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Projective classification of binary and ternary forms

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Abstract

In this paper we study orbits of $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{GL}_3(\mathbb{C})$-actions on the spaces of binary and ternary polynomial as well as rational forms and find criteria for their equivalence. The similar results are also valid for real forms.

Introduction

Denote by $V_n^2 = \mathsf{S}^n(\mathbb{C}^2)^*$ the space of polynomial binary forms of degree n over field $\mathbb{C}$ and consider the following action of group $\mathrm{GL}_2(\mathbb{C})$ on this space : subgroup $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$ acts by linear transformations in the standard way, and center $\mathbb{C}^* \subset \mathrm{GL}_2(\mathbb{C})$ acts by homotheties $f \mapsto \lambda f$, where $f \in V_n^2$ and $\lambda \in \mathbb{C}^*$.

In the first part of the paper we give a solution of classification problem for this action (see also [6]).

This problem is closely related to other classification problems in invariant theory such as classification problem for the action of projective group $\mathrm{PGL}_2(\mathbb{C})$ on finite subsets of projective line and with classification of hyperelliptic curves of genus g (see [1]).

The $\mathrm{SL}_2(\mathbb{C})$-invariant algebras for binary polynomial forms of degree $n \leqslant 8$ are known so far (see [2, 3, 4]).

The case $n = 3$ was solved by Bool in 1841.

The first nontrivial case $n = 4$ was solved by Bool, Cayley and Eisenstein in 1841–1850 and initiated the development of classical invariant theory (note that the problem of classification of binary forms of degree 4 is closely connected with the cross ratio of the four points on projective line, and with j-invariant of elliptic curve; see [1]).

The case $n = 5$ was studied by Cayley. In this case the invariant algebra is generated by four homogeneous polynomials of degrees 4, 8, 12 and 18, which satisfy a homogeneous relation of degree 36 (note that the invariant of degree 18 consists of more than 800 monomials).

The cases $n = 6, 7, 8$ were studied by Gordan, Shioda, Dixmier and Lazard. The explicit form of generators of the invariant algebra for $n = 7$ were found by Bedratyuk in 2007 only (see [4]).

In this paper we suggest a method which classifies $\mathrm{GL}_2(\mathbb{C})$-orbits of polynomial as well as rational binary forms *of any degree*.

This method is based on the interpretation of space V_n^2 as the space of smooth solutions of differential equation $xu_x + yu_y = nu$ (the so called *Euler equation*) and consideration of the $\mathrm{GL}_2(\mathbb{C})$)-action on the solution space of this equation. This interpretation makes it possible to use *differential invariants* instead of algebraic ones. The structure of differential invariant algebra is much simpler than the structure of the classical polynomial invariant algebra, because of a version of *Lie-Tresse's Theorem* (see [8]), which claims that differential invariant algebra of a Lie group acting on solutions of a partial differential equation is generated (under some conditions) by a finite number of differential invariants and invariant derivations. So, to find differential invariant algebra one should find a few basic invariants and invariant derivations.

The similar method may be applied to the problem of classification of *ternary forms* (see also [7]).

The $\mathrm{SL}_3(\mathbb{C})$-invariant algebras of ternary forms of degree $n \leqslant 3$ are known so far.

Thus, in the case, when $n = 2$, the invariant algebra is freely generated by Hessian of ternary form.

The case, when $n = 3$, was studied by Weierstrass. In this case the invariant algebra is equal to $\mathbb{C}[S, T]$, where invariants S and T have degrees 4 and 6 correspondingly (see [1, 2]).

Rational $\mathrm{GL}_3(\mathbb{C})$-invariant $j = \frac{S^3}{T^2}$ is called *j-invariant of ternary cubic form*, and two non-singular forms of degree 3 are $\mathrm{GL}_3(\mathbb{C})$-equivalent if and only if their j-invariants coincide.

It is known that any non-singular ternary form f of degree 3 is equivalent to the *Weierstrass normal form*

$$y^2z + x^3 + pxz^2 + qz^3.$$

Coefficients p and q in this representation are equal to $S(f)$ and $T(f)$ correspondingly.

It is also known (see [2]) that any elliptic curve is isomorphic to the curve $X(f) = \{f = 0\} \subset \mathbb{C}P^2$ for some ternary form f of degree 3 and two elliptic curves $X(f)$ and $X(\widetilde{f})$ are projectively isomorphic if and only if ternary forms f and $\widetilde{f}$ are $\mathrm{GL}_3(\mathbb{C})$-equivalent.

In the case, when $n = 4$, it is known that the invariant algebra contains invariants

$$I_3,\ \ I_6,\ \ I_9,\ \ I_{12},\ \ I_{15},\ \ I_{18},\ \ I_{27},$$

where indexes of invariants show their degrees.

It is worth to note, that Emmy Noether found more than 300 concomitants (see [2]). But it is unknown do they generate the invariant algebra or not.

In this paper we propose a method which classifies $\mathrm{GL}_3(\mathbb{C})$ or $\mathrm{SL}_3(\mathbb{C})$-orbits of polynomial and rational ternary forms *of any degree* $n \neq 0$ with non zero Hessian.

The paper is organized as follows. We begin with a more general problem of classification $\mathrm{GL}_2(\mathbb{C})$-orbits on the space $C^\infty(\mathbb{C}^2)$ of *analytic functions*, which explains the role of the Euler equation. In subsection 1.2 we describe orbits of *polynomial binary forms*. In subsection 1.3 we give $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$-classification of *rational binary forms*. In section 2 we solve the problem of classification of ternary forms: in subsection 2.1 we describe differential invariant algebra structure, and in subsection 2.2 we give a description for orbits of ternary forms.

1 Classification of binary forms

1.1 $\mathrm{GL}_2(\mathbb{C})$-classification of analytic functions

Let $\mathbb{C}^2$ be the plane with the coordinates (x, y). Denote by $J^k\mathbb{C}^2$ spaces of k-jets of analytical functions on the plane. The plane coordinates (x, y) induce the canonical coordinates $(x, y, u, u_{10}, u_{01}, \ldots)$ on the jet spaces.

Group $\mathrm{GL}_2(\mathbb{C})$ acts on the algebra of analytic functions $C^\infty(\mathbb{C}^2)$ in the following way. Namely, subgroup $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$ acts by linear coordinate transformations, and center $\mathbb{C}^* \subset \mathrm{GL}_2(\mathbb{C})$ acts by homotheties $f \mapsto \lambda f$, where $f \in V_n^2$ and $\lambda \in \mathbb{C}^*$. This action naturally prolongs to actions on k-jet spaces $J^k\mathbb{C}^2$.

In general, by a *differential invariant* of order k they understand a function $I \in C^\infty(J^k\mathbb{C}^2)$, which is invariant with respect to prolonged action of group $\mathrm{GL}_2(\mathbb{C})$.

