COLOUR CALCULUS AND COLOUR QUANTIZATIONS

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Abstract. A colour calculus linked with an any discrete group G is developed. Colour differential operators and colour jets are introduced. Algebras colour differential forms and de Rham complexes are constructed. For colour differential equations Spencer complexes are constructed. Relations between colour commutative algebras and quantizations of usual ones are considered.

0.Introduction

Presently there exists a number of different approaches to the construction of a calculus: the universal construction for associative algebras [C],[K], [DV], fermionic and colour calculus [JK],[BMO],[KK],[V] the calculus for quadratic algebras [WZ],[M], etc.

This paper was intended as an attempt to illustrate the general scheme [L1,L2] of braided calculus on the example of colour calculus. The last one is a quantization of usual calculus, determined by some discrete group G of inner symmetries. We restrict ourself to colour calculus over groups only, but it is clear that the constructions may be carried over to the context of Hopf algebras also.

For any Hopf algebra which is not necessarily quasitriangular, Drinfeld [D] suggests to consider the quantum double – a new quasitriangular Hopf algebra. Our discussion in chapter 1 shows a naturality of this construction as a basis for the colour calculus.

Briefly speaking, colour structures related to braidings on the Drinfeld quantum double of a group algebra and a colour is a "diagonalizable" solution of the Yang–Baxter equation. A.B.Sletsjøe [S] recently found the other description of colours in terms of twisted Hochschild complexes.

We start with an analysis of the notion of derivations and yield some structure which we will call a **colour**, arising from a natural requirement to get a reasonable analog of our usual calculus.

In section 1 we introduce the important notion of colour commutative algebra and show that generalized (or colour) derivations of such algebras yield a colour Lie

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algebra structure. The notion of colour is linked with some discrete group G. For the case $G = \{e\}$ we get the usual commutative algebras and for the case $G = \mathbb{Z}_2$ we get the notion of super commutative algebra.

More important examples of colour commutative algebras appear from projective (or twisting) group algebras and crossed products. The crossed products are a natural generalization of group algebras. For example, all finite dimensional division algebras are crossed products. It means that a calculus over these algebras is also a colour calculus.

Other examples are given by the Galois theory of rings where crossed products and skew group algebras may be considered as base objects. Therefore, a calculus in the Galois theory is based on a colour calculus also.

In section 2 we introduce colour differential operators. We show that such objects as modules of colour differential forms and colour jets may be considered as representative objects for functors of colour derivations and colour differential operators. Our construction of colour de Rham complexes over colour commutative algebra A is based on two assumptions:

(1) An algebra $\Omega^*(A)$ of colour differential forms should be an extension of A to a new colour commutative algebra over the group $\bar{G} = \mathbb{Z} \times G$ generated by A and $\Omega^1(A)$.
(2) A colour de Rham differential d should be a colour derivation of the colour commutative algebra $\Omega^*(A)$, such that $d^2 = 0$.

These conditions determine the pair $(\Omega^*(A), d)$ up to some group homomorphism of G, where G is the base group.

Note that in the particular case of our usual calculus $G = \{e\}$ the algebra $(\Omega^*(A), d)$ is a unique, but in the super case $G = \mathbb{Z}_2$, we get two equally possible constructions of super de Rham complexes.

One can continue the procedure and produce a chain of colour commutative algebras: $(\Omega^*(A), \Omega^*(\Omega^*(A))$ and so on. Their colour cohomologies determine a new chain of colour commutative algebras. Remark that our usual calculus actually is the second one and based on calculus over $\Omega^*(A)$, (cf. Lie derivations, Schouten and Nijenhuis brackets etc).

In section 3 we put colour calculus in the framework of category theory. We show that colours are actually symmetries in the special monoidal category. We describe all quantizations in the monoidal category in the sense [L1-L4] and show that colour commutative algebras are quantizations of usual ones. In th particular it yields isomorphisms of algebras of differential operators. This result was noted by Ogievetsky [O] for algebras of quantum hyperplanes.

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1.Derivations

In this chapter we analyse some structures linked with the notion of derivations.

All algebras and rings are assumed to be associative with unit $1 \neq 0$ and all modules are unital.

1.1.Associativity. Let A be an associative algebra over a commutative ring k and let $\lambda : A \longrightarrow A$ be a k–linear map.

We start with the following preliminary naive definition of derivations

A k–linear map $X_\lambda : A \longrightarrow A$ will be called a λ-*derivation* if the following version of *Leibniz rule* holds:

$$X_\lambda(ab) = X_\lambda(a)b + \lambda(a)X_\lambda(b),$$

where $a, b \in A,$.

The associtivity law for A produces the following condition on λ.

We compare

$$X_\lambda(a(bc)) = X_\lambda(a)(bc) + \lambda(a)(X_\lambda(b)c + \lambda(b)X_\lambda(c)),$$

and

$$X_\lambda((ab)c) = (X_\lambda(a)b + \lambda(a)X_\lambda(b))c + \lambda(ab)X_\lambda(c).$$

This gives, $\lambda(ab)X_\lambda(c) = \lambda(a)\lambda(b)X_\lambda(c)$, for all $a, b, c \in A$.

We will suggest that $X_\lambda(c)$ are arbitrary elements of arbitrary algebras or equivalently that our future definition of derivations should work in arbitrary algebras. Therefore, we should assume that λ is an algebra endomorphism.

Moreover, we will **require** λ to be an automorphism of A. In this case the λ–Leibniz rule is compatible with the associativity law.

Examples.

(1) Let $A = C^\infty(M)$ be an algebra of smooth functions on manifold M. It is well known that derivations of A with $\lambda = id$, are vector fields on M.

(2) Let $A = C^\infty(M)$ and $\alpha : M \longrightarrow M$ be a diffeomorphism of M. Denote by $X_\alpha : A \longrightarrow A$ the following "difference" operator: $X_\alpha(f) = \alpha^*(f) - f$.

Then, $X_\alpha(fg) = \alpha^*(fg) - fg = (\alpha^*(f) - f)g + \alpha^*(f)(\alpha^*(g) - g)$, and therefore X_α is an α^*–derivation of A.

1.2.Brackets. Here we amplify our definition of derivations by requirement of some Lie structure analog.

Let X_α and X_β be two derivations of the algebra corresponding to automorphisms $\alpha, \beta \in \mathrm{Aut}(A)$.

Define a commutator $[X_\alpha, X_\beta]$ in the following way:

$$[X_\alpha, X_\beta] := X_\alpha \circ X_\beta - S_{\alpha,\beta} \circ X_\beta \circ X_\alpha,$$

for some k–linear map $S_{\alpha,\beta} : A \longrightarrow A$.

If one requires the commutator to be a derivation corresponding to some automorphism $\sigma \in \mathrm{Aut}(A)$ then

$$\begin{aligned}
[X_\alpha, X_\beta](ab) = {} & X_\alpha X_\beta(a)b + \overset{(4)}{\alpha(X_\beta(a))X_\alpha(b)} + \\
& + \overset{(5)}{X_\alpha(\beta(a))X_\beta(b)} + \overset{(1)}{\alpha\beta(a)X_\alpha X_\beta(b)} - \\
& - \overset{(3)}{S_{\alpha,\beta}(X_\beta X_\alpha(a)b)} - \overset{(5)}{S_{\alpha,\beta}(\beta(X_\alpha(a))X_\beta(b))} - \\
& - \overset{(4)}{S_{\alpha,\beta}(X_\beta(\alpha(a))X_\alpha(b))} - \overset{(2)}{S_{\alpha,\beta}(\beta\alpha(a)X_\beta X_\alpha(b))},
\end{aligned}$$

and on the other hand, by the definition of derivations, we get:

$$[X_\alpha, X_\beta](ab) = [X_\alpha, X_\beta](a)b + \sigma(a)[X_\alpha, X_\beta](b) =$$
$$X_\alpha X_\beta(a)b - S_{\alpha,\beta}(\overset{(3)}{X_\alpha} X_\beta(a))b + \sigma(a)\overset{(1)}{X_\alpha} X_\beta(b) - \sigma(a)S_{\alpha,\beta}(\overset{(2)}{X_\beta} X_\alpha(b)).$$

Comparing terms with mark (1) we get $\sigma = \alpha \circ \beta$. From terms with marks (2) and (3) one has:

$$S_{\alpha,\beta}(a) = s_{\alpha,\beta} \cdot a, \tag{1}$$

where $s_{\alpha,\beta} = S_{\alpha,\beta}(1) \in A$ and

$$s_{\alpha,\beta} \cdot \beta\alpha(a) = \alpha\beta(a) \cdot s_{\alpha,\beta}. \tag{2}$$

for all $a \in A$.

The terms with mark (4) and (5) give

$$\alpha \circ X_\beta = S_{\alpha,\beta} \circ X_\beta \circ \alpha, \tag{3}$$

and

$$X_\alpha \circ \beta = S_{\alpha,\beta} \circ \beta \circ X_\alpha. \tag{4}$$

Therefore, we should assume that

$$s_{\alpha,\beta} \cdot s_{\beta,\alpha} = 1. \tag{4}$$

Then relation (3) is a consequence of (4).

Now we take a derivation X_γ for some $\gamma \in \mathrm{Aut}(A)$ and consider the composition $\alpha \circ X_\beta \circ \gamma$. By formulae (3) and (4) we get the following action on the elements $s_{\beta,\gamma} \in A$:

$$\alpha(s_{\beta,\gamma}) = s_{\alpha,\beta} \cdot s_{\beta,\alpha\gamma}. \tag{5}$$

In particular it follows that

$$s_{\alpha\beta,\gamma} = s_{\alpha,\beta} \cdot s_{\beta,\alpha\gamma} \cdot s_{\alpha,\gamma}. \tag{6}$$

Example. Let X_α and X_β be the derivations corresponding to diffeomorphisms $\alpha : M \longrightarrow M$ and $\beta : M \longrightarrow M$, (see ex.1.1.(2)). To introduce a commutator $[X_\alpha, X_\beta]$ one needs to enlarge the algebra $A = C^\infty(M)$ by means of an element ϑ with new commutative relation: $\vartheta \cdot f = [\alpha^*, \beta^*](f) \cdot \vartheta$, where $f \in A$ and $[\alpha^*, \beta^*]$ is a commutator of automorphisms α^*, β^*.

Actually it means that we should extend our initial algebra A to colour commutative algebra $A[G]$ (see 1.6. below).

1.3. Jacobi identity. Here we yield new properties of $S_{\alpha,\beta}$ by assumption of the following form of Jacobi identity:

$$[X_\alpha, [X_\beta, X_\gamma]] = [[X_\alpha, X_\beta], X_\gamma] + Z_{\alpha,\beta,\gamma} \circ [X_\beta, [X_\alpha, X\gamma]],$$

for some k–linear map $Z_{\alpha,\beta,\gamma} : A \longrightarrow A$.

It follows by the same method as above that we should assume that

$$Z_{\alpha,\beta,\gamma} = S_{\alpha,\beta}, \tag{1}$$

$$X_\alpha(s_{\beta,\gamma}) = o, \tag{2}$$

and the "multiplicative" Jacobi identity

$$s_{\alpha\beta,\gamma} \cdot s_{\gamma\alpha,\beta} \cdot s_{\beta\gamma,\alpha} = 1 \tag{3}$$

hold.

Then the bracket $[X_\alpha, X_\beta]$ is an $\alpha\beta$–derivation and we get Jacobi and skew symmetry properties in the following form:

(1) Jacobi identity:

$$[X_\alpha, [X_\beta, X_\gamma]] = [[X_\alpha, X_\beta], X_\gamma] + s_{\alpha,\beta} \cdot [X_\beta, [X_\alpha, X_\gamma]],$$

(2) skew–symmetry:

$$[X_\alpha, X_\beta] = -s_{\alpha,\beta} \cdot [X_\beta, X_\alpha].$$

Remark. Denote by $Ad_\alpha(\beta) = {}^\alpha\beta = \alpha \cdot \beta \cdot \alpha^{-1}$– the adjoint action of G. Then applying the map $(\quad) \mapsto \gamma \circ (\quad) \circ \gamma^{-1}$ to the Jacobi identity we get the following property of s:

$$\gamma(s_{\alpha,\beta}) = s_{{}^\gamma\alpha,{}^\gamma\beta}.$$

1.4. A module structure. In this section we get an additional structure in the algebra by assuming that the set of all derivations has an A– module structure.

Let X_λ be a λ–derivation and let $a \in A$ be an element such that $a \cdot X_\lambda$ is a ν–derivation for some automorphism $\nu = \nu(a, \lambda)$.

