GRASSMANN CODES AND SCHUBERT UNIONS

by

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Abstract. — We study subsets of Grassmann varieties $G(l, m)$ over a field F, such that these subsets are unions of Schubert cycles, with respect to a fixed flag. We study such sets in detail, and give applications to coding theory, in particular for Grassmann codes. For $l = 2$ much is known about such Schubert unions with a maximal number of F_q-rational points for a given spanning dimension. We study the case $l = 3$ and give a conjecture for general l. We also define Schubert union codes in general, and study the parameters and support weights of these codes.

Résumé. — Soit $G(l, n)$ une variété de Grassmann sur un corps F. Nous étudons les sousensembles de G étant unions de cycles de Schubert, relativement à un drapeau fixe. Nous les étudons en détail, et donnons les applications à la théorie des codes de Grassmann. Dans le cas $l = 2$ on sait beaucoup sur les unions de Schubert ayant un nombre maximal de point F_q-rationnels pour un dimension lineaire donnée. Nous étudons le cas $l = 3$ et faisons une conjecture pour le cas général. Nous définons les codes de unions de Schubert en général, et nous étudons les parametres et poids de support pour ces codes.

1. Introduction

Let $G(l, m) = G_F(l, m)$ be the Grassmann variety of l-dimensional subspaces of a fixed m-dimensional vector space V over a field F. By the standard Plücker coordinates $G(l, m)$ is embedded into $\mathbf{P}^{k-1} = \mathbf{P}_F^{k-1}$ as a non-degenerate smooth subvariety, where $k = \binom{m}{l}$. In [**HJR**] we defined and studied Schubert unions in $G(l, m)$. These were unions of Schubert cycles with respect to a fixed coordinate flag for an m-space V. In this paper we give a more detailed picture of the set of these Schubert unions for some fixed, low values of l, m. We also raise and partly answer som natural questions, concerning properties of the associated Grassmann codes $C(l, m)$ in case the field F is finite.

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In Section 6 we study techniques for finding which Schubert unions that have the maximum F_q-rational points, given their spanning dimension in the Plücker space. For $l = 2$ this issue was treated and clarified in [**HJR**]. In the present paper we investigate the case $l = 3$ where we study an associated "continuous" problem, in the hope of finding an interplay between the issue of finding optimal Schubert unions and questions concerning volume estimates of some natural sets in l-space. These investigations enable us to formulate two natural conjectures about Schubert unions with a maximal number of points, given their spanning dimension.

In Section 7 we define and study properties of Schubert union codes for $l = 2$. These are codes whose generator matrices are formed by Plücker coordinates of the F_q rational points of a given Schubert union.

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2. Basic Description of Schubert Unions

In this section we will recall the well known definition of Schubert cycles in the Grassmann variety $G(l, m)$ over a field F, and describe unions of such cycles. More details can be found in [**HJR**].

Let $B = \{e_1, \ldots, e_m\}$ be a basis of an m-dimensional vector space V over F. Let $A_i = Span\{e_1, \ldots, e_i\}$ in V, for $i = 1, \ldots, m$. Then $A_1 \subset A_2 \subset \ldots \subset A_m = V$ form a complete flag of subspaces of V.

The ordered l-tuples α belong to the grid

$$G_{G(l,m)} = \{\beta = (b_1, \ldots, b_l) \in \mathbf{Z}^l | 1 \leq b_1 < b_2 < \ldots < b_l \leq m\}$$

This grid is partially ordered by $\alpha \leq \beta$ if $a_i \leq b_i$ for $i = 1, \ldots, l$, and it represents the Plücker coordinates (the maximal minors of matrices representing l-spaces, with alternating signs) of the standard embedding of $G(l, m)$ in $\mathbf{P}^{k-1}$. For each $\alpha \in G_{G(l,m)}$ the Schubert cycle S_α is defined as:

$$S_\alpha = \{W | \dim(W \cap A_{a_i}) \geq i, \quad i = 1, \ldots, l\}.$$

Definition 2.1. — $G_S = G_\alpha = \{\beta \in G_{G(l,m)} | \beta \leq \alpha\}$.

Definition 2.2. — *For a subset M of $G(l, m) \subset \mathbf{P}(\wedge^l V)$, let $\mathcal{L}(M)$ be its linear span in the projective Plücker space $\mathbf{P}(\wedge^l V)$, and $L(M)$ the linear span of the affine cone over M in the affine cone over the Plücker space.*

We will consider finite intersections and finite unions of such Schubert cycles S_α with respect to our fixed flag. Set $\alpha_i = (a_{(i,1)}, a_{(i,2)}, \ldots, a_{(i,l)})$, for $i = 1, \ldots, s$. It is clear that: $\cap_{i=1}^s S_{\alpha_i} = S_\gamma$, where $\gamma = (g_1, \ldots, g_l)$, and g_j is the minimum of the set $\{a_{1,j}, a_{2,j}, \ldots, a_{s,j}\}$, for $j = 1, \ldots, l$. Thus the intersection of a finite set of

Schubert cycles S_α is again a Schubert cycle. In particular $\dim L(\cap S_{\alpha_i})$ is equal to the cardinality of G_γ.

For a union $S_U = \cup_{i=1}^s S_{\alpha_i}$ of Schubert cycles, denote by G_U the union $G_U = \cup_{i=1}^s G_{\alpha_i}$, and set $H_U = G_{G(l,m)} - G_U$.

Proposition 2.3. — *Let $S_{\alpha_1}, \ldots, S_{\alpha_s}$ be finitely many Schubert cycles with respect to our fixed flag. Let $S_\gamma = \cap_{i=1}^s S_{\alpha_i}$ be their intersection, and let $S_U = \cup_{i=1}^s S_{\alpha_i}$ be their union.*

1. *The intersection S_γ is itself a Schubert cycle with S-grid $G_\gamma = \cap_{i=1}^s G_{\alpha_i}$.*
2. *$\mathcal{L}(S_U) \cap G(l,m) = S_U$.*
3. *dim $L(S_U)$ equals the cardinality of the grid G_U.*
4. *The number of F_q-rational points on S_U is $\Sigma_{(x_1,\ldots,x_l)\in G_U} q^{x_1+\cdots+x_l-l(l+1)/2}$.*

For Schubert cycles this result was given in [**GT**].

Definition 2.4. — *We denote by $g_U(q) = \Sigma_{(x_1,\ldots,x_l)\in G_U} q^{x_1+\cdots+x_l-l(l+1)/2}$ the number of F_q-rational points on S_U.*

Definition 2.5. — *Let the natural map*

$$rev : G_{G(l,m)} \to G_{G(l,m)^*}$$

be defined as

$$(a_1, a_2, \ldots, a_l) \mapsto (m+1-a_l, \ldots, m+1-a_2, m+1-a_l).$$

Here $G(l,m)^*$ is the dual Grassmannian parametrizing $(m-l)$-spaces in V.

We now recall the dual of a Schubert union U.

