_i = \theta_{n+1} + \displaystyle\sum_{i=1}^{n} s_{ij}\omega_j, \quad i = 1, \ldots, n, \\ \theta_{n+1} = \theta_{n+1}, \\ \theta_{n+2} = \theta_{n+1} + \displaystyle\sum_{i=1}^{n} \frac{a_{i,i}}{a_i}\omega_i. \end{cases}$$

This implies the following results:

Теорема 2. *A torsion-free affine connection, for which an $(n+2)$-web of hypersurfaces is geodesic, is defined by the forms $\theta_1, \ldots, \theta_n$ which can be written as*

$$\theta_i = \theta_{n+1} + \sum_{i=1}^{n} s_{ij}\omega_j, \tag{10}$$

and the corresponding Christoffel symbols are given by formulas (8) *and* (9).

Теорема 3. *Every* $(n+2)$*-web of hypersurfaces determines a unique projective structure, namely a class of projectively equivalent connections defined by forms* (10).

We shall call the only projective structure indicated in Theorem 3 *canonical.*

Note that for any geodesic d-web, where $d \geq n+2$, the canonical projective structures defined by different $(n+2)$-subwebs coincide. Hence in what follows, we will speak on the canonical projective structure of a geodesic d-web, $d \geq n+2$.

We shall call a web with a singled out foliation a *pointed web.*

Consider a pointed d-web and assume that $(n+1)$st foliation is singled out. Choose (locally) a normalization in such a way that $\omega_{n+1} = df$, and choose an affine connection in such a way that the form $\theta_{n+1} \equiv 0$.

Then for this affine connection the function f is affine. Remark that such a function f is defined up to an affine gauge transformation $f \to af + b$.

Теорема 4. *Every pointed* $(n+2)$*-web of hypersurfaces determines a unique affine connection for which the web is geodesic and the singled out foliation is defined by an affine function*

We shall call the only affine structure indicated in Theorem 4 *canonical* too.

6 Conditions of geodesicity of a d-web

Suppose that we have already made normalizations (4) and (3). Consider a foliation given by a form ω, where

$$\omega = b_1\omega_1 + \cdots + b_n\omega_n.$$

This foliation is totally geodesic in the connection ∇ if

$$d^s_\nabla \omega = \theta \cdot \omega,$$

or

$$s^a_{ij} = s^b_{ij}.$$

This produces the following result.

Теорема 5. *Denote by (a^k), where $k = n+2, \dots, d$, a set of basic invariants. Then a d-web of hypersurfaces is geodesic if and only if*

$$s^{(a^k)}_{ij} = s^{(a^l)}_{ij} \;\; \textit{for all} \;\; k, l = n+2, \dots, d.$$

7 Linearizability of webs

It is known that if $\dim M = 2$, then a manifold M is flat if and only if the Liouville tensor vanishes (see [5], [6] or [3]). On the other hand, if $\dim M > 2$, then a manifold M^n is flat if and only if the Weyl tensor vanishes (see [9]). This and the results of Section 4 imply the following theorem.

Теорема 6. ([2]) 1. *If* $\dim M = 2$, *then a d-web, $d \geq 4$, of hypersurfaces is locally linearizable if and only if it is geodesic and the Liouville tensor of the canonical projective structure vanishes.*

2. *If* $\dim M > 2$, *then a d-web, $d \geq n+2$, is locally linearizable if and only if it is geodesic and the Weyl tensor of the canonical projective structure vanishes.*

8 Theorems of Gronwall's type

Theorem 3 implies the following theorems of Gronwall's type (see [4] and [2] for $n = 2$).

Теорема 7. *If $d \geq n + 2$, then any mapping of a geodesic d-web of hypersurfaces onto another geodesic d-web is a projective transformation with respect to the canonical projective structures.*

Theorem 4 implies a stronger theorem of Gronwall's type.

Теорема 8. *If $d \geq n + 2$, then any mapping of pointed geodesic d-webs of hypersurfaces is affine with respect to the canonical affine structures.*

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