SYMMETRY, COMPATIBILITY AND EXACT SOLUTIONS OF PDES

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Abstract. We discuss various compatibility criteria for overdetermined systems of PDEs generalizing the approach to formal integrability via brackets of differential operators. Then we give sufficient conditions that guarantee that a PDE possessing a Lie algebra of symmetries has invariant solutions. Finally we discuss models of equations with large symmetry algebras, which eventually lead to integration in closed form.

Introduction

Overdetermined systems of PDEs always have compatibility conditions. If these are satisfied, the system is called formally integrable (we assume regularity throughout the paper) and we can formally parametrize the space of solutions [C_3, BCG^3, KL_2].

For Frobenius type systems the compatibility conditions are just equalities of mixed derivatives. For instance (we denote the partial derivatives as usual by indices) the system $\mathcal{E}$ on $\mathbb{R}^2(x^1,x^2)$

$$\{F_{ij}:u_{ij}=f_{ij}(x,u,\partial u)\,|\,i,j=1,2\}$$

has compatibility conditions

$$\left\{\begin{array}{l}\mathcal{D}_1(F_{12})=\mathcal{D}_2(F_{11})\\ \mathcal{D}_1(F_{22})=\mathcal{D}_2(F_{12})\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}\mathcal{D}_1(f_{12})=\mathcal{D}_2(f_{11})\\ \mathcal{D}_1(f_{22})=\mathcal{D}_2(f_{12})\end{array}\right. \mod \mathcal{E}.$$

In general this is wrong. For example, consider the following system $\mathcal{E}$ on $\mathbb{R}^4(x^0,x^1,x^2,x^3)$

$$\left\{\begin{array}{l}F_1:\ u_{13}=u_{22}+f_1(x,u,\partial u),\\ F_2:\ u_{12}=u_{03}+f_2(x,u,\partial u),\\ F_3:\ u_{02}=u_{11}+f_3(x,u,\partial u).\end{array}\right. \tag{1}$$

Equalities of the mixed derivatives do not yield compatibility here, because the first compatibility condition involving only two equations is of order 3 (the syzygy operators are of the second order), and so it involves the 2nd derivatives of f.

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There exists however one compatibility condition of order 2 (involving only first derivatives of f):

$$\mathcal{D}_1(F_1) + \mathcal{D}_2(F_2) + \mathcal{D}_3(F_3) \equiv 0 \mod \mathcal{E}.$$

In Section 1 we discuss how to calculate the compatibility conditions. This is related in the next Section to classical and higher symmetries, and we discuss the question of existence of invariant solutions.

Consider a PDE $\mathcal{E}$ and a subalgebra $\mathcal{G} \subset \mathrm{sym}(\mathcal{E})$; at this point we restrict to the local extrinsic (point, contact or higher) symmetries. In general it is not true that $\mathcal{E}$ has $\mathcal{G}$-invariant solutions, as was noticed in [K_1]. The simplest counter-example constitute linear systems with the symmetry being shift by a nonzero solution.

Another example is given by a pair of constant coefficients nonhomogeneous linear scalar PDEs. They commute and so they are symmetries of each other. Generically the PDEs are compatible, but this does not happen always (e.g. $u_{xx} = 1, u_x = 0$). Similar story happens for matrix differential operators and symmetries.

In Section 2 we demonstrate that this is a consequence of either degeneracy or higher dimensionality of the symmetry algebra compared to the amount of independent variables. We will prove that under certain genericity conditions the symmetry is compatible with the equation. This yields (a similar result is proven by another approach in [IV]):

Provided the symbols of $\mathcal{E}$ and $\mathcal{G}$ are generic, the system $\mathcal{E}$ has $\mathcal{G}$-invariant solutions.

In Section 3 we discuss implications that existence of a large symmetry group has on the solution space of a PDE system. At this point we need to consider intrinsic symmetries. Relations between extrinsic and intrinsic symmetries are given by Lie-Bäcklund type theorems, see [AI, KLV, AKO, AK], and this relates this integrability problem with what is discussed above (see also [L_1, K_2, Ga, BCA] for applications of symmetries and generalizations).

We will concentrate on exact solvability of ODEs and PDEs and discuss relations with Darboux integrability via examples. Informally a lot of symmetry implies exact integrability, more precisely this holds true for maximal symmetric models (can fail for sub-maximal cases).

We will briefly discuss some symmetric models. Relations of integrability to transformations and differential substitutions in PDEs is central in [L_2, K_3, K_4] for the case when the system $\mathcal{E}$ depends on 1 function of 1 argument (so called Lie class $\omega = 1$). This applies in the other cases too, but will be considered elsewhere.

The paper is organized so that all sections can be read independently.

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1. Compatibility, differential syzygies and brackets

An overdetermined system of differential equations can be viewed geometrically as a finite sequence of submanifolds $\mathcal{E}_k \subset J^k(\pi)$ in jets with $\mathcal{E}_{k-1}^{(1)} \supset \mathcal{E}_k$, where $\pi : E \to M$ is a vector bundle and $\mathcal{E}_{k-1}^{(1)}$ denotes the prolongation (locus of the derivatives of functions specifying $\mathcal{E}_{k-1}$).

Formal integrability is equivalent to the claim that all projections $\pi_{k+1,k} : \mathcal{E}_{k+1} \to \mathcal{E}_k$ are submersions. Then we can define $\mathcal{E}_\infty = \varprojlim \mathcal{E}_k$.

Let T be the model tangent space for independent variables and N the model tangent space for dependent variables. Define the symbols of order k as $g_k = \operatorname{Ker}(d\pi_{k,k-1} : T\mathcal{E}_k \to T\mathcal{E}_{k-1}) \subset S^kT^* \otimes N$ and let $g = \oplus g_k$ be the symbol bundle over $\mathcal{E}$. For k greater than the (maximal) order l of $\mathcal{E}$ we can define $g_k = S^{k-l}T^* \otimes g_l \cap S^kT^* \otimes N$.

The Spencer δ-cohomology group $H^{*,*}(\mathcal{E})$ is the cohomology of the complex $g_* \otimes \Lambda^*(T^*)$ with symbolic de Rham differential over $\mathcal{E}$ [S].

The formal theory of differential equations identifies obstructions to formal integrability (compatibility conditions) as certain elements $W_k \in H^{k-1,2}(\mathcal{E})$ (structural functions or Weyl tensors). For geometric problems $H^{*,2}(\mathcal{E})$ is the space of curvature/torsion tensors [KL$_3$].

Another way to look at this is to regard compatibility conditions as differential syzygies, i.e. relations between generators of $\mathcal{E}$.