Definition 2.6. — *Let S_U be a Schubert union in $G(l,m)$ with G-grid G_U. Then the dual of S_U is the Schubert union $S_{U^\perp} \subset G(l,m)^*$ whose G-grid $G_{U^\perp}$ is $rev(H_U)$.*

Definition 2.7. — *Let S_U be a Schubert union, and let $g_U(q)$ be its number of F_q-rational points, as given by Proposition 2.3. Let $\delta = l(m-l)$ be the Krull dimension of $G(l,m)$. Denote by $n(q)$ the number of F_q-rational points of $G(l,m)$, and set $h_U(q) = n(q) - g(q)$.*

We have:

Proposition 2.8. — *Let S_U be a Schubert union. The number of F_q-rational points of $S_{U^\perp}$ is $q^\delta h_U(q^{-1})$.*

3. Properties of Grassmann Codes

In this section we define and list some known properties of Grassmann codes.

It is well known that $G_{F_q}(l,m)$ contains n points, where

$$n = \frac{(q^m-1)(q^{m-1}-1)\cdots(q^{m-l+1}-1)}{(q^l-1)(q^{l-1}-1)\cdots(q-1)}. \tag{1}$$

Pick a Plücker representative of each of the n points as a column vector in $(F_q)^k$, for $k = \binom{m}{l}$, and form a $k \times n$-matrix M with these n vectors as columns (in any preferred order). The code $C(l,m)$ is then the code with M as generator matrix. Hence C is a linear $[n,k]$-code (only defined up to code equivalence.)

The higher weights $d_1 < d_2 < \ldots < d_k$ of $G(l,m)$ satisfy:

$$d_r = n - H_r, \tag{2}$$

where H_r is the maximum number of points from S contained in a codimension r subspace of $(F_q)^k$.

We have (in addition to $d_k = n$) the following essentially well-known result:

Proposition 3.1. — *The weights satisfy*
$d_r = q^{\delta} + q^{\delta-1} + \cdots + q^{\delta-r+1}$, *for* $r = 1, \ldots s$, *and*
$d_{k-a} = n - (1 + q + \cdots + q^{a-1})$, *for* $a = 1, \ldots, s$,
where $s = max(l, m-l) + 1$, *and* $\delta = \dim G(l,m) = l(m-l)$.
Moreover, for the code $C(2,5)$ *we have* $d_5 = n-(q^3+2q^2+q+1) = d_4+q^4 = d_6-q^2$.

The result for the lower weights was given in [**N**], the result for the higher weights is just a consequence of the existence of projective spaces within the $G(l,m)$, and the result for $C(2,5)$ was given in [**HJR**]. Studying the proofs of the statements of Proposition 3.1, one observes:

Corollary 3.2. — *For* $m \leq 5$ *all the* d_r *for the* $C(2,m)$ *are computed by Schubert unions.*

Definition 3.3. — *For given* l, m, *set* $\Delta_r = d_r - d_{r-1}$ *for* $r = 1, \ldots., k$. *(*$\Delta_0 = 0$.*)*

We have:

$$C(2,3) : \begin{bmatrix} r: & 1 & 2 & 3 \\ \Delta_r: & q^2 & q & 1 \end{bmatrix}$$

$$C(2,4) : \begin{bmatrix} r: & 1 & 2 & 3 & 4 & 5 & 6 \\ \Delta_r: & q^4 & q^3 & q^2 & q^2 & q & 1 \end{bmatrix}$$

$$C(2,5) : \begin{bmatrix} r: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \Delta_r: & q^6 & q^5 & q^4 & q^3 & q^4 & q^2 & q^3 & q^2 & q & 1 \end{bmatrix}$$

This motivates the following definitions:

Definition 3.4. — *For given l, m, let J_r be the maximum number of points in a Schubert union spanning a linear space of codimension at least r in the Plücker space, and set $D_r = n - J_r$, and $E_r = D_r - D_{r-1}$, for $r = 1, \dots, k$. ($D_0 = 0$.)*

In view of (2) we then have the following obvious, but useful, result:

Proposition 3.5. — *For all l, m, and r we have*

$$d_r \leq D_r.$$

We can in principle calculate all D_r, using Proposition 2.3. It is an open question whether the upper bound D_r is equal to the true value d_r in the cases not determined by Propositions 3.1.

Recall the polynomials $g_U(q)$ defined in Definition 2.7 (and described in Proposition 2.3). In [**HJR**] we gave the following result. The details of the proof can be found in [**HJRp**].

Proposition 3.6. — *Fix a dimension $0 \leq K \leq \binom{m}{2}$, and consider the set of Schubert unions $\{S_U\}_K$ in $G(2, m)$ with spanning dimension K. Among these unions, let S_L be the unique one on the form $S_{(x,m)} \cup S_{(x+1,y)}$, with $1 \leq x \leq m-1$ and $1 \leq y \leq m$, and let S_R be the unique union of the form $S_{(x,x+1)} \cup S_{(a,x+2)}$, with $1 \leq x \leq m-1$ and $1 \leq a \leq x+1$.*

Then S_L or S_R is maximal in $\{S_U\}_K$ with respect to the natural lexicographic order on the polynomials g_U. Furthermore, the one(s) that is(are) maximal with respect to g_U, also has(have) the maximum number of points over F_q for all large enough q.

Hence one only has to check two Schubert of each spanning (co)dimension to find the union maximizing g_U. This result makes it a routine matter to find the J_r, D_r and E_r for $l = 2$ and fixed m if the union(s) U with maximum g_U is also the one with maximal number of points. For large enough q , at least, this always holds.

4. Explicit analysis of Schubert unions for low l and m

In this section we will give a detailed study of Schubert unions in $G(2, m)$ for some low values of m, and we will also study Schubert unions in $G(3, 6)$. For the $G(2, m)$ the Schubert unions form a Boolean algebra $P(M)$ (with 2^{m-1} elements), for $M = \{1, \dots m-1\}$. We identify a Schubert union S_U with an element M_U of $P(M)$ as follows:

Definition 4.1. — *$M_U = \{m_1, \dots, m_r\}$ if there are m_i points (x, y) in G_U with $x = i$, for $i = 1, \dots, r$, and no points with $x = i$, for $i > r$.*

The m_i form a decreasing sequence.

In Figure 1 we list all Schubert unions in $G(2, 5)$.