We consider only invariants which are analytical in x and y and polynomial (or rational) in u_σ, u^{-1} and E^{-1} (see Proposition 1.1).

In a similar way one defines an *invariant derivation* as a linear combination of total derivatives

$$\nabla = A\frac{d}{dx} + B\frac{d}{dy},$$

where A, $B \in C^\infty(J^\infty\mathbb{C}^2)$ (possibly with polynomial or rational behavior in derivatives u_σ) and $\frac{d}{dx}$, $\frac{d}{dy}$ are the total derivatives, which is invariant with respect to the prolonged action of group $\mathrm{GL}_2(\mathbb{C})$.

Note that for such derivations functions $\nabla(I)$ are differential invariants (of order, as a rule, higher than the order of I) for any differential invariant I. This observation allows us to construct new differential invariants from known ones by differentiations only.

The next proposition, which could easily checked by straightforward computations, shows the role of the Euler equation for classification of binary forms.

Proposition 1.1. *The function*

$$E = \frac{xu_{10} + yu_{01}}{u}$$

is the differential invariant of order 1, and functions

$$H = \frac{u_{20}u_{02} - u_{11}^2}{u^2} \qquad \text{and} \qquad F = \frac{u_{01}^2 u_{20} - 2u_{10}u_{01}u_{11} + u_{10}^2 u_{02}}{u^3}$$

are differential invariants of order 2.

Derivations

$$\nabla_1 = x\frac{d}{dx} + y\frac{d}{dy} \qquad \text{and} \qquad \nabla_2 = \frac{u_{01}}{u}\frac{d}{dx} - \frac{u_{10}}{u}\frac{d}{dy}$$

are invariant.

Moreover, the following differential syzygy holds:

$$E_2^2 - E_1 F + HE^2 = 0,$$

where we used notation I_j for $\nabla_j I$.

We begin description of $\mathrm{GL}_2(\mathbb{C})$-orbits in jet spaces with small orders.

First of all let us describe the orbits in the spaces of 0-jets, 1-jets and 2-jets.

There are two orbits of incomplete dimension in the space $J^0\mathbb{C}^2$. Namely, orbits $\{u = 0\}$ and $\{x = y = 0\}$. These orbits will be called *singular orbits*.

In the space $J^1\mathbb{C}^2$, there are two types of orbits of incomplete dimension: orbit $\{E = 0\}$ and orbits, whose projections in $J^0\mathbb{C}^2$ are singular.

In a similar way, the orbits of incomplete dimension in the space $J^2\mathbb{C}^2$ are orbits lying in $\{H = 0\}$ and the orbits, whose projections in $J^1\mathbb{C}^2$ have incomplete dimension. All these orbits are called *singular*.

Finally, $\mathrm{GL}_2(\mathbb{C})$)-orbits in spaces $J^k\mathbb{C}^2$ are called *singular*, if their projections in $J^2\mathbb{C}^2$ are singular.

It is worth to note, that regular orbits in spaces $J^k\mathbb{C}^2$ have dimension $\dim \mathrm{GL}_2(\mathbb{C}) = 4$, when $k \geqslant 1$.

Theorem 1.1. *The algebra of $\mathrm{GL}_2(\mathbb{C})$-differential invariants is locally (i.e. in neighborhoods of the jet spaces) generated by invariants E and F and by invariant derivations ∇_1 and ∇_2.*

This algebra separates regular orbits and its differential syzygies are generated by the following:

$$(E_{22} - F_1)E - 3E_2^2 + 3E_1 F - 4EF = 0.$$

Proof. Let us consider a regular orbit $\mathcal{O}_1 \subset J^1\mathbb{C}^2$.

Its codimension is equal to

$$\dim J^1\mathbb{C}^2 - \dim \mathrm{GL}_2(\mathbb{C}) = 1.$$

On the other hand, we have nonzero invariant E of order 1. This invariant obviously generates differential invariants of order 1 and separates regular orbits.

In a similar way, codimension of a regular orbit in space $J^2\mathbb{C}^2$ is equal to 4 and is equal to the number of independent invariants of order $\leqslant 2$, which we may get from the basic (these invariants are E, $\nabla_1 E$, $\nabla_2 E$ and F).

Finally, in the space of 3-jets there are not more then four independent invariants of pure order 3, but after the derivations of the invariants E and F we get five invariants.

Therefore, there exists a relation among them.

Now, let us consider case $k \geqslant 3$ and a regular orbit $\mathcal{O}_k$ in space $J^k\mathbb{C}^2$.

Then, its projection

$$\mathcal{O}_{k-1} = \pi_{k,k-1}(\mathcal{O}_k) \subset J^{k-1}\mathbb{C}^2$$

is also regular.

Moreover, the dimension of the bundle $\pi_{k,k-1}\colon \mathcal{O}_k \to \mathcal{O}_{k-1}$ is equal to

$$(\dim J^k\mathbb{C}^2 - \dim \mathrm{GL}_2(\mathbb{C})) - (\dim J^{k-1}\mathbb{C}^2 - \dim \mathrm{GL}_2(\mathbb{C})) = k+1.$$

We claim that using the invariants E and F and derivations ∇_1 and ∇_2 it is possible to get $(k+1)$ independent differential invariants of pure order k.

Let d_1 and d_2 be the symbols of the total derivatives $\frac{d}{dx}$ and $\frac{d}{dy}$ correspondingly.

Then, the symbols of invariant derivations ∇_1 and ∇_2 are

$$\Xi_1 = \frac{xd_1 + yd_2}{u}, \qquad \Xi_2 = \frac{u_{01}d_1 - u_{10}d_2}{u},$$

correspondingly.

It is easy to see, that symbols of differential invariants E and F are Ξ_1 and Ξ_2^2.

Note also, that symbols Ξ_1 and Ξ_2 are independent, if $E \neq 0$.

So, the symbol of invariant

$$E_{\sigma_1,\sigma_2} = \nabla_1^{\sigma_1}\nabla_2^{\sigma_2}E$$

is equal to

$$\Xi_1^{\sigma_1+1}\Xi_2^{\sigma_2}$$

and symbol of invariant

$$F_{\tau_1,\tau_2} = \nabla_1^{\tau_1}\nabla_2^{\tau_2}F$$

is

$$\Xi_1^{\tau_1}\Xi_2^{\tau_2+2},$$

where $\sigma_1 + \sigma_2 = k-1$ and $\tau_1 + \tau_2 = k-2$.

Therefore, the vector space generated by invariants $E_{\sigma_1\sigma_2}$ has dimension k, and the vector space generated by invariants $F_{\tau_1\tau_2}$ has dimension $k-2$.

On the other hand, there exist $k-2$ differential syzygies which could be obtained from the syzygy of Proposition 1.1 by derivations.

Thus, the intersection of these two spaces has dimension $k-2$, and there exist $k+(k-1)-(k-2) = k+1$ independent invariants from the complement of this intersection. □

Let $f \in C^\infty(\mathbb{C}^2)$ be an analytical function and let $E(f)$ and $F(f)$ be the values of the differential invariants on this function.

