Global Lie-Tresse theorem

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Abstract

We prove a global algebraic version of the Lie-Tresse theorem which states that the algebra of differential invariants of an algebraic pseudogroup action on a differential equation is generated by a finite number of polynomial-rational differential invariants and invariant derivations. [1]

Introduction

According to Erlangen program of F.Klein a geometry is characterized by the invariants of a transitive transformation group. Finite generation property for algebraic invariants was the topic of D.Hilbert's XIV problem.

For infinite groups of S.Lie and E.Cartan (as well as for usual Lie groups) [Li$_1$, C$_1$] (see also [SS, S$_1$, AM]) the Notherian property generally does not hold for the algebra of differential invariants (of arbitrary order), and instead finiteness is guaranteed by the Lie-Tresse theorem, which uses invariant functions and invariant derivations as generators.

This theorem is a phenomenological statement motivated by Lie and Tresse [Li$_2$, Tr$_1$]. It was rigorously proved for un-constrained actions of pseudogroups (on regular strata in the space of jets) in [Kum], see also [Ov, Ol$_1$, MMR, SSh]. For pseudogroup actions on differential equations (which can be, for instance, singular strata of un-constrained actions) this was proved in [KL$_1$]. Algorithmic construction of differential invariants via Gröbner basis technique (in the case of free actions) is done in [OP].

In all these approaches the regularity issue plays a significant role and the generating property of the algebra $\mathcal{A}$ of differential invariants (as well as the very definition of $\mathcal{A}$) holds micro-locally, i.e. on open domains in the space of infinite jets J^∞ (usually these domains are not Zariski open or G-invariant, and their jet-order and size depend on the element of $\mathcal{A}$).

In this paper we overcome this difficulty by considering *algebraic actions* (essentially all known examples are such) and restricting to differential invariants that are rational functions by jets of certain order $\leq l$ and polynomial by higher jets. These invariants will separate the regular orbits and will be finitely generated in Lie-Tresse sense globally.

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The set of singularities is algebraic in J^∞, i.e. given by a finite number of finite jet-order relations, specifying when the generators are not defined (denominators vanish) or are dependent (some determinants vanish). This is one of our results on *stabilization of singularities*. Then we prove Lie-Tresse theorem in the complement to these singularities. This global result is an enhancement of the micro-local smooth situation since the latter can be easily deduced by an application of the implicit function theorem.

Rational differential invariants are natural in the classification problems, and all known examples are such (see the discussion in §4.3 about appearance of roots). This paper seems to be the first to justify this experimental observation.

It is important to stress that separation property holds globally and this gives a method to distinguish between different regular orbits of the pseudogroup actions. This implies, for instance, possibility to solve algebraic equivalence problems via the differential-geometric technique (see e.g. [BL]).

0.1. Algebraic pseudogroups

A pseudogroup $G \subset \mathrm{Diff}_{\mathrm{loc}}(M)$ acting on a manifold M consists of a collection of local diffeomorphisms φ, each bearing own domain of definition $\mathrm{dom}(\varphi)$ and range $\mathrm{im}(\varphi)$, that satisfies the following properties:

1. $\mathrm{id}_M \in G$ and $\mathrm{dom}(\mathrm{id}_M) = \mathrm{im}(\mathrm{id}_M) = M$,
2. If $\varphi, \psi \in G$, then $\varphi \circ \psi \in G$ whenever $\mathrm{dom}(\varphi) \subset \mathrm{im}(\psi)$,
3. If $\varphi \in G$, then $\varphi^{-1} \in G$ and $\mathrm{dom}(\varphi^{-1}) = \mathrm{im}(\varphi)$,
4. $\varphi \in G$ iff for every open subset $U \in \mathrm{dom}(\varphi)$ the restriction $\varphi|_U \in G$,
5. The pseudogroup is of order l if this is the minimal number such that $\varphi \in G$ whenever for each point $a \in \mathrm{dom}(\varphi)$ the l-jet is admissible: $[\varphi]^l_a \in G^l$.

The latter property means that the pseudogroup is defined by a Lie equation of differential order l, i.e. the embedding $G^l \subset J^l(M, M)$ determines the higher groupoids G^k, $k \geq l$ uniquely through the prolongation technique.

In this paper we assume that the action of G on M is *transitive*, i.e. any two points on M can be superposed by a group element. In this case, provided that the pseudogroup action is local, the above assumption on existence of the order l can be omitted, as follows from Hilbert basis (or Cartan-Kuranishi) theorem.

In the case of intransitive action, the results of this paper hold micro-locally and they also apply in a neighborhood of a regular orbit.

A transformation $\varphi \in G$ defines a map (l-th prolongation) of the space of jets of dimension n submanifolds $\varphi^{(l)} : J^l(M, n) \to J^l(M, n)$, which obeys the following property:

$$(\varphi \circ \psi^{-1})^{(l)} = \varphi^{(l)} \circ (\psi^{(l)})^{-1}.$$

This action on $J^\infty(M, n)$ induces the action of G on the space of functions $\mathcal{F}(J^\infty(M, n))$, which is the main object of our study. The alternatives for $\mathcal{F}$

are: smooth or analytic functions, rational functions or polynomials with respect to jets-variables (there is the natural affine structure on the fibers of the bundle $\pi_{i+1,i}: J^{i+1} \to J^i$). We will use a (reasonably minimal) combination of these.

Let us denote the stabilizer of order k of the point $a \in M$ by $G_a^k = \{\varphi \in G^k : \varphi(a) = a\}$ (in the transitive case dependence on a is not essential: these sub-groups are conjugate). The group G_a^k acts on the k-jets of dimension n submanifolds at a, which is an algebraic manifold $J_a^k(M,n)$, see §1 for details.

Definition 1. *The G action on M is called algebraic if for the order l of the pseudogroup the stabilizer G_a^l is an algebraic group acting algebraically on J_a^l.*

Actually we require that G_a^l is an algebraic subgroup of the differential group D_a^l (see §1.2 for more details). It is then straightforward to check that the prolonged action of G_a^k on J^k is also algebraic for all $k \geq l$.

We want more than this, namely we wish to consider a symmetric differential equation $\mathcal{E}$ as a submanifold in jets, so that $\mathcal{E}^k \subset J^k$ is G-invariant (in a general setup, to which our results extend, the action of G is defined only on $\mathcal{E}$ and is not induced by an ambient action).

In this case we have an action of G_a^k on $\mathcal{E}_a^k = \pi_k^{-1}(a)$ and we accordingly call it *algebraic* if the induced sub-representation is such. This means that $\mathcal{E}_a^k \subset J_a^k$ for any point $a \in M$ is an algebraic (nonsingular) submanifold on which G_a^k acts algebraically. Notice that this property concerns the behavior only with respect to the derivatives, so for instance sin-Gordon $u_{xy} = \sin u$ or Liouville $u_{xy} = e^u$ are algebraic differential equations from this perspective.

Without loss of generality we can assume that the maximal order of $\mathcal{E}$ is $\leq l$. Then the assumption that $\mathcal{E}^l$ is algebraic and *irreducible* implies the same property for the prolongations $\mathcal{E}^k = (\mathcal{E}^l)^{(k-l)}$ provided the system is formally integrable (this is relevant for overdetermined systems only, see §1.1).

0.2. Main results and discussion

Like in the classical invariant theory, the pseudogroup actions possess the algebra $\mathcal{A}$ of differential invariants, which are simply the invariant functions of the prolonged action in jets $\varphi^{(k)} \in \mathrm{Diff}_{\mathrm{loc}}(J^k)$, $\varphi \in G$, for all k. Generally the number of independent differential invariants is infinite. But micro-locally $\mathcal{A}$ can be finitely generated via Lie approach with a finite number of invariant derivations and a finite number of differential invariants, or via Tresse method using differentiation of some invariants by the others (see §1.4).

We call a closed subset $S \subset J^k$ *Zariski closed* if its intersection S_a with every fiber J_a^k, $a \in M$, is Zariski closed. The same concerns subsets $S \subset \mathcal{E}^k$ in algebraic differential equations. If S is G-invariant, its projection to the base shall be the whole M (the action is assumed transitive). Formally integrable equations correspond to compatible systems and are discussed in §1.

Our first result is the analog of Cartan-Kuranishi theorem (that regularity is guaranteed by a finite number of conditions) for pseudogroup actions.

Theorem 1. *Consider an algebraic action of a pseudogroup G on a formally integrable irreducible differential equation $\mathcal{E}$ over M. Suppose G acts transitively on M. Then there exists a number l and a Zariski closed invariant proper subset $S_l \subset \mathcal{E}^l$ such that the action is regular in $\pi_{\infty,l}^{-1}(\mathcal{E}^l \setminus S_l) \subset \mathcal{E}^\infty$, i.e. for any $k \geq l$ the orbits of G^k on $\mathcal{E}^k \setminus \pi_{k,l}^{-1}(S_l)$ are closed, have the same dimension and algebraically fiber the space. Thus there exists a rational geometric quotient*

$$\big(\mathcal{E}^k \setminus \pi_{k,l}^{-1}(S_l)\big)/G^k \simeq Y_k.$$

In other words, we have the following stabilization of singularities: By Rosenlicht theorem the action on $\mathcal{E}^k$ has a geometric quotient outside the proper singularity set S_k for all $k > 0$. Then for some number l (which can exceed the order of the pseudogroup G), we have: $\pi_{k,l}(S_k) \subset S_l$ for $k \geq l$.

Denote by $\mathfrak{P}_l(\mathcal{E})$ the algebra of functions on $\mathcal{E}^\infty$ (this means $\cup_k \pi_{\infty,k}^*(\mathcal{F}(\mathcal{E}^k))$) that are smooth by the base variables, rational by the fibers of $\pi_l : J^l \to M$ and polynomial by higher jets of order $k > l$.[2] We will look for differential invariants among such functions.

Our second result is the global Lie-Tresse theorem for pseudogroup actions on differential equations (it includes the important case of un-constrained action, when $\mathcal{E}^l = J^l$ is the full jet-space).

Theorem 2. *With the assumptions of the previous theorem, there exists a number l and a Zariski closed invariant proper subset $S \subset \mathcal{E}^l$ such that the algebra $\mathfrak{P}_l(\mathcal{E})^G$ of differential invariants separates the regular orbits from $\mathcal{E}^\infty \setminus \pi_{\infty,l}^{-1}(S)$ and is finitely generated in the following sense.*

There exists a finite number of functions $I_1, \dots, I_t \in \mathfrak{P}_l(\mathcal{E})^G$ and a finite number of rational invariant derivations $\nabla_1, \dots, \nabla_s : \mathfrak{P}_l(\mathcal{E})^G \to \mathfrak{P}_l(\mathcal{E})^G$ such that any function from $\mathfrak{P}_l(\mathcal{E})^G$ is a polynomial of differential invariants $\nabla_J I_i$, where $\nabla_J = \nabla_{j_1} \cdots \nabla_{j_r}$ for some multi-indices $J = (j_1, \dots, j_r)$, with coefficients being rational functions of I_i.

The number l is not the same as in Theorem 1 and it is usually greater than the stabilization jet-level of G and $\mathcal{E}$. It is determined by the stabilization in cohomology as shown in Section 2. The set S includes the previous singularity locus S_l but it also contains points where the ranks of the set of invariant derivations and differentials of the invariants drop. The number t can be taken equal to the codimension of the regular orbit in $\mathcal{E}^l$ plus 1. The number s does not exceed the dimension of the affine complex characteristic variety of $\mathcal{E}$.

Let us remark that the derivations ∇_j not always can be taken as Tresse derivatives — in Section 4 we show examples, where the growth of the differential invariants is a polynomial of degree $< n$, and the number of invariant derivations s is less than n.

In the above approach we remove not only the genuine singular orbits but also some regular ones (for instance non-closed orbits). We can minimize the

[2] Alternatively one can study the field of functions that are smooth by the base variables and rational by the fibers. The theory applies to this class as well.

amount of removed regular orbits (shrink S) at the price of increasing the number of basic invariants I_i and possibly the number of derivations ∇_j.

Remark 1. *It is not enough to consider only polynomial invariants, as readily follows from Nagata-Popov counter-examples to Hilbert XIV. Hilbert conjecture holds for reductive groups, but the sub-groups G_x^k arising in the pseudogroup actions are seldom such. In fact, generally we get nilpotent groups which obstruct finite-generation property for polynomial invariants (however regular orbits can be separated by polynomial relative differential invariants).*

Some other results proved in this paper are: finiteness theorem for invariant derivations, differential forms and other natural geometric objects (tensors, differential operators, connections etc). We also prove finiteness for for differential syzygies and higher syzygies, and interpret this as G-equivariant Cartan-Kuranishi theorem.

Let us remark that the results of the paper hold true for some other classes of functions. Namely if the manifold $\mathcal{E}^l$ and the G-action is real or complex analytic, then we can consider the field $\mathfrak{R}(\mathcal{E})$ consisting of functions that are analytic by the base variables of $\pi_\infty : J^\infty \to M$ and rational by the fibers (derivatives). Our theory of differential invariants applies to this class as well, and we can conclude finiteness and separation property.

If $\mathcal{E}^l$ is a complex analytic Stein space (for $l \gg 1$) and G a complex-reductive Lie group acting upon it by symmetries, then results of [GM] imply existence of rational quotient on a finite jet-level. Then the results of Section 3 yield a global structure theorem in the class of meromorphic functions $\mathcal{M}(\mathcal{E})$, i.e. meromorphic rational differential invariants are finitely generated in Lie-Tresse sense and they separate the orbits.

The classes $\mathfrak{P}_l(\mathcal{E})$, $\mathfrak{R}(\mathcal{E})$ and $\mathcal{M}(\mathcal{E})$ are sufficiently rich, yet controllable. Differential invariants in the larger space of smooth functions $C^\infty(\mathcal{E})$ can fail to satisfy the finiteness property of generalized Lie-Tresse theorem globally.

0.3. Overview of the problem and Outline of the paper

Differential invariants play an important role in solution of the equivalence problem for geometric structures (Kähler and Einstein metrics, projective and conformal structures, webs etc) and integration of differential equations.

Their investigations has origin in the works by Lie, Tresse and Cartan, with later advances by Sternberg, Kobayashi, Chern and Tanaka to name a few. This includes the theory of representations and the structure of infinite pseudogroups [C_1, SS, GS_1, Kum, RK].

Structure description of the algebra of differential invariants, in spite of the recent interest and progress, was mainly micro-local or was limited to locally free and un-bounded actions [Th, KJ, KL_3, OP, Man].

Many classical irrational algebraic expressions (containing roots) are local or even micro-local but not global invariants; we claim that all global G-invariants are generated by rational differential invariants.

To see this consider the well-known formula for the curvature, which illustrates the difference between micro-local and global approaches.

Example. The proper motion group $E(2)_+ = SO(2) \ltimes \mathbb{R}^2$ acts on the curves in Euclidean $\mathbb{R}^2(x,y)$. The classical curvature

$$K = \frac{y''(x)}{\sqrt{(1+y'(x)^2)^3}}$$

is not an invariant of $E(2)$ (indeed, the reflection $(x,y)\mapsto(-x,-y)$ preserves the circle $x^2+y^2=R^2$ but transforms $K\mapsto -K$; $K=\pm R^{-1}$), but its square K^2 is a bona fide rational differential invariant.