FIGURE 1. Schubert unions for $G(2,5)$

U	Span	Krull	M_U	number of points	Maximal
$\emptyset$	0	-1	$\emptyset$	0	Yes
$(1,2)$	1	0	$\{1\}$	1	Yes
$(1,3)$	2	1	$\{2\}$	$q+1$	Yes
$(1,4)$	3	2	$\{3\}$	q^2+q+1	Yes
$(1,5)$	4	3	$\{4\}$	q^3+q^2+q+1	Yes
$(2,3)$	3	2	$\{1,2\}$	q^2+q+1	Yes
$(1,4)\cup(2,3)$	4	2	$\{1,3\}$	$2q^2+q+1$	No
$(1,5)\cup(2,3)$	5	3	$\{1,4\}$	q^3+2q^2+q+1	Yes
$(2,4)$	5	3	$\{2,3\}$	q^3+2q^2+q+1	Yes
$(1,5)\cup(2,4)$	6	3	$\{2,4\}$	$2q^3+2q^2+q+1$	No
$(2,5)$	7	4	$\{3,4\}$	$q^4+2q^3+2q^2+q+1$	Yes
$(3,4)$	6	4	$\{1,2,3\}$	$q^4+q^3+2q^2+q+1$	Yes
$(1,5)\cup(3,4)$	7	4	$\{1,2,4\}$	$q^4+2q^3+2q^2+q+1$	Yes
$(2,5)\cup(3,4)$	8	4	$\{1,3,4\}$	$2q^4+2q^3+2q^2+q+1$	Yes
$(3,5)$	9	5	$\{2,3,4\}$	$q^5+2q^4+2q^3+2q^2+q+1$	Yes
$(4,5)$	10	6	$\{1,2,3,4\}$	$q^6+q^5+2q^4+2q^3+2q^2+q+1$	Yes

In the column to the right we indicate whether the Schubert union in question has the maximum possible of points among the Schubert unions of that spanning dimension. The (affine) spanning dimension is given in the column marked "Span". The maximal Krull dimension of a component is given in the column marked "Krull". This Krull dimension is of course equal to the degree of $g_U(q)$, interpreted as a polynomial in q, and this is the polynomial appearing in the column marked "number of points". Moreover it is well known that the Krull dimension of a Schubert cycle $S_{(a_1,\dots,a_l)}$ is $a_1+a_2+\dots+a_l-\frac{l(l+1)}{2}$, so the Krull dimension can be "read off" both from the leftmost and the "number of points" column.

The dual of a Schubert union with a given M_U in $G(2,m)$ is the Schubert union V with $M_V=\{1,\dots,m-1\}-M_U$.

We observe that

Proposition 4.2. — *For the $G(2,m)$ there is no self-dual Schubert union for any $m\geq 2$.*

Proof. — A subset M_U of $M=\{1,\dots,m-1\}$ is never equal to its own complement. □

We shall see below that the situation may be different for $G(l,m)$ with $l=3$.

A table for $G(2,4)$ can derived from the table for $G(2,5)$, roughly speaking by only focusing on those rows where M_U is a subset of $\{1,2,3\}$. A little caution is necessary, though. For $G(2,4)$ one quickly sees that all 8 Schubert unions are maximal for their spanning dimensions. For $G(2,5)$ those with $M_U = \{1,3\}$ or $\{2,4\}$ are not maximal. So it is not an "intrinsic" property of a Schubert union whether it is maximal for its spanning dimension. It depends on the Grassmann variety, in which it sits.

The tables above were produced, mainly by using Corollary 2.3.

Remark 4.3. — Given two Schubert unions U_1, U_2 with corresponding polynomials $g_{U_1}(q)$ and $g_{U_2}(q)$. The issue of which of the two that gives the highest value for given q is in principle a different one, for each q. On the other hand, if we order the Schubert unions, first by degree, and then lexicographically with respect to g_U for each degree, then it is clear that this order is the same as the "number of point"-order for all large enough q. In all the examples we have seen up to now, it is clear by inspection that these orders are the same for all prime powers q. Hence the "Yes" and "No" in the "Max." column can be interpreted in two ways simultaneously (counting points, and ordering with respect to g_U).

In Figure 2 we give a table listing all Schubert unions, is $G(3,6)$. We make the table shorter by listing pairs of dual unions. There is no M_U for these unions.

All Schubert unions with spanning dimension at most 9 can be found in the left half of the table, and unions with spanning dimension at least 11 can be found on the right side (as duals). For spanning dimension 10 all 6 unions are listed on at least one side.

Remark 4.4. — (i) The table reveals a situation different from the case $l = 2$ and $\binom{m}{l}$ even, where no Schubert union is self-dual. Here we see that both $(2,3,6)$ and $(1,3,6) \cup (1,4,5) \cup (2,3,5)$ are self-dual Schubert unions.

(ii) It can be shown that for $l = 2$ we have:

The dual of a Schubert union which is a proper union of s cycles, is a proper union of $s-1$, s or $s+1$ cycles. From the tables above we see that this fails for $l = 3$. The dual of $S_{(1,3,5)}$ is the proper triple union $S_{(1,5,6)} \cup S_{(2,3,6)} \cup S_{(3,4,5)}$ (and vice versa). We encourage the interested reader to reconstruct this situation, and the selfduality described in (i), by playing with cubes, representing the coordinate grids associated to these cycles.

Remark 4.5. — (i) For the Grassmann varieties we have described in the tables above, a Schubert union has a maximal number of points, given its spanning dimension, if and only if its dual union enjoys the same property, so the "Yes" and "No" in the "Max."-column of the last table apply to the left and right half of the table simultaneously. The same property holds for $(l,m) = (2,7)$ and $(2,8)$, but for reasons of space we do not give the full tables here, from which the shorter lists of the E_r at the start of this section were deduced.

FIGURE 2. Schubert unions for $G(3,6)$

U	Span	Dual Schubert union	Max.
$\emptyset$	0	$(4,5,6)$	Yes
$(1,2,3)$	1	$(3,5,6)$	Yes
$(1,2,4)$	2	$(2,5,6)\cap(3,4,6)$	Yes
$(1,2,5)$	3	$(1,5,6)\cap(3,4,6)$	Yes
$(1,3,4)$	3	$(2,5,6)\cap(3,4,5)$	Yes
$(1,2,6)$	4	$(3,4,6)$	Yes
$(2,3,4)$	4	$(2,5,6)$	Yes
$(1,2,5)\cup(1,3,4)$	4	$(1,5,6)\cup(2,4,6)\cup(3,4,5)$	No
$(1,3,5)$	5	$(1,5,6)\cup(2,3,6)\cup(3,4,5)$	Yes
$(1,2,5)\cup(2,3,4)$	5	$(1,5,6)\cup(2,4,6)$	Yes
$(1,2,6)\cup(1,3,4)$	5	$(2,4,6)\cup(3,4,5)$	Yes
$(1,4,5)$	6	$(1,5,6)\cup((3,4,5)$	Yes
$(1,2,6)\cup(1,3,5)$	6	$(1,4,6)\cup(2,3,6)\cup(3,4,5)$	No
$(1,2,6)\cup(2,3,4)$	6	$(2,4,6)$	No
$(1,3,5)\cup(2,3,4)$	6	$(1,5,6)\cup(2,3,6)\cup((2,4,5)$	No
$(1,3,6)$	7	$(2,3,6)\cup(3,4,5)$	Yes
$(2,3,5)$	7	$(1,5,6)\cup(2,3,6)$	Yes
$(1,2,6)\cup(1,4,5)$	7	$(1,4,6)\cup(3,4,5)$	Yes
$(1,4,5)\cup(2,3,4)$	7	$(1,5,6)\cup(2,4,5)$	Yes
$(1,2,6)\cup(1,3,5)\cup(2,3,4)$	7	$(1,4,6)\cup(2,3,6)\cup(2,4,5)$	No
$(1,3,6)\cup(1,4,5)$	8	$(1,3,6)\cup(3,4,5)$	Yes
$(1,4,5)\cup(2,3,5)$	8	$(1,5,6)\cup(2,3,5)$	Yes
$(1,2,6)\cup(2,3,5)$	8	$(1,4,6)\cup(2,3,6)$	No
$(1,3,6)\cup(2,3,4)$	8	$(2,3,6)\cup(2,4,5)$	No
$(1,2,6)\cup(1,4,5)\cup(2,3,4)$	8	$(1,4,6)\cup(2,4,5)$	No
$(1,4,6)$	9	$(1,2,6)\cup(3,4,5)$	Yes
$(2,4,5)$	9	$(1,5,6)\cup(2,3,4)$	Yes
$(1,3,6)\cup(2,3,5)$	9	$(1,4,5)\cup(2,3,6)$	No
$(1,3,6)\cup(1,4,5)\cup(2,3,4)$	9	$(1,3,6)\cup(2,4,5)$	No
$(1,2,6)\cup(1,4,5)\cup(2,3,5)$	9	$(1,4,6)\cup(2,3,5)$	No
$(1,5,6)$	10	$(3,4,5)$	Yes
$(2,3,6)$	10	$(2,3,6)$	No
$(1,2,6)\cup(2,4,5)$	10	$(1,4,6)\cup(2,3,4)$	No
$(1,3,6)\cup(1,4,5)\cup(2,3,5)$	10	$(1,3,6)\cup(1,4,5)\cup(2,3,5)$	No