If the generators are (nonlinear) differential operators F_i, i.e. $\mathcal{E} = \{F_1[u] = 0, \dots, F_r[u] = 0\}$, then the symbolic spaces can be expressed through the symbols of linearizations of these operators (we will use the notation σ for the symbol)

$$g = \operatorname{Ker}\{\sigma(\ell_{F_1}), \dots, \sigma(\ell_{F_r})\} \subset ST^* \otimes N.$$

It turns out that $g^* = \oplus g_k^*$ is an ST-module, where the latter is viewed as the algebra of polynomials on T^*. This is called the symbolic module of the system $g^* = \mathcal{M}_\mathcal{E}$.

Definition 1. *Differential operator G from the left differential ideal $\langle \mathbf{F} \rangle$ is called a differential syzygy for the system $\mathcal{E}$ if its symbol is the usual syzygy for the symbolic module $\mathcal{M}_\mathcal{E}$.*

If $\mathcal{E}$ is linear we can consider $\langle \mathbf{F} \rangle$ as the left module over the algebra $\operatorname{Diff}(\mathbf{1}, \mathbf{1})$ of scalar linear differential operators (on the trivial rank 1 bundle $\mathbf{1}$ over M). In the nonlinear case, the algebra should be changed to the algebra of $\mathscr{C}$-differential operators $\mathscr{C}\,\mathrm{Diff}(\mathbf{1},\mathbf{1}) = C^\infty(\mathcal{E}_\infty)\otimes_{C^\infty(M)}\mathrm{Diff}(\mathbf{1},\mathbf{1})$, see [KLV].

Thus to each differential syzygy there corresponds a symbolic syzygy, but the reverse problem

$$\text{syzygy} \overset{\mathfrak{q}}{\rightsquigarrow} \text{differential syzygy},$$

is not uniquely solvable. We would like to have solutions with nice algebraic properties (quantizations).

Let $f_i = \sigma(\ell_{F_i}) \in ST\otimes N^*$ be symbols of the differential operators defining $\mathcal{E}$ and let $\sum g_i f_i = 0$ be a syzygy, with $g_i \in ST$ being some polynomials on T^* with $k_i = \deg(f_i) = \mathrm{ord}(F_i)$.

Choose any $\mathscr{C}$-differential operators G_i with $\sigma(G_i) = g_i$. Then the corresponding differential syzygy has the form

$$\nabla = \sum G_i \circ F_i.$$

The order of this operator is $k-1$, where $k = \max_i\{\mathrm{ord}(G_i)+k_i\}$ is called the order of the syzygy.

Non-uniqueness comes through the lower order terms in G_i. So the class $\nabla \bmod \mathcal{J}_{k-1}(\mathbf{F})$ is well defined, where

$$\mathcal{J}_t(\mathbf{F}) = \{\sum Q_i\circ F_i : \mathrm{ord}(Q_i)\le t-k_i\}.$$

Denote by $[S]$ the equivalence class $S \bmod \mathcal{J}_{k-1}(\mathbf{F})$ of the differential syzygy $S=\mathfrak{q}(s)$, where k is the order of the syzygy.

Theorem 1 ([KL_4]). *An overdetermined PDE system $\mathcal{E}$ is formally integrable iff for a basis $\{s_j\}$ of symbolic syzygy the corresponding classes of differential syzygies vanish $[S_j]=0$.*

This follows from the fact that the Spencer δ-cohomology group $H^{k-1,2}(\mathcal{E})$ equals the corresponding graded second cohomology group for projective resolution of the symbolic module $\mathcal{M}_\mathcal{E}$, i.e. it can be enumerated via a basis of symbolic syzygies.

There are two basic approaches to construct the arrow $\mathfrak{q}$:

1. Construct differential syzygies successively in order k, i.e. according to passage $\mathcal{E}_k \dashrightarrow \mathcal{E}_{k+1}$. This corresponds to prolongation-projection approach having origin in E.Cartan's equivalence method.

2. Successive identification of differential syzygies involving two operators F_i, F_j, three or more. This represents compatibility as Massey products [KL_4] and is related to deformation of the symbolic module $\mathcal{M}_\mathcal{E}$ (or to the corresponding noncommutative D-module $\mathcal{E}^*$).

We will elaborate the first idea for PDE systems with nice characteristic variety $\mathrm{Char}^\mathbb{C}(\mathcal{E}) = \{\xi\in\mathbb{P}^\mathbb{C}T^* : \mathrm{rank}[\sigma(\ell_F)(\xi)] < m = \dim N\}$.

Now I want to present the explicit form of compatibility conditions for certain generic overdetermined systems of PDEs. We start with the scalar case $m = 1$ (single u); $n = \dim M$ is arbitrary.

Jacobi bracket $\{F, G\}$ of scalar (non-linear) differential operators $F, G \in \mathrm{diff}(\mathbf{1}, \mathbf{1})$ is defined via the linearization operator as follows:

$$\{F, G\} = \ell_F\, G - \ell_G\, F.$$

If $\operatorname{ord} F = k$, $\operatorname{ord} G = l$, then $\operatorname{ord}\{F, G\} = k + l - 1$.

Let (x^i, u_σ), $1 \le i \le n$, $\sigma = (i_1, \dots, i_n)$, be the "canonical" coordinates on the jet-space, $\mathcal{D}_j = \partial_{x^j} + \sum u^i_{\sigma+1_j} \partial_{u^i_\sigma}$ the total derivative w.r.t. x^j and let $\mathcal{D}_\sigma = \mathcal{D}_1^{i_1} \dots \mathcal{D}_n^{i_n}$ be the operator of higher total derivative. Then the Jacobi bracket writes

$$\{F, G\} = \sum \mathcal{D}_\sigma(F) \partial_{u_\sigma}(G) - \mathcal{D}_\sigma(G) \partial_{u_\sigma}(F),$$

We define the Mayer bracket of F_i, F_j by the formula

$$[F_i, F_j]_{\mathcal{E}} = \{F_i, F_j\} \bmod \mathcal{J}_{k_i+k_j-1}(\mathbf{F}).$$

For first order scalar operators these are respectively the classical Lagrange and Mayer brackets.

Theorem 2 ([KL$_1$]). *Consider a scalar system $\mathcal{E} \subset J^k(M)$ given by $r \le n$ differential equations $F_1[u] = 0, \dots, F_r[u] = 0$, such that for each point $x_k \in \mathcal{E}$ the characteristic varieties for the equations F_i are jointly transversal, i.e.* $\operatorname{codim}\bigl[\mathrm{Char}^{\mathbb{C}}_{x_k}(\mathcal{E}) \subset \mathbb{P}\,(T^*_x M)^{\mathbb{C}}\bigr] = r$.

Then the system is formally integrable iff all the Mayer brackets vanish:

$$[F_i, F_j]_{\mathcal{E}} = 0, \;\; 1 \le i < j \le r.$$

Koszul complex is the minimal resolution of the symbolic module $\mathcal{M}_{\mathcal{E}}$ for complete intersections, whence the symbolic syzygies are generated by commutators. Thus the arrow $\mathfrak{q}$ associates the higher Jacobi bracket $\{,\}$ to the commutator $[,]$ (and so can be treated as a quantization).

