HAMMING WEIGHTS AND BETTI NUMBERS OF STANLEY-REISNER RINGS ASSOCIATED TO MATROIDS

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Abstract. To each linear code over a finite field we associate the matroid of its parity check matrix. We show to what extent one can determine the generalized Hamming weights of the code (or defined for a matroid in general) from various sets of Betti numbers of Stanley-Reisner rings of simplicial complexes associated to the matroid.

1. Introduction

Let $\mathbb{F}_q$ be a finite field. A linear code C is a linear subspace of $\mathbb{F}_q^n$ for some $n \in \mathbb{N}$. We usually denote the dimension of the code by k (and it can defined as $k = log_q|C|$ also for non-linear block codes). Such a code is called an $[n,k]$-code over $\mathbb{F}_q$. For $h = 1, 2, \cdots, k$ let D_h be the set of all linear subspaces of the linear code C of dimension h, and let

$$d_h = min\{|Supp(E)||E \in D_h\}.$$

As usual d_1 can be identified with the minimum distance

$$d = min_{\mathbf{x},\mathbf{y}\in C,\mathbf{x}\neq\mathbf{y}} d(\mathbf{x},\mathbf{y}),$$

of the code C, where $d(\mathbf{x},\mathbf{y})$ is the usual Hamming distance. One aim in coding theory is to maximise d given q, n, k. In processes of trellis decoding, (or in certain methods in cryptology, using generator/parity check matrices of C instead as a starting point), it is interesting to study and maximise d_h also for higher values of h. Thus a full determination of the code parameters for a linear code over $\mathbb{F}_q$ can be said to involve finding $n, k, d_1, \cdots, d_k$.

These parameters are completely determined by the underlying matroid structure of the code. All generator matrices G for the code determine the same finite matroid M_G of rank k and cardinality n for an $[n,k]$-code C, while all parity check matrices H determine the same matroid M_H of rank $r = n - k$ and cardinality n.

Given a parity check matrix H it is a well established fact that: d_h is the minimum number s, such that there are s columns of H that form a submatrix of rank $s - h$.

The d_h can be read off from generator matrices of C also, in a quite different way. The ease, by which we can translate question of determining the d_h into properties of the rank function of the matroid associated to the parity check matrices H, however, makes us choose as a standard the matroid $M(C) = M_H$, the matroid derived from the linear code C.

It is a well known fact that the matroids M_H and M_G are matroid duals: hence $M(C)$ and $M(C^\perp)$ are matroid dual (and determine each other), where $C^\perp$ is the orthogonal complement of C. Furthermore the weight hierarchy $d_1, \cdots, d_k$ of C determines the weight hierarchy $d_1^\perp, \cdots, d_r^\perp$ of $C^\perp$ by Wei duality (See [12]), and vice versa.

In this article we show to what extent the code parameters (in particular the generalized Hamming weights) of a linear code are determined by the various sets of Betti numbers one

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can associate to simplicial complexes derived from the underlying matroid $M(C)$, through matroid and Alexander duality.

The main results are Theorems 4.1 and 4.2, and the total picture is summarized in the conclusion at the end of the paper.

In Section 2 we recall the standard definitions of Stanley-Reisner rings of simplicial complexes, and their resolutions and Betti numbers of various kinds. We describe its nullity function n, which will be a central tool, and we use it to define higher weights of matroids so that these weights "match" the weights of codes they represent when they do so. The definition (2.1) also applies to non-vectorial matroids.

In Section 3 we define sets of so-called non-redundant circuits of matroids $(E, \mathcal{I})$ and use properties of matroids to identify the nullity of a subset of E with the maximal number of non-redundant circuits contained in it.

In Section 4 we give our main results in positive direction, explaining how the weights $d_1, \cdots, d_k$ are determined by the $\mathbb{N}$-graded Betti-numbers of the Stanley-Reisner ring of the simplicial complex whose faces are the independant sets of $M(C)$, and derive some consequences. We also give examples which give negative results concerning other types of Betti numbers. while the theory is valid for matroids, and thus codes, the negative examples that we give come from codes giving non-degenerate matroids over the same alphabet and with the same rank.

In Section 5 we investigate Alexander duality, which gives us simpler resolutions, but show that the ungraded and $\mathbb{N}$-graded Betti-numbers of these resolutions are not sufficient to give us the weight hierarchy.

2. Definitions and notation

A simplicial complex Δ on the finite ground set E is a subset of 2^E closed under taking subsets. We refer to [8] for a brief introduction of the theory of simplicial complexes, and we follow their notation. The elements of Δ are called faces, and maximal elements under inclusion facets. The dimension of a face is equal to one less its cardinality. We denote $F_i(\Delta)$ the set of its faces of dimension i. The Alexander dual $\Delta^\star$ of Δ is the simplicial complex defined by

$$\Delta^\star = \{\overline{\sigma}, \sigma \notin \Delta\}.$$

Given a simplicial complex Δ on the ground set E, define its Stanley-Reisner ideal and ring in the following way: let $\Bbbk$ a field and let $S = \Bbbk[\mathbf{x}]$ be the polynomial ring over $\Bbbk$ in $|E|$ indeterminates $\mathbf{x} = \{x_e |\ e \in E\}$. Then the Stanley-Reisner ideal I_Δ of Δ is