We say that function f is *regular in a neighborhood* if

$$dE(f) \wedge dF(f) \neq 0$$

in this neighborhood.

For a regular function f one can (possibly in a smaller neighborhood) express values of the differential invariants $E_1(f), E_2(f)$ and $F_1(f), F_2(f)$ in terms of functions $E(f)$ and $F(f)$.

Let, for example,

$$E_i(f) = A_i(E(f), F(f)), \quad F_i(f) = B_i(E(f), F(f)),$$

for some functions A_i, B_i, where $i = 1, 2$.

Note that these functions have to satisfy the syzygy relation and also the syzygy relation is a formal integrability condition of this system, considered as PDE system for function f.

Theorem 1.2. *The functions (A_1, A_2, B_1, B_2) satisfying the syzygy relation determine (locally) the $\mathrm{GL}_2(\mathbb{C})$-orbit of regular function $f \in C^\infty(\mathbb{C}^2)$.*

Proof. It is obvious that if functions f_1 and f_2 are equivalent in neighborhood $\mathcal{O}$, then the corresponding dependencies coincide.

Let us prove the converse statement. It is easy to check that for a regular function its orbit in the space of 3-jets is a connected set of dimension 6.

Our PDEs system is completely integrable (because of the syzygy condition) and as a submanifold in the space of 3-jets has dimension 6 and its solution space has dimension 4. In other words this submanifold coincides with the orbit of a solution.

So, if two functions have the same set of dependencies, orbits of their graphs in 3-jet space $J^3\mathbb{C}^2$ are locally coincide and these functions are also locally equivalent. □

1.2 $\mathrm{GL}_2(\mathbb{C})$-orbits of polynomial binary forms

In this section we consider the $\mathrm{GL}_2(\mathbb{C})$-action on space V_n^2 of binary polynomial forms.

Recall that space V_n^2 of binary forms of degree n is identified with the space of smooth solutions of the Euler equation $xu_x + yu_y = nu$.

The corresponding algebraic manifold $\mathcal{E} \subset J^1\mathbb{C}^2$ is given by equation

$$xu_{10} + yu_{01} = nu.$$

Prolongations $\mathcal{E}^{(k)} \subset J^k\mathbb{C}^2$, $k = 1, 2, \ldots$ of the Euler equation are given by equations

$$xu_{i,m-i} + yu_{i-1,m-i+1} = (n - m + 1)u_{i-1,m-i},$$

where $i = 1, \ldots, m$, $m = 1, \ldots, k$ and $u_{00} = u$.

Moreover, the projections $\pi_{k+1,k}\colon \mathcal{E}^{(k+1)} \to \mathcal{E}^{(k)}$ are 1-dimensional affine bundles (see, for example,KLV).

Remark also that the prolongations of $\mathrm{GL}_2(\mathbb{C})$-action into jet spaces $J^k\mathbb{C}^2$ leave invariant the prolongations of the Euler equation. Therefore, restrictions of differential invariants on whole jet spaces to the prolongations $\mathcal{E}^{(k)}$ are differential invariants too.

Below by a differential invariant of order k of binary form we mean a $\mathrm{GL}_2(\mathbb{C})$-invariant function on the manifold $\mathcal{E}^{(k)}$, which is polynomial in u_σ and u^{-1}, and by regular orbit on $\mathcal{E}^{(k)}$ we mean the intersection of regular orbit in $J^k\mathbb{C}^2$ with $\mathcal{E}^{(k)}$.

Theorem 1.3. *The differential invariants algebra of the* $\mathrm{GL}_2(\mathbb{C})$*-action on the manifold* $\mathcal{E}^{(\infty)}$ *is freely generated by differential invariant* H *and invariant derivation* $\nabla = \nabla_2 \mid_{\mathcal{E}^{(\infty)}}$.

Proof. First of all, let us note that restrictions of basic invariants F and H and invariant derivation ∇_1 on the Euler equation satisfy the following relations:

$$F = n(n-1)H, \quad \nabla_1(u) = nu.$$

Consider now a regular orbit $\mathcal{O}_{k+1} \subset \mathcal{E}^{(k+1)}$.

The dimension of bundle $\pi_{k+1,k}\colon \mathcal{O}_{k+1} \to \mathcal{O}_k$ is equal to

$$(\dim \mathcal{E}^{(k+1)} - \dim \mathrm{GL}_2(\mathbb{C})) - (\dim \mathcal{E}^{(k)} - \dim \mathrm{GL}_2(\mathbb{C})) = 1.$$

On the other hand, the number of independent invariants of order $k+1$ which we get by differentiations is one more than the number of invariants of order k. Denote by $H_k = (\nabla)^(k - 2)(H)$ k-th order invariant obtained by k-times differentiation of H along ∇. Then $H_(k+1)$ is a non zero affine function along fibres of the bundle $\pi_{k+1,k}\colon \mathcal{E}^{(k+1)} \to \mathcal{E}^{(k)}$.

Let I be a differential invariant of order $(k+1)$ which is polynomial along the fibres of the projection. Then one can find functions $\lambda_1, ..., \lambda_r$ on $\mathcal{E}^{(k)}$ such that

$$\epsilon = I - \lambda_r H_{k+1}^r - .. - \lambda_1 H_{k+1}$$

is a function on $\mathcal{E}^{(k)}$.

Let us now denote by $X^{(k)}$ prolongation of the vector field X from the Lie algebra $\mathrm{gl}_2(\mathbb{C})$ to the k-jet space.

Then, applying vector fields $X^{(k+1)}$ to the above relation, we get that functions

$$X^{(k)}(\lambda_r)H^r_{k+1} + .. + X^{(k)}(\lambda_1)H_{k+1}$$

are functions on $\mathcal{E}^{(k)}$.

Therefore,

$$X^{(k)}(\lambda_r) = ... = X^{(k)}(\lambda_1) = 0,$$

and functions $\lambda_1, ..., \lambda_r$ are differential invariants of order k.

Moreover, if function I is a polynomial in in u_σ and u^{-1}, then the analysis of symbols, similar to the proof of Theorem 1.1., shows us that functions λ_r are also polynomial.

Now the induction in k complete the proof. $\square$

Consider invariants

$$I_1 = H, \quad I_2 = \nabla H \quad I_3 = \nabla^2 H.$$

Their restrictions on graph $L^4_f \subset J^4\mathbb{C}^2$ of a form $f \in V^2_n$ are homogeneous polynomials in x and y.

Therefore they are algebraically dependent and

$$F(I_1(f), I_2(f), I_3(f)) = 0$$

for some irreducible polynomial F.

We order variables I_k by requirement that $I_1 \prec I_2 \prec I_3$ and assume that polynomial F has the minimal degree with respect to this order and is defined up to a non zero scalar.