If the condition of complete intersection is violated, then the conclusion of the theorem may turn to be wrong. Indeed, in example (1) the compatibility conditions $[E_1, E_2]_{\mathcal{E}} = 0$, $[E_2, E_3]_{\mathcal{E}} = 0$, $[E_3, E_1]_{\mathcal{E}} = 0$ should hold, but they do not form a basis in differential syzygy module. There is only one basic vector, as indicated, but it has lower order. In fact, it is easy to see that the above system is not a complete intersection: the characteristic variety is the normal cubic

$$\begin{aligned}\mathrm{Char}^{\mathbb{C}}(\mathcal{E}) &= \{\xi \in \mathbb{C}P^3 \,|\, \xi_1\xi_3 = \xi_2^2, \xi_1\xi_2 = \xi_0\xi_3, \xi_0\xi_2 = \xi_1^2\} \\ &= \{[\lambda^3 : \lambda^2 : \lambda : 1] \,|\, \lambda \in \bar{\mathbb{C}}\}.\end{aligned}$$

We can however have other explicit formulae for scalar non-complete intersections. Let us point out one example. The following statement can be proved similarly to Theorem 2 (and it follows from Theorem 1).

Theorem 3. *Consider the system*

$$\mathcal{E} = \{F_i \circ G_j + \ldots = 0 \,|\, i = 1 \ldots r, j = 1 \ldots s\}$$

where dots stay for the lower order terms. It has reducible characteristic variety $\operatorname{Char}^{\mathbb{C}}(\mathcal{E}) = \{\sigma(\ell_{F_i}) = 0\} \cup \{\sigma(\ell_{G_j}) = 0\}$*, and we suppose that the intersection* $\{\sigma(\ell_{F_i}) = 0\} \cap \{\sigma(\ell_{G_j}) = 0\}$ *has codimension* $r + s$*. Then if* $k_i = \operatorname{ord}(F_i)$*,* $l_j = \operatorname{ord}(G_j)$*, the compatibility conditions are*

$$\{F_i, F_j\}G_k = 0 \bmod \mathcal{J}_{k_i+k_j+l_k-1}(\mathcal{E}), F_i\{G_j, G_k\} = 0 \bmod \mathcal{J}_{k_i+l_j+l_k-1}(\mathcal{E}).$$

Let us now consider vector systems of PDEs on $u = (u^1, \ldots, u^m)$.

Definition 2. *A system* $\mathcal{E} \subset J^k(\pi)$ *of* r *PDEs* $F_i[u] = 0$ *is called a generalized complete intersection if*

(1) $m < r \le n + m - 1$*, where* $n = \dim M$*,* $m = \operatorname{rank}(\pi)$*;*
(2) *The complex projective characteristic variety* $\operatorname{Char}^{\mathbb{C}}(\mathcal{E}) \subset \mathbb{P}^{\mathbb{C}}T^*$ *has codimension* $r - m + 1$*;*
(3) *The characteristic sheaf* $\mathcal{K}$ *over* $\operatorname{Char}^{\mathbb{C}}(\mathcal{E})$ *has fibers of dimension* 1 *everywhere.*

Let us introduce a multi-bracket of linear (scalar) differential operators $\nabla_i \in \operatorname{Diff}(m \cdot \mathbf{1}, \mathbf{1})$ by the formula

$$\{\nabla_1, \ldots, \nabla_{m+1}\} = \sum_{k=1}^{m+1} (-1)^{k-1} \operatorname{Ndet}\big[\nabla_i^j\big]_{i \ne k}^{1 \le j \le m} \cdot \nabla_k,$$

where Ndet is a version of non-commutative determinant. For non-linear differential operators $F_i \in \operatorname{diff}(\pi, \mathbf{1})$ the multi-bracket writes as

$$\{F_1, \ldots, F_{m+1}\} = \frac{1}{m!} \sum_{\alpha \in \mathbf{S}_m, \beta \in \mathbf{S}_{m+1}} (-1)^\alpha (-1)^\beta \ell_{\alpha(1)}(F_{\beta(1)}) \circ \ldots \circ \ell_{\alpha(m)}(F_{\beta(m)}) \left(F_{\beta(m+1)}\right).$$

The we define the reduced bracket

$$[F_{i_1}, \ldots, F_{i_{m+1}}]_{\mathcal{E}} = \{F_{i_1}, \ldots, F_{i_{m+1}}\} \bmod \mathcal{J}_{k_{i_1}+\cdots+k_{i_{m+1}}-1}(\mathbf{F}).$$

Theorem 4 ([KL$_4$])**.** *A system of generalized complete intersection type*

$$\mathcal{E} \subset J^k(\pi) = \{F_1[u] = 0, \ldots, F_r[u] = 0\}$$

is formally integrable iff all the multi-brackets vanish due to the system:

$$[F_{i_1}, \ldots, F_{i_{m+1}}]_{\mathcal{E}} = 0.$$

2. Symmetries and compatibility

Consider a compatible system $\mathcal{E} = \{F_1 = 0, \dots, F_r = 0\}$. Let $\mathcal{G}$ be a subalgebra of the algebra $\mathrm{sym}(\mathcal{E})$ of classical or higher symmetries [KLV], written as another PDE system $\{S_1 = 0, \dots, S_k = 0\}$.

Consider at first the scalar case, when u is a single function. Denoting $k_i = \mathrm{ord}(F_i)$, $l_j = \mathrm{ord}(S_j)$, the symmetry condition writes

$$\{F_i, S_j\} = 0 \mod \mathcal{J}_{k_i+l_j-1}(\mathcal{E}), \tag{2}$$

In addition we have from the symmetry condition that

$$\{S_i, S_j\} = 0 \mod \mathcal{J}_{l_i+l_j-1}(\mathcal{G}). \tag{3}$$

Theorem 5. *Assume that the joint system $\mathcal{E} + \mathcal{G}$: $\{F_i = 0, S_j = 0\}$ is a complete intersection, i.e. its characteristic variety has codimension $r + k \le n$. Then this system is compatible.*

For $k = 1$ this result coincides with Theorem 17 from [KL_1].

Proof. Since the system is a complete intersection, Theorem 2 implies compatibility provided that the Mayer brackets (on the joint system $\mathcal{E} + \mathcal{G}$) vanish. Compatibility of $\mathcal{E}$ yields

$$\{F_i, F_j\} = 0 \mod \mathcal{J}_{k_i+k_j-1}(\mathcal{E}).$$

Vanishing of the other brackets is given by (2) and (3). □

The condition of complete intersection is important, as the following example shows. KdV equation $u_t = u\,u_x + u_{xxx}$ has symmetries: $T_0 = u_t$, $T_1 = u_x$, $S = 3tu_t + xu_x + 2u$, $\Gamma = tu_x + 1$. There are no nontrivial invariant solutions for the following two-dimensional subalgebras of symmetries: 1) T_0, S; 2) T_1, S; 3) T_1, Γ; 4) Γ, S (only zero in the cases 1 & 2 and nothing at all in 3 & 4).