In this case we have

$$\begin{aligned}(a \cdot X_\lambda)(xy) &= a \cdot (X_\lambda(x)y + \lambda(x)X_\lambda(y)) = \\ &= (aX_\lambda)(x)y + \nu(x)(aX_\lambda)(y),\end{aligned}$$

for all $x, y \in A$.

Therefore we should require that $a \cdot \lambda(x) = \nu(x) \cdot a$ for all elements $x \in A$.

To do this we introduce a new version of the commutative law. We say that an element $a \in A$ is a *"simple"* if there exist an automorphism $\sigma = \sigma_a \in Aut(A)$ such that

$$a \cdot x = \sigma_a(x) \cdot a, \tag{1}$$

for all elements $x \in A$.

Denote by $A_\alpha \subset A$ the k–module of all "simple" elements corresponding to the given automorphism α:

$$A_\alpha = \{a \in A \mid ax = \alpha(x)a, \forall x \in A\}.$$

Then $A_* = \sum_\alpha A_\alpha$ is a graded algebra: $A_\alpha A_\beta \subset A_{\alpha\beta}$ and a subalgebra of A.

Let's assume that $G = \{\alpha \in \mathrm{Aut}(A) | A_\alpha \neq 0\}$ is a group and that $A = A_*$.

Then A becomes a G–graded algebra with the given action of group G such that $\alpha(A_\beta) \subset A_{\alpha\beta\alpha^{-1}}$.

Definition. Let G be a multiplicative group with identity element $e \in G$. We say that G–graded G–algebra $A = \sum_{\alpha \in G} A_\alpha$ is a **graded commutative G– algebra** if

(1) A is a G–graded algebra: $A_\alpha A_\beta \subset A_{\alpha\beta}$,
(2) there is a G–action on A such that: $\alpha(A_\beta) \subset A_{\alpha\beta\alpha^{-1}}$,
(3) A is a graded commutative algebra: $a_\alpha \cdot x = \alpha(x) \cdot a_\alpha$,

for all $x \in A, a_\alpha \in A_\alpha; \alpha, \beta \in G$.

1.5. Consider derivations of a graded commutative algebra A. It is natural to assume that derivations are compatible with the graded structure: $X_\lambda(A_\alpha) \subset A_{\Lambda(\alpha)}$, for some map $\Lambda : G \longrightarrow G$.

Comparing the degrees of terms incoming in the Leibniz rule

$$X_\lambda(a_\alpha \cdot a_\beta) = X_\lambda(a_\alpha)a_\beta + \lambda(a_\alpha)X_\lambda(a_\beta),$$

we get $\Lambda(\alpha\beta) = \Lambda(\alpha)\beta$, and therefore $\Lambda(\alpha) = \hat{\lambda} \cdot \alpha$, for some automorphism $\hat{\lambda} \in G$.

Throughout this paper we will identify λ and $\hat{\lambda}$.

In this case the compatibility conditions take the form $X_\lambda(A_\alpha) \subset A_{\lambda\alpha}$.

Remark. Let H be a (discrete) subgroup of $\mathrm{Aut}(A)$.We restrict ourself to derivations X_λ such that $\lambda \in H$.

Then

(1) The Leibniz rule determines a group homomorphism $\varkappa : H \longrightarrow G$, where $\varkappa(\lambda) = \hat{\lambda}$.
(2) The module structure $a_\alpha \cdot X_\lambda = X_\nu, \quad \nu = \alpha \cdot \lambda$ determines an embedding $G \subset H$ such that $\varkappa|_G = id$.
(3) Bracket conditions (1.2.) yield $[H, H] \subset G$, where $[H, H]$ is the commutator group generated by the commutators $[x, y] = xyx^{-1}y^{-1}$.

Moreover, it follows that:

(1) $K = ker\varkappa \subset H$ is an abelian group,
(2) $G \cap K = 1, \quad [G, K] = 1$,

and hence H is a direct product of groups: $H = K \times G$.

1.6. Let X_λ be a derivation of a graded commutative G– algebra A.

Applying the Leibniz rule to products: $\quad a_\alpha x = \alpha(x)a_\alpha, \quad$ where $\alpha \in G$, $a_\alpha \in A_\alpha$ and $x \in A$, we get two terms:

$$X_\lambda(a_\alpha)x + \lambda(a_\alpha)X_\lambda(x)$$

and

$$X_\lambda(\alpha(x))a_\alpha + \lambda(\alpha(x))X_\lambda(a_\alpha).$$

But $X_\lambda(a_\alpha)x = \lambda(\alpha(x))X_\lambda(a_\alpha)$, because A is a graded commutative G–algebra and $X_\lambda(a_\alpha) \in A_{\lambda\alpha}$.

Hence, $\lambda(a_\alpha)X_\lambda(x) = X_\lambda(\alpha(x))a_\alpha$, and by using formulae 1.2.(4) and 1.2.(1) we get

$$\lambda(a_\alpha) = s_{\lambda,\alpha} \cdot a_\alpha, \tag{1}$$

for all $\lambda\alpha \in G, a_\alpha \in A_\alpha$.

We should remark that formula 1.2.(5) means that (1) determines a G–action on A and $s_{\lambda,\alpha} \in A_{[\lambda,\alpha]}$.

Remark. The graded commutative property

$$a_\alpha \cdot a_\beta = s_{\alpha,\beta} \cdot a_\beta \cdot a_\alpha$$

gives the **hexagon equation** on s.

Indeed, if we consider the product $a_\alpha \cdot a_\beta \cdot a_\gamma$ in two different ways we get:

$$\begin{tikzcd}
(a_\alpha \cdot a_\beta)a_\gamma \arrow[r, equal] \arrow[d, equal] & s_{\alpha\beta,\gamma} a_\gamma (a_\alpha a_\beta) \arrow[r, equal] & s_{\alpha\beta,\gamma} a_\gamma a_\alpha a_\beta \arrow[d, equal] \\
a_\alpha (a_\beta a_\gamma) \arrow[r, equal] & (a_\alpha (s_{\beta,\gamma} a_\gamma)) a_\beta \arrow[r, equal] & s_{\alpha,{}^\gamma\beta} s_\beta(\gamma) a_\gamma a_\alpha a_\beta .
\end{tikzcd}$$

Hence, we should require that the following form of hexagon equation holds:

$$s_{\alpha\beta,\gamma} = s_{\alpha,{}^\beta\gamma} \cdot s_{\beta,\gamma}. \tag{2}$$

1.7. Starting from this point we will formalize the above constructions.

Let G be a discrete multiplicative group and $A = \sum_{g\in G} A_g$ be a G–graded algebra. Denote by $U(A)$ the group of units of algebra A.

Definition. A map

$$s : G \times G \longrightarrow U(A),$$

where $s : \alpha \times \beta \mapsto s_{\alpha,\beta} \in A_{[\alpha,\beta]}$, and $s_{e,\alpha} = 1$, for all $\alpha \in G$, is called a **colour on the group** G if the following properties hold:

(1) *multiplicative skew symmetry:*

$$s_{\alpha,\beta} \cdot s_{\beta,\alpha} = 1,$$

(2) the *hexagon* equation:

$$s_{\alpha\beta,\gamma} = s_{\alpha,{}^\beta\gamma} \cdot s_{\beta,\gamma}.$$

(3) the *compatibility* of colour with G–action:

$$s_{\alpha\beta,\gamma} = s_{\alpha,\beta} \cdot s_{\beta,\alpha\gamma} \cdot s_{\alpha,\gamma}.$$

In this case the colour s determines an G–action:

$$\alpha(a_\beta) = s_{\alpha,\beta} \cdot a_\beta, \tag{1}$$

such that

$$\gamma(s_{\alpha,\beta}) = s_{\gamma,\alpha} \cdot s_{\alpha,\gamma\beta}, \tag{2}$$

for all $\alpha, \beta, \gamma \in G; a_\alpha \in A_\alpha$.

Remark. Condition (2) is equivalent to the multiplicative Jacobi identity.

Examples of colours.

(1) Let G be a commutative group. Then $s : G \times G \longrightarrow U(k) \subset A_e$, and conditions (1) and (2) mean that $s_{\alpha,\beta} \cdot s_{\beta,\alpha} = 1$ and s is a group bihomomorphism.

(2) Let $G = \mathbb{Z}$. We have two possible colours: $s_{\alpha,\beta} \equiv 1$–**a trivial colour**, and $s_{\alpha,\beta} = (-1)^{\alpha\beta}$–**a standard colour.**

Note that the trivial colour is the base of our usual calculus and the standard one is a base for super–calculus.

(3) Let $G = \mathbb{Z}^n$. All colours take the following forms:

$$s_{\alpha,\beta} = \prod_{1 \le i,j \le n} a_{ij}^{(\alpha_i\beta_j - \alpha_j\beta_i)} \cdot \prod_{1 \le k \le n} b_k^{\alpha_k\beta_k}, \tag{1}$$

for all $\alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_n) \in \mathbb{Z}^n$, and for some elements $a_{ij} \in k$, and $b_i = \pm 1$.

(4) Let G be an abelian finite generated group, $G = \mathbb{Z}^n/K$, where $K \subset \mathbb{Z}^n$ is a subgroup with generators $\lambda_1 = (\lambda_{11}, ..., \lambda_{1n}), ..., \lambda_r = (\lambda_{r1}, ..., \lambda_{rn})$. Then formula (1) gives a colour on G if $s_{\alpha,\beta} = 1$, for all $\alpha \in K, \beta \in \mathbb{Z}^n$, or equivalently, if

$$\prod_i a_{ij}^{\lambda_{ki}} = 1,$$

for all $i, j = 1, ..., n, \quad k = 1, ..., r.$

1.8.Definition. Let A be a G–graded algebra and $s : G \times G \longrightarrow U(A)$ be a colour. We say that A is a **colour commutative algebra** if A is a commutative graded G–algebra with respect to the action (1.7.(1)).

Examples.

(1) **Group algebras.** Let $k[G]$ be a group algebra of a multiplicative group G with the standard basis $\{\varepsilon_\alpha, \alpha \in G\}$. Then $A = k[G]$ is a colour commutative algebra with respect to the grading: $A_\alpha = k \cdot \varepsilon_\alpha$ and the colour: $s_{\alpha,\beta} = \varepsilon_{[\alpha,\beta]}$.

(2) **Crossed products.** Let G be a finite multiplicative group and k be a commutative G–ring with given action $\nu : G \longrightarrow \mathrm{Aut}(k)$.

In some sense the crossed product of k and G is a generalized group algebra. To define the algebra we consider a free k–module with a basis $\{\varepsilon_\alpha, \alpha \in G\}$, and determine a multiplication as follows:

$$\varepsilon_\alpha \cdot \varepsilon_\beta = \omega(\alpha, \beta)\varepsilon_{\alpha\beta}, \tag{1}$$

and

$$\varepsilon_\alpha \cdot x = \nu_\alpha(x)\varepsilon_\alpha, \tag{2}$$

where $\omega : G \times G \longrightarrow U(k)$ is a **twisting cocycle**, and $\alpha, \beta \in G, \quad x \in k$.

The cocycle conditions on ω we can get from the associativity law for the multiplication:

$$\nu_\alpha(\omega(\beta, \gamma)) \cdot \omega(\alpha, \beta\gamma) \cdot \omega(\alpha\beta, \gamma)^{-1} \cdot \omega(\alpha, \beta)^{-1} = 1. \tag{3}$$

Therefore,

$$\omega \in Z^2_{\text{mult}}(G,k).$$

We should also remark that the scale transformation:

$$\varepsilon_\alpha \mapsto \varepsilon^t_\alpha = t(\alpha)\varepsilon_\alpha,$$

for some function $t : G \longrightarrow U(k)$ determines a new twisting cocycle:

$$\omega^t(\alpha,\beta) = t(\alpha)\nu_\alpha(t(\beta))t^{-1}(\alpha\beta)\omega(\alpha,\beta),$$

or

$$\omega^t = \delta t \cdot \omega,$$

where $\delta t(\alpha,\beta) = t(\alpha)\nu_\alpha(t(\beta))t^{-1}(\alpha\beta)$ is the coboundary of t. Therefore, up to scale automorphisms, properties (1) and (2) together with the cohomology class $[\omega] \in H^2_{\text{mult}}(G,k)$, determine the crossed product algebra $A = k_\omega[G]$.