Notice however that the Lie group $E(2)_+$ is connected, and that K is invariant under the action of its Lie algebra (resp. local Lie group). What happens is that under the lifted action of S^1 on the space of 1st jets, the derivative $y'(x)$ becomes infinite and changes the branch, whence the change of the sign.

In this paper we show that this is the common situation with the algebraic actions and establish the structure theorem for the algebra of differential invariants (with Lie-Tresse generating property). The specification of behavior of differential invariants (rational-polynomial) makes it possible to obtain the first global result on the subject.

Structure of the paper is as follows. In Section 1 we discuss the basics from the geometric theory of differential equations, including symbolic modules, characteristics, Spencer cohomology and prolongations, and we also introduce pseudogroups, actions, and define differential invariants and invariant derivations. Most of these notions are standard, but we make some important specifications for the needs of the present paper.

In Section 2 we introduce our main tools: symbolic calculus of the differential invariants, together with the corresponding Spencer-like cohomology, and we associate an equation that represents the action on submanifolds and counts the differential invariants. We prove Cartan-Kuranishi type theorem for this equation, and deduce stabilization of cohomology. Then, partially following [KL_1], we derive the micro-local Lie-Tresse theorem. We also establish existence of invariant derivations (under no additional assumptions like those of [Kum]), and bound the growth of the Hilbert function. This implies rationality of the Poincaré series, solving the problem of V. Arnold on counting the number of moduli of geometric structures with respect to transitive pseudogroup actions.

In Section 3 we combine Rosenlicht's theorem with the results from Section 2 to obtain our main results. We also generalize the results to obtain finiteness of other geometric quantities for the algebraic actions of the pseudogroups. We briefly indicate how to overcome the restriction that G is algebraic: essentially one needs to separate only the orbits in low jets because the prolonged actions are always algebraic in higher jets.

In Section 4 we perform some calculations. We start by recalling the basic methods to calculate differential invariants, and then we present some new examples. They are chosen to demonstrate importance of our main assumptions. In particular, we show that dropping the assumption of algebraic action or considering non-transitive actions can lead to violation of separation or finite generation property in Lie-Tresse theorem (both local and global versions). The examples are also of independent interest as they are related to classical geometric problems.

1. Pseudogroups actions

In this section we discuss the general introduction of pseudogroups, differential equations and actions developing the ideas of [GS_2, SS, S_2]. In some parts the exposition follows [KL_3].

1.1. Jets and the geometric theory of PDEs

For a smooth manifold M of dimension m denote by $J^k(M,n)$ the space of k-jets $a_k = [N]^k_a$ of n-dimensional submanifolds $N \subset M$ at the point $a \in M$. Every submanifold N of dimension n determines uniquely the jet-extension $j_k(N) = \{[N]^k_a : a \in N\} \subset J^k(M,n)$.

We have $J^0(M,n) = M$ and $J^1(M,n) = \mathrm{Gr}_n(TM)$. The natural projections $\pi_{k,k-1} : J^k(M,n) \to J^{k-1}(M,n)$ allow us to define the projective limit $J^\infty(M,n) = \lim J^k(M,n)$.

The fibers $F(a_{k-1}) = \pi^{-1}_{k,k-1}(a_{k-1})$ for $k > 1$ carry a canonical affine structure [Go, KLV], associated with the vector structure described below.

Denote $\tau = T_a N = [N]^1_a$ and $\nu = T_a M/T_a N$ (both are determined by a_1). Let $a_k \in J^k(M,n)$, $a_{k-1} = \pi_{k,k-1}(a_k)$. Then $T_{a_k} F(a_{k-1}) \simeq S^k\tau^* \otimes \nu$ and we get the exact sequence:

$$0 \to S^k\tau^* \otimes \nu \to T_{a_k} J^k(M,n) \overset{(\pi_{k,k-1})_*}{\longrightarrow} T_{a_{k-1}} J^{k-1}(M,n) \to 0.$$

Thus the affine structure of $F(a_{k-1})$ is modeled on the vector space $S^k\tau^* \otimes \nu$.

If M is the total space of a vector bundle $\pi : M \to B$ of rank $m-n$, the space $J^k\pi \subset J^k(M,n)$ is an open dense subset consisting of jets of sections, i.e. submanifolds N transversal to the fibers of π.

A system of differential equations (PDE) of maximal order l is a sequence $\mathcal{E} = \{\mathcal{E}_k\}_{0\le k\le l}$ of submanifolds $\mathcal{E}_k \subset J^k(M,n)$ with $\mathcal{E}_0 = M$ such that for all $0 \le k < l$ the following conditions hold:

1. $\pi^{\mathcal{E}}_{k+1,k} : \mathcal{E}_{k+1} \to \mathcal{E}_k$ are smooth fiber bundles.
2. The first prolongations

$$\mathcal{E}^{(1)}_k = \{a_{k+1} = [N]^{k+1}_a \,|\, L(a_{k+1}) = T_{a_k} j_k(N) \subset T_{a_k}\mathcal{E}_k\}.$$

are smooth subbundles of π_{k+1} and $\mathcal{E}_{k+1} \subset \mathcal{E}^{(1)}_k$.

The higher prolongations are defined either by requiring the higher contact of $j_k(N)$ with $\mathcal{E}_k$ or inductively. We let $\mathcal{E}_{i+l} = \mathcal{E}_l^{(i)}$ and suppose the projections $\pi_{k+1,k} : \mathcal{E}_{k+1} \to \mathcal{E}_{k+l}$ are (affine) bundles for $k \geq l$ — this is equivalent to formal integrability (compatibility) of the system $\mathcal{E}$ [S$_2$, Go].

Consider a point $a_l \in \mathcal{E}_l$ with $a_i = \pi_{l,i}(a_l)$ for $i < l$. It determines the symbolic system $\{g_i\}_{i=0}^{\infty}$ by the formula

$$g_i = T_{a_i}\mathcal{E}_i \cap F(a_{i-1}) \subset S^i\tau^* \otimes \nu \quad \text{for} \quad i \leq l$$

and $g_i = g_l^{(i-l)} = g_l \otimes S^{i-l}\tau^* \cap S^i\tau^* \otimes \nu$ for $i > l$. By the above conditions the symbols g_i form smooth vector bundles over $\mathcal{E}_l$ and $g_i \subset g_{i-1}^{(1)}$.

The Spencer δ-complex for the PDE system $\mathcal{E}$ at a point $a_l \in \mathcal{E}_l$ (or for the symbolic system g) is

$$\cdots \to g_{i+1} \otimes \Lambda^{j-1}\tau^* \xrightarrow{\delta} g_i \otimes \Lambda^j\tau^* \xrightarrow{\delta} g_{i-1} \otimes \Lambda^{j+1}\tau^* \xrightarrow{\delta} \cdots$$

where δ is the (symbol of) de Rham differential. The cohomology $H^{i,j}(\mathcal{E}; a_l) = H^{i,j}(g)$ is called the Spencer δ-cohomology group. The formal integrability is equivalent to the regularity condition and vanishing of certain curvature type tensors [Ly], which are the elements in Spencer cohomology $W_k(\mathcal{E}) \in H^{k-1,2}(\mathcal{E})$.

In general some W_k can be nonzero, and then one needs to perform Cartan-Kuranishi completion to involution. This usually results in shrinking the equation $\mathcal{E}$ as a submanifold in $J^\infty(M,n)$ and removing some singular pieces. In our algebraic situation this boils down to enlarge the proper Zariski closed subvariety S_l, so we will suppose from the beginning that $\mathcal{E}$ is formally integrable.

1.2. Lie pseudogroups

The construction of the previous Section yields as a partial case the jet space for maps $J^k(M,M) \subset J^k(M \times M, m)$. This contains, as an open dense subset, the jet-space for diffeomorphisms $D^k(M)$ consisting of k-jets of m-dimensional submanifolds $N \subset M \times M$, which diffeomorphically project to both factors.

This $D^k(M)$ equipped with the partially defined composition operation is a *Lie groupoid*, basic for the definition of a general finite order pseudogroup. Its stabilizer at a, D_a^k is called the *differential group* of order k. Let us remark that D_a^k is an affine algebraic group because it is an automorphism group of the space of k-jets with the fixed point.

Definition 2. *A Lie pseudogroup of order l is given by a Lie equation, which is a collection of sub-bundles $G^j \subset D^j(M)$, $0 < j \leq l$, such that the following properties are satisfied:*

1. *If $\varphi_j, \psi_j \in G^j$, then $\varphi_j \circ \psi_j^{-1} \in G^j$ whenever defined,*
2. *$G^j \subset (G^{j-1})^{(1)}$ and $\rho_{j,j-1} : G^j \to G^{j-1}$ is a bundle for every $j \leq l$.*

We will assume *transitivity* of the pseudogroup action, i.e. $G^0 = D^0(M) = M \times M$. Assumption 1 implies that $\mathrm{id}_M^j \in G^j$ and $\varphi_j \in G^j \Rightarrow \varphi_j^{-1} \in G^j$.

Pseudogroups $G = \{G^j\}$ defined by this approach, with $G^{l+j} = (G^l)^{(j)}$ for $j > 0$, can be studied for integrability by the standard prolongation-projection method [GS$_1$, GS$_2$, Kur, KLV, S$_1$]. We will assume formal integrability from the beginning, so that G^k is a bundle over G^{k-1} for any $k > 0$.

Remark 2. *By Cartan's first fundamental theorem [SS] every Lie algebra sheaf is a sheaf of infinitesimal automorphisms of a transitive finite order structure. This is equivalent to the claim that transitive finite order pseudogroups are identical with Lie pseudogroups.*

Denote $G_a^j = \{\varphi_j \in G^j \,|\, \varphi_0(a) = a\}$ the isotropy subgroup of G^j. Transitivity implies that the subgroups $G_a^j \subset D_a^j(M)$ are conjugate for different points $a \in M$. By our assumption these G_a^j are *algebraic subgroups* of D_a^j for $j \le l$. It follows that the prolongations $G_a^k \subset D_a^k(M)$ are also algebraic for all $k > l$.

Definition 3. *The symbol of the pseudogroup G at a point $\varphi_j \in G_a^j$ is*

$$\mathfrak{g}^j(\varphi_j) = \mathrm{Ker}\big[(\rho_{j,j-1})_* : T_{\varphi_j} G^j \to T_{\varphi_{j-1}} G^{j-1}\big] \subset S^j(T_a^* M) \otimes T_a M.$$

Notice that contrary to finite-dimensional Lie groups (which can be abstract), the infinite pseudogroups always come with their natural representations. In our setup G^k acts on $J^k(M,n)$ via the action of local diffeomorphisms on submanifolds $\varphi_k : [N]_a^k \mapsto [\varphi(N)]_{\varphi(a)}^k$.

An important example is given by the natural bundles $\pi : M \to B$, with G being lifted from the pseudogroup $\mathrm{Diff}_{\mathrm{loc}}(B)$ on the base. Here $J^\infty \pi$ is the bundle of formal geometric structures [ALV] and the global Lie-Tresse theorem yields the method for classification of all differential invariants for the geometric structures of a given type.

1.3. Differential invariants and invariant derivations

The equivalence problem is to decide when a submanifold $N_1 \subset M$ can be transformed to a submanifold $N_2 \subset M$ by a map $\varphi \in G$ (if an equation $\mathcal{E}$ is imposed, both submanifolds N_i must be solutions of it)[3]. A sub-problem is related to equivalence of ∞-jets of submanifolds – then in many cases (algebraic, analytic, germs of submanifolds given by elliptic equations etc) the formal equivalence implies the local and eventually the global one.

Thus we have to distinguish between G-orbits in $\mathcal{E}_\infty \subset J^\infty(M,n)$ and this is precisely what the differential invariants (i.e. functions constant on these orbits) do. Consider the algebra of smooth functions $C^\infty(\mathcal{E}_\infty)$, which is the inverse limit of the algebras $C^\infty(\mathcal{E}_k)$ via the maps $\pi_{k,k-1}^*$. In many cases it is convenient to work with the sheaf $C^\infty_{\mathrm{loc}}(\mathcal{E}_k)$ of germs of such smooth functions.

[3] The equivalence problems of geometric structures on a manifold can be set up to be a partial case of our general setting for equivalence of submanifolds.

Definition 4. *A function $f \in C^\infty_{loc}(\mathcal{E}_k)$ is a differential invariant of order k if it is invariant with respect to the G-action on $\mathcal{E}_k$. The algebra of differential invariants of order k is denoted by $\mathcal{A}_k$, and $\mathcal{A} = \lim \mathcal{A}_k$ is the algebra of all smooth differential invariants.*

Thus the algebra of differential invariants is filtered $\mathcal{A} = \cup \mathcal{A}_k$ via the natural inclusions $\pi^*_{k+1,k} : \mathcal{A}_k \to \mathcal{A}_{k+1}$. In addition to the usual algebraic operations $\mathcal{A}$ has the following: for any smooth function Φ with r arguments and a collection $I_1, \dots, I_r \in \mathcal{A}$ we get $\Phi(I_1, \dots, I_r) \in \mathcal{A}$ (whenever the composition is defined).

We will not consider arbitrary smooth invariant functions, but will restrict to rational-polynomial invariant functions, which are defined globally and form a subalgebra $\mathfrak{P}_l(\mathcal{E})^G \subset C^\infty(\mathcal{E}_\infty)$ (as in the Introduction). In this case the above composition Φ must be taken rational-polynomial as well. However the above algebraic operations are not usually enough to finitely generate $\mathcal{A}$ or $\mathfrak{P}_l(\mathcal{E})^G$.

Sophus Lie proposed to produce higher order invariants via invariant derivations ∇ of the algebra of differential invariants. He suggested a theorem that a finite number of them $\nabla_1, \dots, \nabla_n$ is enough to generate the whole algebra $\mathcal{I}$.

Consider the bundle $\tau_k = \pi^*_{k,1}(\tau)$ on the jet-space $J^k(M,n)$, which is the pull-back of the tautological bundle τ over $J^1(M,n)$. Its fiber at the point a_k can be identified with the horizontal space $L(a_k)$ defined in §1.1. Let $v \mapsto \hat{v}$ denote this natural lift $\tau(a_1) \to \tau_k(a_k)$.

Denote by $C^\infty(\tau_k)$ the $C^\infty(J^k)$-module of sections of τ_k, and let $C^\infty(\tau_k|\mathcal{E}_k)$ be its restriction to the equation in order k. Its elements are finite sum $\sum f_i v_i$, where $f_i \in C^\infty(\mathcal{E}_k)$ and v_i are sections of the bundle τ_k.

Projection $\pi_{k+1,k}$ injectively maps these modules and simultaneously extends their rings. The $C^\infty(\mathcal{E}_\infty)$-module of horizontal vector fields is the inductive limit

$$C^\infty(\mathcal{E}_\infty, \tau) = \lim_{\to} C^\infty(\tau_k|\mathcal{E}_k).$$

The $C^\infty(\mathcal{E}_\infty)$-module of horizontal 1-forms $C^\infty(\mathcal{E}_\infty, \tau^*)$ is defined similarly.