Now let us discuss the general case, when the unknown u is a vector-function. Consider at first the following example of matrix linear differential operators:

$$A = \begin{pmatrix} \mathcal{D}_x\mathcal{D}_y & 0 \\ \mathcal{D}_x\mathcal{D}_y - \mathcal{D}_x & \mathcal{D}_x \end{pmatrix} \text{ and } B = \begin{pmatrix} \mathcal{D}_x\mathcal{D}_y + \mathcal{D}_x & 0 \\ \mathcal{D}_x\mathcal{D}_y & \mathcal{D}_x \end{pmatrix}.$$

They commute $[A, B] = 0$ and so do the inhomogeneous operators

$$F = A \cdot \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; G = B \cdot \begin{pmatrix} u \\ v \end{pmatrix} \quad \Rightarrow \quad \{F, G\} = 0.$$

In other words $G \in \mathrm{sym}(F)$. However the operators are incompatible.

We shall show that for two generic nonlinear differential operators the condition $G \in \mathrm{sym}(F)$ implies compatibility of the joint system, and so existence of invariant solutions (in the amount given by the usual formal calculus of dimensions [C$_3$]).

This result was noticed by S.Igonin and A.Verbovetsky and a proof using a different method will appear in [IV]. We would like to give an explicit criterion for this compatibility. At first we notice the following

Lemma . *Consider a system $\mathcal{E}$ of $r \geq m$ differential equations on m unknown functions. Its characteristic variety* $\operatorname{Char}^{\mathbb{C}}(\mathcal{E})$ *has codimension $\leq r-m+1$.*

Notice that codimension cannot exceed n since we adapt the convention $\dim\emptyset = -1$ and the ambient space satisfied $\dim \mathbb{P}^{\mathbb{C}}T^* = n-1$.

Proof. The symbol of the system is a $r\times m$ matrix P with polynomial entries, and the characteristic variety is

$$\{\xi \in \mathbb{P}^{\mathbb{C}}T^* : \operatorname{rank}(P(\xi)) < m\}.$$

Denote by $\Delta_{i_1\dots i_m}$ the determinant of the $m\times m$ minor generated by rows $i_1,\dots,i_m$. $\operatorname{Char}^{\mathbb{C}}(\mathcal{E})$ is given by the conditions $\Delta_{i_1\dots i_m}(\xi)=0$ for all $i_1<\dots<i_m$, but this collection is excessive.

We can suppose that $\Delta_{1\dots m}\not\equiv 0$ (if all the minors are degenerate there is nothing to prove). For $j\geq m$ let us denote by P_j the upper-row submatrix of P of size $j\times m$. By induction we can suppose that the set $\Sigma_j=\{\xi\in\mathbb{P}^{\mathbb{C}}T^* : \operatorname{rank}(P_j(\xi))<m\}$ has codimension $\leq j-m+1$. Consider the subset Σ_j' where the above rank is $<m-1$.

If Σ_j' has a component in Σ_j, then addition of a row to P_j does not increase the rank over $m-1$, and the codimension of Σ_{j+1} containing this component is the same as for Σ_j.

Otherwise $\Sigma_j'\subset\Sigma_j$ has codimension at least 1 and in the complement the matrix $P_j(\xi)$ has rank $(m-1)$. Near every point ξ we can choose $(m-1)\times m$ subminor of maximal rank. Adding to it the row number $(j+1)$ we get a $m\times m$ matrix whose determinant we write as $\tilde{\Delta}_{j+1}$. Then the defining relation for $\Sigma_{j+1}\subset\Sigma_j$ outside Σ_j' is $\tilde{\Delta}_{j+1}=0$, whence the relative codimension is 1.

Thus codimension of Σ_{j+1} in the complex projective variety is at most $j-m$ and this gives the induction step. □

In particular, for $r=2m$ the codimension is at most $m+1$. We wish to refine this in the case the system $\mathcal{E}$ is a PDE with its symmetry.

Consider the PDE system $F=G=0$, where $F,G\in\operatorname{diff}(\pi,\pi)$ are nonlinear differential operators on a rank n bundle π and $G\in\operatorname{sym}(F)$. Denote the symbols of these operators by $P=\sigma(\ell_F)$, $Q=\sigma(\ell_G)$. The symmetry condition $\{F,G\}=0 \bmod F$ implies the following relation with some polynomial matrix K

$$PQ = KP. \tag{4}$$

Proposition 6. *For $m > 1$ let P and Q be two homogeneous polynomial $m \times m$ matrices satisfying (4). Then the characteristic variety*

$$\mathrm{Char}^{\mathbb{C}} = \left\{ \xi \in \mathbb{C}P^{n-1} : \mathrm{rank}\begin{bmatrix} P(\xi) \\ Q(\xi) \end{bmatrix} < m \right\}$$

(characteristic variety of $\mathcal{E} : \{F = G = 0\}$) has codimension $\leq m$.

Proof. Denoting by $R_i(A)$ the row of matrix A, we have from the symmetry condition:

$$\sum_{k=1}^{m} p_{ik} R_k(Q) \in \langle R_1(P), \dots, R_m(P) \rangle \quad \forall i = 1, \dots, m. \tag{5}$$

We can suppose that $\det P(\xi) \not\equiv 0$ (otherwise the claim follows from the Lemma), so that the equation $\det P(\xi) = 0$ determines a subvariety $\Sigma \subset \mathbb{P}^{\mathbb{C}}T^*$ of codimension 1.

Let $\Sigma_0 \subset \Sigma$ be given by the equation $P(\xi) = 0$. First let us study the points $\xi \in \Sigma \setminus \Sigma_0$. At such ξ there exists an entry $p_{ik} \neq 0$. Then we write the conditions $R_j(Q) \in \langle R_1(P), \dots, R_m(P) \rangle$ for $j \neq k$. These are no more than $m-1$ equations and so they specify a subvariety $\mathcal{K} \subset \Sigma$ in a neighborhood of ξ on which also $R_k(Q) \in \langle R_1(P), \dots, R_m(P) \rangle$ by (5). Thus this $\mathcal{K}$ is a part of $\mathrm{Char}^{\mathbb{C}}$ of codimension $\leq m$.

Now let us consider Σ_0. If it has codimension $\geq m$, it is negligible or is a part of $\mathrm{Char}^{\mathbb{C}}$ by the above argument. But if its codimension is $< m$, then we must add the condition $\det Q(\xi) = 0$ specifying a subvariety $\mathcal{K}_0 \subset \Sigma_0$ of codimension ≤ 1. Since $P(\xi) = 0$ for $\xi \in \Sigma_0$ the rank of the matrix $\begin{bmatrix} P(\xi) \\ Q(\xi) \end{bmatrix}$ is $< m$. Thus this $\mathcal{K}_0$ is a part of $\mathrm{Char}^{\mathbb{C}}$ of codimension $\leq m$. ☐

This Proposition is important, so we would like to indicate an idea behind an alternative proof. It will be shown later that condition (4) can be changed to $[P, Q] = 0$ without loss of generality, so we adapt this condition.