(ii)In the table for $G(3,6)$ above study the 16 unions of cycles $S_{(a,b,c)}$ with $c \leq 5$. This gives rise to the corresponding table for $G(3,5)$. But this is isomorphic to $G(2,5)$. It is an amusing exercise to translate all unions in $G(3,5)$ to corresponding ones in $G(2,5)$ and check that the relevant columns of the tables coincide.

Another way to get a picture of the code-theoretical aspects of Schubert unions is to list the E_r for $C(2,m)$, for $m = 6,7,8$, and for $C(3,6)$. The values are determined using a combination of Corollary 2.3 and Proposition 3.6.

$$C(2,6): \begin{bmatrix} r: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ E_r: & \mathbf{q^8} & \mathbf{q^7} & \mathbf{q^6} & \mathbf{q^5} & \mathbf{q^4} & q^6 & q^5 & q^4 & q^3 & q^2 & \mathbf{q^4} & \mathbf{q^3} & \mathbf{q^2} & \mathbf{q} & \mathbf{1} \end{bmatrix}$$

$C(2,7)$:

$$\begin{bmatrix} r: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ E_r: & \mathbf{q^{10}} & \mathbf{q^9} & \mathbf{q^8} & \mathbf{q^7} & \mathbf{q^6} & \mathbf{q^5} & q^8 & q^7 & q^6 & q^4 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ q^5 & q^6 & q^4 & q^3 & q^2 & \mathbf{q^5} & \mathbf{q^4} & \mathbf{q^3} & \mathbf{q^2} & \mathbf{q} & \mathbf{1} \end{bmatrix}$$

C(3,6):

$$\begin{bmatrix} r: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ E_r: & \mathbf{q^9} & \mathbf{q^8} & \mathbf{q^7} & \mathbf{q^6} & q^7 & q^5 & q^6 & q^5 & q^4 & q^3 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ q^6 & q^5 & q^4 & q^3 & q^4 & q^2 & \mathbf{q^3} & \mathbf{q^2} & \mathbf{q} & \mathbf{1} \end{bmatrix}$$

The expressions in boldface indicate values where $E_r = \Delta_r$ because of Proposition 3.1

The expressions not in boldface contribute to upper bounds for "the true values" d_r, when adding monomials from left.

Remark 4.6. — The Schubert unions for the cases studied so far; $(l,m) = (2,m)$, for $m \leq 8$, or $(l,m) = (3,6)$, have in common that the duality operation reverses the lexicographic order on the g_U for the Schubert unions of each fixed spanning dimension. This conclusion is obtained from direct inspection of all Schubert unions appearing, and tables like the ones listed for $C(2,4), C(2,5), C(2,6)$ above. But this does not hold for all (l,m), not even for $l = 2$.

Remark 4.7. — We recall from Proposition 3.6 that for each spanning (co)dimension we need only to check two explicitly defined Schubert unions S_L and S_R, to find one which is maximal with respect to g_U. In the tables below we utilize this fact to give another way to describe Schubert unions for $(2,m)$ for low m. We indicate with an L (go left) if we may use S_L, with an R (go right) if we may use S_R, and with LR if and only if we may use both. The spanning codimension is $r = \binom{m}{2} - K$.

C(2,7):

Codim :	0	1	2	3	4	5	6	7	8	9	10
Direction :	LR	LR	LR	LR	R	R	R	R	R	LR	L

Codim. :	11	12	13	14	15	16	17	18	19	20	21
Direction :	R	LR	L	L	L	L	L	LR	LR	LR	LR

C(2,8):

Codim. :	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Direction :	LR	LR	LR	LR	R	R	R	R	R	R	LR	R	R	R	LR

Codim. :	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Direction :	L	L	L	LR	L	L	L	L	L	L	LR	LR	LR	LR

C(2,9):

Codim. :	0	1	2	3	4	5	6	7	8	9	10	11	12
Direction :	LR	LR	LR	LR	R	R	R	R	R	R	R	R	R

Codim. :	13	14	15	16	17	18	19	20	21	22	23	24
Direction :	R	R	R	R	R	LR	L	L	L	L	L	L

Codim. :	25	26	27	28	29	30	31	32	33	34	35	36
Direction :	L	L	L	L	L	L	L	L	LR	LR	LR	LR

C(2,10):

Codim. :	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Direction :	LR	LR	LR	LR	R	R	R	R	R	R	R	R	R	R	R	R

Codim. :	16	17	18	19	20	21	22	23	24	25	26	27	28	29
Direction :	R	R	R	R	LR	L	R	R	R	LR	L	L	L	L

Codim. :	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
Direction :	L	L	L	L	L	L	L	L	L	L	L	L	LR	LR	LR	LR

It is clear that from these tables one can extract the information giving lists of the E_r as above. Each table starts and ends with 4 occurences of LR. This is because in the three largest and the three smallest spanning dimensions K there is only one Schubert union, and because we have only two Schubert unions with spanning dimension 3, namely $S_{(2,3)}$, in projective terms a β-plane, or $S_{(1,3)}$, an α-plane. Both have $q^2 + q + 1$ points. In codimension 3 we have the duals of these two, of course also with the same number of points.

¿From the tables for $G(2,8)$ and $G(2,9)$ one can conclude without further computations that the E_r are always monomials of type q^i in these cases (See Question (Q7) of Section 5). That is because we never jump directly from an R to an L or vice versa in these cases, we always go via an LR. For $m = 7$ there is a jump between L and R between codimensions 10 and 11, but a calculation reveals that $J_{10} - J_{11} = q^5$. For $m = 10$ we observe the fatal jump from R to L, passing from codimension 22 to 21. Here E_{22} is not a monomial in q. As opposed to the tables above it, the one for $(l, m) = (2, 10)$ is not symmetric in L and R. This proves that the duality operation does not reverse the lexicographical order on the g_U for $(l, m) = (2, 10)$.