$$I_\Delta = < \mathbf{x}^\sigma |\ \sigma \notin \Delta >$$

and its Stanley-Reisner ring is $R_\Delta = S/I_\Delta$. This ring has a minimal free resolution as a $\mathbb{N}^E$-graded module

$$0 \longleftarrow R_\Delta \xleftarrow{\partial_0} P_0 \xleftarrow{\partial_1} P_1 \longleftarrow \cdots \xleftarrow{\partial_l} P_l \longleftarrow 0 \tag{1}$$

where each P_i is of the form

$$P_i = \bigoplus_{\alpha \in \mathbb{N}^E} S(-\alpha)^{\beta_{i,\alpha}}.$$

The $\beta_{i,\alpha}$ are called the $\mathbb{N}^E$-graded Betti numbers of Δ. We have $\beta_{i,\alpha} = 0$ if $\alpha \in \mathbb{N}^E - \{0,1\}^E$. The Betti numbers are independent of the choice of the minimal free resolution. The $\mathbb{N}$-graded and ungraded Betti numbers of Δ are respectively

$$\beta_{i,d} = \sum_{|\alpha| = d} \beta_{i,\alpha}$$

and

$$\beta_i = \sum_d \beta_{i,d}.$$

If we give an ordering ω on E, we can build the reduced chain complex of Δ with respect to ω in the following way: for any $i \in \mathbb{N}$, let V_i be a vector space over $\Bbbk$ whose basis elements e_σ correspond to $\sigma \in F_i(\Delta)$. And if $\sigma \in F_i(\Delta)$, define

$$\partial_{\omega,i}(e_\sigma) = \sum_{x\in\sigma} \epsilon_{\omega,\sigma}(x) e_{\sigma-\{x\}},$$

where $\epsilon_{\omega,\sigma}(x) = (-1)^{r-1}$ if x is the r^{th} element in σ with respect to the ordering ω. The chain complex is

$$0 \longleftarrow V_{-1} \xleftarrow{\partial_{\omega,0}} V_0 \xleftarrow{\partial_{\omega,1}} V_1 \xleftarrow{\partial_{\omega,2}} \cdots \,... \xleftarrow{\partial_{\omega,|E|-1}} V_{|E|-1} \longleftarrow 0.$$

Let

$$\tilde{h}_i(\Delta;\Bbbk) = \dim_{\Bbbk}(\ker(\partial_{\omega,i})/\mathrm{im}(\partial_{\omega,i+1})).$$

This is independant of the ordering ω, so we omit it in the notation.

If $\sigma \subset E$, we denote by Δ_σ the simplicial complex whose faces are $\{\tau \cap \sigma |\ \tau \in \Delta\}$. A result by Hochster [7] says that

$$\beta_{i,\sigma} = \tilde{h}_{|\sigma|-i-1}(\Delta_\sigma;\Bbbk).$$

Let now M be a matroid on the finite ground set E. We refer to [9] for their theory. We will denote $\mathcal{I}_M$ its set of independent sets, $\mathcal{B}_M$ its set of bases, $\mathcal{C}_M$ its set of circuits, and r_M its rank function. The nullity function n_M is defined by $n_M(\sigma) = |\sigma| - r_M(\sigma)$. By abuse of notation, $r(M) = r_M(E)$. A matroid M is also naturally a simplicial complex whose faces are the independent sets. We still denote it by M. As such all the definitions and results mentioned above apply. But in the case of matroids, if $\sigma \subset E$, then M_σ is also a matroid. The rank r_{M_σ} is the restriction of r_M to the subsets of σ, and so is the nullity function. The dual of M is denoted by $\overline{M}$. Note that while $\overline{M}$ still is a matroid, $M^\star$ generally isn't.

Definition 2.1. *The generalized Hamming weights of M are defined by*

$$d_i = \min\{|\sigma||\ n_M(\sigma) = i\}$$

for $1 \leqslant i \leqslant |E| - r_M(M)$.

We will often omit the reference to M when it is obvious from the context.

Remark 2.1. If C is a linear $[n, k]$-code over some finite field $\mathbb{F}_q$, with $(r \times n)$ parity check matrix H (think of r as redundancy of C or rank of H), and $M = M_H$ is the matroid associated to H, then it is well known (see e.g. [12]) that the higher weight heierarchy $d = d_1 < \cdots < d_k$ of C as a linear code is identical to that of M in the sense of Definition 2.1, and this is the motivation of our defintion. Viewing the matroid M as a special case of a so-called demi-matroid, as in [3], the invariants d_i are the same as those called $\overline{\sigma_i}$, for the trivial poset order there.

The goal of this paper is to see the relations between Betti numbers and generalized Hamming weights, for matroids in general, and then automatically, for those matroids associated to (parity check matrices of) linear codes.

3. Relation between the nullity function and the non-redundancy of circuits

We start by giving some definition about the non-redundancy of circuits.