By Gerstenhaber theorem [Ge] every pair of numeric (complex) commuting matrices P, Q is contained in a commutative algebra (with $\mathbf{1}$) of dimension m. If one of them has simple spectrum, say P, then by Cayley-Hamilton theorem this algebra is generated by $\{P^i\}_{i=0}^{m-1}$ [Z].

This being generalized to matrices with polynomial entries, would imply existence of polynomial matrices $Z_0, Z_1, \dots, Z_{m-1}$ and the scalar polynomials a_i, b_j such that

$$P = a_0 Z_0 + a_1 Z_1 + \cdots + a_{m-1} Z_{m-1}, \quad Q = b_0 Z_0 + b_1 Z_1 + \cdots + b_{m-1} Z_{m-1}.$$

Therefore one of the strata of the characteristic variety is given by the condition that the vector $(a_0, \dots, a_{m-1})$ is parallel to $(b_0, \dots, b_{m-1})$ ($m-1$ equations) plus one equation $\det P = 0$ (or $\det Q = 0$), implying $\operatorname{codim} \mathrm{Char}^{\mathbb{C}} \le m$.

Let us demonstrate this in the case $m = 2$. Since identity matrices commute with everything, we can subtract a multiple of them to make $p_{22} = q_{22} = 0$. Then commutativity $[P, Q] = 0$ yields:

$$\left[\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & 0 \end{pmatrix}, \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & 0 \end{pmatrix}\right] = 0 \quad \Rightarrow \quad \frac{p_{11}}{q_{11}} = \frac{p_{12}}{q_{12}} = \frac{p_{21}}{q_{21}},$$

which means that these matrices are proportional to some polynomial matrix. Indeed, the first equality gives $p_{11} = r_1 s_1$, $q_{11} = r_1 s_2$, $p_{12} = r_2 s_1$, $q_{12} = r_2 s_2$ for some polynomials r_i, s_i.

Substituting this into the second equality we get refining: $s_1 = u_1 v_1$, $s_2 = u_1 v_2$, $p_{21} = u_2 v_1$, $q_{21} = u_2 v_2$, whence the claim $P = a_0 \mathbf{1} + a_1 Z$, $Q = b_0 \mathbf{1} + b_1 Z$.

Let us return to sufficient conditions for compatibility of symmetries.

From what was shown above it follows that in non-scalar case $\mathcal{E} = \{F = G = 0\}$ is never a generalized complete intersection (so compatibility cannot be deduced on the basis of Theorem 4). This is a consequence of non-commutativity of the matrix algebra (another manifestation is that generically $\operatorname{ord}\{F, G\} = \operatorname{ord}(F) + \operatorname{ord}(G)$ for $m > 1$).

Theorem 7. *Let $F = 0$ be a determined PDE and $G = 0$ its symmetry (both $m \times m$ systems). If for the joint system $\mathcal{E}$: $F = G = 0$ the variety $\mathrm{Char}^{\mathbb{C}}(\mathcal{E})$ has codimension m, then $\mathcal{E}$ is compatible.*

Proof. Since $F = 0$ is determined ($\det P$ is not identically zero), we can make an invertible transformation to write it in the 1st order evolutionary form. We keep the same letters F and G for the differential operators. This transformation can change m but it does not affect symmetry or compatibility properties, and it preserves $\mathrm{Char}^{\mathbb{C}}(\mathcal{E})$.

Thus $F[u] = u_t - F_0[u]$, where F_0 does not involve $\mathcal{D}_t$ derivatives (u is a vector function with m components). We can substitute $u_t = F_0$ into $G = 0$ and get a PDE that is free of u_t terms. We continue to write G for this new operator.

Denoting the symbols of F and G by P and Q the symmetry condition is $PQ = KP$ for some polynomial matrix K. Since $P = \tau \mathbf{1} + P_0$, where τ stands for the symbol of $\mathcal{D}_t$ differentiation and P_0 (as well as

Q) is free of τ, we conclude that $K = Q$ so that the symmetry condition writes $[P, Q] = 0$.

We claim that this G is determined, i.e. $\det Q \not\equiv 0$. Otherwise the characteristic equation is given by $m-1$ equations at generic point. Indeed, the proof of Lemma/Proposition 6 can be rewritten to start with equation $\det Q = 0$, which is identically true and so the number of defining relations for the variety $\{\xi : \operatorname{rank}\begin{bmatrix} P(\xi) \\ Q(\xi) \end{bmatrix} < m\}$ is by 1 less than in the general case. This yields $\operatorname{codim}\operatorname{Char}^{\mathbb{C}}(\mathcal{E}) < m$ and so contradicts the assumptions.

Thus $G = 0$ is compatible as well as $F = 0$. There are no syzygies between them due to evolutionary form. So the claim follows from Theorem 1. □

As we see from the above proof it is not necessary to require that $\operatorname{codim}\operatorname{Char}^{\mathbb{C}}(\mathcal{E}) = m$. What we want to achieve is that the operator G, when F is written in evolutionary form and $\mathcal{D}_t$ derivatives are removed from G, is determined. This is clearly a generic property.

Let us now give a sufficient condition in a special case when the symbol $P = \sigma(F)$ ($m \times m$ matrix with polynomial entries) can be diagonalized via an invertible transformation over polynomials. As will be seen from the proof, $\operatorname{codim}\operatorname{Char}^{\mathbb{C}}(\mathcal{E}) = 2$ and so this case does not follow from the above theorem (if $m > 2$).

In the proof we will calculate the $\operatorname{Mat}_{m\times m}(ST)$-module

$$\operatorname{Syz}(P, Q) = \{(A, B) \in \operatorname{Mat}_{m\times 2m}(ST) : AP + BQ = 0\}.$$

It encodes all syzygies which are the symbols of differential syzygies, and the latter are the compatibility conditions by Theorem 1.

Theorem 8. *Assume that $P = \sigma(\ell_F)$ is diagonalizable over the algebra ST. Assume also that the characteristic variety of $F = 0$ is irreducible, has codimension 1 (i.e. F is determined) and no multiple components.*

Let $G \in \operatorname{sym}(F)$ be a symmetry (both F, G are $m \times m$ PDE systems). If for the joint system $\mathcal{E} : F = G = 0$ the characteristic variety has codimension > 1, then $\mathcal{E}$ is compatible.

Proof. We can assume from the beginning that P is diagonal $P = \operatorname{diag}(p_1, \dots, p_n)$ for some polynomials on T^*: $p_i \in ST$. Denote the symbol of the symmetry G by Q. This is also a matrix with polynomial entries, $Q = [q_{ij}]_{m\times m}$.