5. Some questions and answers about the Grassmann codes $C(l, m)$

In this section we will study the code-theoretical implications of the observations in the previous two sections.

For fixed l, m, q let $C(l, m)$ be the Grassmann code over F_q described in Section 3. Recall the invariants $d_r, H_r, \Delta_r, J_r, D_r, E_r$ introduceed in Definitions 3.3, and 3.4. Inspired by Proposition 3.1, Proposition 3.6, and the observations of Section 4 we now will formulate some natural questions, which we will also comment on briefly:

For each l, m, q we obviously have:

$$\sum_{r=1}^{k} \Delta_r = d_k = n. \tag{3}$$

Here n and k are the word length and dimension of $C(l, m)$ as before. Moreover it is clear that n is the sum of $k = \binom{m}{l}$ monomials of type q^i. For each l, m one may raise the following questions:

- (Q1) Are the d_r always sums of r monomials of type q^i, for $r = 1, ..., k$?
- (Q2) Is Δ_r always a monomial of the form q^i ?
- (Q3) Is it true that:

  $$\Delta_r(q) = q^{l(m-l)} \Delta_{k+1-r}(q^{-1}),$$

  for all $C(l, m)$, and all r ? This in turn implies that if the answer to question (Q2) is (partly) positive, and $\Delta_r = q^i$ for some i, then $\Delta_{k+1-r} = q^{l(m-l)-i}$.
- Answers to (Q1), (Q2), (Q3): Affirmative for $(l, m) = (2, 3), (2, 4), (2, 5)$ by Proposition 3.1. In other cases we do not know the answers for all r (affirmative for the smallest and biggest r).

  Question (Q3) should be viewed in light of Proposition 2.8 and the duality of Schubert unions.
- (Q4) Is it true that $J_r = H_r$, and therefore $D_r = d_r$, and $E_r = \Delta_r$, for all l, m, r ?
- Answer: Affirmative for $(l, m) = (2, 3), (2, 4), (2, 5)$. In other cases we do not know the answers for all r (affirmative for the smallest and biggest r).

  Taking into account the possibility that the answer to question (Q4) is no, we may phrase similar questions as (Q,1-3) with the J_r, D_r, E_r replacing H_r, d_r, Δ_r, respectively:

- (Q5) Are the D_r and J_r always sums of r monomials of type q^i, for $r = 1, ..., k$?

  Answer: Affirmative for all (l, m) by Proposition 2.3.
- (Q6) Is E_r always a monomial of the form q^i ?
- Answer: Affirmative for $(l, m) = (2, m)$, for $m \leq 9$, and $(l, m) = (3, 6)$. Negative for some r for $(2, m)$, and $m = 10$ (or m big enough, see next section). We have not performed further investigations.
- (Q7) If J_r is computed by a Schubert union S_U, is J_{k-r} then computed by S_{U^*}?
- Answer: Affirmative for $(l, m) = (2, m)$, for $m \leq 8$, and $(l, m) = (3, 6)$. Negative for $(2, m)$, and $m = 10$ (or big enough, see next section). We have not performed further investigations.
- (Q8) Is it true that:

$$E_r(q) = q^{2m-4} E_{k+1-r}(q^{-1})?$$

  for all $C(l, m)$, and all r ? This in turn would imply that if the answer to question (Q6) is (partly) positive, and $\Delta_r = q^i$ for some i, then $\Delta_{k+1-r} = q^{l(m-l)-i}$.
- Answer: Affirmative for $(l, m) = (2, m)$, for $m \leq 9$, and $(l, m) = (3, 6)$. Negative for $(2, m)$, and $m = 10$. We have not performed further investigations.

Remark 5.1. — It follows from the results of Section 3 that all questions have affirmative answers for $l = 2$ and $m \leq 5$. The affirmative answers to (Q6),(Q7), (Q8) for $(l, m) = (2, 6), (2, 7), (2, 8), (3, 6)$ are due to the observations in Section 4. For $(l, m) = (2, 9)$ it is at least clear that (Q6) and (Q8) have affirmative answers. See Remark 4.7. The negative parts of the answers to these questions follow essentially from the analysis in Section 4. For $(l, m) = (2, 10)$ we see from Remark 4.7 and explicit calculations that $E_{22} = J_{21} - J_{22} = q^9 + q^8 - q^6$, so (Q6) has a negative answer. Moreover $E_{24} = J_{23} - J_{24} = q^6$, and hence (Q8) also has a negative answer.

¿From the observations above we may conclude:

Proposition 5.2. — *Neither of the questions question (Q6),(Q7), and (Q8) do always have affirmative answers, and questions (Q1), (Q2), (Q3), and (Q4) do therefore not simultaneously have affirmative answers for all l, m, r, q.*

6. Schubert unions with a maximal number of points

Recall the polynomials $g_U(q)$ defined in Definition 2.7. Moreover, for $l = 2$, and each spanning dimension K, we recall Proposition 3.6, and the two dual Schubert unions that are candidates for maximal Schubert unions with respect to the natural lexicographic order on the polynomials g_U:

$S_L = S_{(x,m)} \cup S_{(x+1,y)}$, with $1 \leq x \leq m - 1$ and $1 \leq y \leq m$, and

$S_R = S_{(x,x+1)} \cup S_{(a,x+2)}$, with $1 \leq x \leq m-1$ and $1 \leq a \leq x+1$.

As usual, let $k = \binom{m}{2}$.

The following result is given in [**HJR**]. The details of the proof can be found in [**HJRp**].:

Proposition 6.1. — *For every $\epsilon > 0$, there exists an M, such that if $m > M$, then*
(i) If $K \leq 0.36k - \epsilon$, then S_L is maximal with respect to g_U.
(ii) If $K \geq 0.36k + \epsilon$, then S_R is maximal with respect to g_U.

A continuous version is the following remark and proposition:

Remark 6.2. — Study the triangle Δ with corners $(0,0),(0,1),(1,1)$. Look at the trapeze T_x with corners $(0,0),(x,x),(x,1),(0,1)$ and area $A = x - \frac{x^2}{2}$. We also study the triangle P_y with corners $(0,0),(y,y),(0,y)$ and area $A = \frac{y^2}{2}$. We get $x = 1 - \sqrt{1-2A}$ for the trapeze, and $y = \sqrt{2A}$ for the triangle. The largest d for which the trapeze T_x intersects a diagonal $x + y = d$ is $d_1(A) = 1 + x = 2 - \sqrt{1-2A}$, where A is the area of T_x. The largest d for which the triangle P_y intersects this diagonal is $d_2(A) = 2y = 2\sqrt{2A}$, where A is the area of P_y. The proof of the following result is a straightforward calculation:

Proposition 6.3. — *We have $d_1(A) > d_2(A)$ iff $0 \leq A < 0.18$, corresponding to 36% of the area of the whole triangle Δ.*

We would be interested in an l-dimensional analogue of these results, both in a discrete, and a continuous setting. A natural strategy for finding "almost" all d_r for all Grassmann codes could be:

a) Prove that $d_r = D_r$, for all l, m, r.

b) Show that for all l, m, r there are essentially two main strategies to find an optimal G_U with $K = k - r$ elements. One may either fill up consecutive "layers" with the first variable x_1 fixed, or fill up layers with the last variable x_l fixed. Only in a small zone around a fixed value of r, depending on the sum $x_1 + \ldots + x_l$ it is hard to decide which of the two strategy to use, if m is big enough compared with (fixed) l.

c) To fill up each layer is essentially equivalent to solving the problem for an l-value which is one smaller.