Definition. We say that binary form $f \in V^2_n$ is *regular*, if

$$(2I_1I_3 - 3I_2^2)(f) \neq 0.$$

The following result, which can be proved by straightforward computations, clarify the above regularity condition.

Proposition 1.2. *Let f be a binary form.*

Then $\mathrm{GL}_2(\mathbb{C})$-orbit of 4-jet $[f]^4_a \in J^4\mathbb{C}^2$ of function f at a point $a \in \mathbb{C}^2$ is transversal to $L^4_f \subset J^4\mathbb{C}^2$ if and only if

$$(2I_1I_3 - 3I_2^2)(f) \neq 0$$

at the point $[f]^4_a$.

Theorem 1.4. *Let f_1, $f_2 \in V^2_n$ be binary forms and let F_1, F_2 be the corresponding dependencies between invariants I_k.*

Then forms f_1 and f_2 are $\mathrm{GL}_2(\mathbb{C})$-equivalent if and only if $F_1 = F_2$.

Proof. It is obvious that for equivalent forms f_1 and f_2, one has $F_1 = F_2$.

Let us prove the converse statement. Consider an arbitrary regular binary form $f \in V^2_n$. It's enough to prove that the orbit $\mathscr{O}_f$ of the graph $L^4_f \subset \mathcal{E}^{(4)}$ of regular form f is the open subset of the irreducible component of algebraic manifold $\mathcal{E}^{(4)} \cap \{F = 0\}$ (then the required statement follows from the Frobenious Theorem of existence and uniqueness solutions for completely integrable systems).

It is obvious that $\mathscr{O}_f \subseteq \mathcal{E}^{(4)} \cap \{F = 0\}$ and $\dim(\mathcal{E}^{(4)} \cap \{F = 0\}) = 6$.

Let us prove that $\dim \mathscr{O}_f = 6$. It's enough to show that 4-prolongations of vector fields from Lie algebra $\mathfrak{gl}_2(\mathbb{C})$ are transversal to the graph L^4_f of form f. The previous proposition shows that this condition may be written as $(2I_1I_3 - 3I_2^2)(f) \neq 0$, and therefore is equivalent to the condition of regularity of the form f.

Therefore, if the form f is regular, then the prolongations of vector fields from $\mathfrak{gl}_2(\mathbb{C})$ are transversal to the graph L_f^4 and $\dim \mathscr{O}_f = 6$.

Thus, $\dim \mathscr{O}_f = 6 = \dim(\mathcal{E}^{(4)} \cap \{F = 0\})$. Moreover, it follows from Theorem 1.3 that $\mathscr{O}_f$ is the open set of irreducible algebraic manifold defined by the system of equations $I_k = I_k(f)$, $k = 1, 2, 3$. Therefore, $\mathscr{O}_f$ is the open subset of the irreducible component of the manifold $\mathcal{E}^{(4)} \cap \{F = 0\}$.

So, it's enough to prove that the graph L_f^4 of any regular form f belongs to the fix irreducible component of manifold $\mathcal{E}^{(4)} \cap \{F = 0\}$.

The last statement is equivalent to the irreducibility of polynomial $F(I_1, I_2, I_3) = F_J = F(x, y, u, \ldots)$ corresponding to f.

Assume the converse. Let $F_J = A \cdot B$ be a decomposition of polynomial F_J into relatively prime multipliers.

Then for each vector field $X \in \mathfrak{sl}_2(\mathbb{C})$ we have

$$X^{(4)}(A) \cdot B + A \cdot X^{(4)}(B) = 0.$$

So, the polynomial $X^{(4)}(A)$ is divided by A and $X^{(4)}(A) = \lambda_X A$, where λ_X is a polynomial.

Compare degrees of left and right parts, we get that $\lambda_X \in \mathbb{C}$ is a constant. Thus λ defines 1-cocycle

$$c\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}, \quad c(X) = \lambda_X.$$

It is known that all 1-cocycles on the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ are trivial, so $\lambda_X = 0$.

Finally, we get that $X^{(4)}(A) = 0$ for all $X \in \mathfrak{sl}_2(\mathbb{C})$.

It implies that the polynomial A is $\mathfrak{sl}_2(\mathbb{C})$- and $\mathfrak{gl}_2(\mathbb{C})$-invariant.

Using Theorem 1.3, we have $A = A(I_1, I_2, I_3)$. Then the polynomial F is reducible. This contradiction completes the proof of the regular case.

Let now $f \in V_n^2$ a non-regular form, i.e. $(2I_1I_3 - 3I_2^2)(f) = 0$.

This relation implies the existence of algebraic dependence between polynomials $I_1(f)$ and $I_2(f)$.

It is easy to see that such dependence has the form $F(I_1, I_2) = I_1^3 - \lambda I_2^2$ where $\lambda \in \mathbb{C}$.

If $I_1(f) \neq 0$, then dimension of the graph's L_f^3 orbit is equal to 5.

If $I_1(f) = 0$, then the dependence has the form $F(I_1) = I_1 - \mu$, and the dimension of the graph's L_f^2 orbit is equal to 4.

It is easy to prove, that in the first case the orbit of the graph L_f^3 coincides with the manifold $\mathcal{E}^{(3)} \cap \{F = 0\}$, and in the second case, the orbit of the graph L_f^2 coincides with the manifold $\mathcal{E}^{(2)} \cap \{F = 0\}$.

The same argumentation as in the regular case completes the proof. □

Remark. It can be proved that the singularity condition $(2I_1I_3 - 3I_2^2)(f) = 0$ is equivalent to the condition that the form $f \in V_n^2$ has no more than two roots without taking multiplicity, or that the form f is equivalent to the form $x^k y^{n-k}$ for some $k \leqslant n$, or that the orbit of form f has the dimension less than 4.

Examples. 1. Consider forms $f_1(x, y) = xy(x+y)(-2x+y)$ and $f_2(x, y) = xy(x+y)(-3x+y)$. The corresponding dependencies between invariants I_k are

$$F_1(I_1, I_2, I_3) = 3087I_2^4 - (12348I_3 + 16464I_1^2)I_1I_2^2 - (800I_3^3 - 7548I_2^2I_3^2 - 23328I_1^4I_3 - 15552I_1^6),$$
$$F_2(I_1, I_2, I_3) = 19773I_2^4 - (79092I_3 + 105456I_1^2)I_1I_2^2 - (9800I_3^3 - 20292I_2^2I_3^2 - 93312I_1^4I_3 - 62208I_1^6).$$

Then $F_1 \neq F_2$ and the forms f_1 and f_2 do not belong to the same orbit. From the point of view of classical invariant theory orbits of binary forms of degree 4 are determined by j-invariant (see [1]). In our case j-invariant of the first form equals $\frac{343}{36}$ and of the second one — $\frac{13^3}{144}$.

Figure 1: Graph of the function $F(1, \xi_1, \xi_2) = 0$ for the form $f(x, y) = xy(x + y)(-2x + y)$, where $\xi_1 = I_2^2/I_1^3$ and $\xi_2 = I_3/I_1^2$.