The horizontal differential $\hat{d}f$ of a function $f \in C^\infty(\mathcal{E}_k)$ at $a_{k+1} \in \mathcal{E}_{k+1}$ is given by $\hat{d}f = d_{a_k} f|_{L(a_{k+1})}$. This gives the map

$$\hat{d} : C^\infty(\mathcal{E}_\infty) \to C^\infty(\mathcal{E}_\infty, \tau^*).$$

The action of horizontal vector fields on functions comes from the natural pairing between τ and τ^*:

$$\xi(f) = i_\xi \hat{d} f = i_{\hat{\xi}} df, \qquad \xi \in C^\infty(\mathcal{E}_\infty, \tau),\ f \in C^\infty(\mathcal{E}_\infty).$$

This action is linear and satisfies the Leibniz rule. Therefore any horizontal vector field ξ is a derivation of the algebra of functions $C^\infty(\mathcal{E}_\infty)$.

Definition 5. *Invariant derivatives are G-invariant elements of the module $C^\infty(\mathcal{E}_\infty, \tau)$ (according to the natural action of the pseudogroup G on it). We denote the space of invariant derivatives by $\mathfrak{D}(\mathcal{E}, \tau)$.*

The space $\mathfrak{D}(\mathcal{E},\tau)$ has the natural structure of a module over the algebra of invariant functions. We have two possibilities for this module: (micro-)local smooth over the algebra $\mathcal{A}$ or global rational-polynomial over $\mathfrak{P}_l(\mathcal{E})^G$.

The space $\mathfrak{D}(\mathcal{E},\tau)$ is also a Lie algebra and as such it acts on the algebra $\mathcal{A}$ from the left. In fact it is a subalgebra of the Lie algebra of derivations.

Proposition 3. *Every $\nabla \in \mathfrak{D}(\mathcal{E},\tau)$ determines a G-invariant derivation $\nabla : \mathcal{A} \to \mathcal{A}$ (if ∇ is rational-polynomial then $\mathcal{A}$ has to be changed to $\mathfrak{P}_l(\mathcal{E})^G$).*

Thus every invariant derivative is an invariant derivation, but in the presence of equation $\mathcal{E}$ and pseudogroup G the inverse implication is not generally true. The relations will be clarified in Section 2.3.

1.4. Particular case: Tresse derivatives

Let us first write the constructions from the previous section in local coordinates. We identify M with the total space of a fiber bundle $\pi : M \to B$ with $\dim B = n$. Then $J^k\pi \subset J^k(M,n)$ is a local chart near $a_k = [N]^k_a$ if the fibers of π are transversal to the germ of the submanifold $N \subset M$ at a.

Choose local coordinates (x^i, u^j) on M adapted to π, i.e. x^i are coordinates on the base and u^j on the fibers. These local coordinates on π induce the canonical coordinates $(x^i, u^j_\sigma)_{0\le|\sigma|\le k}$ on $J^k\pi$, where for a submanifold N given as graph $u = h(x)$ we have: $u^j_\sigma([h]^k_a) = \dfrac{\partial^{|\sigma|}h^j}{\partial x^\sigma}(a)$.

In the local chart $J^\infty(\pi) \subset J^\infty(M,n)$ the derivatives (horizontal vector fields) can be written as $\sum f_i\,\mathcal{D}_i$, where

$$\mathcal{D}_i = \partial_{x^i} + \sum_{j;\sigma} u^j_{\sigma+1_i}\partial_{u^j_\sigma} : C^\infty(J^k\pi) \to C^\infty(J^{k+1}\pi)$$

are the total derivative operators.

The horizontal differential has the following form in these coordinates:

$$\hat{d}f = \sum \mathcal{D}_i(f)\,dx^i.$$

The same formulae apply for the restriction to the equation $\mathcal{E}$.

An important particular case of derivations of the algebra $C^\infty(\mathcal{E}_\infty)$ constitute derivatives a la Tresse, which we now introduce.

Consider n functions $f_1,\dots,f_n$ on $\mathcal{E}_k$ such that the open[4] set

$$\mathcal{E}'_{k+1} = \{a_{k+1} \in \mathcal{E}_{k+1} \,:\, df_1 \wedge \ldots \wedge df_n|_{L(a_{k+1})} \neq 0\} \tag{1}$$

is dense in $\mathcal{E}_{k+1}$. Then we define differential operators

$$\hat{\partial}_i : C^\infty(\mathcal{E}_k) \to C^\infty(\mathcal{E}'_{k+1}),$$

[4]When we turn to the algebraic situation, then 'open' becomes 'Zariski open'.

by the formula

$$df|_{L(a_{k+1})} = \sum_{i=1}^{n} \hat\partial_i(f)(a_{k+1})\, df_i|_{L(a_{k+1})},$$

The expressions $\hat\partial_i(f) = \hat\partial f/\hat\partial f_i$ are called Tresse derivatives of f with respect to f_i. These derivations enjoy the commutativity property $[\hat\partial_i, \hat\partial_j] = 0$.

Proposition 4. *If $f_1, \dots, f_n$ are G-differential invariants, then the operators $\hat\partial_i = \hat\partial/\hat\partial f_i : \mathcal{A}_k \to \mathcal{A}_{k+1}$ are invariant derivations.*

On G-invariant set $\mathcal{E}'_{k+1}$ we can re-write condition in (1) as

$$\hat d f_1 \wedge \dots \wedge \hat d f_n \neq 0,$$

i.e. the Jacobian $DF = \|\mathcal{D}_i(f_j)\|$ is non-degenerate. For any other $f \in \mathcal{A}$ we have:

$$\hat d f = \sum_i \hat\partial_i(f)\, \hat d f_i.$$

Thus

$$\hat d = \sum dx^i \otimes \mathcal{D}_{x^i} = \sum \hat d f_i \otimes \hat\partial/\hat\partial f_i,$$

which yields the expression of Tresse derivatives:

$$\hat\partial_i = \sum_j \bigl(DF^{-1}\bigr)_{ij} \mathcal{D}_{x^j},$$

This formula can be interpreted as the chain rule in total derivatives.

1.5. Characteristics and involutivity

Let $g = \oplus g_k$ be the symbolic system corresponding to a differential equation $\mathcal{E}$ or abstract, which is subject to only one constraint $g_k \subset g_{k-1}^{(1)}$ or $H^{k,0}(g) = 0$ in terms of the Spencer cohomology. For a vector space V let us denote by $SV = \oplus S^i V$ the ring of homogeneous polynomials on V^*.

For $v \in V$ define the map $\delta_v : S^{k+1}\tau^* \otimes \nu \to S^k\tau^* \otimes \nu$ by the formula $\delta_v(p) = \langle v, \delta p\rangle$. More generally given $v_{i_j} \in V$ and $v = \sum v_{i_1} \cdots v_{i_r} \in S^r V$ define $\delta_v = \sum \delta_{v_{i_1}} \cdots \delta_{v_{i_r}} : S^{k+r}\tau^* \otimes \nu \to S^k\tau^* \otimes \nu$.

The $\mathbb{R}$-dual (or $\mathbb{C}$-dual if we work over the field of complex numbers) system $g^* = \oplus g_k^*$ is an $S\tau$-module with the structure given by

$$(v \cdot \kappa)p = \kappa(\delta_v p),\ v \in S\tau,\ \kappa \in g^*,\ p \in g.$$

This module, called the *symbolic module*, is Noetherian and the Spencer cohomology of g dualizes to the Koszul homology of g^*.

The *characteristic ideal* is defined by $I(g) = \mathrm{Ann}(g^*) \subset S\tau$. Notice that passing to the module $\mathcal{M} = S\tau/I(g)$ results in the shift of indices in the homology: $H^{i,j}(g)^* = H_{i,j}(g^*) = H_{i+1,j-1}(\mathcal{M})$ – the Koszul homology of $\mathcal{M}$.

A symbolic system has maximal order l if $H^{k,1}(g) \neq 0$ for $k > l$ (the cohomology $H^{*,1}$ counts the orders). Referring to [KL$_2$] for the general definition in the case of systems of different orders, we adapt the following version here: a symbolic system g is involutive from the order k_0 if $H^{k,j}(g) = 0$ for $k \geq k_0$. By Poincaré δ-lemma [S$_2$] any symbolic system is involutive starting from some order. The bound k_0 can be taken universal, depending only on $n = \dim \tau$ and the orders of g.

The *affine characteristic variety* of g (or of $\mathcal{E}$) is the set of $v \in \tau^* \setminus \{0\}$ such that for every k there exists a $w \in \nu \setminus \{0\}$ with $v^k \otimes w \in g_k$. This is a conical affine variety. If we consider its complexification and then projectivization, we get the *characteristic variety* $\mathrm{Char}^{\mathbb{C}}(g) \subset P^{\mathbb{C}}\tau^*$.

Relation of characteristic variety to the characteristic ideal $I(g) = \oplus I_k$ is given by the formula:

$$\mathrm{Char}^{\mathbb{C}}(g) = \{p \in P^{\mathbb{C}}\tau^* \mid f(p^k) = 0 \,\forall f \in I_k, \forall k\}.$$

Note that dimension d of the affine complex characteristic variety equals to the Chevalley dimension of the symbolic module g^*. Recall also that a sequence of elements $f_1, \dots, f_s \in \tau$ is called regular if f_i is not a zero divisor in the $S\tau$-module $g^*/(f_1, \dots, f_{i-1})g^*$.

An element $v \in \tau$ is regular for sufficiently prolonged system g iff the hyperplane $P\,\mathrm{Ann}(v)^{\mathbb{C}}$ does not contain the characteristic variety $\mathrm{Char}^{\mathbb{C}}(g)$. More generally, a sequence $(v_1, \dots, v_i)$ is regular iff $\mathrm{Char}^{\mathbb{C}}(g)$ meets $P\,\mathrm{Ann}(v_1, \dots, v_i)^{\mathbb{C}}$ transversally. In other words, the projection of the characteristic variety along annihilator to $P(\langle v_1, \dots, v_i\rangle^{\mathbb{C}})^*$ is surjective for $i < d$ and injective elsewise.

It follows that there exists a sequence $(v_1, \dots, v_n)$ in τ that is regular for the module $g^*_{[k_0]} = \oplus_{i \geq k_0} g^*_i$. This implies (see [AB] or the appendix with a letter of Serre in [GS$_1$]) that all Koszul homology of g^* w.r.t. the sequence $(v_1, \dots, v_n)$, or equivalently with coefficients in τ, vanish except perhaps for the zero grading (equivalently $g^*_{[k_0]}$ is a Cohen-Macaulay module over $S\tau$). Dualizing this statement we obtain again that the positive Spencer cohomology $\oplus_{k \geq k_0} H^{k,+}(g)$ vanishes. When $g \subset S\tau^* \otimes \nu$ the zero cohomology vanishes as well.

The above theory is applicable to the pseudogroup G, which is a particular case of an equation (nonlinear Lie equation). The corresponding symbolic group $\mathfrak{g}$ induces the characteristic variety $\mathrm{Char}^{\mathbb{C}}(\mathfrak{g}) \subset P^{\mathbb{C}}T^*$, $T = T_aM$, which (contrary to the case of a general equation $\mathcal{E}$) has the same projective type at different points $\varphi_k \in G^k_a$.

2. Structure of the algebra of scalar differential invariants

In this section we prove that the local behavior of differential invariants is governed by certain differential equations associated to the action, and we deduce from this micro-local Lie-Tresse theorem.

2.1. Automorphic equation associated to regular G-orbits

Consider representation of the pseudogroup G via the action on the jets of submanifolds subject to the equation $\mathcal{E} \subset J^\infty(M,n)$. As the action is transitive by the base, we can restrict to the vertical action of G_a on $\mathcal{E}_a$.

The orbit of this action through the k-jet a_k is $G \cdot a_k \subset \mathcal{E}_k$. Let $\Delta_k(a_k) = T_{a_k}(G \cdot a_k)$ be the tangent differential system. It consists of the evaluations of k-jets of the Lie algebra sheaf $\mathcal{G}$ of the pseudogroup G at the point a_k.

The distribution $\Delta_k \subset T\mathcal{E}'_k$ is obviously integrable on the set of regular orbits $\mathcal{E}'_k \subset \mathcal{E}_k$. Differential invariants of order k are its first integrals.

For a germ of n-dimensional submanifold $N \subset M$ with jet-lift in $\mathcal{E}'_k$ (regularity) consider its orbit in the space of k-jets

$$G_N^k = \bigcup_{a \in N} \{G_a^k \cdot a_k \mid a_k = [N]_a^k\} \subset J^k(M,n).$$

We have obvious projections $\pi_{k,k-1} : G_N^k \to G_N^{k-1}$ and $\pi_k : G_N^k \to N$ since for a pseudogroup element $g \in G$ with k-jet g_a^k at a we have: $g_a^k \cdot [N]_a^k = [g(N)]_a^k$.

Proposition 5. *The collection $\{G_N^k\}_{k=0}^\infty$ is a differential equation over N.*

Assuming regularity, this statement means the inclusion $G_N^{k+1} \subset (G_N^k)^{(1)}$.

Proof. Consider an element $a'_{k+1} = g_a^{k+1} \cdot a_{k+1}$ with $\pi_{k+1,k}(a'_{k+1}) = a'_k = g_a^k \cdot a_k$, where a_k and g_a^k are also the projections. We want to prove $L(a'_{k+1}) \subset T_{a_k}(G_N^k)$. But this follows from equality $L(a'_{k+1}) = (g_a^k)_* L(a_k)$ and the fact that G_N^k is G^k-symmetric. □

If N is a solution of $\mathcal{E}$ this equation is G-automodel, i.e. the symmetry pseudogroup acts transitively on the solutions of G_N^∞. The symbol of this equation is a collection of subspaces $\varpi_k \subset S^k\tau_a^* \otimes \nu_a$ given by

$$\varpi_k = \operatorname{Ker}\bigl(d\pi_{k,k-1} : \Delta_k(a_k) \to \Delta_{k-1}(a_{k-1})\bigr).$$

Proposition 6. *Let us fix a jet $a_\infty \in \mathcal{E}_\infty$. Then $\{\varpi_k\}_{k=0}^\infty$ is a symbolic system.*

Proof. The claim is that the sequence $0 \to \varpi_{k+1} \xrightarrow{\delta} \varpi_k \otimes \tau^*$ is exact. Since the Spencer δ-differential is always injective at the first term, this writes as $\delta_\xi : \varpi_{k+1} \to \varpi_k \ \forall \xi \in \tau$, and so follows from Proposition 5. □

Denote the above symbolic system by $\varpi(a_\infty)$. We can also define the symbolic system $\varpi(a_k)$ by the following truncation rule:

$$\varpi(a_k)_l = \varpi_l \ \text{ for } l \le k \quad \text{ and } \quad \varpi(a_k)_l = \varpi_k^{(l-k)} \ \text{ for } l > k.$$

Proposition 6 re-phrases now so: $\varpi(a_\infty) \subset \varpi(a_k)$ for $a_\infty \in (\pi_{\infty,k}^{\mathcal{E}})^{-1}(a_k)$.