Definition 3.1. *Given a matroid M and $\Sigma \subset \mathcal{C}_M$. We say that the elements in Σ are non-redundant if for every $\sigma \in \Sigma$,*

$$\bigcup_{\tau \in \Sigma - \{\sigma\}} \tau \subsetneq \bigcup_{\tau \in \Sigma} \tau.$$

It is obvious that the elements in Σ are non-redundant if and only if there exists elements $x_\sigma \in E$ for $\sigma \in \Sigma$ with the property that for all $\sigma, \tau \in \Sigma$ we have: $x_\sigma \in \tau \Leftrightarrow \sigma = \tau$.

Definition 3.2. *Let M be a matroid and σ be a subset of the ground set. The degree of non-redundancy of σ is equal to the maximal number of non-redundant circuits contained in σ. It is denoted by $\deg \sigma$.*

We will now see that the degree of non-redundancy of a subset is equal to its nullity.

Lemma 3.1. *Let M be a matroid and let $\tau_1, ..., \tau_n$ be non-redundant circuits. Then*

$$n(\bigcup_{1 \leqslant i \leqslant n} \tau_i) \geqslant n.$$

Proof. This is obvious for $n = 1$. There is actually equality in that case. Suppose that this is true for $n \geqslant 1$, and we prove that it is also true for $n+1$. Let $x_{n+1} \in \tau_{n+1}$ but in no other τ_i. Of course, $\tau_1, ..., \tau_n$ are non-redundant, and by induction hypothesis,

$$n(\bigcup_{1 \leqslant i \leqslant n} \tau_i) \geqslant n.$$

We know that for any given two subsets A, B of the ground set, we have

$$n(A \cup B) \geqslant n(A) + n(B) - n(A \cap B),$$

since this is equivalent to the matroid axiom:

$$r(A \cup B) + r(A \cap B) \leqslant r(A) + r(B).$$

If we apply it to $A = \bigcup_{1 \leqslant i \leqslant n} \tau_i$ and $B = \tau_{n+1}$, noticing that $n(B) = n(\tau_{n+1}) = 1$, $n(A) \geqslant n$, we see that

$$n(\bigcup_{1 \leqslant i \leqslant n+1} \tau_i) \geqslant n + 1 - n(A \cap B).$$

But in this case, $A \cap B \subset \tau_{n+1} - \{x_{n+1}\}$ has to be independant, and therefore $n(A \cap B) = 0$ and the lemma follows. □

Corollary 3.1. *Let M be a matroid and σ a subset of the ground set. Then*

$$n(\sigma) \geqslant deg(\sigma).$$

Proof. Let $d = deg(\sigma)$, and $\tau_1, ...\tau_d$ be d non-redundant circuits included in σ. Since n is growing, we have

$$n(\sigma) \geqslant n(\bigcup_{1 \leqslant i \leqslant n} \tau_i) \geqslant d.$$

□

Lemma 3.2. *Let M be a matroid and $\tau_1, ..., \tau_m$ be m non-redundant circuits with union $\tau = \bigcup_{1 \leqslant i \leqslant m} \tau_i$. Let ρ be another circuit such that $\rho \not\subset \tau$. Let $x \in \rho - \tau$. Then there exists a circuit τ_{m+1} such that $x \in \tau_{m+1}$ and such that $\tau_1, ..., \tau_{m+1}$ are non-redundant.*

Proof. As usual, we fix x_i such that $x_i \in \tau_j$ for $1 \leqslant i, j \leqslant m$ if and only if $i = j$. Consider the set of circuits that contain x. It is by hypothesis not empty. Consider an element τ_{m+1} in this set that contains fewest x_i. We claim that this number is 0. If not, then there exists a $i \leq m$ be such that $x_i \in \tau_{m+1}$. Consider the two circuits τ_{m+1} and τ_i. They have the element x_i in common, and the element $x \in \tau_{m+1} - \tau_i$; by the strong elimination axiom for circuits of a matroid (see [13, Theorem 1.9.2]), we can find a circuit σ such that

$$x \in \sigma \subset \tau_{m+1} \cup \tau_i - \{x_i\}.$$

It is easy to see that σ has fewer x_i than τ_{m+1}, which is absurd. This means that $\tau_1, ..., \tau_{m+1}$ are non-redundant. □

Corollary 3.2. *Let M be a matroid, and $\tau_1, ..., \tau_m$ be a maximal set of non-redundant circuits. Then*

$$\bigcup_{1 \leqslant i \leqslant m} \tau_i = \bigcup_{\tau \in \mathcal{C}} \tau.$$

Lemma 3.3. *Let M be a matroid and σ a subset of the ground set. Let $d = n(\sigma)$. Then there exists d non-redundant circuits in σ. Thus $deg(\sigma) \geqslant n(\sigma)$.*

Proof. This is obviously true for $d = 0$, and for $d = 1$. Suppose the lemma doesn't hold. Let σ minimal such that it doesn't hold. Then $d = n(\sigma) \geqslant 2$, and we can find a circuit $\tau \subset \sigma$. Let $x_d \in \tau$, and consider $\sigma' = \sigma - \{x_d\}$. By minimality of σ, there exists $n(\sigma')$ non-redundant circuits in σ'. And since the lemma doesn't hold for σ, $n(\sigma') < d$. But by the property of n, this implies that $n(\sigma') = d - 1$. Let $\tau_1, ..., \tau_{d-1}$ be $d - 1$ non-redundant circuits in σ'. By lemma 3.2, we can find a circuit $\tau_d \subset \sigma$ such that $x_d \in \tau_d$ and $\tau_1, ..., \tau_d$ are non-redundant, and this contradicts the fact that the lemma doesn't hold for σ. □