The symmetry condition gives the following matrix syzygy:

$$PQ = KP \;\Leftrightarrow q_{ij}p_i = k_{ij}p_j \text{ (no summation).}$$

Because the characteristic variety is irreducible (without multiple components) we conclude from here $q_{ij} = r_{ij}p_j$, $k_{ij} = r_{ij}p_i$ for $i \neq j$. This means that $Q = RP + D$ and $K = PR + D$ for some diagonal matrix D. Since $\operatorname{codim} \operatorname{Char}^{\mathbb{C}}(\mathcal{E}) \geq 2$, p_i and q_{ii} have no common factors.

Then the syzygy $AP+BQ = 0$ is $(A+BR)P+BD = 0$, which yields $B = -LP$ and $A = LD-BR = LK$. Consequently the $\operatorname{Mat}_{m\times m}(ST)$-module $\operatorname{Syz}(P, Q)$ is 1-dimensional and generated by $(K, -P)$. The result follows from Theorem 1. □

Remark 1. *Basing on the generators of the syzygy module* $\operatorname{Syz}(P, Q)$ *one can check that multi-brackets of the rows of the* $2m \times m$ *matrix differential operator* $\begin{bmatrix} F \\ G \end{bmatrix}$ *vanish as a consequence of the symmetry condition.*

A similar argument shows that an analog of Theorem 5 holds for vector nonlinear differential operators, provided the characteristic variety is generic (this implies that n shall be sufficiently big compared to the dimensions of $\mathcal{G}$, amount of equations in $\mathcal{E}$ and $m > 1$).

3. Exact solvability of differential equations

Solutions to a compatible system of differential equations $\mathcal{E}$ are formally parametrized by p functions of g arguments (and some functions with fewer arguments), where $g = \dim_{\mathbb{C}} \operatorname{Char}^{\mathbb{C}}(\mathcal{E}) + 1$ is the dimension of the affine characteristic variety and $p = \deg \operatorname{Char}^{\mathbb{C}}(\mathcal{E})$ its degree [BCG^3, KL$_2$]. Denote the abstract space of such functions by $\mathfrak{S}^g_p$.

Closed form solutions refer to parametrization of the generic stratum of the solutions space by a differential operator $S : \mathfrak{S}^g_p \xrightarrow{\sim} \operatorname{Sol}(\mathcal{E})$. According to Cartan [C$_2$] for underdetermined ODEs ($g = 1$) this is tantamount[1] to internal equivalence of the equation equipped with the contact distribution $(\mathcal{E}, \mathcal{C}_{\mathcal{E}})$ to some mixed jet space $J^\sigma(\mathbb{R}, \mathbb{R}^p)$ equipped with the canonical distribution. Here $\sigma = (i_1, \dots, i_p)$ is a multi-index characterizing the orders of the dependent variables.

Example 1. Consider the equation for null curves in Lorentzian space of $1 + 2$ dimensions

$$x'(t)^2 + y'(t)^2 = 1 \quad \Leftrightarrow \quad dt^2 - dx^2 - dy^2 = 0.$$

An obvious solution involves one arbitrary function and the quadrature: $x = \int \cos\phi(t)dt$, $y = \int \sin\phi(t)dt$. But it can also be integrated in the

[1]In loc.cited only the case $p = 1$ – the classical Monge case of 1 equation on 2 unknowns – was treated, but the general case is similar.

closed form:

$$t=\sigma''(\tau)-\sigma(\tau),\quad x=\sigma''(\tau)\cos\tau+\sigma'(\tau)\sin\tau,$$
$$y=-\sigma''(\tau)\sin\tau+\sigma'(\tau)\cos\tau.$$

This equation in $1+n$ dimensions cannot be solved in closed form for $n>2$. Indeed internally

$$\mathcal{E}=\mathbb{R}^{n+1}\times S^{n-1}=\{(t,x_1,\dots,x_n,x_1',\dots,x_n'):\sum_{i=1}^n(x_i')^2=1\}.$$

The Cartan distribution on $\mathcal{E}$ is generated by $\mathcal{D}_t=\partial_t+\sum_1^n x_i'\partial_{x_i}$ and the subspace $\Pi=TS^{n-1}=\langle\partial_{x_1'},\dots,\partial_{x_n'}\,|\sum x_i'\partial_{x_i'}=0\rangle$: $\mathcal{C}_\mathcal{E}=\mathbb{R}\mathcal{D}_t+\Pi$.

Let $\Delta_1=\mathcal{C}_\mathcal{E}$ and $\Delta_2=[\Delta_1,\Delta_1]$ be its derived distribution. Then $\dim\Delta_1=n$, $\dim\Delta_2=2n-1$ and we have $\Delta_2=\Delta_1+\langle x_i'\partial_{x_j}-x_j'\partial_{x_i}\rangle$. The next derived distribution is $\Delta_3=[\Delta_1,\Delta_2]=T\mathcal{E}$.

Thus by dimensional reasons the only possible corresponding jet-space is $J^\sigma(\mathbb{R},\mathbb{R}^{n-1})$, $\sigma=(1,\dots,1,2)$, with coordinates $t,u_1,\dots,u_{n-1}$, $u_1',\dots,u_{n-1}',u_{n-1}''$. Its Cartan distribution is

$$\tilde\Delta_1=\langle\mathcal{D}_t,\partial_{u_1'},\dots,\partial_{u_{n-2}'},\partial_{u_{n-1}''}\rangle,\ \text{ where }\ \mathcal{D}_t=\partial_t+\sum_{i=1}^{n-1}u_i'\partial_{u_i}+u_{n-1}''\partial_{u_{n-1}'}.$$

The derived distribution is equal to $\tilde\Delta_2=\tilde\Delta_1+\langle\partial_{u_1},\dots,\partial_{u_{n-2}},\partial_{u_{n-1}'}\rangle$ and $\tilde\Delta_3=TJ^\sigma$.

Though the dimensions coincide, the distributions on $\mathcal{E}$ and J^σ are not equivalent. The reason is that Π, which is the maximal involutive space of the bracket $\Lambda^2\Delta_1\to\Delta_2/\Delta_1$, is not the Cauchy characteristic space for Δ_2. But in the second case $\tilde\Pi=\langle\partial_{u_1'},\dots,\partial_{u_{n-2}'},\partial_{u_{n-1}''}\rangle$, which is the maximal involutive space of the bracket $\Lambda^2\tilde\Delta_1\to\tilde\Delta_2/\tilde\Delta_1$, is the Cauchy characteristic space for $\tilde\Delta_2$.

Example 2. Consider the Monge equation $w'(x)=(z'(x))^2$. The general solution depends on 1 function of 1 variable and the form via quadrature is obvious, but here is the closed form solution:

$$x=\sigma''(\tau),\ w=\tau^2\sigma''(\tau)-2\tau\sigma'(\tau)+2\sigma(\tau),\ z=\tau\sigma''(\tau)-\sigma'(\tau).$$

The explanation behind this is the Engel normal form for rank 2 distributions in $\mathbb{R}^4$. However the next candidate – the Hilbert-Cartan equation

$$w'=(z'')^2 \tag{6}$$

is no longer integrable in closed form (without quadratures) as was demonstrated in 1912 by D.Hilbert. In 1914 E.Cartan gave a criterion for resolution of underdetermined ODEs in closed form [C$_2$], which we referred to above.