Theorem 7. *There exists a number l and a G-invariant Zariski open set $\mathcal{E}''_l \subset \mathcal{E}_l$ such that $\varpi(a_\infty) = \varpi(a_l)$ for $a_\infty \in (\pi_{\infty,l}^{\mathcal{E}})^{-1}(a_l)$, $a_l \in \mathcal{E}''_l$.*

Proof. Let $q_k = \text{ess.sup}\{m : H^{m-1,1}(\varpi(a_k)) \neq 0\}$ (essential supremum is the infimum by Zariski open non-empty sets $R_k \subset \mathcal{E}_k$ of the supremum on R_k). We claim that eventually $q_k < k$.

Otherwise we get a sequence of regular points a_k, $\pi_{k+1,k}(a_{k+1}) = a_k$, with strict inequality $\varpi(a_{k+1}) \subset \varpi(a_k)$, namely $\varpi(a_{k+1})_{k+1} \subsetneqq \varpi(a_k)_k^{(1)}$. Then for the projective limit a_∞ of this sequence we obtain that the symbolic system $\varpi = \varpi(a_\infty)$ has infinite-dimensional cohomology group $H^{*,1}(\varpi)$. This contradicts the Hilbert basis theorem.

Starting from the specified level k, all the first Spencer δ-cohomology vanish. Let $\mathcal{E}_k''$ be the set of $a_k \in \mathcal{E}_k$ such that the symbolic system $\varpi(a_k)$ is regular. Since the latter is a finite set of algebraic conditions, $\mathcal{E}_k''$ is Zariski open, and for every point a_{k+s} over it the symbolic system coincides with $\varpi(a_k)$ (i.e. is obtained only by prolongations). □

Thus we proved that for a Zariski open set in $\mathcal{E}_\infty$ the symbolic system $\varpi(a_\infty)$ is stable. By Poincaré δ-lemma [S$_2$] we can also assume that starting from l for all points over $\mathcal{E}_l''$ the Spencer cohomology groups $H^{i,j}(\varpi)$, $i \geq l$, $j \geq 0$, vanish.

2.2. Lie equation associated to the pseudogroup action

Now we discuss the Lie equation corresponding to the equation G_N of Proposition 5. It is important in practice for calculation of the number of independent differential invariants on the level of k-jets.

Consider a germ of submanifold $N \subset M$ of dimension n, which is a solution of $\mathcal{E}$ (this condition relaxes to the requirement that N is a formal solution at a, i.e. the corresponding jet $a_\infty \in \mathcal{E}_\infty$). It determines linearization of the equation $\ell_N(\mathcal{E}) \subset J^\infty(\nu_N)$, where $\nu_N = (TM|_N)/TN$ is the normal bundle over N.

For every germ of the vector field $X \in \mathfrak{D}(M)$ we associate its generating function $\varphi_X \in C^\infty(J^1(M,n),\nu)$ given by $\varphi_X(a_1) = X_a \bmod \tau_a$, where $a_1 = [N]_a^1$, $\tau_a = T_aN$. In canonical coordinates on J^1 the generating function of the point symmetry $X = \alpha^i \partial_{x^i} + \beta^j \partial_{u^j}$ has the usual formula $\varphi_X^j = \beta^j - p_i^j \alpha^i$.

Evaluating the jets of this map for the Lie algebra sheaf $\mathcal{G}$ of G on the submanifold N we get the subbundle

$$\mathcal{R}_N^k = \{[\varphi_{X_k}]_a^k : X_k \in \mathcal{G}_a^k\} \subset \ell_N^k(\mathcal{E}) \subset J^k(\nu_N).$$

We again have the natural projections $\pi_{i,j} : \mathcal{R}_N^i \to \mathcal{R}_N^j$.

Notice that the fiber $(\mathcal{R}_N^k)_a$ over $a \in M$ depends on the $(k+1)$-jet of the submanifold N, while $T_a\mathcal{R}_N^k = (\mathcal{R}_N^k)_a + T_aN$ depends on the k-jet $a_k = [N]_a^k$ (due to linearity we can embed N to $\mathcal{R}_N^k$ as the zero section). This latter can be thus identified with the space $\Delta_k(a_k)$, so that ϖ_k is the symbol of $\mathcal{R}_N$ at a_k.

Proposition 8. *The collection $\{\mathcal{R}_N^k\}_{k=0}^\infty$ is an equation, i.e. $\mathcal{R}_N^{k+1} \subset (\mathcal{R}_N^k)^{(1)}$.*

The proof of this is similar and, in fact, follows from Proposition 5.

Remark 3. *Both Propositions 5 and 8 describe the equations determined by their solutions and so are formally integrable by the construction.*

Cartan-Kuranishi theorem claims that eventually, i.e. starting with some number k

$$\mathcal{R}_N^{k+1} = (\mathcal{R}_N^k)^{(1)}.$$

This theorem holds under the regularity assumptions [Kur, Se] which are not easy to check (apriori there are infinite number of constraints). But in our case this is a consequence of Theorem 7: N is regular if $j_k(N) \subset \mathcal{E}_k''$ for any $k \le l$.

Moreover this statement holds uniformly for Zariski-generic jet of the submanifold N. In order to see this, let us extend the picture as follows.

A submanifold N and a vector field X along it form an object $\tilde{N} = (N, X|_N)$, which is a submanifold of dimension $n+m$ in TM. Restricting the jet of N by the equation $\mathcal{E}$ and the jet of X by the equation $\mathcal{G}$ we get a new equation (checking it is an equation is similar to the proof of Proposition 8)

$$\hat{\mathcal{R}} \subset J^\infty(TM, n+m).$$

The equation $\mathcal{R}$ is obtained from $\hat{\mathcal{R}}$ by fixing $[N]_a^\infty$ and projecting the second component of $[\tilde{N}]_a^\infty$ to the normal bundle ν_N. The kernel of this projection is the stabilizer of the jet of the submanifold in the pseudogroup G, and this gives a method to calculate dimensions of ϖ_k (in particular the quotient of the pseudogroup symbol $\mathfrak{g}^k$ by the stabilizer embeds into ϖ_k).

Theorem 9. *There exists a number l and a Zariski open set $\mathcal{U}_l \subset \hat{\mathcal{R}}_l$ such that $\hat{\mathcal{R}}_{k+1} = \hat{\mathcal{R}}_k^{(1)}$ for all $k \ge l$ over $\mathcal{U}_l$.*

Proof. Since the pseudogroup G acts transitively on the base, all fibers behave similarly. The equation $\hat{\mathcal{R}}$ is algebraic in these fiber variables, and hence the Malgrange version of Cartan-Kuranishi theorem [Ma] applies.

Alternatively, the claim follows from Theorem 7. $\square$

Remark 4. *Singularities in $\hat{\mathcal{R}}$ play the same role as in Thom-Boardman theory, and a submanifold N whose lift misses them yields the regular equation $\mathcal{R}_N$.*

Consider now the space of orbits $\mathfrak{O}^k = \mathcal{E}^k/G$. In general this orbifold has a complicated structure, but since our action is algebraic and transitive, then $\mathfrak{O}^k = \mathcal{E}^k/G_a^k$ and the singularities are nowhere dense (these will be treated in more details in Section 3).

At regular points the fiber of the projection $\mathfrak{O}^k \to \mathfrak{O}^{k-1}$ has the natural affine structure, and the corresponding vector space will be denoted by $\mathfrak{d}_k$. Thus we have the following exact 3-sequence

$$0 \to \mathfrak{d}_k \to \mathfrak{O}^k \to \mathfrak{O}^{k-1} \to 0.$$

It is easier to study this through the action λ of the Lie algebra sheaf $\mathcal{G}$. It gives the following exact sequence

$$0 \to \varpi_k \xrightarrow{\lambda} g_k \longrightarrow \mathfrak{d}_k \to 0. \qquad (2)$$

Let us notice that (see [KL$_1$])

$$\mathfrak{d}_k = (\Theta_k/\Theta_{k-1})^*,$$

where $\Theta_j(a_k) = \{d_{a_k} f \mid f \in \mathcal{A}_j\} \subset T^*_{a_k}\mathcal{E}_k$. With this $\mathfrak{d}_k^*$ can be interpreted as the space of symbols of differential invariants (at the regular points).

Then from the epimorphism $S^k\tau_a \otimes \nu_a^* \to \mathfrak{d}_k^*$ we deduce the natural map

$$\delta^* : \mathfrak{d}_k^* \otimes \tau \to \mathfrak{d}_{k+1}^*, \tag{3}$$

which can be viewed as the symbol of invariant derivations.

2.3. Existence of invariant derivations

Recall that $\tau_k = \pi_{k,1}^*\tau$ at the point $a_k \in \mathcal{E}_k$ can be identified with the horizontal plane $L(a_k) \subset T_{a_{k-1}}\mathcal{E}_{k-1}$.

Let $\alpha_k \subset \tau_k$ be the subspace of vectors invariant with respect to the stabilizer G_{a_k}. Since $\tau_k \simeq \tau$ via the natural projection π_k, we have the following: $a_{k+1} \in F(a_k)$ implies $\alpha_{k+1} \supset \alpha_k$ at the corresponding points. Thus we have the canonically defined limit

$$\alpha(a_\infty) = \cup \alpha_k(a_k).$$

Moreover the stabilization occurs on a finite jet-level.

Proposition 10. *There exists a number l and a Zariski open non-empty subset $\mathcal{E}_l' \subset \mathcal{E}_l$ such that $\alpha_l \subset \tau_l|_{\mathcal{E}_l'}$ is a regular smooth sub-distribution and that rank of α_k at $a_k \in \pi_{k,l}^{-1}(\mathcal{E}_l')$ is independent of $k \geq l$ and the choice of point a_k.*

The proof uses the same arguments as in Theorem 7 and so is omitted. Let $s = \dim \alpha$ be the rank of $\alpha_l|_{\mathcal{E}_l'}$. Then we get that $\dim_{\mathcal{A}} \mathfrak{D}(\mathcal{E}, \tau) = s$.

Corollary 11. *There exists s invariant derivatives of the G-action, and they generate the module $\mathfrak{D}(\mathcal{E}, \tau)$ over $\mathcal{A}$ (resp. $\mathfrak{P}_l(\mathcal{E})^G$), upon restriction to $\mathcal{E}_l'$.*

We would like to have a criterion for the condition $s = n$. For this we define the following notion.

Definition 6. *An equation $\mathcal{E} \subset J^\infty(M, n)$ is called* ample *if for almost all $a_k \in \mathcal{E}_k$ (where $k \geq l$ starts with the order of involutivity) no irreducible component of the characteristic variety* $\mathrm{Char}^{\mathbb{C}}(\mathcal{E}, a_k) \subset \mathbb{P}^{\mathbb{C}}\tau_a^*$ *belongs to one hypersurface.*

Notice that in this definition we can restrict to $k = l$ being the (maximal) order of $\mathcal{E}$ (if it is involutive), since prolongation does not change the characteristic variety. In the case $\mathcal{E}^l = J^l$ (no equation) the condition of ampleness clearly holds, and it also holds for sufficiently non-linear equations $\mathcal{E}$ (ampleness is the condition of general position for the projective variety $\mathrm{Char}^{\mathbb{C}}(\mathcal{E}, a_k)$ provided its dimension is positive; the opposite case is reducible to the usual semi-algebraic action of a Lie group on a finite-dimensional manifold).

Theorem 12. *Assume that $\mathcal{E}$ is ample and that the number of G-differential invariants is infinite* $\dim\mathcal{A}=\infty$. *Then there exist n independent invariant derivatives* $\nabla_1,\dots,\nabla_n\in\mathfrak{D}(\mathcal{E},\tau)$.

The case $\dim\mathcal{A}<\infty$, when the G-action is eventually transitive (orbits have finite codimension in $\mathcal{E}_\infty$), was investigated in [KL$_1$] (in fact it is enough to assume $\dim\mathcal{A}>n$ in the theorem).

Proof. Let the maximal rank of the sub-distribution $\alpha^*=\{\hat{d}f:f\in\mathcal{A}_k\}\subset\tau_k^*$ be s (it is attained on a Zariski open set in $\mathcal{E}_{k+1}$). We choose k sufficiently large, so that this rank is stable (for instance, k shall be greater than the involutivity order of the equation $\mathcal{R}$ from §2.2). We shall prove $s=n$.

Assume the opposite. Then for some independent differential invariants f_i, $1\le i\le s+1$, we have: $\omega_s=\hat{d}f_1\wedge\dots\wedge\hat{d}f_{s-1}\wedge\hat{d}f_s\neq0$ at generic point $a_{k+1}\in\mathcal{E}_{k+1}$, and $\omega_s\wedge\hat{d}f_{s+1}=0$. Moreover by the assumptions for all $i\le n$: $\hat{d}f_1\wedge\dots\wedge\check{df_i}\wedge\dots\wedge\hat{d}f_{s+1}=\sigma_i\omega_s$, where σ_i are some differential invariants.

Let $\sigma_i^0=\sigma_i(a_{k+1})$ be the (constant) values. Changing f_{s+1} to another differential invariant $f_{s+1}-\sum_{i=1}^s(-1)^{s-i}\sigma_i^0f_i$ we get the following equalities at a_{k+1}: $\hat{d}f_1\wedge\dots\wedge\check{df_i}\wedge\dots\wedge\hat{d}f_{s+1}=0$ for $i\le n$.

We can suppose (by changing k and using $\dim\mathcal{A}=\infty$) that $df_{s+1}|g_k\neq0$. Then there exists $\theta\in g_{k+1}$ such that $\delta_v\theta\notin\operatorname{Ker}(df_{s+1}|_{g_k})$ for all $v\in\tau$.[5]

Moreover such θ can be taken as $\theta=p^{k+1}\otimes\zeta$, where the covector $p\in\tau^*$ is characteristic (ζ is from the kernel bundle). Indeed, it is known (see e.g. [S$_2$]) that the characteristic covectors generate the symbol (over $\mathbb{C}$: every element in g_k is a finite linear combination $\sum\lambda_ip_i^k\otimes\zeta_i$ with $p_i\in\operatorname{Char}^{\mathbb{C}}(\mathcal{E})$ and $\lambda_i\in\mathbb{C}$).

By a small perturbation we can achieve $p|_{\operatorname{Ker}\omega_s}\neq0$ (because $\mathcal{E}$ is ample) and we also have $\rho=df_{s+1}(\bar\theta)\neq0$ for $\bar\theta=p^k\otimes\zeta\in g_k$ by the above assumption. Finally we can change the differential invariants $\{f_i\}_{i=1}^s$ in such a way that the above constraints hold, and in addition $df_i(\bar\theta)=0$, $1\le i\le s$, at the point a_{k+1}.

Let us now deform the point $a_{k+1}\in\mathcal{E}_{k+1}$ to $a_{k+1}^\epsilon=a_{k+1}+\epsilon\,\theta$ ($\theta\in S^{k+1}\tau^*\otimes\nu$ is as above and we use the affine structure in the fibers). Then the horizontal $(s+1)$-form $\omega_s\wedge\hat{d}f_{s+1}$ at the point a_{k+1}^ϵ equals

$$\hat{d}f_1\wedge\dots\wedge\hat{d}f_s\wedge\hat{d}f_{s+1}=\epsilon\rho(k+1)\,\omega_s\wedge p\neq0$$

for $\epsilon\neq0$. This contradicts the choice of s.