2. Consider forms $f_1(x, y) = xy(x + y)(-x + y)$ and $f_2(x, y) = xy(x + y)(2x + y)$. The corresponding dependencies between invariants I_k are

$$F_1(I_1, I_2, I_3) = F_2(I_1, I_2, I_3) = 3I_2^2 - 6I_1I_3 - 8I_1^3.$$

Therefore, these forms belong to the same orbit. In this case j-invariants of forms equal $\frac{27}{4}$.

1.3 Binary rational forms

The results obtained above may be generalized on some other actions.

We will consider two cases:

- $\mathrm{GL}_2(\mathbb{C})$-action on the space of binary rational forms, and
- $\mathrm{SL}_2(\mathbb{C})$-action on the space of binary polynomial and rational forms.

1.3.1 $\mathrm{GL}_2(\mathbb{C})$-orbits of binary rational forms

Let R_n^2 be the space of binary rational forms of degree n (i.e. the difference between the degrees of numerator and denominator equals n) equipped with the above $\mathrm{GL}_2(\mathbb{C})$-action. It can be easily checked that all proofs given in the previous section are valid for this action too. The main result of this section can stated as follows.

Theorem 1.5. *1. Let f_1, $f_2 \in R_n^2$ be rational forms of degree $n \neq 0, 1$ and F_1, F_2 are the corresponding dependencies between the invariants I_k. Then the rational forms f_1 and f_2 are equivalent if and only if $F_1 = F_2$.*

2. Rational forms f_1 and f_2 of degree 1 are equivalent if and only if the forms f_1^{-1} and f_2^{-1} of degree -1 are equivalent.

1.3.2 $\mathrm{SL}_2(\mathbb{C})$-orbits of binary rational forms

Finally, we consider $\mathrm{SL}_2(\mathbb{C})$-action on space R_n^2. The main difference between this action and the $\mathrm{GL}_2(\mathbb{C})$-action is the existence of the invariant $U = u$ of degree 0. Then the differential invariant algebra is generated by invariants U and $H_0 = u^2 H$ and derivation $\nabla_0 = u\nabla$.

Figure 2: Graph of the function $F(1, \xi_1, \xi_2) = 0$ for the form $f(x,y) = xy\frac{(x+y)(x+2y)}{(x-y)(x-2y)}$, where $\xi_1 = I_2^2/I_1^3$ and $\xi_2 = I_3/I_1^2$.

Consider invariants $J_1 = U$, $J_2 = H_0$ and $J_3 = \nabla_0 H_0$ and algebraic dependency between their values on a form $f \in R_n^2$: $F(J_1(f), J_2(f), J_3(f)) = 0$.

Theorem 1.6. *1. Let f_1, $f_2 \in R_n^2$ be rational forms of degree $n \neq 0, 1$ and F_1, F_2 are the corresponding dependencies between the invariants J_k. Then this forms are $\mathrm{SL}_2(\mathbb{C})$-equivalent if and only if $F_1 = F_2$.*

2. Rational forms f_1 and f_2 of degree 1 are equivalent if and only if the forms f_1^{-1} and f_2^{-1} of degree -1 are equivalent.

Finally, let us note that Theorem 1.6 can be generalized for the case of real numbers. Namely, the following theorem holds.

Theorem 1.7. *1. Let f_1, $f_2 \in R_n^2$ be rational real forms of degree $n \neq 0, 1$ and F_1, F_2 are the corresponding dependencies between the invariants I_k. Then the forms f_1 and f_2 are equivalent if an only if $F_1 = F_2$ and 3-jets $j_3(f_1)$ and $j_3(f_2)$ belong to the same irreducible component of the manifold $\mathcal{E}^{(3)} \cap \{F_1 = 0\}$.*

2. Rational real forms f_1 and f_2 of degree 1 are equivalent if and only if the forms f_1^{-1} and f_2^{-1} of degree -1 are equivalent.

2 Ternary forms

In this section we use the differential invariant method to describe $\mathrm{GL}_3(\mathbb{C})$-orbits of ternary forms.

2.1 Differential invariant algebra

Let $\mathbb{C}^3$ be the space with the coordinates (x, y, z). As in the previous section, we identify ternary forms with solutions of the Euler differential equation:

$$xf_x + yf_y + zf_z = nf.$$

Denote by $J^k\mathbb{C}^3$ spaces of k-jets of functions with canonical coordinates $(x, y, z, u, u_{100}, u_{010}, u_{001}, \ldots)$.

Let $\mathcal{E}_1 \subset J^1\mathbb{C}^3$ be the algebraic manifold, corresponding to the Euler equation, and let $\mathcal{E}_k \subset J^k\mathbb{C}^3$ be its $(k-1)$-th prolongation.

It is easy to see that dimensions of these manifolds are equal to

$$\dim \mathcal{E}_k = 3 + \binom{k+2}{2}.$$

Recall that by a *differential invariant* of order k we understand a function $I \in C^\infty(\mathcal{E}_k)$, which is invariant with respect to prolonged action of group $\mathrm{GL}_3(\mathbb{C})$. We consider only invariants which are rational in u_σ.

In a similar way one defines an *invariant derivation* as a linear combination of total derivatives

$$\nabla = A\frac{d}{dx} + B\frac{d}{dy} + C\frac{d}{dy}$$

which is invariant with respect to the prolonged action of group $\mathrm{GL}_3(\mathbb{C})$.

Here A, B, $C \in C^\infty(\mathcal{E}_\infty)$ are functions, defined on prolongations of the Euler equation, and $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ are the total derivatives,

It is to check that the following modification of the Hessian

$$H = \frac{1}{u^3} \cdot \begin{vmatrix} u_{200} & u_{110} & u_{101} \\ u_{110} & u_{020} & u_{011} \\ u_{101} & u_{011} & u_{002} \end{vmatrix}$$

is a differential invariant of order 2.

Moreover, the relation

$$\widehat{d}I_1 \wedge \widehat{d}I_2 \wedge \widehat{d}I_3 = \{I_1, I_2, I_3\} d\Omega,$$

where $d\Omega = dx \wedge dy \wedge dz$ is the volume form, defines triple Nambu bracket

$$\{I_1, I_2, I_3\}$$

on the differential invariant algebra.

This bracket is three-linear, skew-symmetric, and is a derivation in each argument.

The bracket also satisfies the Filippov identity

$$\{I_1, I_2, \{J_1, J_2, J_3\}\} = \{\{I_1, I_2, J_1\}, J_2, J_3\} + \{J_1, \{I_1, I_2, J_2\}, J_3\} + \{J_1, J_2, \{I_1, I_2, J_3\}\}.$$

Taking $I_1 = \ln|u|$ and $I_2 = H$, we get invariant derivative $\nabla : I \longmapsto \nabla(I)$, where the function $\nabla(I)$ is defined by the relation:

$$\frac{1}{u}\widehat{d}u \wedge \widehat{d}H \wedge \widehat{d}I = (\nabla I)\widehat{d}\Omega.$$

The radial derivative $r = x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}$ is also invariant and one has $rI = nI$, for each homogenous function I.