The philosophy is as follows, if we assume that a) holds: For each K one wants to find the Schubert union with this spanning dimension, with the maximum number of points. We are happy if we can find one which is maximal with respect to the lexicographic order on the g_U. A necessary condition for being maximal with respect to g_U is being maximal with respect to Krull dimension. The Krull dimension of a Schubert union is defined to be the biggest Krull dimension of a Schubert cycle appearing in the union. So we are interested in: For a given "cost" or spanning dimension K: How big Krull dimension can you obtain with a Schubert cycle of that spanning dimension ? This motivates the following definition:

Definition 6.4. — *(i) The cost $C(x_1, \dots x_l)$ is the spanning dimension of the Schubert cycle $S_{(x_1,\dots x_l)}$.*

(ii) An admissible point is a point $\mathbf{x}$ in $G_{G(l,m)}$ such that Krull-dim($S_{\mathbf{x}}$) $\geq$Krull-dim($S_{\mathbf{y}}$) for all $\mathbf{y}$ such that $C(\mathbf{x}) \geq C(\mathbf{y})$.

Equivalently: $C(\mathbf{x}) < C(\mathbf{y})$ for all $\mathbf{y}$ with Krull-dim($S_{\mathbf{y}}$) >Krull-dim($S_{\mathbf{x}}$).

A formula for the value of $C(x_1, \dots x_l)$ was given in Theorem 7 of [**GT**]. It is also the cardinality of the associated G-grid G_U. It is clear that any Schubert union that maximizes the Krull dimension for given K must contain a Schubert cycle $S_{\mathbf{x}}$ for admissible $\mathbf{x}$.

It is straightforward to see that for $l = 2$, the admissible points on each level diagonal for $x_1 + x_2$ are located close to the end points of the diagonal segments of the coordinate grid. For small values of $x_1 + x_2$ the upper points are admissible, for big $x_1 + x_2$ the lower ones are admissible. This determines whether we go left or right to find optimal Schubert unions. Hence we understand that it is instrumental to identify the admissible poits in order to find optimal Schubert unions. Proposition 6.2 suggests that the essential picture is captured by studying an analogous continuous problem. Letting m go to infinity for fixed l, and scaling down a factor m in all directions, we obtain a polyhedron $\mathcal{G}$ with corners $(0,0,\dots,0), (0,0,\dots,1)(0,\dots,1,1), \dots, (0,1,\dots,1,1), (1,1,\dots,1,1)$, and volume $\frac{1}{l!}$, which is the "limit" or continuous analogue of $G_{G(l,m)}$.

Definition 6.5. — *(i) The continuous cost function of a point $\mathbf{x} = (\mathbf{x_1}, \dots, \mathbf{x_l})$ in $\mathcal{G}$ is*

$$V(\mathbf{x}) = \int_{\mathcal{S}_{x_1,x_2,\dots,x_l}} dV = \int_0^{x_1} \int_{y_1}^{x_2} \dots \int_{y_{l-1}}^{x_l} dy_l \dots dy_2 dy_1$$

This is the multivolume of $\mathcal{G}_{\mathbf{x}}$ which consists of those $\mathbf{y}$ in $\mathcal{G}$ with $\mathbf{y} \leq \mathbf{x}$. :

(ii)An admissible point of $\mathcal{G}$ is a point $\mathbf{x}$ such that $y_1 + \dots + y_l$ is not bigger than $x_1 + \dots + x_l$ for any point $\mathbf{y}$ in $\mathcal{G}$ with $V(\mathbf{y}) \leq V(\mathbf{x})$.

6.1. A continuous analysis for $l = 3$. — We will study the continuous problem for $l = 3$. Now we study the tetrahedron $\mathcal{G}$ with corners $(0,0,0), (0,0,1)(0,1,1), (1,1,1)$.

The analogue of the G-grid of the Schubert cycle $S_{(a,b,c)}$ is

$$\mathcal{G}_{(x,y,z)} = \{(s,t,u) \in \mathcal{G} | s \leq x, t \leq y, u \leq z\}.$$

The discrete cost function is

$$\mathcal{C}(a,b,c) = a(b-1)(c-2) - \frac{a(b-1)(b-2)}{2} + \frac{a(a-1)(a-2)}{6} - \frac{a(a-1)(c-2)}{2},$$

while the continuous version is

$$V(x,y,z) = \int_{\mathcal{S}_{x,y,z}} dV = \int_0^x \int_s^y \int_t^z du\ dt\ ds = xyz - \frac{xy^2}{2} - \frac{x^2z}{2} + \frac{x^3}{6}.$$

(We observe that only the homogeneous part of degree l, in this case 3, of the discrete cost function survives). On $\mathcal{G}$ we study the level triangles $x+y+z=d$ for various d.

The homogenous cost function, restricted to a level triangle, is

$$f_d(x,y,z) = \frac{2x^3}{3} - \frac{x^2y}{2} - \frac{3xy^2}{2} + dxy - \frac{dx^2}{2}.$$

We are only interested in the cases $2 \leq d \leq 3$, since it possible to use an arbitrary small volume $V = \epsilon$ and find a $\mathcal{G}_{x,1,1}$ with volume less than ϵ, and even this volume-small piece reaches the level triangle $d = 2 + x > 2$. We now study the stationary points of $f_d(x,y,z)$ on the respective level triangles, and find that they have no local minima in the interiors, if $2 \leq d \leq 3$. Restricting f_d, to each of the three edges of the triangles, and calculating, we conclude similarly that there are no minima, except at the corners. Hence the minimum of f_d on a level triangle is always one of the 3 corner points. Hence we conclude:

Proposition 6.6. — *For $l = 3$ all continuous-admissible points are located at the line segments $\overline{(\frac{2}{3},\frac{2}{3},\frac{2}{3})(1,1,1)}$ and $\overline{(\frac{1}{2},\frac{1}{2},1)(1,1,1)}$ and $\overline{(0,1,1)(1,1,1)}$.*

These are located on the lines $x = y = z$, and $x = y, z = 1$ and $y = z = 1$, respectively. Hence it is clear that the volume- or (cost-)cheapest way to reach a level triangle for $d \geq 2$, and also the furthest you can reach with a given volume at disposal, with some "asymptotic Schubert union grid", in other words a finite union of sets $\mathcal{G}_{(x,y,z)}$, is to use one of type $\mathcal{G}_{(z,z,z)}$ or $\mathcal{G}_{(y,y,1)}$ or $\mathcal{G}_{(x,1,1)}$. These can be viewed as limits of usual Schubert cycle grids $G_{(c-2,c-1,c)}, G_{(b-1,b,m)}, G_{(a,m-1,m)}$, and gives a three-dimensional analogue of the cycles $S_{(b-1,b)}$ and $S_{(a,m)}$ which typified the process of "going right" and "going left" in the two-dimensional case.