Thus there exist n differential invariants $f_1,\dots,f_n$ with $\hat{d}f_1\wedge\dots\wedge\hat{d}f_n\neq0$ in an open dense neighborhood U, which implies existence of n invariant derivatives $\nabla_i=\hat\partial/\hat\partial f_i$, $i=1,\dots,n$. □

In general we can have $s=\dim\alpha<n$ (see example in §4.2). Let $\alpha^\vee\subset\tau_k$ be the subspace of vectors which annihilate the sub-distribution α^* of invariant horizontal exact 1-forms on $\mathcal{E}_{k+1}$. By the proof of the above theorem its elements annihilate the characteristic variety $\operatorname{Char}^{\mathbb{C}}(\mathcal{E},a_k)$ for every point a_k. These

[5] The ampleness condition can be reformulated as $\delta_v=i_v\circ\delta:g_{k+1}\to g_k$ is onto for all non-zero $v\in\tau$.

vectors act trivially on the symbolic module of the equation $\mathcal{E}$:

$$\alpha^\vee \subset \{v \in \tau : v^\perp \supset \mathrm{Char}^{\mathbb{C}}(\mathcal{E})\}$$

(in other words, v to the right are all non-regular elements for the module g^*).

Theorem 13. *Let $s = \operatorname{codim} \alpha^\vee = \dim(\tau/\alpha^\vee)$. Then on an open dense subset of $\mathcal{E}$ there exist s independent invariant derivations $\nabla_i : \mathcal{A} \to \mathcal{A}$.*

Proof. The natural pairing between $\alpha \stackrel{\text{def}}{=} \tau/\alpha^\vee$ and α^* corresponds to derivations: for $v \in \alpha$ and $\hat{d}f \in \alpha^*$ their product is $v(f)$. This paring is non-degenerate since the symbol of $v(f)$ corresponds to the product of the symbols of v and f in the symbolic module $\mathcal{M}_\mathcal{E} = g^*$ of $\mathcal{E}$, and v does not annihilate the support of $\mathcal{M}_\mathcal{E}$.

Thus any basis $\hat{d}f_1, \dots, \hat{d}f_s$ of α^* (with some choice of differential invariants $\{f_i\}_{i=1}^s$) gives rise to a basis of α, which consists of invariant derivations. The elements identified as sections of $\tau_\infty = \pi^*_{\infty,1}\tau$ are defined only modulo $\alpha^\vee$, but their action on $\mathcal{A}$ does not depend on this. $\square$

Remark 5. *We do not identify the derivations ∇_i from Theorem 13 with invariant derivatives, interpreted as G-stable horizontal vector fields on $\mathcal{E}_\infty$. By the last example of §4.2 the corresponding horizontal fields are not G-invariant (though the non-invariant part acts trivially).*

The derivations obtained in the above theorem are not arbitrary[6]*: they preserve the Cartan ideal in the space of differential forms on $\mathcal{E}_\infty$ and they shift by 1 the filtration $\mathcal{A}_k$ in the algebra $\mathcal{A}$ of invariant functions on $\mathcal{E}_\infty$. Thus they can be called $\mathcal{C}$-derivations (operators in total derivatives).*

We will denote the space of such derivations by $\mathfrak{Der}(\mathcal{E}, \tau)$.

It is possible to identify the invariant derivations with invariant horizontal vector fields under some additional assumptions, like those of Theorem 12. One occasion is this: If G^k is reductive, then its stabilizer at generic point a_k is such (theorems 7.12 and 7.15 in [PV]) and so there exists an invariant complement to $\alpha^\perp$ (lifted to $L(a_{k-1})$ at $a_k \in \mathcal{E}_k$).

Another possibility is given by a condition from [KL$_1$] similar to Kumpera's hypothesis H$_3$ [Kum]. If

$$\mathrm{Reg}^2_k(\mathcal{E}, G) = \{a_k \in \mathcal{E}_k \mid \exists a_{k+1} \in \mathcal{E}_{k+1} : \Delta_k(a_k) \cap L(a_{k+1}) = 0\}$$

is open and dense in $\mathcal{E}_k$ (for $k = l$ and so for all larger k), then the number of invariant derivatives (horizontal vector fields) is $s = n$.

[6] Some general invariant derivations are not very useful. For instance the Lie fields from the normalizer of G are G-invariant, but cannot be used to produce new differential invariants.

2.4. Micro-local Lie-Tresse theorem

We can combine exact 3-sequences (2) with the corresponding Spencer δ-complexes into the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varpi_k & \xrightarrow{\delta} & \varpi_{k-1}\otimes\tau^* & \xrightarrow{\delta} & \varpi_{k-2}\otimes\Lambda^2\tau^* & \xrightarrow{\delta} & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_k & \xrightarrow{\delta} & g_{k-1}\otimes\tau^* & \xrightarrow{\delta} & g_{k-2}\otimes\Lambda^2\tau^* & \xrightarrow{\delta} & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{d}_k & \xrightarrow{\delta} & \mathfrak{d}_{k-1}\otimes\tau^* & \xrightarrow{\delta} & \mathfrak{d}_{k-2}\otimes\Lambda^2\tau^* & \xrightarrow{\delta} & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proposition 14. *There exists a number l and a Zariski open subset $\mathcal{E}_l'' \subset \mathcal{E}_l$ such that for all points a_k with $\pi_{k,l}(a_k) = a_l \in \mathcal{E}_l''$ the δ-cohomology groups $H^{i,j}(\mathfrak{d})$ vanish in the range $i \geq l, j \geq 0$.*

Proof. Take l to be the maximum of the number given by Theorem 7 (to be more precise that from the end of §2.1) and the level, where $\mathcal{E}$ becomes involutive. Then the δ-cohomology groups $H^{i,j}(\varpi)$ and $H^{i,j}(g)$ vanish in the range $i \geq l, j \geq 0$. The standard diagram chase implies the result. □

Theorem 15. *There exists a number l, a Zariski open subset $\mathcal{E}_l'' \subset \mathcal{E}_l$ and a finite or countable open cover $\mathcal{E}_l'' = \cup_\beta U_l^\beta$ such that for every β the structure of the algebra of G-invariant functions on $U_\infty^\beta = \pi_{\infty,l}^{-1}(U_l^\beta) \subset \mathcal{E}_\infty$ is as follows.*

There exist t differential invariants $f_1, \dots, f_t \in \mathcal{A}$ on U_∞^β and s invariant derivations $\nabla_1, \dots, \nabla_s \in \mathfrak{Der}(\mathcal{E}, \tau)$ such that all differential invariants in U_∞^β can be expressed via the derived differential invariants $\nabla_J(f_j)$.[7]

Notice that the sets U_l^β (and so U_∞^β) are not claimed to be G-invariant. This is a feature of the micro-local analysis.

Proof. Let l be the same integer as in Proposition 14. We can suppose that from this level the number of independent invariant derivations stabilizes.

Thus we can find a maximal collection $\nabla_1, \dots, \nabla_s$ of independent invariant derivations and a maximal collection $f_1, \dots, f_t$ of functionally independent differential invariants on $\mathcal{E}_l$. We claim that for $k \geq l$ all differential invariants are functions of $\nabla_J(f_j)$, where the length $|J| \leq k - l$.

[7] for some for a multi-indices $J = (j_1, \dots, j_r)$. Since the derivations ∇_i do not commute in general, the representation can be sensitive to their ordering.

We prove this by induction in k with the base $k = l$. Suppose we are good on the level k and consider the invariants on the level $k + 1$. The fibers of the G-action are affine over $\mathcal{E}''_k$. Consequently the differential invariants of order $\le k + 1$ are generated by the G-invariant functions affine in pure $(k + 1)$-jets, i.e. on the fibers $F(a_k) \cap T_{a_{k+1}}\mathcal{E}_{k+1}$.

By the assumption map (3) is epimorphic, and so the set $\{\nabla_i(\mathcal{A}_k)\}_{i=1}^s$ contains the maximal possible number of functions affine in pure $(k + 1)$-jets (the same amount as in $\mathcal{A}_{k+1}$). Thus any differential invariant $g \in \mathcal{A}_{k+1}$ can be written as $g = \sum_{i=1}^r h_i \nabla_i(g_i) + h_0$, where $g_i \in \mathcal{A}_k$ and h_j are some functions on k-jets. Now independence of $\nabla_i(g_i)$ in pure $(k + 1)$-jets implies that g is a differential invariant iff $h_j \in \mathcal{A}_k$.

Notice that expression of g is global in the jets of order $\ge l$ due to affine behavior. Finally we remark that the commutators of the invariant derivations can decompose

$$[\nabla_i, \nabla_j] = \sum \varrho_{ij}^k \nabla_k$$

(in the case $s < n$ we have to take the expression modulo $\alpha^\vee$ if we consider this as the equality of horizontal fields, but considered as equality of derivations it holds as it is), where ϱ_{ij}^k are differential invariants of order $\le l + 1$. Therefore any composition $\nabla_{i_1} \cdots \nabla_{i_q}(f_j)$ can be algebraically expressed via $\nabla_J(f_j)$.

We conclude that the set $\{\nabla_J(f_j) : |J| \le k-l\}$ contains a maximal collection of functionally independent differential invariants on $\mathcal{E}_k$ (some of the invariants from the set can be omitted due to differential syzygies among them).

By the implicit function theorem any other smooth differential invariant from $\mathcal{A}_k$ can be expressed as a function of these in a neighborhood U_k^β in the space of jets. This neighborhood is cylindrical over $U_l^\beta \subset \mathcal{E}_l$, but in general we cannot make it G-invariant. We need no more than a countable collection of such U_l^β to cover $\mathcal{E}''_l$. □

An alternative approach via investigation of the growth of dimensions of stabilizers and vanishing of the corresponding cohomology is contained in [KL$_1$].

Remark 6. *In the proof we have chosen a collection $f_1, \dots, f_t$, which taking into account the invariant derivations $\nabla_1, \dots, \nabla_s$ is often excessive. There always exists a minimal collection of differential invariants and invariant derivations generating the algebra $\mathcal{A}$.*

2.5. Asymptotic of the dimensions

The micro-local Lie-Tresse theorem implies formulas for growth of the dimensions of the space of differential invariants (there are two different versions – micro-local and the global, but the qualitative conclusion is the same).

Recall that $\dim \mathfrak{d}_k$ counts the number of differential invariants of pure order k (maximal number of functionally independent invariants of order $\le k$ modulo those that have order $< k$) in a neighborhood of a regular point. The Hilbert function (dimension being calculated at a regular point)

$$P_G^{\mathcal{E}}(k) = \dim \mathfrak{d}_k$$

is a polynomial for large $k \gg 1$ by virtue of Proposition 14. We extend it to the polynomial $P_G^{\mathcal{E}}(t)$, $t \in \mathbb{C}$. From the exact sequence (2) we conclude that

$$P_G^{\mathcal{E}}(t) = P_{\mathcal{E}}(t) - P_{\mathcal{R}}(t), \tag{4}$$

where the right hand side contains the Hilbert polynomials of the equations $\mathcal{E}$ and $\mathcal{R}$.

Remark 7. *Following [KL$_1$] we can write the Hilbert polynomial for differential invariants via the equation G for the pseudogroup and the equation $\mathcal{H}$ for the stabilizer:*

$$P_G^{\mathcal{E}}(t) = P_{\mathcal{E}}(t) - P_G(t) + P_{\mathcal{H}}(t),$$

Let $d_{\mathcal{E}}$ and $d_{\mathcal{R}} \le d_{\mathcal{E}}$ be the dimensions of the affine characteristic varieties of $\mathcal{E}$ and $\mathcal{R}$ and $c_{\mathcal{E}}$, $c_{\mathcal{R}}$ be their degrees. Then (4) implies

$$P_G^{\mathcal{E}}(t) = (c_{\mathcal{E}}\, t^{d_{\mathcal{E}}} + \dots) - (c_{\mathcal{R}}\, t^{d_{\mathcal{R}}} + \dots)$$

and in particular $P_G^{\mathcal{E}}$ is a polynomial of degree $d \le d_{\mathcal{E}}$ (the latter estimate explains why we can restrict to $\le d$ invariant derivations).

This implies rationality of the Poincaré function, answering the question of Arnold [A, Problem 1994-24] (cf. [Sar] for the un-constrained case):

Theorem 16. *For transitive actions the Poincaré series $\sum_{k=0}^{\infty} \dim \mathfrak{d}_k \cdot z^k$ equals $\dfrac{R(z)}{(1-z)^{d+1}}$ for some polynomial $R(z)$ and the same number d as above.*

3. Global results on differential invariants

Let us now prove finiteness (in Lie-Tresse sense) for the algebra of (usually infinite) rational-polynomial differential invariants.

3.1. Stabilization of singularities and separation of orbits

Since the action is transitive, all orbits project onto M. So to study the space of orbits on the level of k-jets $\mathfrak{O}^k$ it is enough to restrict to one fiber $\mathcal{E}_a^k$ and the action of G_a^k on it.

Moreover the algebraic manifold $\mathcal{E}_a^k$ is irreducible for all k. Indeed it is obtained by the prolongation (we assumed $\mathcal{E}$ formally integrable), so it is an affine bundle over $\mathcal{E}_a^{k-1}$ and we can apply the induction. Our original assumption is that the real algebraic variety $\mathcal{E}_a^k$ is irreducible, but this implies that its complexification is irreducible as a complex algebraic variety.

The argument for the last statement is as follows. Consider the real ideal $I(\mathcal{E}^k_a) \subset S(\mathbb{R}^N)^*$ (where $N = \dim J^k_a$) corresponding to $\mathcal{E}^k_a$. The corresponding complex ideal $I(\mathcal{E}^k_a) \otimes \mathbb{C} \subset S(\mathbb{C}^N)^*$ has the zero-locus ${}^{\mathbb{C}}\mathcal{E}^k_a \subset {}^{\mathbb{C}}J^k_a$, which is the complexification of $\mathcal{E}^k_a$. This complex variety is irreducible if $I(\mathcal{E}^k_a) \otimes \mathbb{C}$ is prime, i.e. the algebra $S(\mathbb{C}^N)^*/(I(\mathcal{E}^k_a) \otimes \mathbb{C})$ is a field. Assume that $F \cdot G = 0 \mod (I(\mathcal{E}^k_a) \otimes \mathbb{C})$ for two complex polynomials F, G. Multiplying this by $\bar{F}$ and restricting to the real variety $\mathcal{E}^k_a$ we get $|F|^2 \cdot G = 0$ along $\mathcal{E}^k_a$. By real irreducibility this yields the required alternative $F = 0$ or $G = 0$.

Thus by the Rosenlicht's theorem [R_1, R_2], the algebraic action has a geometric quotient outside a G-stable Zariski closed non-dense set S_k: $\bigl(\mathcal{E}^k \setminus S_k\bigr)/G^k \simeq Y_k$. In fact, while in the original paper the field of definition is allowed to be $\mathbb{R}$, in other sources [PV, SR] the standing assumption is that the field is algebraically closed. However the statement over $\mathbb{C}$ implies the one over $\mathbb{R}$ via the complexification argument.

Indeed, since the action is real, the invariants for the complexification can be chosen rational functions with real coefficients. If they form a coordinate system near a regular orbit in the complexification, they will also form a coordinate system when restricted to the totally real part (a Zariski open set of which consists of regular orbits by analytic continuation argument).

The above conclusion holds for all $k \le l$, where l is the integer from Theorem 7. We can also assume that l exceeds the maximal order of $\mathcal{E}$, the involutivity level from Proposition 14 and the number from the end of §2.1.