Moreover, the standard affine connection on $\mathbb{C}^3$ is $\mathrm{SL}_3(\mathbb{C})$-invariant and defines decomposition of the jet spaces into the direct sum of symmetric tensors $\mathsf{S}^j(T^*\mathbb{C}^3)$.

Therefore, the connection defines a $\mathrm{GL}_3(\mathbb{C})$-invariant maps:

$$Q_k : J^k\left(\mathbb{C}^3\right) \to S^k\left(T^*\mathbb{C}^3\right),$$

where values of Q_k at points of $J^k\left(\mathbb{C}^3\right)$ are the following

$$Q_k = \sum_{p+q+s=k} \frac{u_{pqs}}{u} \frac{(\widehat{d}x)^p(\widehat{d}y)^q(\widehat{d}z)^s}{p!q!s!} \in \mathsf{S}^k(T^*\mathbb{C}^3).$$

Obviously, the value of k-tensors Q_k on a set of k invariant derivatives is a differential invariant.

Put

$$I := Q_3(\nabla, \nabla, \nabla).$$

This is a differential invariant of order 3.

Let us find one more invariants derivative.

To this end we consider a graph $L_f^\infty \subset \mathcal{E}_\infty$ of ternary form f.

Then the derivations r and ∇ defines vector fields on L_f^∞, and it is easy to see that these vector fields are orthogonal with respect to the invariant quadric Q_2.

Below we shall consider only such points in $\mathcal{E}_\infty$ where forms Q_2 are non degenerate.

Thus, $\det Q_2 \,|_{L_f} = H(f) \not\equiv 0$.

As $\dim L_f^\infty = 3$, then there exists a unique (up to non zero factor) vector field δ orthogonal to r and ∇. Normalizing it by the condition

$$\begin{vmatrix} r(x) & r(y) & r(z) \\ \nabla(x) & \nabla(y) & \nabla(z) \\ \delta(x) & \delta(y) & \delta(z) \end{vmatrix} = I,$$

we get the invariant derivative linear independent with r and ∇.

Direct computations shows that

$$\delta H = -I/n.$$

Using this derivative we get three more differential invariants of order 3:

$$J := Q_3(\nabla, \nabla, \delta)I, \quad K := Q_3(\nabla, \delta, \delta)I^2, \quad L := Q_3(\delta, \delta, \delta)I^3.$$

Remark that implicit formulae of invariants J, K and L in the jet space coordinates are very long.

We give here the formula for invariant$I$:

$$\begin{aligned} I = \big(&u_{300}u_{100}^3 + u_{030}u_{010}^3 + u_{003}u_{001}^3 + 6u_{111}u_{100}u_{010}u_{001} + \\ &+ 3u_{210}u_{100}^2u_{010} + 3u_{201}u_{100}^2u_{001} + 3u_{102}u_{100}u_{001}^2 + \\ &+ 3u_{120}u_{100}u_{010}^2 + 3u_{021}u_{010}^2u_{001} + 3u_{012}u_{010}u_{001}^2\big)/u^4. \end{aligned}$$

Now let us discuss now the separation of $\mathrm{GL}_3(\mathbb{C})$-jet orbits by differential invariants.

First of all we consider $\mathrm{GL}_3(\mathbb{C})$-orbits in jet spaces of dimension 0, 1, 2 and 3. There are two orbits of incomplete dimension in the space $J^0\mathbb{C}^3$ (namely, orbits $\{u = 0\}$ and $\{x = y = z = 0\}$).

These orbits will be called *singular orbits*.

On the manifold $\mathcal{E}_1 \subset J^1\mathbb{C}^3$, the orbits of incomplete dimension are the only orbits, whose projections in $J^0\mathbb{C}^3$ are singular.

In a similar way, the orbits of incomplete dimension in the manifold $\mathcal{E}_2 \subset J^2\mathbb{C}^3$ are orbits lying in $\{H = 0\}$ and the orbits, whose projections in $\mathcal{E}_1$ have incomplete dimension.

All these orbits are called *singular*.

Finally, $\mathrm{GL}_3(\mathbb{C})$)-orbits in manifolds $\mathcal{E}_k$ are called *singular*, if their projections in $\mathcal{E}_2$ are singular.

It is worth to note, that regular orbits in manifolds $\mathcal{E}_k$ have dimension $\dim \mathrm{GL}_3(\mathbb{C}) = 9$, when $k \geqslant 3$.

Theorem 2.1. *The field of differential invariant for the* $\mathrm{GL}_3(\mathbb{C})$*-action on the manifold* $\mathcal{E}_\infty$ *is algebraically generated by invariants* H, K, L *and invariant derivations* $\gamma = \nabla + Jr$ *and* δ. *This field separates non-singular orbits.*

Proof. Counting of the dimension of non-singular orbit shows that there exists one independent differential invariant of order 2.

It follows from the decomposition of jet space $J^k\mathbb{C}^3$ into direct sum of k-forms Q_k, that every differential invariant of order k is an algebraic function of coefficients of the forms Q_k in the "invariant basis" $\{\nabla^*, \delta^*\}$, when $k > 2$.

Moreover, I, J, K and L are coefficients of the form Q_3 in the "invariant basis"and therefore, every differential invariant of order 3 is an algebraic function, depending on H, I, J, K, L.

Finally, the invariant derivations of these invariants have the same symbols as the coefficients of the form Q_k in "invariant basis".

Therefore, due to the Rosenlicht theorem (see, for example,??), every differential invariant is an algebraic function on the invariant derivatives of H, I, J, K, L.

□

Remark. This proof shows, that the following version of the above theorem for differential invariant algebra is valid.

Consider differential invariants which are rational functions on u_σ, when $|\sigma| \leq 3$ and polynomial in u_τ, when $|\tau| > 3$. Then, the above theorem shows, that the field $\mathcal{F}_3$ of differential invariants of order less then 4 is algebraically generated by invariants H, K, L, I.

Moreover, similar to the proof of Theorem 1.1, one can show that differential invariant algebra, as an algebra over field $\mathcal{F}_3$, is generated by by invariants H, K, L and their invariant derivations.

Differential syzygies are generated by one relation for invariants of order 3 and by three relations for invariants of order 4.

2.2 $\mathrm{GL}_3(\mathbb{C})$-orbits of ternary forms

In this subsection we describe $\mathrm{GL}_3(\mathbb{C})$-orbits of ternary forms with non zero Hessian.

Consider differential invariants

$$H,\ I,\ J,\ K,\ L,\ \nabla I,\ \nabla J,\ \nabla K,\ \nabla L,\ \delta L.$$

Their restrictions on the graph $L_f^4 \subset \mathcal{E}_4$ of 4-jet of a ternary form f are homogenous rational functions in x, y, z.