If we consider A as a k^G–algebra and define a grading by $A_\alpha = k\cdot\varepsilon_\alpha, \forall\alpha \in G$, and a G–action by

$$\alpha(\varepsilon_\beta) = \frac{\omega(\alpha,\beta)}{\omega({}^\alpha\beta,\alpha)}\varepsilon_{{}^\alpha\beta},$$

we get a colour algebra with colour

$$s_{\alpha,\beta} = \chi(\alpha,\beta)\varepsilon_{[\alpha,\beta]},$$

where the function $\chi : G\times G \longrightarrow k$ is given by the formula

$$\chi(\alpha,\beta) = \frac{\omega(\alpha,\beta)}{\omega({}^\alpha\beta,\alpha)\omega([\alpha,\beta],\alpha)}. \tag{4}$$

(3) We get **projective** or **twisting** group algebras if the action ν is trivial. If the twisting is trivial we get a **skew** group algebra.

(4) Let $A = \sum_{\alpha\in G} A_\alpha$ be a G–graded algebra such that $A_e = k$.

Assume that any A_α contains an invertible element and denote this element by g_α and define a G–action on k as follows:

$$\alpha(x) = g_\alpha \cdot x \cdot g_\alpha^{-1},$$

for any $x \in k = A_e$.

Remark that any element $a_\alpha \in A_\alpha$ can be represented as $a_\alpha = g_\alpha \cdot x_\alpha$, with $x_\alpha \in k$. It is show that action (1) is independent of the choice of g_α. If we define a twisting 2–cocycle ω by the formula

$$\omega(\alpha,\beta) = g_\alpha \cdot g_\beta \cdot g_\alpha^{-1} \cdot g_\beta^{-1} \in A_{[\alpha,\beta]},$$

then

$$g_\alpha \cdot g_\beta = \omega(\alpha,\beta)g_\beta \cdot g_\alpha,$$

and we get an algebra isomorphism $\psi : k_\omega[G] \longrightarrow A$, such that $\psi(\varepsilon_\alpha) = g_\alpha$.

(5) Let $A = \sum_{\alpha \in G} A_\alpha$ be a *strongly* G–graded algebra [Da] with $A_e = k$. Then by definition $A_\alpha \cdot A_\beta = A_{\alpha\beta}$ and we can use the above procedure.

(6) Let $k = \mathbb{R}$ and $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Denote by $a = (\bar{1}, \bar{0})$ and $b = (\bar{0}, \bar{1})$ the generators in the group and by

$$\omega(\alpha, \beta) = (-1)^{\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_2\beta_1},$$

a twisting cocycle.

As $\mathbb{R}$–vector space the projective group algebra $\mathbb{R}_\omega[G]$ is generated by $\varepsilon_0 = 1, \varepsilon_a, \varepsilon_b, \varepsilon_{a+b}$ with relations $\varepsilon_a^2 = \varepsilon_b^2 = \varepsilon_{a+b}^2 = -1$, and $\varepsilon_a\varepsilon_b = \varepsilon_b\varepsilon_a = \varepsilon_{a+b}$.

Hence, if we define the map ψ by

$$\varepsilon_a \mapsto \imath, \quad \varepsilon_b \mapsto \jmath, \quad \varepsilon_{a+b} \mapsto k,$$

we get an isomorphism between $\mathbb{R}_\omega[G]$ and the quaternion algebra $\mathbb{H}$.

Therefore, $\mathbb{H}$ is a colour commutative algebra.

(7) Let $G = (\mathbb{Z}_2)^n$.

Identify elements $\alpha = (\alpha_1, ..., \alpha_n) \in G$ with sequences $\hat{\alpha}_1 < ... < \hat{\alpha}_k$, where $1 \leq \alpha_i \leq n$ and $\alpha_{\hat{\alpha}_i} = 1$.

We define a twisting 2–cocycle on G as follows:

$$\omega(\alpha, \beta) = (-1)^{\pi(\alpha,\beta)}.$$

Here $\pi(\alpha, \beta)$ is a number of inversions in the sequence $(\hat{\alpha}_1, ..., \hat{\alpha}_k, \hat{\beta}_1, ..., \hat{\beta}_l)$.

Then $\mathbb{R}_\omega[G]$ is a Clifford algebra.

(8) Let $k = \mathbb{R}$, $\quad G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $A = \mathrm{Mat}_2(\mathbb{R})$.

Similarly to the construction of ex.6 we define a map

$$\psi : \mathbb{R}_\omega[G] \longrightarrow \mathrm{Mat}_2(\mathbb{R}),$$

for some cocycle ω as follows:

$$\varepsilon_a \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \varepsilon_b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\varepsilon_{a+b} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \varepsilon_0 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now we define a cocycle ω in such a way that ψ becomes an algebra morphism. Finally we get the following conditions on ω:

$$\omega(a, a) = \omega(b, b) = \omega(a, b) = \omega(a, a + b) = -\omega(a + b, a + b) = 1,$$

and $\omega(x, y) = -\omega(y, x)$, if $x \neq y$.

Tensoring of the algebras $\mathrm{Mat}_2(\mathbb{R})$ we get a colour commutative structure on the algebras $\mathrm{Mat}_{2^n}(\mathbb{R})$, for all $n = 1, 2, ...$

(9) Any matrix algebra $A = \mathrm{Mat}_n(\mathbb{C})$ is a colour commutative algebra.

To see this consider matrixes

$$p = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \eta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \eta^{n-1} \end{bmatrix} \quad q = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ \eta & 0 & \dots & 0 & 0 \\ 0 & \eta^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \eta^{n-1} & 0 \end{bmatrix}$$

where $\eta = \exp(\frac{\pi \imath}{n})$. Then,

$$pq = \eta \cdot qp,$$

and the set $\{p^a q^b; a, b = 0, 1, ..., n-1\}$ is an $\mathbb{R}$–vector space basis of the algebra $\mathrm{Mat}_n(\mathbb{C})$.

Consider now the group $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ and the twisting cocycle

$$\omega(\alpha, \beta) = \begin{cases} \eta^{\bar{\alpha}_1 \bar{\beta}_2}, & \text{if} \quad \alpha_2 + \beta_2 < n, \\ (-1)^{n-1} \eta^{\bar{\alpha}_1 \bar{\beta}_2}, & \text{if} \quad \alpha_2 + \beta_2 \geq n, \end{cases}$$

where $\alpha = (\bar{\alpha}_1, \bar{\alpha}_2), \beta = (\bar{\beta}_1, \bar{\beta}_2) \in G$, and $\alpha_i, \beta_j = 0, 1, ..., n-1$.

An isomorphism ψ between $\mathbb{C}_\omega[\mathbb{Z}_n \oplus \mathbb{Z}_n]$ and $\mathrm{Mat}_n(\mathbb{C})$ is given by the formula:

$$\psi : \varepsilon_\alpha \mapsto q^{\alpha_2} \cdot p^{\alpha_1},$$

for all $\alpha \in G$.

(10) Let $G = \mathbb{Z}^n$, $k = \mathbb{C}$ and $\Theta = \|\theta_{ij}\| \in \mathrm{Mat}_n(\mathbb{R})$ be a matrix such that $\theta_{ij} = 0$, if $i \leq j$. Taking the twisting cocycle:

$$\omega(x, y) = \exp(\pi \imath \langle Ax, y \rangle),$$

where $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{Z}^n, \langle x, y \rangle = \sum_i x_i y_i$, we get the algebra $\mathbb{C}_\omega[\mathbb{Z}^n]$ called a **quantum torus.**

1.9. Now we suggest our main notion.

Definition. A k–linear map $X_\lambda : A \longrightarrow A$ is said to be a **colour λ–derivation,** $\lambda \in G$, if

(1) X_λ is a graded derivation: $X_\lambda(A_\alpha) \subset A_{\lambda\alpha}$,

(2) λ–Leibniz rule holds:

$$X_\lambda(a_\alpha \cdot a_\beta) = X_\lambda(a_\alpha) a_\beta + \lambda(a_\alpha) X_\lambda(a_\beta),$$

(3) values $s_{\alpha,\beta}$ of the colour map are constants: $X_\lambda(s_{\alpha,\beta}) = 0$,

where $a_\alpha \in A_\alpha, a_\beta \in A_\beta; \alpha, \beta \in G$.

Remark. Property (3) of colour derivations is equivalent to the following interaction between the G–action and derivations:

$$X_\lambda \circ \beta = s_{\lambda,\beta} \cdot \beta \circ X_\lambda.$$

Summarizing results 1.1-1.6., we have the following

Theorem. *Let A be a colour commutative algebra and let $\mathrm{Der}_\lambda(A)$ be a k-module of all colour λ–derivations $\lambda \in G$. Then*

(1) *A G–graded k– module $\mathrm{Der}_*(A) = \sum_{\lambda\in G} \mathrm{Der}_\lambda(A)$ is a left G–graded A–module:*

$$A_\alpha \cdot \mathrm{Der}_\lambda(A) \subset \mathrm{Der}_{\alpha\lambda}(A), \quad \alpha, \lambda \in G.$$

(2) *$\mathrm{Der}_*(A)$ is a G–module with respect to the action:*

$$\alpha(X_\lambda) = \alpha \circ X_\lambda \circ \alpha^{-1} = s_{\alpha,\lambda} \cdot X_\lambda,$$

and $\alpha(\mathrm{Der}_\lambda(A) \subset \mathrm{Der}_{^\alpha\lambda}(A)$.

(3) *The bracket*

$$[X_\alpha, X_\beta] = X_\alpha \circ X_\beta - s_{\alpha,\beta} \cdot X_\beta \circ X_\alpha$$

determines a **colour Lie algebra** *structure on $Der_*(A)$:*

$$[\mathrm{Der}_\alpha(A), \mathrm{Der}_\beta(A)] \subset \mathrm{Der}_{\alpha\beta}(A), \tag{1}$$

$$[X_\alpha, X_\beta] = -s_{\alpha,\beta} \cdot [X_\beta, X_\alpha], \tag{2}$$

$$[X_\alpha, [X_\beta, X_\gamma]] = [[X_\alpha, X_\beta], X_\gamma] + s_{\alpha,\beta} \cdot [X_\beta, [X_\alpha, X_\gamma]]. \tag{3}$$

Remark. Note that defining in an obvious way from the theorem colour Lie algebra structures are particular case of generalized Lie algebras introduced by D.Gurevich [G].

1.10. Now we analyze a notion of derivation with values in A–modules. Let P be an A–module. Before we consider derivations with values in P we should remark that the definition of maps satisfying Leibniz rule requires an $A - A$ bimodule structure on P. Moreover, working with graded algebras we should consider graded modules too.

So we have to start with a notion of graded bimodule over a graded algebra.

Let $A = \sum_{\alpha\in G} A_\alpha$ be a G–graded G–algebra and let $P = \sum_{m\in M} P_m$ be a M–graded k–module where M is an arbitrary grading set.

To define an $A - A$ bimodule structure on P we must require that:

$$A_\alpha \cdot P_m \subset P_{\alpha \cdot m}, \quad P_m \cdot A_\alpha \subset P_{m \cdot \alpha}, \tag{1}$$

and

$$(A_\alpha \cdot P_m) \cdot A_\beta = A_\alpha \cdot (P_m \cdot A_\beta). \tag{2}$$

Therefore, we need a left and a right G–action on the set M.

Definitions.

(1) A set M with left $G \times M \longrightarrow M, \alpha \times m \mapsto \alpha \cdot m$, and right $M \times G \longrightarrow M, m \times \beta \mapsto m \cdot \beta$, G–actions is called a G–**bispace** if $(\alpha \cdot m) \cdot \beta = \alpha \cdot (m \cdot \beta)$, for all $\alpha, \beta \in G, m \in M$.

(2) Let M be a bispace and P be an $A - A$–bimodule. We say that P is a M–**graded** bimodule if conditions (1) and (2) hold.

1.11. It is natural to consider derivations compatible with the grading only. From the beginning we define derivations $X : A \longrightarrow P$ with values in M–graded $A - A$ bimodule P as k–linear maps of degree $m \in M$; $X = X_m : A_\alpha \longrightarrow P_{m \cdot \alpha}$ satisfying the following form of Leibniz rule:

$$X_m(a_\alpha \cdot a_\beta) = X_m(a_\alpha)a_\beta + \tilde{\sigma}_m(a_\alpha)X_m(a_\beta).$$

Denote by $\mathrm{Aut}_\Phi(A)$ the set of automorphisms of the algebra A over the group automorphism $\Phi : G \longrightarrow G$, that is the set of automorphisms $\tilde{\sigma}$ of A such that $\tilde{\sigma}(A_\alpha) \subset A_{\Phi(\alpha)}$.