Then vanishing of the Spencer cohomology implies that over the regular points $\mathcal{E}^l \setminus S_l$ the fibers and the orbits have constant dimensions, and so the quotient $Y_{l+1} \to Y_l$ is an affine bundle. Similarly, $Y_{k+1} \to Y_k$ is an affine bundle for all $k \ge l$.

In particular, there are no singularities over the set of regular points $\mathcal{E}^l \setminus S_l$. The singular orbits belong to the stratum $\pi^{-1}_{\infty,l}(S_l) \subset \mathcal{E}_\infty$ and so have finite codimension. The variety S_l is the usual locus of singularities for algebraic actions, see [SR].

Since the differential invariants can be chosen affine in the fibers of $Y_{k+1} \to Y_k$, $k \ge l$, they definitely separate the orbits in higher jets provided the orbits of G^l_a-action in $\mathcal{E}^l_a$ are separated by the rational invariants. But this latter is guaranteed by the Rosenlicht's theorem (we again use separation of complex generic orbits by complex rational invariants, and then take real and imaginary parts of the latter to separate the real orbits). Actually, the field of rational differential invariants $\mathfrak{P}_k(\mathcal{E})^G$ coincides with the field of rational functions on the quotient variety $\mathfrak{R}(Y_k)$.

This proves Theorem 1 and the first part of Theorem 2.

3.2. Global Lie-Tresse theorem

Let us now give the proof of the main result. We let l be the order from which both the equation $\mathcal{E}$ and the equation $\mathcal{R}$ are involutive (in particular for $k \ge l$ the cohomology $H^{k,*}(\mathfrak{d})$ for differential invariants vanish, see §2.4).

Since the action is transitive, the differential invariants of order $k \leq l$ are bijective with the usual algebraic invariants of the prolonged action G_a^k on $\mathcal{E}_a^k$ for any point $a \in M$.

Therefore the field of rational differential invariants $\mathfrak{P}_l(\mathcal{E})^G \cap \mathcal{A}_k$ consists of rational algebraic invariants of the k-prolonged action to the stalk over a. The latter can be interpreted as a field of rational functions on $\bar{Y}_k = \mathcal{E}_a^k // G_a^k$. Thanks to Rosenlicht's theorem, the latter categorical quotient can be identified with geometric quotient.

Since the orbifold $\bar{Y}_k$ is finite-dimensional, the transcendence dimension of $\mathfrak{P}_l(\mathcal{E})^G \cap \mathcal{A}_k$ is finite. For $k = l$ denote it by $(t-1)$. There exist rational differential invariants $I_1, \dots, I_{t-1}$ of order l generating a subfield in $\mathfrak{P}_l(\mathcal{E})^G \cap \mathcal{A}_l$ such that any other element is obtained by a (finite) algebraic extension. By the theorem on primitive element there exists an element I_t such that $I_1, \dots, I_{t-1}, I_t$ rationally generate the whole field $\mathfrak{P}_l(\mathcal{E})^G \cap \mathcal{A}_l$.

From §2.5 we know that dimension of the space of the symbols of differential invariants grows polynomially $P_G^{\mathcal{E}}(k) = c\,k^d + \dots$, where d does not exceed the dimension of the characteristic variety $d_{\mathcal{E}}$. Since the action (together with its prolongations) is algebraic this quantity $P_G^{\mathcal{E}}(k)$ equals to the transcendental dimension of the field of rational differential invariants on $\mathcal{E}_k$. However, for $k \geq l$ the differential invariants from $\mathfrak{P}_l(\mathcal{E})^G \cap \mathcal{A}_k$ are not rational functions but polynomial by coordinates in the fiber $\pi_{k,l}^{-1}(a_l)$.

According to §2.3 there exist $s \geq d_{\mathcal{E}}$ invariant derivations, where $s = \operatorname{codim} \alpha^\vee$ is the number from Theorem 13. By the construction they can be taken rational (see also §3.3), and so they define the derivations $\nabla_1, \dots, \nabla_s$ of the rational-polynomial algebra $\mathfrak{P}_l(\mathcal{E})^G$ of differential invariants.

By the arguments of the previous section ∇_i map the G-invariant functions from $\mathcal{E}_k \setminus \pi_{k,l}^{-1}(S_l)$ to the G-invariant functions from $\mathcal{E}_{k+1} \setminus \pi_{k+1,l}^{-1}(S_l)$, $k \geq l$.

For a multi-index J the iterated derivations map the algebra of differential invariants to itself $\nabla_J : \mathcal{A}_k \cap \mathfrak{P}_l(\mathcal{E})^G \to \mathcal{A}_{k+|J|} \cap \mathfrak{P}_l(\mathcal{E})^G$. We claim that the union of their images gives the right hand side for $k \geq l$. This will be proved by induction in k assuming $|J| = 1$.

By Proposition 14 we have vanishing of the cohomology $H^{k,*}(\mathfrak{d}) = 0$, $k \geq l$. In particular, the Spencer differential $\delta : \mathfrak{d}_{k+1} \to \mathfrak{d}_k \otimes \tau^*$ is injective, implying that the dual map is surjective:

$$\mathfrak{d}_k^* \otimes \tau \longrightarrow \mathfrak{d}_{k+1}^*.$$

Since the symbol v_i of the invariant derivation ∇_i is dual to $\delta_{v_i} : \mathfrak{d}_{k+1} \to \mathfrak{d}_k$, we conclude that at regular points the elements $v_i(\mathfrak{d}_k^*)$ span the space $\mathfrak{d}_{k+1}^*$.

This implies for $k \geq l$

$$\mathcal{A}_{k+1} \cap \mathfrak{P}_l(\mathcal{E})^G = \langle \pi_{k+1,k}^*(\mathcal{A}_k \cap \mathfrak{P}_l(\mathcal{E})^G), \nabla_i(\mathcal{A}_k \cap \mathfrak{P}_l(\mathcal{E})^G) | i = 1, \dots, s \rangle.$$

In other words, any differential invariant I of order $(k+1)$, which is rational in jets of order $\leq l$ and polynomial in jets in order $k \geq l$, can be represented as

$$I = \sum q_i \left(\prod I_{\beta_i}\right),$$

where $I_\beta = \nabla_{i_\beta}(g_\beta)$ and $g_\beta, q_i \in \mathcal{A}_k \cap \mathfrak{P}_l(\mathcal{E})^G$.

To see this let us split the affine coordinates in $F(a_k) \subset \mathcal{E}_{k+1}$ so: $p = (p', p'')$, where p' are expressible via the invariants $\nabla_i(\mathcal{A}_k \cap \mathfrak{P}_l(\mathcal{E})^G)$ affine on $F(a_k)$ and p'' are from the complement. Then we can express in a Zariski open set (local dependence implies the global one in $F(a_k)$ due to affine behavior): $I = \tilde{I}(\theta_x, \nabla_i(g_\alpha), p'')$, where θ_x are coordinates on $(\mathcal{E}_k)_x$.

Since $\tilde{I}$ is polynomial in p'' and no such term can occur on the symbolic level, it does not actually occur and we get: $I = \tilde{I}(\theta_x, \nabla_i(g_\alpha))$. Moreover this latter relation is polynomial in the derivatives of order $(k+1)$, so the coefficients are differential invariants from $\mathcal{A}_k \cap \mathfrak{P}_l(\mathcal{E})^G$.

Indeed, if $I = \sum h_\sigma J^\sigma$, where J is a collection of differential invariants of order $(k+1)$ and of type $\nabla_i(g_\alpha)$ affine in the highest jets, σ a multi-index and $h_{i\sigma}$ some functions on k-jets, then for a vector field $X \in \mathcal{G}$ prolonged to $\mathcal{E}$ we have:

$$L_{\hat{X}}(I) = \sum L_{\hat{X}}(h_\sigma) J^\sigma = 0,$$

which implies that $h_\sigma \in \mathcal{A}_k \cap \mathfrak{P}_l(\mathcal{E})^G$. This proves Theorem 2. □

Remark 8. *It is possible to decrease the number s of invariant derivations to $d = \deg P_G^{\mathcal{E}}$ as well as to decrease the number of differential invariants to t by allowing algebraic extensions in addition to rational functions.*

Thus one has an alternative: either to represent $\mathfrak{P}_l(\mathcal{E})^G$ with the minimal number of basic invariants and derivations, or to reduce S in size (by decreasing the locus of dependencies for generators of the algebra of differential invariants).

3.3. Invariant derivations and other geometric structures

The module $\mathfrak{D}(\mathcal{E}, \tau)$ over the algebra $\mathfrak{P}_l(\mathcal{E})^G$ is finite-dimensional, and we expect $\dim_{\mathfrak{P}_l(\mathcal{E})^G} \mathfrak{D}(\mathcal{E}, \tau) = s$ (where s is the number from Corollary 11).

It can be also obtained as the space of invariants of an algebraic action of the pseudogroup G. Indeed, consider the G^k-action on the bundle τ_k over $\mathcal{E}_k$, which we denote by $\mathcal{E}_k \times \tau$ (we can again consider the vertical parts over a point $a \in M$ only). It is clearly algebraic and by Rosenlicht's theorem [R$_1$, SR] there is a geometric quotient (the singularity locus $\hat{S}$ fibers over the singularity locus S for the differential invariants)

$$(\mathcal{E}^k \times \tau \setminus \hat{S})/G^k = (\mathcal{E}^k_a \times \tau_a \setminus \hat{S}_a)/G^k_a \simeq \hat{Y}_k.$$

There is a natural algebraic morphism $\hat{Y}_k \to Y_k$ and dimension of its fibers corresponds to the number of invariant derivatives on the level k.

Consequently we obtain the following statement (s is as in Corollary 11).

Theorem 17. *The module $\mathfrak{D}(\mathcal{E}, \tau)$ is generated by s rational invariant derivatives over the algebra $\mathfrak{P}_l(\mathcal{E})^G$.* □

Yet another construction of the Tresse type is available on the basis of §2.3, which we now enhance with Rosenlicht-like arguments. Consider the G^k action on the bundle τ_k^* over $\mathcal{E}$. Taking the annihilator $\alpha^\vee$ of the invariant spaces in τ_k^* and passing to the quotient $\tau/\alpha^\vee$ we get the space of rational invariant derivations $\mathfrak{Der}(\mathcal{E},\tau) \supset \mathfrak{D}(\mathcal{E},\tau)$.

For this (possibly bigger) space we also get finite generation property (but now with the number s given by Theorem 13).

Theorem 18. *The module $\mathfrak{Der}(\mathcal{E},\tau)$ is generated by s rational invariant derivations over the algebra $\mathfrak{P}_l(\mathcal{E})^G$.* □

Remark 9. *In particular we see that by dimensional reasons invariant derivations appear on the lower jet-level k than differential invariants – the fact familiar from the experience [OP, KL$_3$, Man] that finally gets explanation.*

Similar as invariant derivatives (and derivations) are finitely generated, we can consider other invariant tensors. For instance, invariant 1-forms form a module over the algebra $\mathfrak{P}_l(\mathcal{E})^G$ of rational-polynomial differential invariants (among them are exact 1-forms $\hat{d}f$, where f is a differential invariants). They correspond to the limit of the G^k-invariant forms τ_k^* over $\mathcal{E}_k$. We have already seen that this module is finitely generated in Lie-Tresse sense.

Similar we can prove finiteness of the module of q-forms for any q (an important example: closed $(n-1)$-forms, which are invariant conservation laws). More generally any tensorial module of fixed type, for instance $\bigl((\otimes^p\tau)\otimes(\otimes^q\tau^*)\bigr)^G$ with fixed p,q, is finitely generated.

What about the whole tensor algebra (all possible valencies)? Let us restrict, for instance, to $\sum_{p,q}((\otimes^p\tau)\otimes(\otimes^q\tau^*))^G$ (a combination of symmetric and skew-symmetric powers is also possible).

Is such a module finitely generated in Lie-Tresse sense? We do not know the answer in general, but understand the generic situation.

Theorem 19. *Let $\mathcal{E}$ be ample (for instance the whole space of jets), and the number of scalar differential invariants of the pseudogroup G action be infinite. Then the differential tensors on $\mathcal{E}_\infty$ invariant with respect to G are finitely generated over the algebra $\mathfrak{P}_l(\mathcal{E})^G$.*

Proof. Indeed, by Theorem 12 there are n invariant derivatives $e_i = \nabla_i$, so by dualization there are n invariant 1-forms $e^i = \omega^i$. We can decompose any (p,q)-tensor through these

$$T = T^{j_1\dots j_p}_{i_1\dots i_q} e_{j_1}\otimes\dots\otimes e_{j_p}\otimes e^{i_1}\otimes\dots\otimes e^{i_q}.$$

The coefficients are scalar differential invariants iff T is an invariant differential tensor. Thus the claim follows from the global Lie-Tresse theorem. □

In a similar manner we can prove finiteness of other natural geometric objects: lifted connections, higher order differential operators etc.

3.4. Differential syzygies etc

The relations in the algebra $\mathfrak{P}_l(\mathcal{E})^G$ (generated by invariants and derivations) are called *differential syzygies*. They have the form $\sum f_i \nabla_{J_i}(g_i) = 0$, where f_i, g_i are differential invariants and ∇_{J_i} are iterated invariant derivations.

To our knowledge the first result on finiteness (and rationality) of differential syzygies (for free un-constrained actions of the pseudogroups) is due to [OP]. We generalize it.

Theorem 20. *The module $\mathfrak{syz}_\infty^G$ of differential syzygies over the algebra $\mathfrak{P}_l(\mathcal{E})^G$ is finite-dimensional.*

Proof. The syzygies are controlled by the same symbolic sequence for differential invariants (in fact by the dual) as in §2.4

$$0 \to \mathfrak{d}_k \to \mathfrak{d}_{k-1} \otimes \tau^* \to \mathfrak{d}_{k-2} \otimes \Lambda\tau^* \to \dots$$

but now the finite-dimensionality is equivalent to vanishing of the first cohomology $H^{k,1}(\mathfrak{d}) = 0$, $k \geq l$ (the group $H^{*,0}(\mathfrak{d})$ is responsible for finiteness of the differential invariants).

By this we mean that the symbols of the differential syzygy is given by the symbolic syzygy module (encoded in the sequence), and the tails are reconstructed recursively.

Since we established vanishing of the cohomology in the stable range in Proposition 14, we can use the arguments from the beginning of the section to conclude the claim. □

Remark 10. *We can interpret Theorem 20 as the statement that the algebra of differential invariants is an equation with dependent variables $I_1, \dots, I_t$, derivations $\nabla_1, \dots, \nabla_s$ (if these are Tresse derivatives we can talk of independent and dependent variables) and the relations being the syzygies.*

Treating the latter as differential equations on I_j, the finite generation property becomes an invariant version of the Cartan-Kuranishi theorem.

In the same way we can establish finiteness of higher G-invariant syzygies (i.e. relations between relations etc). Notice that the second syzygies can be interpreted as the module of compatibility conditions for the invariant differential equations from Remark 10.

We can also study relations in the $\mathfrak{P}_l(\mathcal{E})^G$-module $\mathfrak{Der}(\mathcal{E}, \tau)$ of the type $\sum f_i \nabla_i = 0$, but these are easily seen to be finitely generated over $\mathfrak{P}_l(\mathcal{E})^G$.