Define the rational morphism $\pi \colon \mathbb{C}^3 \to \mathbb{C}^{10}$, as follows :

$$\pi(a) = (H(j_a^4(f)), I(j_a^4(f)), \ldots, \delta L(j_a^4(f))).$$

Then there exist algebraic dependencies between the restrictions of these differential invariants on the ternary form f.

Denote by $\mathscr{D}_f$ the set of these dependencies and by Σ_f image of morphism π.

Theorem 2.2. *1. Ternary forms f and $\widetilde{f}$, having non zero Hessian, are* $\mathrm{GL}_3(\mathbb{C})$*-equivalent if and only if* $\Sigma_f = \Sigma_{\widetilde{f}}$.

2. Ternary forms f and $\widetilde{f}$, having non zero Hessian, are $\mathrm{GL}_3(\mathbb{C})$*-equivalent if and only if* $\mathscr{D}_f = \mathscr{D}_{\widetilde{f}}$.

Proof. It is obvious that $\Sigma_f = \Sigma_{\widetilde{f}}$ and $\mathscr{D}_f = \mathscr{D}_{\widetilde{f}}$, for equivalent forms f and $\widetilde{f}$.

Let us prove the converse statement for the first statement of the Theorem.

Consider an arbitrary ternary form f with non zero Hessian and denote $\mathrm{GL}_3(\mathbb{C})$-orbit of the graph L_f^4 by $\mathscr{O}_f$.

Assume that for two ternary forms f and $\widetilde{f}$ one has $\Sigma_f = \Sigma_{\widetilde{f}}$.

Differential invariant algebra separates non-singular jet orbits, therefore sets $\mathscr{O}_f$ and $\mathscr{O}_{\widetilde{f}}$ has non empty intersection, and their intersection includes a Zarissky-open subset.

Hence, there exists an element $g \in \mathrm{GL}_3(\mathbb{C})$, such that graphs gL^4_f and $L^4_{\widetilde{f}}$ also intersect, and their intersection includes a Zarissky- open subset.

We claim that $gL^4_f = L^4_{\widetilde{f}}$ and $gf = \widetilde{f}$.

Consider the Cartan distribution $\mathcal{C}$ on manifold $\mathscr{O}_f$ (see [5]).

Then $\dim \mathcal{C} = 3$ on the open subset of $\mathscr{O}_f$ because $\dim \mathscr{O}_f = \dim \mathscr{O}^{(1)}_f$.

Thus, the Cartan distribution on $\mathscr{O}_f$ is completely integrable. It follows from the Frobenious theorem, that in a neighborhood of a non singular point there exists a unique maximal integral manifold. Thus, manifolds gL^4_f and $L^4_{\widetilde{f}}$ locally coincide.

But these manifolds are open subsets of irreducible algebraic manifolds, so $gL^4_f = L^4_{\widetilde{f}}$ and $gf = \widetilde{f}$.

Let us prove the converse statement of the second statement of the Theorem.

Consider algebra $B_f = \mathbb{C}[H(f), \ldots, \delta L(f)]$ and its spectrum $X_f = \mathrm{Spec}(B_f) = V(\mathscr{D}_f) \subset \mathbb{C}^{10}$.

Then the rational map $\pi \colon \mathbb{C}^3 \to X_f$ is dominant because $\pi^* \colon B_f \hookrightarrow \mathbb{C}(x, y, z)$ is an injection.

Therefore, its image $\mathrm{Im}(\pi) = \Sigma_f$ includes a Zarissky-open subset of X_f.

Now assume that for two ternary forms f and $\widetilde{f}$ one has $\mathscr{D}_f = \mathscr{D}_{\widetilde{f}}$. Then corresponding spectrums X_f and $X_{\widetilde{f}}$ coincide. So, $\Sigma_f \subset X_f$ and $\Sigma_{\widetilde{f}} \subset X_{\widetilde{f}}$ intersect and their intersection $\Sigma_f \cap \Sigma_{\widetilde{f}}$ includes an open subset of X_f. The rest of the proof repeats the proof of first statement of the Theorem verbatim. □

Remark. Theorem 2.2 is valid also for rational ternary forms of degree $n \neq 0, 1$. In case $n = 1$ rational ternary forms f and $\widetilde{f}$ of degree 1 are equivalent if and only if forms f^{-1} and $\widetilde{f}^{-1}$ of degree -1 are equivalent. In case $n = 0$ rational ternary forms $f = f_1/f_2$ and $\widetilde{f} = \widetilde{f}_1/\widetilde{f}_2$ of degree 0 are equivalent if and only if ternary forms $\lambda_1 f_1 + \lambda_2 f_2$ and $\lambda_1 \widetilde{f}_1 + \lambda_2 \widetilde{f}_2$ are equivalent (here λ_1, λ_2 are considered as the formal variables).

So, Theorem 2.2 makes it possible to classify $\mathrm{GL}_3(\mathbb{C})$-orbits of pairs of ternary forms, whose quotient has non-zero Hessian.

3 Appendix

In this section we give computer programs for Maple.

3.1 Binary forms

In this subsection we give a computer program in Maple which solves the question "*are two given rational forms f and g equivalent?*". Here $\xi_1 = I_2^2/I_1^3$ and $\xi_2 = I_3/I_1^2$, and polynomial F is presented as polynomial in ξ_1 and ξ_2.

```
>with(DifferentialGeometry): with(JetCalculus): with(Tools):
>with(LinearAlgebra): with(PolynomialIdeals): with(PDETools):
>Preferences(<<JetNotation>>,<<JetNotation2>>): DGsetup([x,y],[u],Bin,4):
>Tools:-DGinfo(Bin,<<FrameJetVariables>>):
>
> # INVARIANT AND INVARIANT DERIVATIVE
>H := (u[2,0]*u[0,2]-u[1,1]^2)/u[0,0]^2:
>nabla := proc(f)
  simplify((u[0,1]*TotalDiff(f,x)-u[1,0]*TotalDiff(f,y))/u[0,0])
 end proc:
>L := proc(h) {
  u[0,0]=h, u[1,0]=diff(h,x), u[0,1]=diff(h,y), u[2,0]=diff(h,x,x),
  u[1,1]=diff(h,x,y), u[0,2]=diff(h,y,y), u[3,0]=diff(h,x,x,x),
```

```
  u[2,1]=diff(h,x,x,y), u[1,2]=diff(h,x,y,y), u[0,3]=diff(h,y,y,y),
  u[4,0]=diff(h,x,x,x,x), u[3,1]=diff(h,x,x,x,y), u[2,2]=diff(h,x,x,y,y),
  u[1,3]=diff(h,x,y,y,y), u[0,4]=diff(h,y,y,y,y)}
 end proc:
>J[1] := proc(f) simplify(eval(H, L(f))) end proc:
>J[2] := proc(f) simplify(eval(nabla(H), L(f))) end proc:
>J[3] := proc(f) simplify(eval(nabla(nabla(H)), L(f))) end proc:
>
> # INVARIANT OF RATIONAL BINARY FORM
>First := proc(f) simplify(eval(J[2](f)^2/J[1](f)^3, y=t*x)) end proc:
>Second := proc(f) simplify(eval(J[3](f)/J[1](f)^2, y=t*x)) end proc:
>Singular := proc(f) simplify(2*J[1](f)*J[3](f)-3*J[2](f)^2) end proc:
>Invariant := proc (f) local P, Q, R, T, G
  if f = 0 then return 0; fi;
  if J[1](f) = 0 then return h; fi;
  if Singular(f) = 0 then return xi[1]-First(f); fi;
  P := numer(First(f))-xi[1]*denom(First(f));
  Q := numer(Second(f))-xi[2]*denom(Second(f));
  R := factor(resultant(P, Q, t));
  T := simplify(Radical(<R>));
  G := Generators(T); G[1];
 end proc:
>
> # EQUIVALENCE OF TWO BINARY FORMS
>equivalence := proc(f,g) local Id;
  if degree(f) <> degree(g)
    then return false; fi;
  Id := <Invariant(f)>;
  IdealMembership(Invariant(g), Id);
 end proc:
```