Then for the automorphism $\tilde{\sigma}_m$ from the Leibniz rule one has: $\tilde{\sigma}_m \in \mathrm{Aut}_{\varphi_m}(A)$, for some automorphism $\varphi_m \in \mathrm{Aut}\, G$.

Comparing the degrees of terms interring in the Leibniz rule one gets the following interaction between the left and the right G–structures:

$$m \cdot \alpha = \varphi_m(\alpha) \cdot m. \tag{1}$$

Moreover, if we analogously to (1.4.) require an A–module structure in the set of all derivations, we must assume that A is a commutative G–graded G–algebra and

$$\tilde{\sigma}_{\alpha m} = \alpha \circ \tilde{\sigma}_m, \quad \varphi_{\alpha m} = Ad_\alpha \circ \varphi_m. \tag{2}$$

1.12. Assume now that A is a colour commutative algebra and consider the action of X_m on the products $a_\alpha \cdot a_\beta = \alpha(a_\beta) \cdot a_\alpha$:

$$X_m(a_\alpha \cdot a_\beta) = X_m(a_\alpha) \cdot a_\beta + \tilde{\sigma}_m(a_\alpha) \cdot X_m(a_\beta),$$

and

$$X_m(\alpha(a_\beta) \cdot a_\alpha) = X_m(\alpha(a_\beta)) \cdot a_\alpha + \tilde{\sigma}_m(\alpha(a_\beta)) \cdot X_m(a_\alpha).$$

Comparing the terms with $X_m(a_\alpha)$ one gets the following symmetric properties for the bimodule P:

$$z_m \cdot a_\beta = \tilde{\sigma}_m(a_\beta) \cdot z_m,$$

and

$$\tilde{\sigma}_{m\alpha} = \tilde{\sigma}_m \circ \alpha, \quad \varphi_{m\alpha} = \varphi_m \circ Ad_\alpha. \tag{1}$$

To compare the terms with $X_m(a_\beta)$ we will require the presence of some G–action $G \times P \longrightarrow P$, such that $\alpha : P_m \longrightarrow P_{\alpha m \alpha^{-1}}$, and the following compatibility of the derivations with the group action:

$$X_m \circ \alpha = \sigma_m(\alpha) \cdot \alpha \circ X_m,$$

for some map $\sigma : M \times G \longrightarrow U(A)$, where $\sigma : m \times \alpha \mapsto \sigma_m(\alpha) \in A_{\varphi_m(\alpha)\alpha^{-1}}$.

In this case we get

$$\tilde{\sigma}_m(a_\alpha) = \sigma_m(\alpha) \cdot a_\alpha. \tag{1}$$

Definitions.

(1) Let M be a G–bispace and $\varphi : M \longrightarrow \mathrm{Aut}(G)$ a map. We say that M is a φ–**commutative** G–space if relations 1.11 (1) and (2), on automorphisms φ_m hold.

(2) Let M be a commutative G–bispace. A map $\sigma : M \times G \longrightarrow U(A)$,
$\sigma : m \times \alpha \longrightarrow \sigma_m(\alpha) \in A_{\varphi_m(\alpha)\alpha^{-1}}$, we will call **a colour** on the bispace M over a colour $s : G \times G \longrightarrow U(A)$ on the group G if

$$\sigma_m(\alpha\beta) = \sigma_m(\alpha)\alpha(\sigma_m(\beta)), \quad \sigma_m(e) = 1, \tag{1}$$

$$\sigma_{\alpha m} = \alpha(\sigma_m(\beta))s_{\alpha,\beta} \tag{2}$$

for all $\alpha, \beta \in G, m \in M$.

In other words M is a commutative bispace if M ia a left G–space considered together with a map φ. In this case 1.11.(1) may be considered as the definition of the right G–action.

1.13. Let M be a commutative G–bispace with a colour σ and let A be a colour commutative G–algebra.

Definition. We say that an M–graded $A-A$ bimodule $P = \sum_{m\in M} P_m$ is a **colour symmetric bimodule** if there is a G– action on P, such that

$$\alpha(x_m) = \sigma_m(\alpha)^{-1} \cdot x_m,$$

and

$$x_m \cdot a_\alpha = \sigma_m(\alpha) \cdot \alpha_\alpha \cdot x_m,$$

for all $\alpha \in G, a_\alpha \in A_\alpha, x_m \in P_m$.

We should remark that any M–graded A–module, where M is a commutative G–bispace with a colour σ, may be considered as a colour symmetric bimodule if the properties from the definition one looks at as the definition of G–action and right multiplication.

1.14. Let E be a colour symmetric $A - A$ bimodule.

Definition. A graded k–linear morphism $X : A \longrightarrow P$ a degree $m \in M$, $X(A_\alpha) \subset P_{m\alpha}$ will be called **a colour derivation** with values in the bimodule P if the following properties hold:

(1) colour Leibniz rule:

$$X(a_\alpha \cdot a_\beta) = X(a_\alpha)a_\beta + \sigma_m(\alpha)a_\alpha X(a_\beta),$$

(2) constancy of colours:

$$X(\sigma_m(\alpha)) = X(s_{\alpha,\beta}) = 0,$$

or

$$X_m \circ \alpha = \sigma_m(\alpha) \cdot \alpha \circ X_m,$$

for all $m \in M; \alpha, \beta \in G; a_\alpha \in A_\alpha, a_\beta \in A_\beta$.

1.15. Denote by $\mathrm{Der}_m(A, P)$ the module of all m–degree colour derivations with values in P and by

$$\mathrm{Der}_*(A, P) = \sum_{m \in M} \mathrm{Der}_m(A, P)$$

a M–graded k–module of all colour derivations.

Theorem. $\mathrm{Der}_*(A, P)$ *is a* M*–graded* A*–module:*

$$A_\alpha \cdot \mathrm{Der}_m(A, P) \subset \mathrm{Der}_{\alpha m}(A, P),$$

for all $\alpha \in G, m \in M$.

2.Colour differential operators

In this chapter we define colour differential operators in a category of colour symmetric bimodules and introduce the main functors of colour calculus. We show that such objects as modules of colour differential forms and colour jets may be considered as representative objects of functors differential operators and derivations. We should stress also that the group G in the colour differential calculus are considered in the triple role:

(1) G is a group controlling the commutative laws,
(2) G is a grading group of the base algebra A,
(3) G is an action group.

2.1. We start with a reformulation of the notion of derivation. To do this we consider the Leibniz rule

$$X_m(a_\alpha a_\beta) = X_m(a_\alpha)a_\beta + \sigma_m(\alpha)a_\alpha X_m(a_\beta)$$

in the following form:

$$\left(X_m \circ l_{a_\alpha} - \sigma_m(\alpha) l_{a_\alpha} \circ X_m\right)(a_\beta) = l_{X_m(a_\alpha)}(a_\beta), \tag{1}$$

where $l_a : A \longrightarrow A$ is an operator of the left multiplication:
$l_a(b) = ab, \quad \forall a, b \in A$.

Denote by $[X_m, l_{a_\alpha}]$ the operator on the left hand side of (1). Then the Leibniz rule takes the form:

$$[X_m, l_{a_\alpha}] = l_{X_m(a_\alpha)}.$$

We should note that the operator $\tilde{l} = l_{X_m(a_\alpha)} : A_\beta \longrightarrow E_{(m\alpha)\beta}$ has a degree $m\alpha$ and satisfies the relation:

$$[\tilde{l}, l_{a_\beta}] = \tilde{l} \circ l_{a_\beta} - \sigma_{m\alpha}(\beta) l_{a_\beta} \circ \tilde{l} = 0. \tag{2}$$

2.2. To define differential operators one can use the usual inductive definition of differential operators (see, for example [KLV]) with an evident modification of the commutator notion.

To do this we start with a notion of 0-order differential operators or "colour" homomorphisms.

Let $P = P_M = \sum_{m\in M} P_m, \quad Q = Q_N = \sum_{n\in N} Q_n$ be graded modules over the colour algebra A.

Here M and N are G–bispaces.

Let $\sigma : A \longrightarrow A$ be a graded automorphism. We say that a k–linear graded morphism $f : P \longrightarrow Q$ is a σ–morphism if $f(ax) = \sigma(a)f(x), \quad$ for all $a \in A$, $x \in P$.

Below we will consider automorphisms σ which are generated by a colour:

$$\sigma(a_\beta) = s_{\alpha,\beta} a_\beta,$$

for some elements $\alpha = \alpha(\sigma) \in G$.

Any σ–morphism f determines a map $\hat{f} : M \longrightarrow N$, such that $f(P_m) \subset Q_{\hat{f}(m)}$. Compare degrees of $f(a_\beta x_m)$ and $\sigma(a_\beta)f(x_m)$ one gets:

$$\hat{f}(\beta m) = {}^{\alpha}\beta \hat{f}(m).$$

Therefore, the colour dictates a special type of G–morphisms between grading sets.

Returning to the definition of colour derivations we see that one needs special types of commutative bispaces and colours.

More precisely, we will require that in the definition of commutative bispace morphism φ has the following form:

$$\varphi_m(\beta) = {}^{\hat{m}}\beta,$$

for some map $\hat{}: M \longrightarrow G$, satisfying the condition analogous 1.11.: $\widehat{\alpha m} = \alpha\hat{m}$ and $\sigma_m(\alpha) = s_{\hat{m},\alpha}$, for all $\alpha, \beta \in G, m \in M$.

Definitions.

(1) Let M be a G–bispace. We say that M is a **symmetric bispace** if there is a G–bispace map $\hat{}: M \longrightarrow G$ such that $m\cdot\alpha = {}^{\hat{m}}\alpha\cdot m$, for all $m \in M, \alpha \in G$.

(2) Let $\alpha \in G$. A map $\varphi : M \longrightarrow N$ of symmetric bispaces will be called an α– **map** if $\varphi(\beta m) = {}^{\alpha}\beta\varphi(m)$.

(3) A k–morphism $f : P_M \longrightarrow Q_N$ of colour symmetric bimodules over symmetric bispaces will be called φ–**morphism** if $f(P_m) \subset Q_{\varphi(m)}$, where φ is an α–map, and the following properties hold:

$$f(a_\beta \cdot x_m) = s_{\alpha,\beta} \cdot a_\beta \cdot f(x_m), \tag{1}$$

$$f \circ \beta = s_{\alpha,\beta} \cdot \beta \circ f, \tag{2}$$

for all $\beta \in G, x_m \in P_m, m \in M$.

2.3. Let $\mathrm{Map}_\alpha(M,N)$ be a set of all α–maps and

$$\mathrm{Map}_*(M,N) = \bigcup_{\alpha\in G} \mathrm{Map}_\alpha(M,N).$$

Denote by $\mathrm{Hom}_\varphi(P_M, Q_N)$ a set of all φ–morphisms, where $\varphi \in \mathrm{Map}_\alpha(M, N)$, and

$$\mathrm{Hom}_*(P_M, Q_N) = \sum_{\varphi \in \mathrm{Map}_*(M,N)} \mathrm{Hom}_\varphi(P_M, Q_N).$$

The set $\mathrm{Map}_*(M, N)$ is a symmetric bispace with respect to G–actions

$$(\beta \cdot \varphi)(m) = \beta(\varphi(m)), \quad (\varphi \cdot \beta)(m) = \varphi(\beta m)$$

and a map $\hat{}: \mathrm{Map}_*(M, N) \longrightarrow G$, where $\hat{\varphi} = \alpha$, if $\varphi \in \mathrm{Map}_\alpha(M, N)$.

The k–module $\mathrm{Hom}_*(P_M, Q_N)$ is a $\mathrm{Map}_*(M, N)$–graded module by definition.

Moreover, if one defines the left and right A–structures by

$$(a_\beta \cdot f)(x_m) = a_\beta \cdot (f(x_m)), \quad (f \cdot a_\beta)(x_m) = f(a_\beta \cdot x_m),$$

where $a_\beta \in A_\beta, x_m \in P_m, f \in \mathrm{Hom}_\varphi(P_M, Q_N)$, one gets a symmetric A–module with

$$A_\beta \cdot \mathrm{Hom}_\varphi(P_M, Q_N) \subset \mathrm{Hom}_{\beta\varphi}(P_M, Q_N),$$

$$\mathrm{Hom}_\varphi(P_M, Q_N) \cdot A_\beta \subset \mathrm{Hom}_{\varphi\beta}(P_M, Q_N),$$

and

$$f \cdot a_\beta = s_{\alpha,\beta} \cdot a_\beta \cdot f,$$

if $\varphi \in \mathrm{Map}_\alpha(M, N)$.

Summarizing, we have the following result.