Proposition 3.1. *Let M be a matroid, and let σ be a subset of the ground set. Then we have*

$$deg\ \sigma = n(\sigma).$$

4. Betti numbers and generalized Hamming weights

Let M be a matroid on the ground set E. For any integer $0 \leqslant d \leqslant |E| - r(M)$, let $N_d = n^{-1}(d)$. Note that $N_0 = \mathcal{I}$. We will now prove the following:

Theorem 4.1. *Let M be a matroid on the ground set E. Let $\sigma \subset E$. Then*

$$\beta_{i,\sigma} \neq 0 \Leftrightarrow \sigma \text{ is minimal in } N_i.$$

And in the case where $\sigma \in N_i$, we have $\beta_{i,\sigma} = (-1)^{r(\sigma)-1}\chi(M_\sigma)$.

Proof. The matroid M_σ has rank $r(\sigma)$, and thus by [1, th. 7.8.1], we know that M_σ might have reduced homology just in degree $r(\sigma) - 1$. We know that $\beta_{i,\sigma} = \tilde{h}_{|\sigma|-i-1}(M_\sigma, \mathbb{k})$. So $\beta_{i,\sigma} = 0$ except may be when $i = n(\sigma)$. In this case, we have by [1, th 7.4.7 and 7.8.1]

$$\beta_{i,\sigma} = \tilde{h}_{r(\sigma)-1}(M_\sigma, \mathbb{k}) = (-1)^{r(\sigma)-1}\chi(M_\sigma).$$

It remains to prove that this is non-zero if and only if σ is minimal in N_i. It is well know that for a matroid N, $\chi(N) = 0$ if and only if N has an isthmus, that is, if and only if $\overline{N}$ has a loop. This follows for example from [1, Exerc. 7.39]. But we have the equivalences:

- The matroid $\overline{N}$ has a loop,
- There exists an element which is in no base of $\overline{N}$,
- There exists an element which is in all the bases of N,
- There exists an element which is in no circuit of N,

- The underlying set of N is not equal to the union of its circuits.

And Corollary 3.2 just says that σ is minimal in N_i if and only if it is equal to the union of its circuits. $\square$

Corollary 4.1. a) *Let M be a matroid on the ground set E. Then*

$$\beta_{0,\sigma} = \begin{cases} 1 & \text{if } \sigma = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta_{1,\sigma} = \begin{cases} 1 & \text{if } \sigma \text{ is a circuit} \\ 0 & \text{otherwise} \end{cases}$$

b) *The resolution has length exactly $k = |E| - r(M)$, that is: $N_k \neq 0$, but $N_i = 0$, for $i > k$.*

Proof.

a) is immediate from Theorem 4.1.

b) There exists a σ such that $|\sigma| - rk(\sigma) = n - r$ (for example $\sigma = E$) but no σ with $|\sigma| - r_M(\sigma) > n - r$. Hence $N_{n-r} \neq \emptyset$, but $N_i = \emptyset$ if $i > n - r$.

$\square$

Corollary 4.2. *A matroid M is entirely given by its $\mathbb{N}^E$-graded Betti numbers in homology degree 1.*

Remark 4.1. Part b) of Corollary 4.1 is not new. It shows that the projective dimension of R as an S-module is $\dim S - \dim R = n - r$. In [11, Theorem 3.4] it is shown that the Stanley-Reisner ring R of a matroid complex is level, in particular it is a Cohen-Macaulay graded algebra over k, and then the projective dimension is $n - r$. That R is level also means that the rightmost term P_{n-r} of its minimal resolution is pure, that is of the form $S(-b)^a$ for some non-negative integers a, b.

We are now able to give and prove the following relation between $\mathbb{N}$-graded Betti numbers of a matroid M and its Hamming weights:

Theorem 4.2. *Let M be a matroid on the ground set. Then the generalized Hamming weights are given by*

$$d_i = \min\{d |\ \beta_{i,d} \neq 0\} \text{ for } 1 \leqslant i \leqslant |E| - r(M).$$

Proof. Let σ minimal such that $n(\sigma) = i$. Then $\beta_{i,\sigma} \neq 0$ which implies $\beta_{i,d_i} = \beta_{i,|\sigma|} \neq 0$, and thus

$$d_i \geqslant \min\{d |\ \beta_{i,d} \neq 0\}.$$

Let now d minimal such that $\beta_{i,d} \neq 0$. This means that there exists a subset σ of E of cardinality d such that $\beta_{i,\sigma} \neq 0$. Then σ is (minimal) in N_d, and thus

$$d_i \leqslant \min\{d |\ \beta_{i,d} \neq 0\}.$$

$\square$

Corollary 4.3. *Let M be a matroid on the ground set E, of rank r. Then*

$$d_{|E|-r} = \left| \bigcup_{\tau \in \mathcal{C}} \tau \right| = |E| - |\{\text{loops of } \overline{M}\}|.$$

Remark 4.2. When $M = M_H$, the matroid of some parity check matrix for a linear code C, this number is just the cardinality of the support of C, since each loop of $\overline{M} = M_G$ for any generator matrix G of C, corresponds to a coordinate position where all code-words are zero.