Below we give an example of application of this program.

```
>f:=(x^4+y^4)/(x^2+y^2):
>g:=(2*x^4+4*x^3*y+30*x^2*y^2+28*x*y^3+17*y^4)/(2*x^2+2*x*y+5*y^2):
>equivalence(f,g);
```

$$true$$

```
>Invariant(f);
```

$$\begin{aligned} &-147456\xi_1^2 - 36864\xi_2^2\xi_1 + 165888\xi_2\xi_1^2 - 28032\xi_2^3\xi_1 + \\ &\qquad + 178848\xi_2^2\xi_1^2 + 90396\xi_2\xi_1^3 + 2808\xi_2^4\xi_1 - 15552\xi_2^3\xi_1^2 - \\ &\qquad\qquad - 1236384\xi_1^3 - 2304\xi_2^4 - 2187\xi_1^4 - 2208\xi_2^5 + 200\xi_2^6 \end{aligned}$$

3.2 Ternary forms

Here we give a computer program in Maple which counts invariants I, J, K and L.

```
>with(DifferentialGeometry): with(JetCalculus): with(Tools):
>with(LinearAlgebra): with(PolynomialIdeals): with(PDETools):
>Preferences(<<JetNotation>>,<<JetNotation2>>): DGsetup([x,y,z],[u],Inv,3):
>Tools:-DGinfo(Inv,<<FrameJetVariables>>):
>
> # INVARIANTS H, K AND DERIVATION NABLA
>Hessian := Matrix([[u[2,0,0], u[1,1,0], u[1,0,1]],
                  [u[1,1,0], u[0,2,0], u[0,1,1]],
                  [u[1,0,1], u[0,1,1], u[0,0,2]]])/u[0,0,0]:
>H := Determinant(Hessian):
>
>nabla := proc (f) local M;
   M := Matrix([[TotalDiff(f,x), TotalDiff(u[0,0,0],x), TotalDiff(H,x)],
               [TotalDiff(f,y), TotalDiff(u[0,0,0],y), TotalDiff(H,y)],
               [TotalDiff(f,z), TotalDiff(u[0,0,0],z), TotalDiff(H,z)]]):
   simplify(Determinant(M))/u[0,0,0];
 end proc:
>
>Nx := simplify(nabla(x)):
>Ny := simplify(nabla(y)):
>Nz := simplify(nabla(z)):
>
>K := simplify(Matrix([[Nx, Ny, Nz]]).Hessian.Matrix([[Nx], [Ny], [Nz]]))[1][1]:
>
> # DERIVATION DELTA
>N := Matrix([[Nx], [Ny], [Nz]]):
>eq1 := simplify(Matrix([[X, Y, Z]]).Hessian.N)[1][1]:
>eq2 := factor(Determinant(Matrix([[x, y, z], [Nx, Ny, Nz], [X, Y, Z]]))):
>sol := simplify(solve([eq1, eq2-1,
       u[1,0,0]*X+u[0,1,0]*Y+u[0,0,1]*Z], [X, Y, Z])):
>
>Dx:=factor(eval(X,sol[1]))*K:
>Dy:=factor(eval(Y,sol[2]))*K:
>Dz:=factor(eval(Z,sol[3]))*K:
>
>delta := proc (f)
   Dx*TotalDiff(f,x)+Dy*TotalDiff(f,y)+Dz*TotalDiff(f,z);
 end proc:
>
> # INVARIANTS I AND J
>Mat := <<a,p,x>|<b,q,y>|<c,r,z>|<symb[radial],symb[nabla],symb[delta]>>:
>M_sol := LinearSolve(Mat):
>d[1] := M_sol[1]: d[2] := M_sol[2]: d[3] := M_sol[3]:
>
>Q[3] := (u[3,0,0]*d[1]^3/(3!*0!*0!)+u[2,1,0]*d[1]^2*d[2]^1/(2!*1!*0!)+
 u[2,0,1]*d[1]^2*d[3]^1/(2!*0!*1!)+u[1,2,0]*d[1]^1*d[2]^2/(1!*2!*0!)+
 u[1,1,1]*d[1]*d[2]*d[3]/(1!*1!*1!)+u[1,0,2]*d[1]^1*d[3]^2/(1!*0!*2!)+
 u[0,3,0]*d[2]^3/(0!*3!*0!)+u[0,2,1]*d[2]^2*d[3]/(0!*2!*1!)+
 u[0,1,2]*d[2]^1*d[3]^2/(0!*1!*2!)+u[0,0,3]*d[3]^3/(0!*0!*3!))/u[0,0,0]:
>
> inv_I := factor(eval(coeff(eval(Q[3], [symb[radial]=0, symb[nabla]=1]),
         symb[delta]^3), [a=Nx,b=Ny,c=Nz,p=Dx,q=Dy,r=Dz])):
```

```
> inv_J := factor(eval(coeff(eval(Q[3], [symb[radial]=0, symb[nabla]=1]),
          symb[delta]^2), [a=Nx,b=Ny,c=Nz,p=Dx,q=Dy,r=Dz])*K):
> inv_K := factor(eval(coeff(eval(Q[3], [symb[radial]=0, symb[nabla]=1]),
          symb[delta]^1), [a=Nx,b=Ny,c=Nz,p=Dx,q=Dy,r=Dz])*K^2):
> inv_L := factor(eval(coeff(eval(Q[3], [symb[radial]=0, symb[nabla]=1]),
          symb[delta]^0), [a=Nx,b=Ny,c=Nz,p=Dx,q=Dy,r=Dz])*K^3):
```

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