Theorem. *Let M and N be symmetric bispaces and P_M, Q_N be M and N–graded A–modules respectively. Then:*

1. *$\mathrm{Map}_*(M, N)$ is a symmetric bispace,*
2. *$\mathrm{Hom}_*(P_M, Q_N)$ is a symmetric $A - A$ bimodule, and*
3. *$f \in \mathrm{Hom}_\varphi(P_M, Q_N), \varphi \in \mathrm{Map}_\alpha(M, N)$, and $g \in \mathrm{Hom}_\psi(Q_N, R_K)$, $\psi \in \mathrm{Map}_\beta(M, N)$, then $\varphi \circ \psi \in \mathrm{Map}_{\alpha\beta}(M, K)$ and $g \circ f \in \mathrm{Hom}_{\varphi\psi}(P_M, R_K)$.*

2.4. Denote by $\mathrm{Hom}^\varphi_k(P_M, Q_N)$ the set of all k–linear morphisms $\Delta : P_M \longrightarrow Q_N$ of degree $\varphi \in \mathrm{Map}_\alpha(M, N)$, i.e. $\Delta(P_m) \subset Q_{\varphi(m)}$, and such that $\Delta \circ \beta = s_{\alpha,\beta}\beta \circ \Delta$ for all $\beta \in G$.

For any homogeneous element $a_\beta \in A_\beta$ we denote by l_{a_β}, r_{a_β} and δ_{a_β} the following operators:

$$l_{a_\beta}(\Delta) = a_\beta \cdot \Delta, \quad r_{a_\beta}(\Delta) = \Delta \cdot a_\beta, \quad \delta_{a_\beta} = r_{a_\beta}(\Delta) - l_{\alpha(a_\beta)}(\Delta).$$

Definition. The element $\Delta \in \mathrm{Hom}^\varphi_k(P_M, Q_N)$ is a **colour homomorphism** if $\delta_{a_\beta}(\Delta) = 0$, for all homogeneous elements $a_\beta \in A_\beta$ and $\beta \in G$.

Denote by $\mathrm{Hom}^\varphi_A(P_M, Q_N)$ the set of all colour homomorphisms and define colour differential operators by induction on the order.

Definition.

(1) Colour differential operators of order 0 are colour homomorphisms.
(2) The element $\Delta \in \mathrm{Hom}_k^{\varphi}(P_M, Q_N)$ is a **colour differential operator** over colour commutative algebra A of order $\leq l+1$, $\quad l = 0, 1, \ldots$ and degree φ acting from the colour symmetric module P to the colour symmetric module Q if $\delta_{a_\beta}(\Delta)$ is a colour differential operator of order $\leq l$ and degree $\beta \cdot \varphi$ for any homogeneous element $a_\beta \in A_\beta, \beta \in G$.

Denote by $\mathrm{Diff}_l^{\varphi}(P_M, Q_N)$ the set of all colour differential operators of order $\leq l$ and degree φ acting from P_M to Q_N.

The set of colour differential operators of order $\leq l$,

$$\mathrm{Diff}_l(P_M, Q_N) = \sum_{\varphi \in \mathrm{Map}(M,N)} \mathrm{Diff}_l^{\varphi}(P_M, Q_N)$$

is stable with respect to left l_{a_β} and right r_{a_β} multiplications in

$$\mathrm{Hom}_k(P_M, Q_N) = \sum_{\varphi \in \mathrm{Map}(M,N)} \mathrm{Hom}_k^{\varphi}(P_M, Q_N)$$

and therefore possesses an $A - A$ bimodule structure.

The definition of colour differential operators implies the existence of imbeddings: $\mathrm{Diff}_l^{\varphi}(P_M, Q_N) \subset \mathrm{Diff}_{l+1}^{\varphi}(P_M, Q_N)$.

Denote by

$$\mathrm{Diff}_*^{\varphi}(P_M, Q_N) = \bigcup_{l \geq 0} \mathrm{Diff}_l^{\varphi}(P_M, Q_N)$$

the k–module of all colour differential operators of degree φ and by

$$\mathrm{Diff}_*(P_M, Q_N) = \sum_{\varphi \in \mathrm{Map}(M,N)} \mathrm{Diff}_*^{\varphi}(P_M, Q_N)$$

the filtered $A - A$ bimodule of all colour differential operators.

2.5. Let $\Delta_1 \in \mathrm{Diff}_l^{\varphi}(P_M, Q_N)$, $\quad \Delta_2 \in \mathrm{Diff}_t^{\psi}(Q_N, R_K)$ be colour differential operators of degrees $\varphi \in \mathrm{Map}_\alpha(M, N)$, $\quad \psi \in \mathrm{Map}_\beta(N, K)$ respectively.

An easy computation shows that for any element $a = a_\gamma \in A_\gamma$ we have the following identity:

$$\delta_a(\Delta_2 \circ \Delta_1) = \Delta_2 \circ \delta_a(\Delta_1) + \delta_{\alpha(a)}(\Delta_2) \circ \Delta_1.$$

This implies the following

Proposition. *The composition $\Delta_2 \circ \Delta_1$ of colour differential operators $\Delta_1 \in \mathrm{Diff}_l^{\varphi}(P_M, Q_N)$ and $\Delta_2 \in \mathrm{Diff}_t^{\psi}(Q_N, R_K)$ is a colour differential operator $\Delta_2 \circ \Delta_1 \in \mathrm{Diff}_{l+t}^{\psi\circ\varphi}(P_M, R_K)$.*

Any morphism $\varphi \in \mathrm{Map}(G, M)$ can be determined by the element $m = \varphi(e) \in M$. We denote by $\mathrm{Diff}_l^{m}(P_M) = \mathrm{Diff}_l^{\varphi}(A, P_M)$ the corresponding module of colour differential operators, and

$$\mathrm{Diff}_l(P_M) = \sum_{m \in M} \mathrm{Diff}_l^{m}(P_M), \quad \mathrm{Diff}_*(P_M) = \bigcup_{l \geq 0} \mathrm{Diff}_l(P_M).$$

The composition of colour differential operators defines an associative algebra structure in $\mathrm{Diff}_*(A)$ and right $\mathrm{Diff}_*(A)$–module structure in the module $\mathrm{Diff}_*(P_M)$.

Moreover, the rule $\alpha(\Delta_m) = s_{\alpha,\hat{m}} \cdot \Delta_m$, where $\alpha \in G$, $\Delta_m \in \mathrm{Diff}_*^m(P_M)$, defines a G–module structure in $\mathrm{Diff}_*(P_M)$.

Suppose $\Delta : P_M \longrightarrow Q_N$ is a colour differential operator of order $\leq l$. Then the composition $f_\Delta : \nabla \in \mathrm{Diff}_t^m(P_M) \mapsto \Delta \circ \nabla \in \mathrm{Diff}_{l+t}^{\varphi(m)}(Q_N)$ defines a homomorphism of filtered right $\mathrm{Diff}_*(A)$–modules:

$$f_\Delta : \mathrm{Diff}_*(P_M) \longrightarrow \mathrm{Diff}_*(Q_N).$$

Example. Let A be the algebra of the quantum hyperplane generated by elements $x_1, \dots x_n$ and relations $x_i x_j = \omega_{ij} x_j x_i$, for some 2-cocycle ω (see examples 1.8,2.8.). Then the module $\mathrm{Der}_*(A, A) = \sum_{\bar{n} \in \mathbb{Z}^n} \mathrm{Der}_{\bar{n}}(A, A)$ is generated by the partial derivations $\partial/\partial x_i \in \mathrm{Der}_{-1_i}(A, A)$, where $1_i = (0, ..., o, \underset{i}{1}, 0, ..., 0) \in \mathbb{Z}^n$, and $\partial/\partial x_i(x_j) = \delta_{ij}$.

The colour Leibniz rule gives the relations:

$$\partial/\partial x_i \cdot x_j - \omega_{ij}^{-1} x_j \cdot \partial/\partial x_i = \delta_{ij},$$

and

$$\partial/\partial x_i \cdot \partial/\partial x_j - \omega_{ij} \partial/\partial x_j \cdot \partial/\partial x_i = 0.$$

The algebra of colour differential operators $\mathrm{Diff}_*(A, A)$ is a $\mathbb{Z}^n$– graded algebra generated by elements x_i of degree 1_i, $i = 1, ...n$, and $\partial/\partial x_j$ of degree -1_j, $j = 1, ...n$, and the relations above.

2.6. The graded module associated with the filtered module $\mathrm{Diff}_*(P_M)$,

$$\mathrm{Smbl}_*(P_M) = \sum_{t \geq 0, m \in M} \mathrm{Smbl}_t^m(P_M),$$

where

$$\mathrm{Smbl}_t^m(P_M) = \frac{\mathrm{Diff}_t^m(P_M)}{\mathrm{Diff}_{t-1}^m(P_M)},$$

is the **module of colour symbols** of P_M.

Since the map f_Δ is compatible with the filtration, it defines a morphism

$$\mathrm{smbl}_*(\Delta) : \mathrm{Smbl}_*(P_M) \longrightarrow \mathrm{Smbl}_*(Q_N)$$

and k–morphisms

$$\mathrm{smbl}_t(\Delta) : \mathrm{Smbl}_t^m(P_M) \longrightarrow \mathrm{Smbl}_{l+t}^{\varphi(m)}(Q_N)$$

called **t–symbols** of the operator Δ.

Proposition. *Let* $\Delta_\alpha \in \mathrm{Diff}_t^\alpha(A)$, $\quad \Delta_\beta \in \mathrm{Diff}_s^\beta(A)$. *Then the* **colour commutator**

$$[\Delta_\alpha, \Delta_\beta] = \Delta_\alpha \circ \Delta_\beta - s_{\alpha,\beta} \cdot \Delta_\beta \circ \Delta_\alpha$$

is a colour differential operator of order $\leq s+t-1$ *and degree* $\alpha\beta$.

Proof. We have

$$\delta_a(\Delta_\alpha) = [\Delta_\alpha, l_a]$$

if $a = a_\gamma \in A_\gamma$.

Therefore, by using the colour Jacobi identity, we get

$$\begin{aligned}\delta_a[\Delta_\alpha, \Delta_\beta] =& [[\Delta_\alpha, \Delta_\beta], l_a] = \\ & [\Delta_\alpha, [\Delta_\beta, l_a]] + s_{\alpha,\beta} \cdot [\Delta_\beta, [\Delta_\alpha, l_a]] = \\ & [\Delta_\alpha, \delta_a(\Delta_\beta)] + s_{\alpha,\beta} \cdot [\Delta_\beta, \delta_a(\Delta_\alpha)].\end{aligned}$$

□

Denote by $\mathrm{smbl}(\Delta) = \Delta \mod \mathrm{Diff}_{t-1}^\alpha(A)$, if $\Delta \in \mathrm{Diff}_t^\alpha(A)$, the **symbol** of the colour differential operator.

Then the following theorem is a direct consequence of the proposition above.

Theorem.