As an other comment, not directly related to coding, we add that since R (see Remark 4.1) is level, we have by [11, Prop. 3.2,f.] that $P_{n-r} = S(-d_{n-r})^{h_s}$, where s is maximal such that $h_s \neq 0$. For a Cohen-Macaulay Stanley-Reisner ring the h_i can be defined by $\sum_{i=0}^{r} f_{i-1}(t-1)^{r-i} = \sum_{i=0}^{r} h_i t^{d-i}$, where f_i is the number of independent sets of cardinality $i+1$ in the matroid M (See [6, Formula (1)]). Here $s \leq n-r$, and we see that

$$h_{n-r} = \sum_{i=0}^{r} f_{i-1}(-1)^{r-i}.$$

Example 4.1. It is well known (see e.g. [10, (Text (following) Proof of Lemma 5.1]) that the resolution of the uniform matroid $U(r,n)$ corresponding to MDS-codes of length n and dimension $k = n - r$ (since we are studying the rank function of the parity matrix/matroid) is:

$$0 \longleftarrow R_{U(r,n)} \longleftarrow S \longleftarrow S(-(r+1))^{\binom{r}{r}\binom{n}{r+1}} \longleftarrow S(-(r+2))^{\binom{r+1}{r}\binom{n}{r+2}} \longleftarrow$$
$$S(-(r+3))^{\binom{r+2}{r}\binom{n}{r+3}} \longleftarrow \cdots \longleftarrow S(-(n-1))^{\binom{n-2}{r}\binom{n}{n-1}} \longleftarrow S(-n)^{\binom{n-1}{r}\binom{n}{n}} \longleftarrow 0.$$

Hence the weight hierarchy is $\{n-k+1, \cdots, n-1, n\}$. We see from the rightmost part of the resolution that $h_{n-r} = \binom{n-1}{r}$, while

$$f_{r-1} - f_{r-2} + \cdots + (-1)^r f_{-1} = \binom{n}{r} - \binom{n}{r-1} + \cdots + (-1)^r \binom{n}{0},$$

which is also $\binom{n-1}{r}$.

Example 4.2. Let C be the binary non-degenerate $[6,3]$-code with parity check matrix

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The matroid $M_1 = M_{H_1}$ has $E = \{1,2,3,4,5,6\}$ and maximal independent sets (basis):

$$\mathcal{B}_1 = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{2,3,4\},\{2,3,6\},$$
$$\{2,4,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\}.$$

We then (using [2]) have the resolution:

$$0 \longleftarrow R_{M_1} \longleftarrow S \longleftarrow S(-2) \oplus S(-3)^3 \oplus S(-4)^2 \longleftarrow S(-4)^2 \oplus S(-5)^7 \longleftarrow S(-6)^4 \longleftarrow 0.$$

Hence the d_i are $2,4,6$.

Remark 4.3. We have just seen that the $\mathbb{N}$-graded Betti-numbers of a matroid decide the generalized Hamming weights. The converse is not true, as the following example shows: consider the binary non-degenerate $[4,2]$-codes with parity check matrices

$$H_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

and

$$H_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Their associated matroids are on $E = \{1, 2, 3, 4\}$ with basis sets:

$$\mathcal{B}_2 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$$

and

$$\mathcal{B}_3 = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$$

respectively. Their weight hierarchy is $(2, 4)$, while their ungraded Betti numbers are $(1, 3, 2)$ and $(1, 2, 1)$ respectively.

Remark 4.4. Moreover, the ungraded Betti numbers don't give the weight hierarchy, as the following example shows: consider the binary non-degenerate $[4, 2]$-codes with parity check matrices

$$H_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

and

$$H_5 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Their associated matroids are on $E = \{1, 2, 3, 4\}$ with basis sets:

$$\mathcal{B}_4 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$$

and

$$\mathcal{B}_5 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$$

respectively. Their ungraded Betti numbers are $(1, 3, 2)$ while their weight hierarchies are $(2, 3)$ and $(2, 4)$ respectively.

In the rest of this section we derive and study some consequences of Theorem 4.2. Recall that a linear code C of length n and dimension k is called h-MDS if its higher support weight d_h satisfies $d_h = n - k + h$ for some $h \in \{1, \cdots, k\}$. If C is h-MDS, then C is i-MDS for all $i \in \{h, \cdots, k\}$. We recall that C is called MDS if C is 1-MDS, that is $d = n - k + 1$. For a length n and dimension k code, the matroid $M = M(C^{\perp})$ has rank $r = n - k$ and is equipped with invariants $d_1, \cdots, d_k$ which have the same values as the support weights of C. For the resolution (1) of the Stanley-Reisner ring R of (the simplicial complex of independent sets of) M we then have:

Corollary 4.4. *We have the following:*

a) *C is h-MDS if and only if the right part*

$$P_h \longleftarrow P_{h+1} \longleftarrow \cdots \longleftarrow P_k$$

of the resolution is linear, and M has no isthmus.