Another possibility is to consider the relations in the Lie algebra $\mathfrak{Der}(\mathcal{E}, \tau)$. This latter module of differential syzygies for invariant derivations is also finitely generated.

Indeed it is generated by the commutator relations of the generators (Maurer-Cartan equations for the pseudogroup) and the differential syzygies $\mathfrak{syz}_\infty^G$.

4. Examples and counter-examples

In this section we give new examples illustrating importance of our assumptions, and discuss how to calculate the algebra of scalar differential invariants.

4.1. On calculation of differential invariants

There are two approaches to calculate differential invariants of a pseudogroup action. Both are essentially microlocal, as one uses the Lie algebra sheaf while the other the local pseudogroup (a germ of unity).

The first method is related to solution of the Lie equation $L_{\hat{X}}(I) = 0$, where $X \in \mathcal{G}$ is the element of the Lie algebra sheaf and I a function on $\mathcal{E}$. This is a linear PDE, and in some cases with additional symmetry it can be effectively solved (one does not need to find all solutions, but only some solutions of lower degree, as the Lie-Tresse theorem will yield then the rest). But in general to find the solutions is a difficult task.

The other approach is to use parametrization of the pseudogroup (the method of moving frames) and to exclude the parameters along the orbit, to obtain the expressions of differential invariants. The calculations here are more algorithmic as the action is affine in the highest derivatives, but there are serious restrictions of the method: the group G shall be nicely parametrized (which is not the case, for instance, with the symplectic and contact pseudogroups), and the action has to be locally free (the standard technical assumption).

To calculate the global rational-polynomial invariants any combination of these methods can be applied, but more important shall be the methods from the invariant theory. Indeed, the action of G prolonged to the k-jets, namely that of G_a^k on $\mathcal{E}_a^k$, is an algebraic action of a Lie group on an algebraic manifold. One searches for rational invariants of this action.

This is still a difficult task, but much more feasible. In a forthcoming publication we will explain how the use of relative invariants further simplifies this problem and makes its solution algorithmic. In a similar way one finds invariant derivations, and then the Lie-Tresse theorem finishes the story (the symbolic calculus helps finding the number of necessary invariants and derivations).

4.2. Examples of calculations

As we mentioned in the Introduction most of the classical examples are related to calculation of micro-local differential invariants. To establish equivalence it is necessary that they are preserved, but it is not always sufficient.

The first non-trivial equivalence result involving an infinite pseudogroup was the classification of 2nd order ODEs with respect to point transformations. This was initiated by S.Lie and R.Liouville and essentially finished by A.Tresse, see [Tr$_2$] and [Kr]. In the latter reference the invariants are written via rational generators, and this implies solution of the global equivalence problem (for non-singular ordinary differential equations).

Differential invariants for some other problems involve algebraic roots, and so can be re-written via rational generators. For instance the celebrated Cartan's 5-variables paper [C_2, p.170] contains a relative invariant I such that only I^4 is a bona fide differential invariant. More on this will be said in the next section.

In some purely algebraic problems differential invariants turns out useful, see [Ol_2]. It is important to use global invariants (as algebraic classifications are always global). On this way the classical equivalence problem for binary and ternary forms was solved recently [BL].

Let us discuss some other examples illustrating our conditions and methods.

Example. Consider the flux equation $\mathcal{E}$

$$u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0.$$

The group $G = \mathrm{SL}_3 \times \mathrm{Diff}(\mathbb{R})$ acts on $\mathcal{E}$ by symmetries, where the first factor corresponds to projective transformations of $B = \mathbb{R}^2(x, y)$ and the second factor corresponds to invertible changes of the dependent variable $u \mapsto U(u)$.

Here the characteristic variety at the point $a_1 = (x, y, u, u_x, u_y)$ (since the equation is quasi-linear $\mathrm{Char}^{\mathbb{C}}(\mathcal{E}, a_2) \subset {}^{\mathbb{C}}\mathbb{P}T^*B$ does not depend on a choice of $a_2 \in F(a_1)$) is given by the linear equation

$$\mathrm{Char}^{\mathbb{C}}(\mathcal{E}, a_1) = \{[p_x : p_y] \,|\, u_y p_x = u_x p_y\} \subset \mathbb{C}P^1.$$

However even though this variety belongs to a hypersurface (in our case: a point), there is no such horizontal direction at $a_0 = (x, y, u)$ that annihilates all characteristics at all points $a_1 \in F(a_0)$. And this implies existence of 2 invariant derivatives.

To simplify the argument let us change G to its second component $G_2 = \mathrm{Diff}(\mathbb{R})$. Then the invariant derivatives are $\mathcal{D}_x, \mathcal{D}_y$ and the basic differential invariant is $w = u_x/u_y$ with the only relation (differential syzygy which is the flux equation written via w) – the equation of gas dynamics

$$w_x = w\, w_y. \tag{5}$$

To include the first component $G_1 = \mathrm{SL}_3$ we re-write the Lie algebra of the first component via w. Here are the generators of $\mathfrak{g}_1 = \mathrm{Lie}(G_1)$:

$$\partial_x,\ \partial_y,\ x\,\partial_x - w\,\partial_w,\ y\,\partial_y + w\,\partial_w,\ x\,\partial_y - \partial_w,\ y\,\partial_x + w^2\partial_w,$$
$$x^2\partial_x + xy\,\partial_y - (xw + y)\,\partial_w,\ xy\,\partial_x + y^2\partial_y + (xw^2 + yw)\,\partial_w.$$

Thus we have an action of a finite-dimensional Lie group on the equation (5) and the validity of Lie-Tresse theorem in such situation is known.

For completeness let us provide the exact formulae for differential invariants. They are micro-locally generated by

$$I_6 = \frac{9w_2^3 w_6 - 200w_3^4 - 72w_2^2 w_3 w_5 + 300w_2 w_3^2 w_4 - 45w_2^2 w_4^2}{R^{4/3}}$$

and

$$\nabla_1 = \frac{6w_2}{R^{1/3}}\mathcal{D}_y - \frac{w_2w_4 - (4/3)w_3^2}{w_2^2 R^{1/3}}(\mathcal{D}_x - w\mathcal{D}_y)$$

where $R = 9w_2^2w_5 - 45w_2w_3w_4 + 40w_3^3$ and $w_i = \mathcal{D}_y^i(w)$.

The module of invariant derivations is generated by ∇_1 and

$$\nabla_2 = \frac{R^{1/3}}{w_2^2}(\mathcal{D}_x - w\mathcal{D}_y)$$

The last derivative is trivial in the sense that it does not generate new invariants (derivation along the Cauchy characteristic). It is interesting to note that $\nabla_1(\operatorname{mod}\nabla_2)$ coincides with the classical Study derivative from the theory of curves on the projective plane.

The global rational differential invariants are generated by I_6^3 and $I_6^{-1}\nabla_1$.

Example. Consider the natural action of the pseudogroup $G = \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}$ on $\mathbb{R}^3 = \mathbb{R}^2(x,y) \times \mathbb{R}^1(u)$. The corresponding LAS $\mathfrak{g}$ is transitive and has generators

$$f(x)\,\partial_x,\ \partial_y, \partial_u.$$

For the action of G on un-bounded jets $J^\infty(\mathbb{R}^2)$ there are two invariant derivatives $\frac{1}{u_x}\mathcal{D}_x, \mathcal{D}_y$ and the basic differential invariant u_y, so that the algebra of differential invariants $\mathfrak{P}_1^G = \langle u_y, u_{xy}/u_x, u_{yy}, \dots\rangle$ is generated by them.

The differential equation $\mathcal{E} = \{u_x = 0\} \subset J^\infty(\mathbb{R}^2)$ is G-invariant, and the global Lie-Tresse theorem applies. But for the action on $\mathcal{E}$ there is only one invariant derivative $\mathcal{D}_y$ (together with its multiples by differential invariants). The algebra of differential invariants $\mathfrak{P}_1(\mathcal{E})^G = \langle u_y, u_{yy}, u_{yyy}, \dots\rangle$ is freely generated by $\mathcal{D}_y$ and u_y.

Remark 11. *This example shows that the number of invariant derivatives needs not to be $n = \dim\tau$.*

On the other hand there can be more invariant derivatives than one can expect from characteristic variety of $\mathcal{E}$.

Indeed, if we consider the subgroup $\mathbb{R}^3 \subset G$ of translations, then $\mathcal{D}_x$ is an invariant derivative on $\mathcal{E}$, but it acts trivially on $\mathcal{A}$.

Example. Consider now a bigger pseudogroup G on $\mathbb{R}^3 = \mathbb{R}^2(x,y) \times \mathbb{R}^1(u)$ with the transitive LAS $\mathfrak{g}$ given by

$$f(x,y)\,\partial_x,\ \partial_y, \partial_u.$$

This algebra acts transitively in the complement to the equation $\mathcal{E} = \{u_x = 0\}$, so there are no differential invariants. There are however invariant horizontal fields in accordance with Theorem 12 (of course it is not possible to find them neither as derivations, as the latter are trivial, not as Tresse derivatives). They form a 2-dimensional commutative Lie algebra (indeed, any third field should be a linear combination of these with coefficients being differential invariants)

$$\mathfrak{D}(J^\infty,\tau)^G=\Bigl\langle \frac{1}{u_x}\mathcal{D}_x,\mathcal{D}_y-\frac{u_y}{u_x}\mathcal{D}_x\Bigr\rangle.$$

Restriction to $\mathcal{E}$ yields non-trivial algebra of differential invariants $\mathfrak{P}_1(\mathcal{E})^G=\langle u_y,u_{yy},u_{yyy},\dots\rangle$, it is generated almost in the same way as above, except that we shall take $\mathcal{D}_y \mod\langle\mathcal{D}_x\rangle$ as an invariant derivation. This is the only invariant derivation up to multiplication by a differential invariant, $\mathfrak{Der}=\langle[\mathcal{D}_y]\rangle$.

But it is invariantly defined only as the equivalence class – there are no invariant horizontal vector fields at all $\mathfrak{D}(\mathcal{E},\tau)=0$!

4.3. Roots in differential invariants

By our main theorem we can use only rational functions in description of global differential invariants. However books are full of examples of differential invariants with algebraic roots, see e.g. [Th, Ol_1, Ol_2, KJ, KL_3, Man].

This visible contradiction can be resolved in two ways. First of all many of the wide-spread invariants with roots are not invariants in the global sense, but only micro-local invariants. Equivalently they are invariants of the Lie algebra (satisfy the linear PDE $L_X(I)=0$) or invariants of the local Lie group (where the size of the neighborhood of unity in $U\subset G$, for which $g^*I(a_\infty)=I(a_\infty)$ $\forall g\in U$, depends on the point $a_\infty\in\mathcal{E}_\infty$). See the example from the Introduction.

Another possibility is that the roots are in proper place, but the classification deals with coverings in global problem like orientations or spin. Action of quotients of algebraic groups which are not themselves algebraic can lead to similar problems. Let us explain appearance of roots in two examples.

Example. Consider the action of the proper motion group $E(2)_+=SO(2)\ltimes\mathbb{R}^2$ on the space of oriented curves in $M=\mathbb{R}^2$. The action lifts to the space of jets $J^\infty(M,1)$. We can chose local coordinates on M such that dx gives positive orientation and dy positive co-orientation on the graph-curves $y=y(x)$. These induce canonical coordinates $(x,y,y_1,y_2,\dots)$ in the open chart $J^\infty(\mathbb{R},\mathbb{R})\subset J^\infty(M,1)$, and we write the curvature and the invariant derivation

$$K=\frac{y_2}{(1+y_1^2)^{3/2}},\qquad \frac{d}{ds}=\frac{1}{\sqrt{1+y_1^2}}\frac{d}{dx}. \tag{6}$$

Both are invariant with respect to $E(2)_+$ and they generate the algebra of differential invariants.

If the curves are not orientable or we act by the whole motion group $E(2)=O(2)\ltimes\mathbb{R}^2$ (both conditions imply that the curves are not co-orientable), then K is not an invariant and we shall change the above pair of generators to

$$K^2=\frac{y_2^2}{(1+y_1^2)^3},\qquad \nabla=K\frac{d}{ds}=\frac{y_2}{(1+y_1^2)^2}\frac{d}{dx}. \tag{7}$$

It seem that (7) satisfies but (6) contradicts our version of Lie-Tresse theorem. Of course, (7) are also rational invariants of $E(2)_+$ action, but they do not distinguish orbits (for instance upper and lower half-circles both oriented from left to right). What is the problem?

To understand this consider the stabilizer in $S^1 = SO(2) \subset E(2)_+$ of the point $(0,0) \in M$. It acts on the fiber of the projection $J^2 \to M$, which is $S^1 \times \mathbb{R}$ in the case of oriented curves and is $\mathbb{R}P^1 \times \mathbb{R}$ in the case of non-oriented curves. The action is $(\phi, h) \mapsto (\phi + t, h)$ and so $h \in \mathbb{R}$ is (a rational) invariant.

However this is the case of the jet-space $J^2(M,1)$, while in the affine chart the action writes $(y_1, y_2) \mapsto (\frac{y_1 \cos t - \sin t}{y_1 \sin t + \cos t}, \frac{y_2}{(y_1 \sin t + \cos t)^3})$, $t \in S^1$. The affine chart consists of one piece $\mathbb{R}P^1 \setminus \{1\}$ in the non-oriented case, but it has two pieces $S^1 \setminus \{\pm 1\}$ in the oriented case. Since the two half-circles constitute a reducible variety, there is no contradiction with our Lie-Tresse theorem. In fact on one component (that's to say e.g. dx gives the orientation) we can use the rational generators (7) and they do separate orbits.

Example. Another occasion of roots is the classical algebraic problem of characterizing quadrics on the plane:

$$u = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0. \tag{8}$$

The group $G = E(2) \times \mathbb{R}^*$ acts on the plane (x, y) by the first component, and on the function u (quadric) by the second (rescalings). The invariants are (in [PV] the cubic roots of these are given, but then they are ill-defined over $\mathbb{C}$)

$$I_1 = \frac{(ac - b^2)^3}{\Delta^2},\ I_2 = \frac{(a + c)^3}{\Delta}, \qquad \text{where } \Delta = \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}.$$

We can obtain the invariants also through the action of G on the Monge equation $\mathcal{E}$ characterizing quadrics (as functions $y = y(x)$)

$$V = y_5 y_2^2 - 5 y_2 y_3 y_4 + \tfrac{40}{9} {y_3}^3 = 0.$$

Remark 12. *This V together with $U = y_2$ are the basic projective relative differential invariants of the curves on the projective plane. The basic absolute invariant $R \cdot V^{-8/3}$ (with $R = U^4 V \cdot y_7 + \dots$ a differential polynomial of order 7 [H], see also [KL$_3$]) in this case contains the cubic root, but can be changed to the global rational differential invariant R^3/V^8.*

Indeed, the action has 3 (micro-local) differential invariants (they contain roots – which can be eliminates as above – but will be of temporal use)

$$K = \frac{y_2}{(1 + y_1^2)^{3/2}},\ K' = \frac{d}{ds}K,\ K'' = \frac{d^2}{ds^2}K; \qquad \frac{d}{ds} = \frac{1}{\sqrt{1 + y_1^2}} \frac{d}{dx}.$$

They satisfy the following differential syzygy (our equation $\mathcal{E}$)

$$K''' = 5\,\frac{K' K''}{K} - \frac{40}{9} \frac{K'^3}{K^2} - 4K^2 K'$$

with the solution given implicitly $s = \int (C_1 K^{8/3} + C_2 K^{10/3} - 9K^4)^{-1/2} dK$. Extracting constants (first integrals) from this expression we get two invariants of the solution space $\mathrm{Sol}(\mathcal{E})$:

$$j_1 = \frac{3KK'' - 5K'^2 + 9K^4}{K^{8/3}}, \quad j_2 = \frac{3KK'' - 4K'^2 + 18K^4}{K^{10/3}}.$$

Expressed in jet-coordinates they equal

$$j_1 = \frac{3y_2 y_4 - 5y_3^2}{y_2^{8/3}}, \quad j_2 = \frac{3y_1^2 y_2 y_4 - 4y_1^2 y_3^2 - 6y_1 y_2^2 y_3 + 9y_2^4 + 3y_2 y_4 - 4y_3^2}{y_2^{10/3}}.$$

The bona fide rational invariants of this algebraic problem are $J_1 = j_1^3$, $J_2 = j_2^3$ (they separate orbits and correspond to the previous invariants I_1, I_2, when y is expressed through x from (8)), while roots are temporal results of calculations.