(1) *The symbol algebra* $\mathrm{Smbl}_*(A)$ *is a colour commutative algebra:*

$$\mathrm{smbl}(\Delta_\alpha) \cdot \mathrm{smbl}(\Delta_\beta) = s_{\alpha,\beta} \cdot \mathrm{smbl}(\Delta_\beta) \cdot \mathrm{smbl}(\Delta_\alpha).$$

(2) *The bracket*

$$\{\mathrm{smbl}(\Delta_a), \mathrm{smbl}(\Delta_\beta)\} = [\Delta_\alpha, \Delta_\beta] \mod \mathrm{Diff}_{s+t-1}^{\alpha\beta}(A),$$

where $\Delta_\alpha \in \mathrm{Diff}_t^\alpha(A)$ *and* $\Delta_\beta \in \mathrm{Diff}_s^\beta(A)$, *determines a* **colour Poisson structure.**

It means that the bracket

$$\{\quad,\quad\} : \mathrm{Smbl}_t^\alpha(A) \times \mathrm{Smbl}_s^\beta(A) \longrightarrow \mathrm{Smbl}_{s+t-1}^{\alpha\beta}(A)$$

satisfies the following conditions:

[2.1.] *the* **colour skew symmetry***:*

$$\{\mathrm{smbl}(\Delta_\alpha), \mathrm{smbl}(\Delta_\beta)\} = -s_{\alpha,\beta} \cdot \{\mathrm{smbl}(\Delta_\beta), \mathrm{smbl}(\Delta_\alpha)\},$$

[2.2.] *the* **colour Jacobi identity***:*

$$\{\mathrm{smbl}(\Delta_\alpha), \{\mathrm{smbl}(\Delta_\beta), \mathrm{smbl}(\Delta_\gamma)\}\} =$$
$$\{\{\mathrm{smbl}(\Delta_\alpha), \mathrm{smbl}(\Delta_\beta\}, \mathrm{smbl}(\Delta_\gamma)\} + s_{\alpha,\beta} \cdot \{\mathrm{smbl}(\Delta_\beta), \{\mathrm{smbl}(\Delta_\alpha), \mathrm{smbl}(\Delta_\gamma)\}\},$$

[2.3.] *the* **action condition***:*

$$\mathrm{smbl}(\alpha(\Delta_\beta)) = s_{\alpha,\beta} \cdot \mathrm{smbl}(\Delta_\beta),$$

[2.4.] *the* **colour derivation condition***:*

$$\{\mathrm{smbl}(\Delta_\alpha), \mathrm{smbl}(\Delta_\beta) \cdot \mathrm{smbl}(\Delta_\gamma)\} =$$
$$\{\mathrm{smbl}(\Delta_\alpha), \mathrm{smbl}(\Delta_\beta)\} \cdot \mathrm{smbl}(\Delta_\gamma) + s_{\alpha,\beta} \cdot (\mathrm{smbl}(\Delta_\beta)) \cdot \{\mathrm{smbl}(\Delta_\alpha), \mathrm{smbl}(\Delta_\gamma)\}.$$

2.7. In this section we build up the representative object for the functor of colour derivations.

Denote by

$$\Omega^1(A) = \sum_{\alpha\in G} \Omega^1_\alpha(A)$$

the A–module generated by elements $a_\beta da_\gamma$ of degree $\beta\gamma$, with

(1) the usual relations:

$$d(a_\alpha + a_\beta) = da_\alpha + da_\beta, \quad d(a_\alpha \cdot a_\beta) = da_\alpha \cdot a_\beta + a_\alpha \cdot da_\beta,$$

and

(2) the relations of colour constancy, $ds_{\alpha,\beta} = 0$.

Note that relation (1) can be considered as the definition of a right A– module structure on $\Omega^1(A)$.

Let $d : A \longrightarrow \Omega^1(A)$ be the operator: $d : a_\alpha \mapsto da_\alpha$.

Theorem. *For any colour derivation $X_m : A \longrightarrow P_M$ of degree $m \in M$ there is a homomorphism $f_m \in \mathrm{Hom}_m(\Omega^1(A), P_M)$ of degree $m \in M$ such that*

$$X_m = f_m \circ d,$$

The homomorphism f_m is uniquely determined, and the correspondence $X_m \mapsto f_m$ establishes an isomorphism between $\mathrm{Der}_m(A, P_M)$ and $\mathrm{Hom}_m(\Omega^1(A), P_M)$.

Proof. If we set

$$f_m(a_\alpha da_\beta) = (s_{\hat{m},\alpha} \cdot a_\alpha) \cdot X_m(a_\beta),$$

we transform the usual Leibniz rule for the operator d into the colour Leibniz rule for the derivation X_m. □

2.8. Starting from $A - A$ bimodule $\Omega^1(A)$ and a colour commutative algebra $\Omega^0(A) = A$ we build up an algebra of colour differential forms over A.

This algebra will be a new colour commutative algebra

$$\Omega^*(A) = \sum_{i\in\mathbb{N},\alpha\in G} \Omega^i_\alpha(A),$$

graded by the group $\bar{G} = \mathbb{Z}\times G$ and generated by elements $a_\alpha \in A_\alpha = \Omega^0_\alpha$ of degree $(0, \alpha)$ and their differentials $da_\alpha \in \Omega^1_\alpha$ of degree $(1, \alpha)$.

We will require also that the universal derivation $d : A \longrightarrow \Omega^1(A)$ can be extended to a colour derivation of the algebra $\Omega^*(A)$ with degree $(1, e)$ in such a way that $d^2 = 0$.

The last condition on d dictates some restrictions on a colour in the algebra $\Omega^*(A)$. To get the restrictions we consider a some colour $\bar{s} : \bar{G} \times \bar{G} \longrightarrow U(A)$ on $\Omega^*(A)$ such that $\bar{s}|_{G\times G} = s$.

Hexagon condition 1.6.(2) for $\bar{s}$ may be written in the form

$$\bar{s}_{(n+m,\alpha\beta),(k,\gamma)} = \bar{s}_{(n,\alpha),(k,{}^\beta\gamma)} \cdot \bar{s}_{(m,\beta),(k,\gamma)}, \tag{1}$$

where $k, m, n \in \mathbb{N}, \quad \alpha, \beta, \gamma \in G$. Denote by $\omega \wedge \theta \in \Omega^{n+m}_{\alpha\beta}(A)$ the product of forms $\omega \in \Omega^n_\alpha(A), \theta \in \Omega^m_\beta$ in the algebra $\Omega^*(A)$. Then,

$$d(\omega \wedge \theta) = d\omega \wedge \theta + \bar{s}_{(1,e),(n,\alpha)} \cdot \omega \wedge d\theta,$$

and

$$d^2(\omega \wedge \theta) = \bar{s}_{(1,e),(n+1,\alpha)} \cdot d\omega \wedge d\theta + \bar{s}_{(1,e),(n,\alpha)} \cdot d\omega \wedge d\theta = 0.$$

Hence, we should require that

$$\bar{s}_{(1,e),(n+1,\alpha)} + \bar{s}_{(1,e),(n,\alpha)} = 0. \tag{2}$$

From these relations it follows that

$$\bar{s}_{(1,e),(k,\alpha)} = (-1)^k \varphi(\alpha),$$

where $\varphi : G \longrightarrow U(A_e), \quad \varphi(\alpha) = \bar{s}_{(1,e),(0,\alpha)}$.

Taking $n = m = 0, k = 1, \gamma = e$ in hexagon equation (1), one gets that φ is a group homomorphism. So, if we fix a colour $s : G \times G \longrightarrow U(A)$ and a group homomorphism $\varphi : G \longrightarrow U(A_e)$, then from equation (1) we get

$$\bar{s}_{(n,\alpha),(m,\beta)} = (-1)^{nm} \varphi(\alpha)^{-m} \cdot s_{\alpha,\beta} \cdot \varphi(\beta)^n, \tag{3}$$

and compatibility conditions 1.7.(2) are fulfilled.

Proposition. *Let A be a colour commutative algebra with a colour s. Then any colour $\bar{s}$ on the group $\bar{G} = \mathbb{Z} \times G$ with conditions $\bar{s}|_{G\times G} = s$ and 2.7.(2) are given by formula (3) for some group homomorphism $\varphi : G \longrightarrow U(A_e)$.*

Below we will denote by $\Omega^*(A, \varphi)$, or simply by $\Omega^*(A)$, the colour commutative $\bar{G}$–graded algebra with the colour derivation $d = d_\varphi$ of degree $(1.e)$.

Therefore for any colour commutative algebra A, a group homomorphism $\varphi : G \longrightarrow U(A_e)$, and an element $\alpha \in G$, we have the complex:

$$0 \longrightarrow A \xrightarrow{d_\varphi} \Omega^1_\alpha(A, \varphi) \xrightarrow{d_\varphi} \Omega^2_\alpha(A, \varphi) \xrightarrow{d_\varphi} \cdots \xrightarrow{d_\varphi} \Omega^i_\alpha(A, \varphi) \xrightarrow{d_\varphi} \Omega^{i+1}_\alpha(A, \varphi) \xrightarrow{d_\varphi} \cdots.$$

The cohomology of the complexe at the term $\Omega^i_\alpha(A, \varphi)$ we will denote by $H^i_\alpha(A, \varphi)$, and will be called as the *colour de Rham cohomology* of the algebra A.

Because of d_φ is a colour derivation of the colour commutative algebra $\Omega^*(A, \varphi)$ then

$$H^*(A, \varphi) = \sum_{i\in\mathbb{N}, \alpha\in G} H^i_\alpha(A, \varphi)$$

is a $\mathbb{Z} \times G$–graded colour commutative algebra with respect to the colour $\bar{s}$.

Examples.

(1) Let $G = \{e\}$. Then homomorphism φ is a also trivial, and we get the standard colour on the algebra of differential forms: $\bar{s}_{m,n} = (-1)^{mn}$.

(2) Let $G = \mathbb{Z}_2$. There are two possibilities for φ: (1) φ is a trivial homomorphism, and (2) $\varphi(\bar{1}) = -1$. Thus there are two algebras of differential forms linked with a commutative $\mathbb{Z}_2$–algebras.

(3) Consider $A = \mathrm{Mat}_n(\mathbb{C})$ as $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$–graded colour commutative algebra. Then $\varphi \in \bar{G} = \mathrm{Hom}(G, \mathbb{C}^*)$ is a character of the group. Therefore, any character on G gives us the algebra of colour differential forms on $\mathrm{Mat}_n(\mathbb{C})$.

(4) The quantum hyperplane [cf.WZ,BP] is given by the following data:
$k = \mathbb{C}, G = \mathbb{Z}^n$, and the twisted 2-cocycle:

$$\omega(\bar{a}, \bar{b}) = q^{\langle \vartheta\, \bar{a}, \bar{b} \rangle},$$

where ϑ is a skew-symmetric $n \times n$ matrix, $q \in \mathbb{C}^*$, $\quad \bar{a}, \bar{b} \in \mathbb{Z}^n$.

Let A be the algebra generated by elements $x_1, \ldots x_n$ and the relations: $x_i x_j = \omega_{ij} x_j x_i$, where ω_{ij} are matrix elements of ω. A is obviously a colour commutative algebra and we can build up the algebra of colour differential forms on A. For this end we note that any group homomorphism
$\varphi : G \longrightarrow \mathbb{C}^*$ has the form $\varphi(\bar{a}) = z^{\bar{a}}$, for some complex vector
$z = (z_1, \ldots z_n) \in (\mathbb{C}^*)^n$.

The algebra $\Omega^*(A, \varphi)$ is now a colour commutative algebra generated by elements $x_i, y_j = dx_j, \quad 1 \le i, j \le n$, and the relations:

$$x_i x_j = \omega_{ij} x_j x_i, \quad x_i y_j = z_i \omega_{ij} y_j x_i, \quad y_i y_j = -z_i z_j \omega_{ij} y_j y_i.$$

2.9. Here we outline the construction of Lie derivations and Nijenhuis brackets over colour commutative algebras.

Let's describe derivations of the algebra $\Omega^*(A)$. Denote by $\mathrm{Der}^{alg}_* \Omega^*(A)$ submodule of all algebraic derivations, i.e. colour derivations X of $\Omega^*(A)$ such that $X|_{\Omega^0(A)} = 0$.

Since any algebraic derivation is determined by values on $\Omega^1(A)$ we get the isomorphism:

$$\mathrm{Der}^{alg}_{(k,\alpha)} \Omega^*(A) \simeq \mathrm{Hom}_\alpha(\Omega^1(A), \Omega^{k+1}(A)) \simeq \mathrm{Der}_\alpha(\Omega^{k+1}(A)). \tag{1}$$

For any derivation $X \in \mathrm{Der}_\alpha(\Omega^{k+1}(A))$ we will denote by $\imath_X$ the corresponding algebraic (*inner*) derivation of $\Omega^*(A)$.

In other words the operator may be defined as the colour derivation in $\Omega^*(A)$ such that:

(1) $\imath_X : \Omega^j_\beta(A) \longrightarrow \Omega^{j+k}_{\alpha\beta}(A)$,
(2) $\imath_X(\omega_1 \wedge \omega_2) = \imath_X(\omega_1) \wedge \omega_2 + (-1)^{jk} \varphi(\alpha)^{-j} \cdot s_{\alpha,\beta} \cdot \varphi(\beta)^k \cdot \omega_1 \wedge \imath_X(\omega_2)$,
(3) $\imath_X(a_\beta) = 0, \quad \imath_X(da_\beta) = X(a_\beta)$,

where $j, k \in \mathbb{N}, \quad \alpha, \beta \in G, \omega_1 \in \Omega^j_\beta(A), a_\beta \in A_\beta$.

The module of colour algebraic derivations is obviously closed with respect to the colour commutator of derivations.