Already in 1910 Cartan found that the symmetry group of (6) is G_2 (though it was not written like this in [C$_1$], he surely knew this) and proved that it is the most symmetric equation among such Monge equations with finite-dimensional symmetry groups (linearizable equations have infinite-dimensional group of symmetries).

The more general problem when a PDE is integrable in closed form via solutions of a simpler equation (usually ODE) is known as the method of Darboux (that is we allow for the operator S above to involve quadratures and some other nonlocalities). For instance, Liouville equation

$$u_{xy} = e^u \quad \Longrightarrow \quad u = \log \frac{2f'(x)g'(y)}{(f(x)+g(y))^2}$$

is Darboux integrable, while sin-Gordon $u_{xy} = \sin u$ is not.

Example 3: The following overdetermined system of PDE on $\mathbb{R}^2$ appeared in [C$_1$] (λ is a parameter to be excluded):

$$u_{xx} = \tfrac{1}{3}\lambda^3,\ u_{xy} = \tfrac{1}{2}\lambda^2,\ u_{yy} = \lambda. \tag{7}$$

It is a compatible involutive system. The general solution is parametrized by 1 function of 1 argument

$$x = x,\ y = z''(\tau) + x\tau,$$
$$u = xz(\tau) + z'(\tau)z''(\tau) - \tfrac{1}{2}w(\tau) - \tfrac{1}{2}\tau\, z''(\tau)^2 - \tfrac{1}{2}\tau^2 x\, z''(\tau) - \tfrac{1}{6}\tau^3 x^2,$$

where $w'(\tau) = (z''(\tau))^2$. In fact, this system has a common characteristic $\partial_x - \lambda\partial_y$, which lifts to the Cauchy characteristic of the Cartan distribution (of rank 3). The quotient by the Cauchy characteristic is a 5-dimensional manifold with rank 2 distribution equivalent to (6).

By a Lie-Bäcklund type theorem [C$_1$] the contact symmetry group of (7) coincides with the internal symmetry group of (6), and so is G_2.

According to Goursat [Gou] the general form of overdetermined involutive (in this case: compatible with a common characteristic) system of 2nd order PDE on the plane is

$$r + 2\lambda s + \lambda^2 t = 2\psi,\ s + \lambda t = \psi_\lambda,\ t = \psi_{\lambda\lambda}, \tag{8}$$

where we use the classical notations $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$ and suppose $\psi_{\lambda\lambda\lambda} \neq 0$ (nonlinearity). System (7) corresponds[2] to $\psi = \lambda^3/3!$

Removing the last equation from this system we obtain a determined parabolic PDE $\mathcal{E}$ of the 2nd order. It has the largest contact symmetry group (among non-linear equations) for $\psi = \lambda^3/3!$ in which case

[2]One also need to change $y \mapsto -y$ to match the sign.

excluding λ we obtain the Goursat equation

$$4(2s-t^2)^3+(3r-6st+2t^3)^2=0. \tag{9}$$

This equation has the same symmetry group G_2 and it can be parametrized as the 2D tangent cone $\rho(\lambda)+\mu\,\rho'(\lambda)$ to the twisted cubic $\rho(\lambda)$:

$$r=\tfrac13\lambda^3+\lambda^2\mu,\ s=\tfrac12\lambda^2+\lambda\mu,\ t=\lambda+\mu.$$

Excluding λ and μ we get equation (9).

It has the following geometric reduction to (6), giving exact solutions for (9). The double characteristic $\partial_x-\lambda\partial_y$ lifts to

$$\xi=\mathcal{D}_x-\lambda\mathcal{D}_y=\partial_x-\lambda\partial_y+(p-\lambda q)\partial_u-\tfrac16\lambda^3\partial_p-\tfrac12\lambda^2\partial_q,$$

which together with $\partial_\mu=\lambda^2\partial_r+\lambda\partial_s+\partial_t$ forms the integrable characteristic rank 2 distribution $\Pi\subset\mathcal{C}_\mathcal{E}$. Quotient by it maps the Cartan distribution $\mathcal{C}_\mathcal{E}$ (of rank 4) to a rank 3 distribution $\bar\Delta$ on a 5-dimensional manifold M^5. This rank 3 distribution is the derivative distribution of the unique rank 2 distribution Δ which maps to zero under restriction of the natural bracket $\Lambda^2\bar\Delta\to TM/\bar\Delta$.

If we identify M^5 with $\mathbb{R}^5(y,u,p,q,\lambda)$, then $\Delta=\langle\partial_y+q\partial_u+\frac12\lambda^2\partial_p+\lambda\partial_q,\partial_\lambda\rangle$ and $\bar\Delta=\bar\Delta+\langle\partial_q+\lambda\partial_p\rangle$. Thus we see that (M^5,Δ) is equivalent to Hilbert-Cartan equation (6).

The following deformation $\mathcal{E}_\epsilon$ of the Goursat equation was studied in [T]:

$$(4+\epsilon)(2s-t^2)^3+(3r-6st+2t^3)^2=0. \tag{10}$$

Here $\epsilon>0$ and for every such number (10) is hyperbolic[3] and it has maximal symmetry algebra of dimension 9 among all hyperbolic 2nd order PDEs on the plane, which are neither of Monge-Ampere nor of Goursat type (see [T] for details). The Lie algebra of symmetries is of the type $\mathfrak{g}=\mathrm{sl}_2\ltimes\mathfrak{r}$ (radical $\mathfrak{r}$ is 6-dimensional).

Except for this family there is one more hyperbolic equation with 9-dimensional symmetry algebra ($\mathfrak{g}$ has the same type but different $\mathfrak{r}$)

$$3rt^3+1=0. \tag{11}$$

All these equations are Darboux integrable. The last one was studied in Goursat [Gou]. Here is one intermediate integral of order 2:

$$s\,t+1=0. \tag{12}$$

The system (11)+(12) is involutive and so allows reduction to a rank 2 distribution. Indeed this system can be re-written in the form similar to (7) — using Goursat representation (8) with $\psi=-\frac43\lambda^{3/2}$ we get[2]

$$u_{xx}=\tfrac13\lambda^{3/2},\ u_{xy}=\lambda^{1/2},\ u_{yy}=-\lambda^{-1/2}. \tag{13}$$

[3]Outside the submanifold in $\mathcal{E}_\epsilon$ given by Cartan equation (7)!

This system is compatible (compatibility writes formally as $\lambda_x = \lambda\lambda_y$ but this relation expresses the prolongation) and the characteristic field is $\xi = \mathcal{D}_x - \lambda\mathcal{D}_y$, which is also the Cauchy characteristic field for the Cartan distribution $\mathcal{C}_\mathcal{E}$ on the equation [K_4].