4.4. Non-algebraic situation

If we drop the requirement of algebraic action, some of the results continue to hold. Namely the prolonged action is affine and so algebraic in higher jets. Thus the finite-generation property will hold over a neighborhood in finite jets, where we have the property for the corresponding Lie group action.

However the separation property for the orbits can fail, so that the algebra of differential invariants will distinguish between only the closure of the orbits.

Example. Consider the action of the 5-dimensional group G on $M = \mathbb{R}^4(x^1, x^2, x^3, x^4)$ with the Lie algebra $\mathfrak{g} = \langle \partial_{x^1}, \dots, \partial_{x^4}, \xi \rangle$, where $\xi = (x^2 \partial_{x^1} - x^1 \partial_{x^2}) - \lambda (x^4 \partial_{x^3} - x^3 \partial_{x^4})$ and λ is a generic irrational number.

The action is transitive and the stabilizer group G_x is one-dimensional. We embed G into the diffeomorphism group of $\mathbb{R}^5(x^1, x^2, x^3, x^4, u)$ or equivalently let G act on the space $J^\infty(M)$ of jets of functions on M. The space $J_x^k(M)$ has coordinates u_σ with the multi-index σ of length $|\sigma| \le k$.

The first of them $u = u_\emptyset$ is obviously an invariant of G. On the jet-level $k = 1$ we have 3 more differential invariants: $I_1' = u_1^2 + u_2^2$, $I_1'' = u_3^2 + u_4^2$ and $I_1''' = \arctan(u_4/u_3) + \lambda \arctan(u_2/u_1)$. The last invariant I_1''' is however micro-local and has to be omitted in the list of global invariants.

Geometric explanation of the absence of one invariant in the global sense is the following: $T^* = \mathbb{R}^4 = \mathbb{R}^2(u_1, u_2) \times \mathbb{R}^2(u_3, u_4)$ has an invariant foliation by tori which are the product of concentric circles in the factors, and the orbit of ξ on every torus is an irrational winding with the slope λ, whence the closure of almost every orbit is 2-dimensional.

Thus already on this step we observe that the algebra of micro-local invariants differs from the algebra of global differential invariants. Also notice that the omitted non-global invariant I_1''' is not a rational function and this is in correspondence with the fact that the field $\xi \in \mathfrak{g}$ is not a replica in the sense of Chevalley [Ch] and G is not algebraic.

Consider the prolongations $\xi^{(k)}$ of the last vector field of $\mathfrak{g}$. This field represent the action of G in $J_x^k = \oplus_0^k S^k T^*$, where $T = T_x M$. It is semi-simple, and every summand is invariant. Restriction to T has the purely imaginary spectrum $\mathrm{Sp}(\xi^{(1)}|T) = \{\pm i, \pm\lambda\, i\}$, and consequently the spectrum on S^jT is the j-multiple Minkowski sum of $\mathrm{Sp}(\xi^{(1)}|T)$ with itself (elements in the sum enter with multiplicity): $\mathrm{Sp}(\xi^{(k)}|J^k_x) = \{\pm s\, i \pm \lambda\, t\, i \,|\, 0 \le s, t \le s + t \le k\}$ (some elements enter with multiplicity).

In complex coordinates $z = x_1 + ix_2$, $w = x_3 + x_4$ the first order invariants are $I_1' = |u_z|^2$ and $I_1'' = |u_w|^2$. The higher order invariants are

$$I_{pqrs} = \frac{1}{u_z^p u_{\bar z}^q u_w^r u_{\bar w}^s} \frac{\partial^{p+q+r+s} u}{\partial z^p \partial \bar z^q \partial w^r \partial \bar w^s}, \quad p + q + r + s \ge 2$$

(to get real invariants one has to take the real and complex parts unless $p = q$, $r = s$; also we can let $p \le q$ and $r \le s$ if $p = q$). The algebra of differential invariants $\mathfrak{I}$ is generated by these, and has the following invariant derivatives (again to stay on the real side one separates the real and imaginary parts):

$$\nabla_z = \frac{1}{u_z}\mathcal{D}_z,\ \nabla_{\bar z} = \frac{1}{u_{\bar z}}\mathcal{D}_{\bar z},\ \nabla_w = \frac{1}{u_w}\mathcal{D}_w,\ \nabla_{\bar w} = \frac{1}{u_{\bar w}}\mathcal{D}_{\bar w}.$$

Then $\mathfrak{I}$ is finitely generated by $I_1', I_1'', I_{1100}, I_{1010}, I_{0011}$ and $\nabla_z, \nabla_{\bar z}, \nabla_w, \nabla_{\bar w}$ (the minimal set of generators is I_1', I_2' and $\nabla_z, \nabla_{\bar z}, \nabla_w, \nabla_{\bar w}$ since 6 of the 10 2nd order differential invariants occur as the coefficients of the commutators of the invariant derivatives).

Thus we see that the Lie-Tresse theorem holds, though the invariants from the algebra $\mathfrak{I}$ do not separate the G-orbits.

4.5. Singular systems

In classification of differential invariants we describe most of those that characterize regular orbits. The complement to regular set consists of singular orbits, but can be stratified into invariant equations $\mathcal{E}(\alpha)$, for each of which the G-action produces its own algebra of differential invariants. Provided the equation is regular and the action is transitive, the Lie-Tresse theorem applies in our global version, and the description of this algebra is finite.

However, some singularities can be more complicated, for instance if the projection of the equation to M is not surjective or the action of G is not transitive, and for them the theorem could fail. To illustrate this we discuss an example of conservative vector fields with isolated singularities.

Example. Consider the equivalence problem for Hamiltonian systems near a non-degenerate linearly stable equilibrium point (the point is isolated and we take it to be 0). In this equivalence problem the pseudogroup consists of local diffeomorphisms of the germ of 0.

Equivalently the problem is this: the pseudogroup G consists of germs of symplectic diffeomorphisms of $(\mathbb{R}^{2n}, \omega_0)$ preserving 0. Its subgroup acts on the equation $\mathcal{E}$ consisting of the germs of functions H vanishing at 0 to order 2 with the operator $A_H = \omega^{-1} d_0^2 H$ having purely imaginary spectrum. In addition, we assume that $\mathrm{Sp}(A_H) = \{\pm i\lambda_1, \dots, \pm i\lambda_n\}$ has no resonances of any order.

The equation is indeed singular, as it is projected to $J^1(\mathbb{R}^{2n})$ to one point, $\mathcal{E}\subset\pi_{2,1}^{-1}(0)$. But in addition, the condition of absence of resonances is non-algebraic (but the closure of $\mathcal{E}$ is semi-algebraic).

We change, as usual, germs to jets to get the formal normal form of H. This is the well-known Birkhoff normal form [B]: there are canonical coordinates (Darboux coordinates for the symplectic form ω_0) $x_1,y_1,\dots,x_n,y_n$ such that with the notations $\tau_i=x_i^2+y_i^2$, $\tau^\sigma=\tau_1^{\sigma_1}\cdots\tau_n^{\sigma_n}$, the infinite jet of H is

$$j_0^\infty(H)=\sum a_\sigma\tau^\sigma.$$

The coefficients a_σ are the differential invariants of the problem.

One can write them in jet coordinates, though the formulae are rather complicated. For instance, for n=1 the first invariant a_1 corresponds to the Hessian

$$I_2=H_{11}H_{22}-H_{12}^2$$

(we write x^1,x^2 instead of x_1,y_1 and denote derivatives by the subscripts). The next differential invariant is

$$\begin{aligned}I_4={}&(3H_{22}H_{2222}-5H_{222}^2)H_{11}^3+\big(-3H_{12}^2H_{2222}+(30H_{122}H_{222}-12H_{22}H_{1222})H_{12}\\&+6H_{22}^2H_{1122}+(-9H_{122}^2-6H_{112}H_{222})H_{22}\big)H_{11}^2+\Big(12H_{12}^3H_{1222}+\\&(-24H_{112}H_{222}+6H_{22}H_{1122}-36H_{122}^2)H_{12}^2+\big(-12H_{22}^2H_{1112}+(54H_{122}H_{112}\\&+6H_{222}H_{111})H_{22}\big)H_{12}+3H_{22}^2(-3H_{112}^2-2H_{122}H_{111}+H_{22}H_{1111})\Big)H_{11}\\&-12H_{12}^4H_{1122}+(36H_{122}H_{112}+12H_{22}H_{1112}+4H_{222}H_{111})H_{12}^3-\\&3H_{22}(12H_{112}^2+H_{22}H_{1111}+8H_{122}H_{111})H_{12}^2+30H_{22}^2H_{12}H_{112}H_{111}-5H_{22}^3H_{111}^2\end{aligned}$$

and we get 1 new invariant in any even order. Thus the Lie-Tresse derivative does not exist in this case (otherwise it would produce invariants in any sufficiently high order).

One might get an impression that there is an invariant 2nd order differential operator (which does the job of the usual invariant derivative in this case), and indeed the operator

$$\begin{aligned}\Delta={}&H_{22}\mathcal{D}_1^2-2H_{12}\mathcal{D}_1\mathcal{D}_2+H_{11}\mathcal{D}_2^2+\\&I_2^{-1}(-H_{11}H_{122}H_{22}+H_{12}H_{11}H_{222}-H_{22}^2H_{111}+3H_{12}H_{112}H_{22}-2H_{12}^2H_{122})\mathcal{D}_1\\&+I_2^{-1}(-2H_{12}^2H_{112}+3H_{12}H_{122}H_{11}+H_{111}H_{22}H_{12}-H_{11}H_{22}H_{112}-H_{11}^2H_{222})\mathcal{D}_2\end{aligned}$$

satisfies the identity

$$[\Delta,\hat{X}_F]|_{\mathcal{E}}=0$$

for any function $F\in C^\infty(\mathbb{R}^2)$ vanishing at 0 to the 2nd order (X_F is the Hamiltonian field with the generating function F).

Remark 13. *The symbol of Δ is canonical: using ω_0 it is identified with the operator A_H, and further raising of the indices yields the Hessian 2-form d_0^2H.*

However the restrictions of the fields from the pseudogroup $\hat{X}_F$ to $\mathcal{E}$ do no commute with Δ, and this is the reason this operator does not map $I_{2k} \to I_{2k+2}$.

Proposition 21. *There does not exist a differential operator which maps the algebra of differential invariants to itself.*

Proof. Let us discuss the case of the 2nd order, the general case is similar.

The space of differential operators of order $\le 2k$ is $\mathcal{J}_{2k} = C^\infty(I_2, \dots, I_{2k})$ (here we can equally well consider the spaces of smooth or rational or polynomial functions). If the operator exists it maps $\Delta : \mathcal{J}_{2k} \to \mathcal{J}_{2k+2}$.

In particular, for $k = 1$ we get the following. Let us decompose $\Delta = \Delta_2 + \Delta_1 + \Delta_0$, where Δ_i are differential operators of pure order i (working in coordinates such a splitting is possible). Then $\Delta(1) = \Delta_0$, so this latter is a differential invariant, and can be omitted. The symbol of $\Delta_2 = \sum a_{ij}\mathcal{D}_i\mathcal{D}_j$ is also an invariant, and so is a multiple of the operator from Remark 13: $a_{ij} = (-1)^{i+j}\lambda\, H_{3-i,3-j}$. Then we have:

$$\Delta f(I_2) = f'(I_2)\Delta(I_2) + f''(I_2)\sum a_{ij}\mathcal{D}_i(I_2)\mathcal{D}_j(I_2).$$

This implies that the third order expression $\sum a_{ij}\mathcal{D}_i(I_2)\mathcal{D}_j(I_2)$ is a differential invariant (in general, we shall also consider the case $\mathrm{ord}(\lambda) > 3$ but this leads to the same result by restricting to larger k), which is impossible unless $\lambda = 0$.

Thus $\Delta = \Delta_1$, but there are no invariant differential operators of order 1 because its symbol would have been invariant and hence zero (alternatively: it would have mapped $\mathcal{J}_{2k} \to \mathcal{J}_{2k+1} = \mathcal{J}_{2k}$). $\square$

For $n > 1$ the algebra of differential invariants of order $\le k$ is $\mathcal{J}_{2k} = C^\infty(I_2^{i_1}, I_4^{i_1 i_2}, \dots, I_{2k}^{i_1\dots i_k})$ ($1 \le i_j \le n$ numerate different differential invariants in different orders), and we get a similar result. Hence we conclude:

Corollary 22. *The space of differential invariants $\mathcal{J}_\infty$ of formal Hamiltonian systems at a non-degenerate linearly stable non-resonant equilibrium point is not finitely generated, even in a generalized Lie-Tresse sense.*

Let us also notice that the Poincaré function of this equivalence problem

$$P_G^{\mathcal{E}}(z) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^{2k} = \frac{1}{(1-z^2)^n}$$

does not satisfy the conclusion of Theorem 16. This is often the case with classification problems arising in singularity theory.

5. Conclusion

The structure of the algebra of differential invariants encodes the structure of the pseudogroup and its action. It plays the same role in differential geometry as the invariant theory in the classical algebra. With this in mind we can informally summarize the main results of the present paper as follows:

Hilbert and Rosenlicht theorems allow to treat the quotient of an algebraic variety by an algebraic group action as an algebraic variety. Our global version of the Lie-Tresse theorem allows to treat the quotient of a differential equation by an algebraic pseudogroup action as a differential equation.

The meaning of the latter is explained in Section 3.4.

Calculation of differential invariants leads one to find symmetric exact solutions, and even can be used to study complete solvability.

We did not touch in this paper the computational aspect of the theory. It uses several algebraic methods (including Gröbner basis and differential algebra, see e.g. [HK]). In practice symbolic packages are helpful. For some calculations in Section 4 we used Maple package *DifferentialGeometry* by I. Anderson.

Our examples were chosen simple to illustrate the power and limitation of the main results. More substantial calculations will appear in future publications.

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