Therefore we get a colour Lie algebra structure on

$$\mathcal{N}ij(A) = \sum_{k \in \mathbb{Z}, \alpha \in G} \mathrm{Hom}_\alpha(\Omega^1(A), \Omega^{k+1}(A)).$$

The colour Lie algebra $\mathcal{N}ij(A)$ will be called a *Nijenhuis algebra* of the colour commutative algebra A and the bracket will be called a *colour algebraic Nijenhuis bracket.*

By definition the Nijenhuis bracket of elements $X \in \mathrm{Hom}_\alpha(\Omega^1(A), \Omega^{k+1}(A))$ and $Y \in \mathrm{Hom}_\beta(\Omega^1(A), \Omega^{l+1}(A))$ is given by the formula

$$[X,Y](\omega) = \imath_X(Y(\omega)) - (-1)^{kl} \cdot \varphi(\alpha)^{-l} \cdot s_{\alpha,\beta} \cdot \varphi(\beta)^k \cdot \imath_Y(X(\omega)),$$

for all $\omega \in \Omega^1(A)$.

Any derivation $X \in \mathrm{Der}_\alpha(A)$ determines an inner derivation $\imath_X \in \mathrm{Der}_{(-1,\alpha)}(\Omega^*(A))$ and a *Lie derivation*: $\mathcal{L}_X = [\imath_X, d]$.

For Lie derivations we have the same properties as for the usual ones.

Theorem.

$\mathcal{L}_X$ is a colour (k,α)– derivation of the algebra of colour differential forms $\Omega^*(A)$:

$$\mathcal{L}_X(\omega_1 \wedge \omega_2) = \mathcal{L}_X(\omega_1) \wedge \omega_2 + \varphi(\alpha)^{-k} \cdot s_{\alpha,\beta} \cdot \omega_1 \wedge \mathcal{L}_X(\omega_2).$$

(1) *The bracket $[\mathcal{L}_X, \mathcal{L}_Y]$ is a Lie derivation $\mathcal{L}_Z$ for some element $Z = [X,Y]$, and is called the* **Frölicher - Nijenhuis bracket.**
(2) *The Frölicher-Nijenhuis bracket*

$$\mathrm{Hom}_\alpha(\Omega^1(A), \Omega^{k+1}(A)) \times \mathrm{Hom}_\beta(\Omega^1(A), \Omega^{l+1}(A)) \longrightarrow \mathrm{Hom}_{\alpha\beta}(\Omega^1(A), \Omega^{k+l+2}(A))$$

determines a $\bar{G}$–graded $\bar{s}$–colour Lie algebra structure in the Nijenhuis algebra.

Here $\omega_1 \in \Omega^k_\beta(A), \omega_2 \in \Omega^l_\gamma(A)$.

2.10. In this section we outline the construction of the modules of colour jets as representative objects for the functors of colour differential operators (see [KLV] for the usual case).

Let P_M be a symmetric colour module. Consider the tensor product $A \underset{k}{\otimes} P_M$ as a colour M–graded module too. To do this we assume that elements $a_\alpha \otimes x_m, \quad a_\alpha \in A_\alpha, x_m \in P_m$, have the degree αm and the G–action is given by the formula $\beta(a_\alpha \otimes x_m) = \beta(\alpha) \otimes \beta(x_m)$. The left A–module structure in $A \underset{k}{\otimes} P_M$ is induced by multiplication in the first factor.

For any element $a_\alpha \in A_\alpha$ we define the endomorphism $\delta^{a_\alpha} : A \underset{k}{\otimes} P_M \longrightarrow A \underset{k}{\otimes} P_M$ as follows:

$$\delta^{a_\alpha}(a_\beta \otimes x_m) = a_\beta \cdot a_\alpha \otimes x_m - a_\beta \otimes a_\alpha x_m.$$

Then δ^{a_α} are endomorphisms such that $\delta^{a_\alpha}(A_\beta \underset{k}{\otimes} P_m) \subset (A \underset{k}{\otimes} P_M)_{\beta\alpha m}$, and $\delta^{a_\alpha}(a_\beta \otimes z) = a_\beta \otimes \delta^{a_\alpha}(z)$.

Let μ_{l+1} be the M–graded submodule of $A \underset{k}{\otimes} P_M$ generated by all the elements of the form

$$\delta^{a_{\alpha_0}} \circ \delta^{a_{\alpha_1}} \circ \cdots \circ \delta^{a_{\alpha_l}}(a_\beta \otimes x_m).$$

Denote by $\mathcal{J}^l(P_M)$ the quotient module $A \underset{k}{\otimes} P_M/\mu_{l+1}$ and by $j_l : P_M \longrightarrow \mathcal{J}^l(P_M)$ the map: $j_l(x_m) = (1 \otimes x_m) \mod \mu_{l+1}$.

We will call the modules $\mathcal{J}^l(P_M)$, $\quad l = 0, 1...$ as **modules of colour jets** and j_l as **colour l-jets operators.**

One has

$$\begin{aligned}&\delta_{a_\alpha}(j_l)(x_m) = j_l(a \cdot x_m) - a_\alpha \cdot j_l(x_m) =\\ &(1 \otimes a_\alpha \cdot x_m - a_\alpha \otimes x_m) \mod \mu_{l+1} = -\delta^{a_\alpha}(1 \otimes x_m) \mod \mu_{l+1}\end{aligned}$$

and $j_l \circ \beta = \beta \circ j_l$.

Therefore, $\delta_{a_{\alpha_0}} \circ \cdots \circ \delta_{a_{\alpha_l}}(j_l) = 0$, and $j_l : P_M \longrightarrow \mathcal{J}^l(P_M)$ is a colour differential operator of degree $id : M \longrightarrow M$ and order l.

Theorem. *For any colour differential operator $\Delta : P_M \longrightarrow Q_N$ of degree $\varphi \in \mathrm{Map}_\alpha(M, N)$ and order $l \geq 0$ there is a colour homomorphism $f^\Delta : \mathcal{J}^l(P_M) \longrightarrow Q_N$ of degree φ such that $\Delta = f^\Delta \circ j_l$. The homomorphism f^Δ is uniquely determined, and the correspondence $\Delta \mapsto f^\Delta$ establishes an isomorphism between $\mathrm{Diff}_l^\varphi(P_M, Q_N)$ and $\mathrm{Hom}_\varphi(\mathcal{J}^l(P_M), Q_N)$.*

Proof. The uniqueness of the morphism f^Δ is obvious. In order to show existence we define $\bar{f}^\Delta : A \underset{k}{\otimes} P_M \longrightarrow Q_N$ by putting

$$\bar{f}^\Delta(a_\beta \otimes x_m) = \alpha(a_\beta) \cdot \Delta(x_m).$$

Then

$$\begin{aligned}&\bar{f}^\Delta(\delta^{a_\beta}(a_\gamma \otimes x_m)) = \bar{f}^\Delta(a_\gamma \otimes a_\beta x_m - a_\gamma a_\beta \otimes x_m) =\\ &\alpha(a_\gamma)\Delta(a_\beta x_m) - \alpha(a_\gamma a_\beta)\Delta(x_m) = \alpha(a_\gamma)\delta_{a_b e}(\Delta)(x_m).\end{aligned}$$

Therefore, $\bar{f}^\Delta|_{\mu_{l+1}} = 0$ and we get the morphism $f^\Delta : \mathcal{J}^l(P_M) \longrightarrow Q_N$, such that $\Delta = f^\Delta \circ j_l$. $\square$

2.11. Let $\Delta : P_M \longrightarrow Q_N$ be a colour differential operator of order l and degree $\varphi \in \mathrm{Map}_\alpha(M, N)$. We define the *t-jet prolongation* of Δ in the usual way as the composition $\Delta^{(t)} = j_t \circ \Delta : P_M \longrightarrow \mathcal{J}^t(Q_N)$.

Denote by $f^{\Delta^{(t)}}$ the corresponding homomorphisms: $\Delta^t = f^{\Delta^t} \circ j_{l+t}$.

The embeddings $\mathrm{Diff}_t^m(P_M) \subset \mathrm{Diff}_u^m(P_M)$, $\quad t \leq u$, generate epimorphisms $\pi_{u,t} : \mathcal{J}_m^u(P_M) \longrightarrow \mathcal{J}_m^t(P_M)$, $\quad m \in M$, such that $\pi_{u,t} \circ j_u = j_t$.

Moreover, $\pi_{u,t} \circ \pi_{v,u} = \pi_{v,t}$, for all $t \leq u \leq v$.

Denote by $\mathrm{Cosmbl}_m^t(P_M)$ the kernel of the projection $\pi_{t,t-1} : \mathcal{J}_m^t(P_M) \longrightarrow \mathcal{J}_m^{t-1}(P_M)$ and by

$$\mathrm{Cosmbl}_*(P_M) = \sum_{t \geq 0, m \in M} \mathrm{Cosmbl}_m^t(P_M)$$

the *cosymbol module* of P_M.

One has $\pi_{t,t-1} \circ f^{\Delta^{(t)}} = f^{\Delta^{t-1}} \circ \pi_{t+l,t+l-1}$ and therefore any colour differential operator Δ determines a homomorphism

$$\mathrm{Cosmbl}_m^{l+t}(\Delta) : \mathrm{Cosmbl}_m^{t+l}(P_M) \longrightarrow \mathrm{Cosmbl}_{\varphi(m)}^t(Q_N)$$

of degree φ.

Definitions.

(1) Let $\Delta : P_M \longrightarrow Q_N$ be a colour differential operator of order l and degree $\varphi \in \mathrm{Mor}_\varphi(M,N)$. Then M–graded submodule $\mathcal{R}_l = \mathrm{Ker}\, f^\Delta \subset \mathcal{J}^l(P_M)$ is called a *colour differential equation.* $\mathcal{R}_{l+t} = \mathrm{Ker}\, f^{\Delta^{(t)}} \subset \mathcal{J}^{l+t}(P_M)$ is called *t–prolongations* of $\mathcal{R}_l$.

(2) The M–graded module $g_l = \sum_{m\in M} g_l^m$, where $g_l^m = \mathrm{Ker\,Cosmbl}_m^l(\Delta)$, is called a *symbol of the equation* $\mathcal{R}_l$, and $g_{l+t} = \sum_{m\in M} g_{l+t}^m$, where $g_{l+t}^m = \mathrm{Ker\,Cosmbl}_{l+t}^m(\Delta)$, is called *$t$–prolongations* of the symbol.

Similarly ([KLV]) by using of colour de Rham operator $d = d_\varphi$, we can define **Spencer colour operators**

$$S = S_\varphi : \Omega^i(A,\varphi) \underset{A}{\otimes} \mathcal{J}^k(P_M) \longrightarrow \Omega^{i+1}(A,\varphi) \underset{A}{\otimes} \mathcal{J}^{k-1}(P_M),$$

such that $S(\omega \otimes j_k(p)) = d\omega \otimes j_{k-1}(p)$, for all $\omega \in \Omega^i(A), p \in P_M$.

For any colour differential equation we have $S(\mathcal{R}_{l+t} \subset \Omega^1(A) \underset{A}{\otimes} \mathcal{R}_{l+t-1}$.

Therefore, Spencer operators produce the complexes:

$$0 \longrightarrow \mathcal{R}_{l+t} \longrightarrow \Omega^1(A,\varphi) \underset{A}{\otimes} \mathcal{R}_{l+t-1} \longrightarrow \Omega^2(A,\varphi) \underset{A}{\otimes} \mathcal{R}_{l+t-2} \longrightarrow \cdots,$$

which cohmologies are **Spencer colour cohomologies** of colour differential equation $\mathcal{R}_l$.

3.Graded monoidal categories and colour quantizations

In this chapter we compare the colour calculus developed above with the calculus in braided tensor categories suggested in [L1]. To do this we describe two monoidal categories linked with colour calculus. The first one is an underlying category of grading spaces and the second one is an corresponding category of graded k–modules.

3.1. Denote by $\mathcal{S}ym_G$ the category of symmetric G–bispaces. Thus, an object X in the category is a symmetric G–bispace, or in other words a left G–space X endowed with G–map $\hat{}: X \longrightarrow G$. A morphism in the category is a G–map $f : X \longrightarrow Y$ of left G–spaces such that $\widehat{f(x)} = \hat{x}$, for all $x \in X$.

One can convert $\mathcal{S}ym_G$ into a monoidal category by introducing a "tensor product" of G–spaces.

At first we define the tensor product to be the pushout $X \underset{G}{\times} Y$ of X and Y, where X is considered as right G–space and Y as left G–space, respectively. Therefore, elements of $X \underset{G}{\times} Y$ can be identified with equivalence classes $x \otimes y$ of $(x,y) \in X \times Y$ with respect to the following equivalence relation:
$(x\alpha, y) \sim (x, \alpha y), \quad \forall \alpha \in G.$

Examples.