b) *C is MDS if and only if the entire resolution is linear (that is linear from P_1 and rightover), and M has no isthmus (C is non-degenerate).*

c) *If C is non-degenerate, then it is MDS if and only if the Alexander dual of M is also (the set of independent sets of) a matroid.*

Proof. It is clear from Corollary 4.1 and the fact that $N_i = \emptyset$ for all $i > k = |E| - rk(E)$ that $\beta_{i,j} = 0$ for all j for $i > k$. From Theorem 4.2 it is clear that $d_h = n - k + h$ if and only if $\beta_{i,j} = 0$ for $j < n - k + i$, and $\beta_{i,n-k+i} \neq 0$, for each $i \in \{h, h+1, \cdots, k\}$ (in particular for $i = h$, and thus implying the statements for the remaing i, by standard facts from coding theory). Since there is no facet with more than n elements it is clear that $\beta_{k,j} = 0$ for $j > n$. Let

$$\nu_i (= d_i) = \min\{j | \beta_{i,j} \neq 0\} \text{ and } \mu_i = \max\{j | \beta_{i,j} \neq 0\}.$$

We have just shown that $\nu_k = \mu_k = n$ if and only if $d_k = n$. Since by [11, Theorem 3.4] R is a level ring, in particular Cohen-Macaulay, it is true that the μ_i form a strictly increasing sequence (See [4, Prop. 1.1]. It is therefore also clear that if M has no isthmus, that is $d_k = n$, then $\mu_i \leq \mu_k - [k-i] = n-k+1$. Hence $\beta_{i,j} = 0$ for $j > n-k+i$. This gives a).

The statement of b) is just the special case $h = 1$ of a). It is clear that if C is an $[n,k]$ MDS-code, then M is the uniform matroid $U(r,n)$, and then its Alexander dual is the matroid $U(k-1,n)$. If on the other hand C is not MDS, then the Stanley-Reisner ring does not have a linear resolution, and then it's not the Alexander dual of any matroid (by [6, Proposition 7]). □

Example 4.3.

- Let C be the linear binary $[6,3]$-code with parity check matrix

$$H_6 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

For the matroid $M_6 = M_{H_6}$ we have $E = \{1,2,3,4,5,6\}$ and maximal independent sets

$$\begin{gathered}\mathcal{B}_6 = \{\{1,2,3\},\{1,2,4\},\{1,2,6\},\{1,3,5\},\{1,3,6\},\\ \{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\\ \{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\}.\end{gathered}$$

Using [2] we obtain the resolution:

$$0 \longleftarrow R_{M_6} \longleftarrow S \longleftarrow S(-3)^4 \oplus S(-4)^3 \longleftarrow S(-5)^{12} \longleftarrow S(-6)^6 \longleftarrow 0.$$

Here $\{d_1, d_2, d_3\} = \{3,5,6\}$, so C is 2-MDS, and we see that the part of the resolution consisting of the two rightmost terms (corresponding to d_2 and d_3) is linear as described.

- Let C be the linear $[6,3]$-code over F_5 with generator matrix

$$G_7 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 & 4 & 1 \end{pmatrix}.$$

Here the maximal independent sets of M_{G_7} are precicely the 10 subsets of $\{1,2,3,4,5,6\}$ of cardinality 3, not containing 2. Let H_7 be any parity check matrix. For the matroid $M_7 = M_{H_7}$ we have $E = \{1,2,3,4,5,6\}$ and the maximal independent sets are precisely the 10 subsets of E of cardinality 3 containing 2. Using [2] this gives the resolution:

$$0 \longleftarrow R_{M_7} \longleftarrow S \longleftarrow S(-3)^{10} \longleftarrow S(-4)^{15} \longleftarrow S(-5)^6 \longleftarrow 0.$$

Hence the d_i are $3,4,5$. We see that $d_3 = 5$, and not 6, since C is degenerate, and M_7 has the isthmus 2 (loop of $\overline{M_7}$). The resolution is linear, but M_7 does not correspond to an MDS-code. The code obtained by truncating the second position is MDS of word length 5.

Example 4.4. Let X be an algebraic curve of genus g defined over $\mathbb{F}_q$, and embed X into $\mathbb{P}^{g-1}$ by use of the complete linear system $L(K)$. Let $\{P_1, \cdots, P_n\}$ be a set of $\mathbb{F}_q$-rational points (of degree 1) on X. We form the matrix H where column nr. i consists of the coordinates of (the image of) P_i for each $i = 1, \cdots, n$. Each column is then determined up to a non-zero multiplicative constant; fix a choice for each i. Now we let H be the parity check matrix of a linear code C, and let M be the matroid associated to C. Different choices of multiplicative constants give equivalent linear codes and therefore the same code parameters, and even the same matroid M. If the chosen points fail to span all of $\mathbb{P}^{g-1}$, we replace H with a suitable matrix with fewer rows (that are linear combinations of those in H). Set $D = P_1 + \cdots + P_n$. (The code is also code equivalent to the algebraic-geometric code $C(D, D)$ in standard terminology, provided one is able to define such a code properly. As one understands, this is not a strongly algebraic-geometric code $C(D, G)$, since for such a code one demands $2g - 2 < deg(G) < deg(D)$.) Let the ground set E be the set of subdivisors of D, corresponding to all subsets of $\{1, 2, \cdots, n\}$, representing sets of columns of H. Let A be a subdivisor of D, and let $r(A)$ be the value at A of the rank function associated to the matroid M. It is a consequence of the geometric version of the Riemann-Roch theorem that:

$$r(A) = l(K) - l(K - A) = g - h^1(A).$$

Moreover, for the nullity $n(A)$, the Riemann-Roch theorem gives:

$$n(A) = l(A) - 1.$$

These rank and nullity functions are described in detail in [5, Section 5], which provides the inspiration for this example.