It remains, to get an analogue of Proposition 6.3, to stratify the interval $I = [0, \frac{1}{l!}]$ into intervals or subsets I_i, for $i = 1, ..., l$, such that if $V \in I_i$, then the most distance-potent choice is $\mathcal{S}_{(x_i,...,x_i,1,..,1)}$ (i copies of x_i). Most probably each I_i is just an interval.

To find the sets I_1, I_2, I_3 for $l = 3$, we find the volumes of $\mathcal{G}_{(x,1,1)}, \mathcal{G}_{(y,y,1)}, \mathcal{G}_{(z,z,z)}$:

$V_x = \frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2}$, $V_y = \frac{y^2}{2} - \frac{y^3}{3}$, $V_z = \frac{z^3}{6}$, respectively. Inverting expressions, we get: $x = 1-(1-6V)^{\frac{1}{3}}$ and $z = (6V)^{\frac{1}{3}}$. To find $y(V)$ one would like to solve the cubic equation $2y^3 - 3y^2 + 6V = 0$. The point with largest d-value on $\mathcal{G}_{(x,1,1)}$ is $(x,1,1)$, and the value is $2 + x = 3 - (1-6V)^{\frac{1}{3}}$. The point on $\mathcal{G}_{(y,y,1)}$ with largest d-value is $(y,y,1)$ with value $2y+1 = 2y(V)+1$. The point on $\mathcal{G}_{(z,z,z)}$ with largest d-value is (z,z,z) with value $3z = 3(6V)^{\frac{1}{3}}$. Hence for each V we must simply find out which value is the largest,

$$d_1(V) = 3 - (1-6V)^{\frac{1}{3}}, \text{ or } d_2(V) = 2y(V)+1 \text{ or } d_3(V) = 3(6V)^{\frac{1}{3}}.$$

We denote by $V_1(d), V_2(d), V_3(d)$ the inverses of these functions in d. The equations $2y^3 - 3y^2 + 6V = 0$ and $d_2 = d_2(y) = 2y + 1$ give $(d_2)^3 - 6(d_2)^2 + 9d_2 - 4 = -24V$, and hence $V = V_2(d) = \frac{-d^3}{24} + \frac{d^2}{4} - \frac{9d}{24} + \frac{1}{6}$.

Both $d_2(V)$ and $d_3(V)$ are increasing functions in V, and hence the inequality $d_2(V_0) > d_3(V_0)$ is equivalent to: The point V_1, such that $d_2(V_1) = d_3(V_0)$, is smaller than V_0 (this is true if V_1 can be chosen within an interval where d_2 is increasing). We write this statement as: $V_1 = V_2(d_2(V_1)) = V_2(d_3(V_0)) < V_0$. Using only the last inequality, and calling the variable V instead of V_0, we obtain the condition:

$$V_2(d_3(V)) < V.$$

Comparison of $d_1(V)$ and $d_2(V)$ yields $V_2(d_1(V)) < V$ in the same way, while $d_1(V)$ and $d_3(V)$ can be compared directly.

We now check the condition $d_2(V) > d_3(V)$. It is clear that $d_2(0) = 1$, and $d_3(0) = 0$, so for small $V < d_3^{-1}(1) = \frac{1}{162}$ we see that $d_2(V)$ is the larger value. Assume $V \geq \frac{1}{162}$. There we may apply the criterion $V_2(d_3(V)) < V$ to check $d_2(V) > d_3(V)$. This amounts to

$$-31T^3 + 54T^2 - 27T + 4 < 0,$$

where $T = (6V)^{\frac{1}{3}}$.

Since d_2 and d_3 have the same value for $T = 1$, we may divide by $T - 1$. This gives the criterion: $31T^2 - 23T + 4 < 0$. Combined with $d_2 > d_3$ if $V \leq \frac{1}{162}$ we see that $d_2 > d_3$ iff $T < 0.463$, that is $6V < (0.463)^3 = 0.0992$. This small range $6V \in [0, 0.0992)$ is the only one where $d_2(V) > d_3(V)$, while $d_3(V) > d_2(V)$, for $6V \in [0.0992, 1)$.

We now check the condition $d_2(V) > d_1(V)$. We do this by checking when $V_2(d_1(V)) < V$. This gives

$$U^2(5U - 3) \leq 0,$$

where $U = (1 - 6V)^{\frac{1}{3}}$. This is the same as:

$$6V > \frac{5^3 - 3^3}{5^3} = 0.784.$$

Hence $d_1(V) > d_2(V)$ for $6V \in [0, 0.784)$, and $d_2(V) > d_1(V)$ for $6V \in (0.784, 1)$.

We now check when $d_1(V) > d_3(V)$. This can be done directly and gives:

$$26T^2 - 55T + 26 > 0,$$

where $T = (6V)^{\frac{1}{3}}$. This holds iff $T < 0.713$, which is equivalent to $6V < 0.713^3 = 0.362$. The exact value is $(\frac{55-\sqrt{321}}{52})^3$. Hence $d_1(V) > d_2(V)$ for $6V \in [0, 0.362)$, and $d_2(V) > d_1(V)$ for $6V \in (0.362, 1)$. We observe that the interval $[0, 0.783)$, where $d_1(V) > d_2(V)$, contains the interval $[0, 0.0992)$ where $d_2(V) > d_3(V)$. Hence $d_2(V)$ is never largest of all the $d_i(V)$, and we conclude that also for $l = 3$ there are only two optimal strategies for the continuous problem.

Hence, for $6V \in [0, 0.362)$ the distance $d_1(V)$ is largest, and for $6V \in (0.362, 1)$ the distance $d_3(V)$ is largest. The numerical value 0.362 is strikingly similar to 0.36 in the case $l = 2$, but the exact values are different. We obtain:

Proposition 6.7. — *If $V < \frac{(\frac{55-\sqrt{321}}{52})^3}{6}$, then the unique point (x, y, z) in $\mathcal{G}$ with largest $(x+y+z)$-value among those with cost at most V, is of the form $(x, 1, 1)$.*

If $V > \frac{(\frac{55-\sqrt{321}}{52})^3}{6}$, then the unique point (x, y, z) in $\mathcal{G}$ with largest $(x+y+z)$-value among those with cost at most V, is of the form (z, z, z).

Inspired by Propositions 3.6, 6.1, and 6.7, we now formulate two natural conjectures for optimal Schubert unions.

Conjecture 6.8. — *Fix a natural number K less than $\binom{m}{l}$, and consider the set of Schubert unions $\{S_U\}_K$ in $G(l, m)$ with spanning dimension K. We have the following recursive procedure to find an optimal Schubert union in this set.*

(i) Let x be the largest x_1 such that $C(x_1, m-l+2, \dots, m-1, m) \leq K$. Set $K' = K - C(x, m-l+2, \dots, m-1, m)$. Let G' be the subset of $G_{G(l,m)}$ with $x_1 = x+1$. Identify G' with $G_{G(l-1,m-x-1)}$ via the bijection $f(x+1, x_2, \dots x_l) = (x_2 - x - 1, \dots x_l - x - 1)$. Let $G_{U'}$ be the grid of an optimal Schubert union U' for $G_{G(l-1,m-x-1)}$ for the spanning dimension K', and let G'_L be the inverse image by f of $G_{U'}$. Let G_L be the union of G'_L and $G_{(x,1,\dots,1)}$, and let S_L be the Schubert union with $G_{S_L} = G_L$.

