UNIQUENESS IN TANAKA THEORY

BORIS KRUGLIKOV

Abstract. In this paper we prove that provided the maximal symmetry algebra of a vector distribution on a manifold is finite-dimensional, it is unique and graded. In the maximal-symmetric case the distribution is flat in Tanaka sense.

1. The main result

Let Δ be a regular completely non-holonomic distribution on a connected smooth manifold M, i.e. $\Delta \subset TM$ is a vector subbundle. This paper concerns the Lie algebra $\mathrm{sym}(\Delta)$ of its symmetries.

In [T] N. Tanaka introduced a graded nilpotent Lie algebra (GNLA) $\mathfrak{m}_x$ and its algebraic prolongation $\mathfrak{g}_x = \hat{\mathfrak{m}}_x$ at every point $x \in M$.

The Lie algebra $\mathrm{sym}(\Delta)$ of the symmetries of Δ satisfies:

$$\dim \mathrm{sym}(\Delta) \leq \sup_M \dim \mathfrak{g}_x. \tag{1}$$

This was proved in [T] in the case Δ is strongly regular (which means the GNLA $\mathfrak{m}_x$ does not depend on x) and in [Kr] in the general case.

Theorem. *Suppose all Lie algebras $\mathfrak{g}_x$ are finite-dimensional. Then equality in (1) is attained only in one case: The distribution Δ is flat. Consequently the algebra* $\mathrm{sym}(\Delta)$ *of maximal dimension is graded.*

Recall that the distribution is flat if the canonical absolute parallelism on the prolongation bundle over the manifold has zero curvature, or vanishing structure functions [T] (originally this was defined for strongly regular systems, but this restriction was relaxed in [M2]). Equivalently a flat distribution Δ is locally diffeomorphic to the standard model on the Lie group corresponding to $\mathfrak{m}$, see [T].

Some remarks concerning this theorem are of order. First, it is not true in general that the maximal symmetric model of a geometric structure is unique. A counter-example is given by the Riemannian metrics on surfaces. The symmetry algebra $\mathfrak{s}$ (space of isometries) depends on the curvature K: It is 3-dimensional for the round sphere ($K = 1$), the Euclidean plane ($K = 0$) and

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the Lobachevsky plane ($K=-1$), but the structure of Lie algebra depends on the signature of K (notice that $\mathfrak{s}$ is graded only for $K=0$).

Second, the known criteria for flatness are based on the deformation theory of Lie algebras and use the cohomological obstructions [M1]. But in the case of Tanaka algebra the cohomology group $H^2(\mathfrak{m},\mathfrak{g})_+$ (plus refers to non-negative grading) is usually non-zero, as it numerates differential invariants fundamental for the equivalence problem.

Third, the Lie algebra of symmetries is seldom graded, see [KN]. For a homogeneous Δ with $\mathrm{sym}(\Delta)$ of non-maximal dimension the symmetry algebra is usually not a subalgebra of $\mathfrak{g}$ and not graded (but it is filtered and the corresponding graded Lie algebra embeds into $\mathfrak{g}$).

An example is the split real G_2 which is the symmetry group of the Hilbert-Cartan equation $y'=(z'')^2$ (maximally symmetric model among all nondegenerate Monge equations $y'=F(x,y,z,z',z'')$) – it is graded, but the compact version of the Lie group G_2 is irreducible, i.e. preserves no non-holonomic distribution[1], and is not graded.

Finally, the above theorem has two versions — local and global, both correct if we assume the manifold M simply connected. Otherwise, we have several models, like the nilpotent model $\exp(\mathfrak{m})$ and the homogenous Cartan model G/P, where $G=\exp(\mathfrak{g})$ is the Lie group of the Tanaka algebra $\mathfrak{g}$ and $P=\exp(\mathfrak{g}_+)$ its parabolic subalgebra.

The proof that we give below is simple, but this shall be compared to the other particular cases, which were performed "by brute force" (see e.g. [DZ, AK] for the case of rank 2 distributions).

2. Review of Tanaka theory

Given a distribution $\Delta\subset TM$, its weak derived flag $\{\Delta_i\}_{i>0}$ is given via the module of its sections by $\Gamma(\Delta_{i+1})=[\Gamma(\Delta),\Gamma(\Delta_i)]$ with $\Delta_1=\Delta$.

The distribution will be completely non-holonomic, i.e. $\Delta_\kappa=TM$ for some (minimal) κ. We will assume also that the flag of Δ is regular, i.e. the ranks of Δ_i are constant.

The quotient sheaf $\mathfrak{m}=\oplus_{i<0}\mathfrak{g}_i$, $\mathfrak{g}_i=\Delta_{-i}/\Delta_{-i-1}$ (we let $\Delta_0=0$), has a natural structure of GNLA at any point $x\in M$. The bracket on $\mathfrak{m}$ is induced by the commutator of vector fields on M.

[1]As a historical remark let us mention that the first G_2 appeared in 1910 in [C1], while the compact case came later in 1914 [C2] as the symmetry group of octonions $\mathrm{Aut}(\mathbb{O})$. These two realizations by Cartan complimented the abstract Killing-Cartan classification for the first exceptional Lie group.

The Tanaka prolongation $\mathfrak{g} = \hat{\mathfrak{m}}$ is the graded Lie algebra with negative graded part $\mathfrak{m}$ and non-negative part defined successively by

$$\mathfrak{g}_k = \{u \in \bigoplus_{i<0} \mathfrak{g}_{k+i} \otimes \mathfrak{g}_i^* : u([X,Y]) = [u(X),Y] + [X,u(Y)],\ X,Y \in \mathfrak{m}\}.$$

Since Δ is bracket-generating, the algebra $\mathfrak{m}$ is fundamental, i.e. $\mathfrak{g}_{-1}$ generates the whole GNLA $\mathfrak{m}$, and therefore the grading k homomorphism u is uniquely determined by the restriction $u : \mathfrak{g}_{-1} \to \mathfrak{g}_{k-1}$.