We define the t_D-gonality of X as the minimal degree of a subdivisor A of D such that $l(A) = t + 1$. Hence the t_D-gonality of X is the minimal cardinality of a subset $A \subset E$ such that $n(A) = t$, in other words d_t. By Theorem 4.2 $t_D = min\{j | \beta_{t,j} \neq 0\}$.

We also define the D Clifford index $Cl_D(A)$ of a subdivisor A of E as $deg(A) - 2(l(A) - 1)$. Regarding A as a subset of $\{1, 2, \cdots, n\}$ we obtain that this number is $|A| - 2n(A)$. The D Clifford index $Cl_D(X)$ of X is $min\{Cl_D(A)\}$, where A is taken only over those A with $h^0(A) \geq 2$ and $h^1(A) \geq 2$. We have $h^0(A) = n(A) + 1 \geq 2$ iff $n(A) \geq 1$, and $h^1(A) = l(K - A) = l(A) - |A| - 1 + g = n(A) - |A| + g \geq 2$ if and only if $|A| \leq n(A) + g - 2$. This, in combination with Theorem 4.2, gives:

$$Cl_D(X) = min\{d_i - 2i | i \geq 1; j \leq g - 2 + i\}.$$

Hence these kinds of Clifford indices can be read off from these kinds of Betti numbers. The most interesting case is perhaps when one lets D be the sum of all $\mathbb{F}_q$ rational points of X (and the rank of the matroid is typically g then). Then the t_D and Cl_D are close to being the usual t-gonality and Clifford index of X restricted to $\mathbb{F}_q$. But these usual definitions also include divisors with repeated points.

Another example is $D = K$ as in [5, Section 5]. Then M is a self dual matroid of rank $g - 1$ (so H will have to be processed a little to be a parity check matrix of the code). In [5] one shows that $Cl_K(A) \geq 0$ for all A, using only properties of matroids.

Since the t_D-gonalities thus have natural generalizations to all finite matroids and linear codes in form of the d_t, one might ask if the Clifford index also has such a generalization. Since $r = g$ for the particular matroid above (assuming the images of the points of $Supp(D)$ span $\mathbb{P}^{g-1}$), one might define $Cl(M)$ of any matroid M as:

$$Cl(M) = min\{d_i - 2i | i \geq 1; j \leq r - 2 + i\}.$$

This is, however, only defined if the set we are taking the minimum over, is non-empty, and this happens iff $d \leq r - 1$. The Singleton bound only gives $d \leq r + 1$. Hence this is not defined for MDS-codes (uniform matroids, $d = r + 1$), and almost-MDS-codes (almost-uniform matroids, $d = r$). It is unclear to us whether such an Clifford index says something useful and/or interesting about linear codes or matroids, and in that case, if its definition can be relaxed to apply to MDS-codes and almost-MDS codes also. In general such a $Cl(M)$ can be negative. (Take a linear code with only zeroes in 2 positions, but MDS after these two positions have been truncated. Then $d = r - 1$, and for any parity check matrix H we see that $C(M_H)$ is computed by $d_{n-r} - 2(n-r) = 2r - n - 2$, which is negative for some r, n.)

5. Relationship with other resolutions

From Wei duality (generalized to matroids, as in say [3]) it is clear that the higher weight hierarchy of a matroid M determines and is determined by that of its matroid dual $\overline{M}$. But since M is also a simplicial complex, through its set $\mathcal{I}_M$ of independent sets, there is also another duality, Alexander duality, which comes into play. It is well known, for example from [6], that for the underlying simplicial complex of M, the $\mathbb{N}$-graded resolution of the Stanley-Reisner ring of the Alexander dual complex, $M^\star$, has a particularly nice form. It is linear and of the form

$$0 \longleftarrow R_{M^\star} \xleftarrow{\partial_0} S \xleftarrow{\partial_1} P_1 \longleftarrow \cdots \xleftarrow{\partial_l} P_l \longleftarrow 0$$

where each P_i is of the form

$$P_i = S(-r(\overline{M}) - 1 - i)^{\beta_i},$$

for some l.

We thus study the minimal resolutions of the Stanley-Reisner rings of the Alexander duals $M^\star$ or $\overline{M}^\star$ and investigate whether the Betti numbers β_i of such resolutions determine the higher weight hierarchy of M (and $\overline{M}$).