(ii)Let z be the largest x_l such that $C(x_l - l + 1, \dots, x_l - 1, x_l) \leq K$. Set $K'' = K - C(z-l+1, \dots, z-1, z)$. Let G'' be the subset of $G_{G(l,m)}$ with $x_l = z+1$. Identify G'' with $G_{G(l-1,z)}$ via the bijection $h(x_1, \dots x_{l-1}, z+1) = (x_1, \dots x_{l-1})$. Let $G_{U''}$ be the grid of an optimal Schubert union U'' for $G_{G(l-1,m-x-1)}$ for the spanning dimension K'', and let G''_R be the the inverse image by h of $G_{U''}$. Let G_R be the union of G''_R and $G_{(z,\dots,z)}$ and let S_R be the Schubert union with $G_{S_R} = G_R$.

Then either S_L or S_R is an optimal Schubert union for spanning dimension K (with $G_S = G_L$ or $G_S = G_R$, respectively).

We also claim:

Conjecture 6.9. — *Given $l \geq 2$. For each $m \geq 2$ set $k = \binom{m}{l}$. Then there exists a real positive number P such that for every $\epsilon > 0$, there exists an M, such that if $m > M$, then*

(i) If $K \leq Pk - \epsilon$, then S_L is maximal with respect to g_U.

(ii) If $K \geq Pk + \epsilon$, then S_R is maximal with respect to g_U.

(iii) For $l = 3$ we have $P = (\frac{55-\sqrt{321}}{52})^3$.

7. Codes from Schubert unions

In earlier sections we have studied the impact of Schubert unions to Grassmann codes in order to make the bound $d_r \leq D_r$ explicit. Now we will study codes made from a Schubert union S_U in the same way as the codes $C(l,m)$ are made from the $G(l,m)$. In other words; For a given Schubert union S_U and prime power q denote the (affine) spanning dimension of S_U by $K_U = K$. Then the Plücker coordinates of all points of S_U have only zeroes in all the coordinates corresponding to the $k - K$ points of H_U, so we delete them. Choose coordinates for each point, and make the corresponding K-tuples columns of a $k \times g_U(q)$-matrix $\mathcal{G}$. This matrix will be the generator matrix of a code. If we change coordinates for a point by multiplying by a factor, the code changes, but its equivalence class and code parameters do not, so by abuse of notation we denote all equivalent codes appearing this way by C_U.

In [**HC**] it was shown that if $l = 2$, and we simply have a Schubert cycle S_α, then the minimum distance $d_1 = d$ of the code is q^δ, where δ is the Krull dimension of the Schubert cycle. We will use this result to give the following generalization:

Proposition 7.1. — *For a Schubert union S_U in $G(2,m)$, which is the proper union of s Schubert cycles S_i with Krull dimensions δ_i, for $i = 1,..,s$, the minimum distance of C_U is the smallest number among the q^{δ_i}.*

Proof. — Let S_α be one of the cycles in the given union with minimal Krull dimension δ. We now intersect S_U with the coordinate hyperplane X_α (restricted to the K-space in which S_U sits, if one prefers). Since α is not contained in the G_β of any Schubert union S_β different from S_α appearing in the union, this coordinate hyperplane contains all these S_β.

There are exactly q^δ points from S_α that are not contained in this hyperplane (all these points are then of course outside all the other S_β): If $\alpha = (a,b)$, then this hyperplane cuts out $S_{(a-1,b)} \cup S_{(a,b-1)}$, with exactly one point (a,b) less in its G-grid, and by Corollary 2.3 we must then subtract q^{a+b-3} to obtain the number of points. On the other hand it is clear that if we intersect S_U with an arbitrary hyperplane H in K-space (or an arbitrary hyperplane in the Plücker space, not containing S_U), then there is at least one S_i, which is not contained in H. Now the maximal number of points of any hyperplane section of S_i is equal to the cardinality of S_i minus q^{δ_i}, so there are at least q^{δ_i} points of $S_i - H$. Hence there are at least q^{δ_i} points of $S_U - H$ also. Hence the maximal number of points of $S_U \cap H$ is $g_U(q) - q^\delta$, where δ is the smallest δ_i, and $d = d_1$ is computed by X_{α_i} for such a corresponding i. □

We may also mimick the contents of Proposition 3.1. Let α be such that S_α is one of the Schubert cycles S_i with minimal Krull dimension in S_U, and set $\delta = \delta_i$

Proposition 7.2. — *(i)* $d_r = q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1}$*, for* $r = 1, \ldots s$*, where* s *is the largest natural number such that* $(a-s+1), (a-s+2, b), ..., (a-1, b), (a, b)$ *all are contained in* G_U*.*

(ii) Let $S_U = S_{(a_1,b_1)} \cup \cdots \cup S_{(a_s,b_s)}$*, and let* b *be the largest* b_i*. Then* $d_K = g_U(q)$*, and*

$$d_{k-a} = g_U(q) - (1 + q + \ldots q^{a-1}),$$

for $a = 1, \ldots, b_s - 1$*.*

Proof. — (i) $d_r \geq q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1}$ is the Griesmer bound. The opposite inequalities follow if we can exhibit linear spaces with increasing codimension, which intersect S_U in an appropriate number of points. We intersect with:

$$X_{(a,b)} = X_{(a-1,b)} = X_{(a-2,b)} = ... = X_{(a-r+1,b)} = 0$$

Then, as intersections we obtain smaller successive Schubert unions. Their cardinalities are determined by Corollary 2.3 and the fact that we peel off points one by one to obtain the successive G-grids.

(ii) S_U contains a projective space of dimension $b_s - 2$. □

Of course we also have a relative bound, analogous to $d_r \leq D_r$

Proposition 7.3. — *Let* S_U *be a Schubert union in* $G(l, m)$*, and let* M_r *be the maximum cardinality of a Schubert union that is contained in* U*, and whose spanning dimension is* r *less than that of* S_U*. Then* $d_r \leq g_U(q) - M_r$*.*

The proof is obvious.

Example 7.4. — *In Section 4 we listed Schubert unions that compute the* d_r *for the Grassmann code* $C(2, 5)$ *from* $G(2, 5)$*. We leave it to the reader to find the full weight hierarchy for all* C_U*, for all* 15 *non-empty Schubert unions* U *of* $G(2, 5)$*, using the results above and the table for* $G(2, 5)$ *in the appendix.*

For $l \geq 3$ the expected result $d = d_1 = q^\delta$ for Schubert cycles has not yet been shown. If it is shown, we see that we can extend it to Schubert unions as in the case $l = 2$, and also a variant of Proposition 7.2 will then follow.

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