Quotient of this rank 3 distribution $\mathcal{C}_\mathcal{E}$ by ξ gives a rank 2 distribution on 5-manifold M^5, which corresponds to the Monge equation

$$w' = (z'')^{1/3}. \tag{14}$$

This latter is equivalent to (6) and has Lie(G_2) algebra of symmetries. Thus by Cartan's version of Lie-Bäcklund theorem [C_1] the contact symmetry algebra of the involutive system (11)+(12) is the same as for (14) — it's one more representation of G_2.

Remark 2. *These all are realizations of non-compact (split) form of G_2 as the maximal symmetric model. The compact form of G_2 is realized by the automorphism group of the Calabi almost complex structure (S^6, J). By* [K_5] *this is the maximal symmetric model that acts on non-linear overdetermined non-integrable*[4] *Cauchy-Riemann equations*

$$\zeta_{\bar z} = \Phi(z, w, \zeta),\ \ \zeta_{\bar w} = \Psi(z, w, \zeta).$$

Higher Monge equations were studied in [AK], and all maximal symmetric models were identified as the following underdetermined ODEs: $y^{(m)}(x) = (z^{(n)}(x))^2$ (these again cannot be solved without quadratures according to Cartan's criterion).

The symmetries here can be thought of as internal or external – they coincide by a version of Lie-Backlund theorem from [AK].

There are also PDE models for these according to [K_4]. We will demonstrate some in examples, which also indicate a relation to the projective geometry of curves (the tool from [DZ]) — in our case (most symmetric models) these are the rational normal curves.

Example 4: The following system is involutive[5]

$$u_{xxx} = \tfrac{1}{4}\lambda^4, u_{xxy} = \tfrac{1}{3}\lambda^3, u_{xyy} = \tfrac{1}{2}\lambda^2, u_{yyy} = \lambda \tag{15}$$

It has type $3E_3$ in notations of [K_3]. Quotient by Cauchy characteristic $\xi = \mathcal{D}_x - \lambda\mathcal{D}_y$ yields a rank 2 distribution on a manifold of dimension 8. The weak growth vector of this distribution (we refer for the definition and properties to [AK]) is $(2, 1, 2, 3)$ and the corresponding Monge system is

$$y'' = \tfrac{1}{2}(z''')^2,\ \ w' = \tfrac{1}{3}(z''')^3. \tag{16}$$

[4]This means the corresponding Nijenhuis tensor N_J is non-degenerate.

[5]It has Lie class $\omega = 1$, i.e. the solutions depend upon 1 function of 1 argument.

The corresponding graded nilpotent (Carnot-Tanaka) Lie algebra is free truncated of length 4 with 2-dimensional fundamental space $\mathfrak{g}_{-1}$.

Similarly we reduce $4E_4$

$$u_{xxxx} = \tfrac{1}{5}\lambda^5, u_{xxxy} = \tfrac{1}{4}\lambda^4, u_{xxyy} = \tfrac{1}{3}\lambda^3, u_{xyyy} = \tfrac{1}{2}\lambda^2, u_{yyyy} = \lambda$$

to a rank 2 distribution with growth $(2,1,2,3,4)$ and the Monge equation

$$y''' = \tfrac{1}{2}(z^{iv})^2,\ v'' = \tfrac{1}{3}(z^{iv})^3,\ w' = \tfrac{1}{4}(z^{iv})^4.$$

We can modify the symmetric models without destroying the symmetry algebra. For the above $3E_3$ we get its tangent 2D cone

$$u_{xxx} = \tfrac{1}{4}\lambda^4 + \lambda^3\mu,\ u_{xxy} = \tfrac{1}{3}\lambda^3 + \lambda^2\mu,$$
$$u_{xyy} = \tfrac{1}{2}\lambda^2 + \lambda\mu,\ u_{yyy} = \lambda + \mu.$$

This type $2E_3$ system is compatible (the prolongation obeys $\lambda_x = \lambda\lambda_y$), and its general solution depends on $\omega = 2$ functions of 1 argument.

The characteristic is still $\mathcal{D}_x - \lambda\mathcal{D}_y$ (with multiplicity two) and the contact symmetry algebra is the same as the contact external algebra for (15) and the same as the internal algebra for (16): it has dimension 12 and Levi decomposition $\mathfrak{g} = \mathrm{sl}_2 \ltimes \mathfrak{r}$ (radical $\mathfrak{r}$ is 9-dimensional and it consists of 1-dimensional center and 8-dimensional nil-radical).

Further modification gives the 3D tangent cone of the normal curve, which is a strictly parabolic 3rd order determined PDE with the same 12-dimensional symmetry algebra $\mathfrak{g}$. Denoting by $\alpha, \beta, \gamma, \delta$ the 3rd derivatives $u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy}$ we can write this system of equations parametrically

$$\alpha = \tfrac{1}{4}\lambda^4 + \lambda^3\mu + 3\lambda^2\nu, \quad \gamma = \tfrac{1}{2}\lambda^2 + \lambda\,\mu + \nu,$$
$$\beta = \tfrac{1}{3}\lambda^3 + \lambda^2\mu + 2\,\lambda\,\nu, \quad \delta = \lambda + \mu,$$

or in the implicit form

$$8\alpha^3 - 18\alpha^2(4\beta\delta + 8\gamma^2 - 12\gamma\delta^2 + 3\delta^4) + 27\alpha\beta^2(18\gamma - \delta^2) + 4\alpha\beta\gamma\delta(3\delta^2 - 10\gamma)$$
$$+ 8\alpha\gamma^3(3\gamma - \delta^2) + 27\beta^2(27\beta\gamma\delta - 8\beta\delta^3 - 18\gamma^3 + 6\gamma^2\delta^2) = 2187\,\beta^4/8.$$

Its solution space depends on $\omega = 3$ functions of 1 argument and is intrinsically related to the Monge system (16).

Higher analogs of the above are valid on the basis of works [AK, K_4].

4. Conclusion

In this paper we discussed compatible overdetermined as well as underdetermined systems $\mathcal{E}$ of differential equations. We indicated that symmetries generically produce particular 'automodel' solutions to $\mathcal{E}$. But if symmetries are few, they do not allow complete integration.

On the other hand abundance of symmetries is a sign of integrability. In Section 3 we briefly indicated via examples how large algebras of symmetries make models unique. These symmetries often do not live on the equation-manifold, and a covering is required to see them. In this way Darboux integrability [D, F] was recast into the language of group quotients in [AFV] and new examples were integrated, see [AF].

Integration in closed form is too restrictive in the context of PDEs, and instead one considers reductions to ODEs. These latter are not arbitrary in the maximal symmetric cases, as they also possess symmetries. This restricts the models (like those Monge equations from [AK]) and gives a method to understand integrability.

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