Unfortunately it is clear however that even for two matroids of the same cardinality n and rank k the Betti-numbers of $M^\star$ and/or $\overline{M}^\star$ do not in general determine the higher weight hierarchy. An indication of this is given in , say a combination of Formula (1) and Theorem 4 (second part), of [6], where one shows that determining the β_i of the Alexander dual of the complex of independent sets of a matroid (which in virtue of being so, carries a simplicial complex of independent sets which is Cohen-Macaulay in virtue of Theorem 7, [6]), is equivalent to finding the f_j of the matroid, that is the number of independent sets of cardinality $j + 1$, for each j, for the matroid. Finding the f_j, although an important characterization of a matroid, is not enough to find the higher weights of a code giving rise to the matroid (if such a code exists).

An even more important characterization of a matroid is its Whitney polynomial

$$W(x, y) = \sum_{X \subset E} x^{r(E)-r(X)} y^{|X|-r(X)}.$$

The information about the f_i can then be read off from the coefficients of the pure x-part $W(x, 0)$ of the Whitney polynomial. How the d_i can be read off is described in [5, p. 131]. Even if the procedures for reading off the d_j from $W(x, y)$ are very different, it could of course a priori happen that the β_i determined the d_j. The following example shows, however, that Alexander duals of matroids M may have the same β_i, but different d_j

Example 5.1. Consider the non-degenerate $[6, 3]$-code over $\mathbb{F}_5$ with parity check matrices

$$H_8 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

and

$$H_9 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 4 & 4 & 1 & 0 \end{pmatrix}$$

Their associated matroids $M_8 = M_{H_8}$ and $M_9 = M_{H_9}$ are on $R = \{1, 2, 3, 4, 5, 6\}$ with the basis sets

$$\mathcal{B}_8 = \{\{1,2,3\},\{1,2,4\},\{1,2,6\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\},$$
$$\{2,3,4\},\{2,3,5\},\{2,4,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\}$$

and

$$\mathcal{B}_9 = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},$$
$$\{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\},\{4,5,6\}\}$$

respectively. The both give rise to the following Stanley-Reisner resolution

$$0 \longleftarrow R_{(\overline{M_i})^\star} \longleftarrow S \longleftarrow S(-3)^{16} \longleftarrow S(-4)^{33} \longleftarrow S(-5)^{24} \longleftarrow S(-6)^6 \longleftarrow 0$$

for $i \in \{8, 9\}$, while their respective weight hierarchies are $(3, 5, 6)$ and $(2, 5, 6)$.

Also it is clear from the following very simple examples that the d_j do not in general determine the β_i (of the Alexander dual of $\overline{M}$):

Example 5.2. Consider the codes from remark 4.3. We have already seen that they have the same weight hierarchies. But they give rise to the following resolutions:

$$0 \longleftarrow R_{(\overline{M_2})^\star} \xleftarrow{\partial_0} S \xleftarrow{\partial_1} S(-2)^5 \xleftarrow{\partial_2} S(-3)^6 \xleftarrow{\partial_3} S(-4)^2 \xleftarrow{\partial_4} 0$$

and

$$0 \longleftarrow R_{(\overline{M_3})^\star} \xleftarrow{\partial_0} S \xleftarrow{\partial_1} S(-2)^4 \xleftarrow{\partial_2} S(-3)^4 \xleftarrow{\partial_3} S(-4) \xleftarrow{\partial_4} 0.$$

6. Conclusion

Summing up we can roughly say that there are 12 sets of Betti-numbers that we have considered in this paper: The three sets of $\mathbb{N}^E$-graded, $\mathbb{N}$-graded, and ungraded Betti numbers, for each of the four simpicial complexes $M, \overline{M}, M^\star$, and $(\overline{M})^\star$. Two of these sets, the $\mathbb{N}^E$-graded ones for $M, \overline{M}$ determine M, and therefore the weight hierarchy in a trivial way, since $\beta_{1,\sigma} \neq 0$ iff σ is a circuit of the matroid in question.

Likewise two other sets, the $\mathbb{N}^E$-graded ones for $M^\star, (\overline{M})^\star$ determine M, and therefore the weight hierarchy in a trivial way, since $\beta_{1,\sigma} \neq 0$ iff σ is a basis of the matroid dual of the matroid in question.

Two other sets, $\mathbb{N}$-graded ones for $M, \overline{M}$ determine the d_i in (what we dare to consider) a non-trivial way, and this is the main result of our paper (Theorem 4.2).

The two sets of ungraded Betti numbers for $M, \overline{M}$ do not in general determine the d_i, since we have presented examples of pairs of codes with different sets of d_i, but the same Betti numbers (Remark 4.4).

The two sets of $\mathbb{N}$-graded Betti-numbers for the Alexander duals $M^\star, (\overline{M})^\star$ do not in general determine the d_i, since we have presented examples of pairs of codes with different sets of d_i, but the same sets of Betti numbers (Example 5.1).

The two sets of ungraded Betti-numbers for the Alexander duals $M^\star, (\overline{M})^\star$ are the same as the sets of $\mathbb{N}$-graded ones (since the resolutions are linear), so the same conclusion applies to them.

Finally, Example 5.2 and Remark 4.3 show that the weight hierarchy never decides the Betti numbers.

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