Alternatively the above symbolic prolongation can be defined via Lie algebra cohomology with coefficients: $\mathfrak{g}_0 = H^1_0(\mathfrak{m},\mathfrak{m})$, $\mathfrak{g}_1 = H^1_1(\mathfrak{m},\mathfrak{m}\oplus\mathfrak{g}_0)$ etc, where the subscript indicates the grading [AK].

The prolongation of $\mathfrak{m}$ is $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots$. The space $\mathfrak{g} = \oplus \mathfrak{g}_i$ is naturally a graded Lie algebra, called the Tanaka algebra of Δ. For further properties of $\mathfrak{g}$ we refer to [Y].

We will need the following simple observation: $\mathfrak{g}_0$ equals the space of grading preserving derivations of $\mathfrak{m}$. Consequently it contains the Euler vector ϵ given by the commutation relation $[\epsilon, v] = j\,v$ for $v \in \mathfrak{g}_j$ (this is true for negative, zero and positive j). If we identify $\mathfrak{g}_0 \subset \mathrm{gl}(\mathfrak{g}_{-1})$, then ϵ corresponds to the (minus) identity element.

3. The proof

Let $\mathfrak{s} = \mathrm{sym}(\Delta)$ be the symmetry algebra of Δ. It follows from [Kr] (Theorem 1 and Remark 1) that in the case (1) is an equality, the distribution Δ is strongly regular[2].

Moreover Δ is homogeneous. Indeed, there is a canonical absolute parallelism (e-structure) on the prolongation manifold $\mathcal{E}$ of the structure Δ ($\dim\mathcal{E} = \dim\mathfrak{g}$). Equality in (1) implies that the structure functions of this parallelism are constants [T] and then the symmetry group is a Lie group [Ko] of dimension $\dim\mathfrak{s} = \dim\mathfrak{g}$.

This symmetry algebra $\mathfrak{s}$ is filtered – the filtration is induced at any point $x \in M$ by the weighted stabilizers. This is explained in details in Section 5 in [Kr], where $\mathfrak{s}$ was identified with the Lie algebra sheaf $\mathscr{L}$ at x. So we get

$$\mathfrak{F}_r \subset \mathfrak{F}_{r-1} \subset \dots \subset \mathfrak{F}_0 \subset \mathfrak{F}_{-1} \subset \dots \subset \mathfrak{F}_{-\kappa} = \mathfrak{s}$$

and the corresponding graded Lie algebra is

$$\mathrm{gr}(\mathfrak{s}) = \oplus \mathfrak{F}_i/\mathfrak{F}_{i+1} \simeq \mathfrak{g}.$$

If we establish $\mathfrak{s} \simeq \mathfrak{g}$, then we conclude that the Lie algebra $\mathfrak{m} = \mathfrak{s}_-$ acts transitively and effectively on M, and so we can locally identify M with $\exp(\mathfrak{m})$.

[2] The present paper clearly removes the flatness condition mentioned in [Kr], in relation to the criterion in [M1].

Under this diffeomorphism Δ corresponds to $\mathfrak{g}_{-1}$, and this implies that Δ is flat. Thus to finish the proof we need

Proposition. *The filtered Lie algebra $\mathfrak{s}$ is isomorphic to $\mathfrak{g}$, so that $\mathfrak{F}_i = \oplus_{j\geq i}\mathfrak{g}_i$. Thus $\mathfrak{s}$ is graded.*

Proof. Denote by $\pi_i : \mathfrak{F}_i \to \mathfrak{F}_i/\mathfrak{F}_{i+1} = \mathfrak{g}_i$ the canonical projection.

Let $E \in \pi_0^{-1}(\epsilon)$. Then we can inductively (by decreasing j) define $\mathfrak{s}_j = \{V \in \mathfrak{F}_j : [E,V] = j\,V\}$. This space has dimension $\dim \mathfrak{g}_j$ and we have the direct sum of vector spaces $\mathfrak{F}_j = \mathfrak{s}_j \oplus \mathfrak{F}_{j+1}$.

But since $[E,[V,W]] = k\,[V,W]$ for $V \in \mathfrak{s}_i$, $W \in \mathfrak{s}_j$ and $k = i+j$ we conclude that $[V,W] \in \mathfrak{s}_k$ (in particular if $k \notin [-\kappa, r]$, then this commutator is zero).

We claim that the map $h : \mathfrak{s} \to \mathfrak{g}$ given on the pure grade components $V \in \mathfrak{s}_i$ by $h(V) = \pi_i(V) \in \mathfrak{g}_i$ and extended by linearity, is a homomorphism of Lie algebras.

Indeed, by the filtration reasons $[V,W] \mod \mathfrak{F}_{k+1} = [v,w] \in \mathfrak{g}_k$ for $v = \pi_i(V)$, $w = \pi_j(W)$. Since $[V,W] \in \mathfrak{s}_k$ we conclude $h[V,W] = [v,w]$.

Thus by linearity $h[V,W] = [h(V), h(W)]$ for all vectors $V, W \in \mathfrak{s}$ and since h is obviously bijective, our claim is proved. $\square$

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Department of Mathematics and Statistics, University of Tromsø, Tromsø, 9037, Norway. *boris.kruglikov@uit.no*