(1) Let $X = G$ be a standard symmetric G–bispace. Then $G \underset{G}{\times} Y = Y$, for all $Y \in \mathcal{O}b(\mathcal{S}ym_G)$, and the isomorphism above is given by the formula $(\alpha, y) \mapsto \alpha y, \quad \forall \alpha \in G, y \in Y.$

(2) Let $Y = *$ be a trivial G–bispace. Then, $X \underset{G}{\times} * = X/G$ is the orbit space.

We will consider $X \underset{G}{\times} Y$ as a G–bispace with left and right multiplications $\alpha \cdot (x \otimes y) = \alpha x \otimes y$ and $(x \otimes y) \cdot \alpha = x \otimes y\alpha$, and with the symmetry function: $\widehat{x \otimes y} = \hat{x} \cdot \hat{y}$.

3.2. Let $\mathcal{C}$ be an arbitrary monoidal category with a tensor product $\otimes$. Recall the following

Definition[McL]. A *symmetry* for monoidal category $\mathcal{C}$ is a natural isomorphism $\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$, for all $X, Y \in \mathcal{O}b(\mathcal{C})$, such that

(1) the *hexagon condition:* $\sigma_{X \otimes Y, Z} = (\sigma_{X,Z} \otimes id_Y) \circ (id_X \otimes \sigma_{Y,Z})$
and
(2) the *unitary condition:* $\sigma_{X,Y} \circ \sigma_{Y,X} = id_{X \otimes Y}$

hold.

Theorem. *Any symmetry in the monoidal category $\mathcal{S}ym_G$ has the form*

$$\sigma_{X,Y}(x \otimes y) = [\hat{x}, \hat{y}] \cdot y \otimes x, \tag{1}$$

for all $X, Y \in \mathcal{O}b(\mathcal{S}ym_G), \quad x \in X, y \in Y.$

Proof. We will identufy elements $x \in X$ with morphisms $t_x : G \longrightarrow X$ in the category such that: $t_x(\alpha) = (\alpha \hat{x}^{-1}) \cdot x$.

From the naturality of the symmetry σ it follows that the diagram

$$\begin{tikzcd}
X \underset{G}{\times} Y \arrow[r, "\sigma_{X,Y}"] & Y \underset{G}{\times} X \\
G \underset{G}{\times} G \arrow[u, "t_X \underset{G}{\times} t_Y"] \arrow[r, "\sigma_{G,G}"] \arrow[d, equal] & G \underset{G}{\times} G \arrow[u, "t_Y \underset{G}{\times} t_X"'] \arrow[d, equal] \\
G \arrow[r, "\lambda"] & G
\end{tikzcd}$$

commutes.

But $\lambda = \sigma_{G,G}$ is a morphism in the category. Hence $\lambda = id$, and we get formula (1). An easy computation shows that the hexagon and the unitary conditions are valid for the given σ. $\square$

Remark. The same result we get if we consider the new tensor produt $X \underset{[G,G]}{\times} Y$, where $[G, G]$ is the commutator group.

3.3. Denote by $\mathcal{G}Mod_k$ the category of graded k–modules equipped with a special G–action.

Thus objects in $\mathcal{G}Mod_k$ are graded k–modules $P_M = \sum_{m \in M} P_m$, graded by G–bispaces $M \in \mathcal{O}b(\mathcal{S}ym_G)$, and endowed with a left $k[G, G]$–module structure such that $\alpha(P_m) \subset P_{\alpha m}$ for all $\alpha \in [G, G]$.

The $k[G, G]$–module structure determine a special G–module structure on P_M :

$$\alpha(x_m) = [\alpha, \hat{m}] \cdot x_m,$$

and a right $k[G, G]$–module structure

$$x_m \cdot \beta = {}^{\hat{m}}\alpha \cdot x_m,$$

where $\alpha \in [G,G], \beta \in G, x_m \in P_m$.

Morphisms in the category are graded $k[G,G]$–morphisms $f : P_M \longrightarrow Q_N$ covering morphisms in the category $\mathcal{S}ym_G$.

We convert $\mathcal{G}Mod_k$ into a monoidal category by letting $P_M \otimes Q_N$ be the usuall $k[G,G]$–tensor product graded by the space $M \underset{G}{\times} N$.

Any object $M \in \mathcal{O}b(\mathcal{S}ym_G)$ determines an object $k(M) \in \mathcal{O}b(\mathcal{G}Mod_k)$, where $k(M)$ is the algebra of k–valued functions on M with basis $\{\delta_m\}, m \in M$, such that $\delta_m(m) = 1$, and $\delta_m(m') = 0$, otherwise. We will consider $k(M)$ as M–graded k–module, where functions δ_m have degree m, the G–action is given by $\alpha(\delta_m) = \delta_{\alpha m \alpha^{-1}}$, and the left $k[G,G]$– module structure is given by $\alpha \cdot \delta_m = \delta_{\alpha m}$.

Any element $x_m \in P_m$ we will identify with morphism: $t(x_m) : k(M) \longrightarrow P_M$, such that $t(x_m)(\delta_m) = x_m, \quad t(x_m)(\delta_{\alpha m \alpha^{-1}}) = \alpha(x_m), \forall \alpha \in G$, and $t(x_m)(\delta_{m'}) = 0$ otherwise.

Let σ be a symmetry in the monoidal category. By using of naturality of symmetries we get the following commutative diagram:

$$\begin{array}{ccc} P_M \otimes Q_N & \xrightarrow{\sigma_{P,Q}} & Q_N \otimes P_M \\ \Big\uparrow{\scriptstyle t(x_m)\otimes t(y_n)} & & \Big\uparrow{\scriptstyle t(y_n)\otimes t(x_m)} \\ k(M) \otimes k(N) & \xrightarrow{\sigma_{k(M),k(N)}} & k(N) \otimes k(M). \end{array}$$

Any symmetry σ covers the symmetry in the category $\mathcal{S}ym_G$. Therefore,

$$\sigma_{k(M),k(N)} : \delta_m \otimes \delta_n \mapsto \chi(\hat{m}, \hat{n})[\hat{m}, \hat{n}]\delta_n \otimes \delta_m,$$

for some function $\chi : G \times G \longrightarrow U(k)$, and

$$\sigma_{P,Q}(x_m \otimes y_n) = \chi(\hat{m}, \hat{n})[\hat{m}, \hat{n}] y_n \otimes x_m. \tag{1}$$

The condition that σ is a morphism in the category yield the following condition on χ :

$$(1) \quad \chi(\alpha\alpha_0, \beta) = \chi(\alpha, \alpha_0\beta),$$

for all $\alpha, \beta \in G, \alpha_0 \in [G,G]$.

The unitary equation produces the multiplicative skew–symmetry property on χ

$$\chi(\alpha, \beta) \cdot \chi(\beta, \alpha) = 1,$$

and the hexagon equation yields

$$\chi(\alpha\beta, \gamma) = \chi(\alpha, \gamma) \cdot \chi(\beta, \gamma),$$

for all $\alpha, \beta, \in G$.

Summarizing we get the following

Theorem. *Any symmetry σ in the monoidal category $\mathcal{G}Mod_k$ is given by formula (1), where $\chi : G \times G \longrightarrow U(k)$ is a skew- symmetric bihomomorphism such that property (1) holds.*

3.4. Let $A = \sum_{\alpha\in G} A_\alpha$ be an algebra in the category $\mathcal{G}Mod_k$ with multiplication $\mu : A \otimes A \longrightarrow A, \quad \mu(a \otimes b) = a \cdot b$. Therefore $A_\alpha \cdot A_\beta \subset A_{\alpha\beta}$, and there is a left $[G, G]$–action such that $\alpha(A_\beta) \subset A_{\alpha\beta}, \quad \forall\alpha \in [G, G]$.

By using the action we can identify elements α of the group with the elements $\tilde{\alpha} = \alpha(1) \in A_\alpha$ of the algebra.

Let A be a σ–commutative algebra in the category with respect to some symmetry σ. Then from 3.3.(1) we get the following law of commutativity:

$$a_\alpha \cdot a_\beta = \chi(\alpha\beta)[\tilde{\alpha}, \tilde{\beta}] \cdot a_\beta \cdot a_\alpha, \tag{1}$$

for all $a_\alpha \in A_\alpha, a_\beta \in A_\beta$.

Theorem.

(1) *Any G–graded σ–commutative algebra $A = \sum_{\alpha\in G} A_\alpha$ in $\mathcal{G}Mod_k$ with given symmetry σ, is a colour commutative algebra with respect to the colour $s_{\alpha,\beta} = \chi(\alpha\beta)[\tilde{\alpha}, \tilde{\beta}]$.*

(2) *Let A be a σ–commutative algebra in the category. Then any σ–symmetric $A - A$ bimodule in the category is colour symmetric.*

3.5. We start with recalling the definition [L1,L2] of quantizations in monoidal categories.

Let $\mathcal{C}$ be a monoidal category with

(1) a bifunctor of tensor product: $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}, \quad \otimes : X \times Y \longrightarrow X \otimes Y$, where $X, Y \in \mathcal{O}b(\mathcal{C})$,

(2) a trivial associativity constraint: $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$,

(3) a unit object k, with natural isomorphisms: $\eta^l_X : k \otimes X \longrightarrow X$,
$\eta^r_X : X \otimes k \longrightarrow X$, such that
$id_X \otimes \eta^l_Y = \eta^r_X \otimes id_Y : X \otimes k \otimes Y \longrightarrow X \otimes k \otimes Y$.

Definition. A quantization $\mathcal{Q}$ of a monoidal category $\mathcal{C}$ is a natural isomorphism of tensor product bifunctor $\mathcal{Q}_{X,Y} : X \otimes Y \longrightarrow X \otimes Y$, such that the coherence conditions

(1) $\mathcal{Q}_{X,Y\otimes Z} \circ (id_X \otimes \mathcal{Q}_{Y,Z}) = \mathcal{Q}_{X\otimes Y,Z} \circ (\mathcal{Q}_{X,Y} \otimes id_Z)$,

(2) $\eta^r_X \circ \mathcal{Q}_{X,k} = \eta^r_X, \quad \eta^l_X \circ \mathcal{Q}_{k,X} = \eta^l_X$,

hold.

A quantization in monoidal category $\mathcal{C}$ preserves all natural algebraic structures linked with the tensor product in $\mathcal{C}$, [L1,L2]. Thus, if $\mu : A \otimes A \longrightarrow A$ is an algebra in the category we can define a *quantization* (A_q, μ^q) of the algebra as the object $A_q = A$ with new multiplication $\mu^q = \mu \circ \mathcal{Q}_{A,A}$. In the same way, for any left A–module X with multiplication $\mu_X : A \otimes X \longrightarrow X$ we can define the A_q–module (X_q, μ^q_X), a *quantization* of X, as the object $X_q = X$ with new multiplication $\mu^q_X = \mu \circ \mathcal{Q}_{A,X}$. It is important that (A_q, μ^q) is an algebra and that (X_q, μ^q_X) is an A_q–module in the category too.

Moreover, if σ is a symmetry (or braiding) in the monoidal category then $\sigma^q_{X,Y} = \mathcal{Q}^{-1}_{Y,X} \circ \sigma_{X,Y} \circ \mathcal{Q}_{X,Y}$ is a symmetry too [L1]. One can define σ–differential operators in braided tensor categories in such a way that quantizations generate transformations of modules of σ–differential operators [L1,L2].

3.6. Here we describe quantizations in the category $\mathcal{G}Mod_k$.

Let $\mathcal{Q}$ be a quantization. Then by using the naturality of $\mathcal{Q}$ we get the following commutative diagram (see 3.3.):

$$\begin{tikzcd}
X_M \underset{G}{\otimes} Y_N \arrow[r, "\mathcal{Q}_{X,Y}"] & X_M \underset{G}{\otimes} Y_N \\
k[G] \underset{G}{\otimes} k[G] \arrow[u, "t(x_m)\otimes t(y_n)"] \arrow[r, "\mathcal{Q}_{k[G],k[G]}"] \arrow[d, equal] & k[G] \underset{G}{\otimes} k[G] \arrow[u, "t(x_m)\otimes t(y_n)"'] \arrow[d, equal] \\
k[G] \arrow[r, "q"] & k[G],
\end{tikzcd}$$

where $X_M, Y_N \in \mathcal{O}b(\mathcal{G}Mod_k)$, $\quad x_m \in X_M, y_n \in Y_N$.

Because of $\hat{q} = id$, we have $q(\varepsilon_\alpha) = q(\alpha)\varepsilon_\alpha$, for some function $q : G \longrightarrow k^*$, and $\mathcal{Q}_{X,Y}(x_m \otimes y_n) = q(\hat{m}\hat{n}) x_m \otimes y_n$ From condition 3.5.(1) we get $q \equiv 1$, and therefore $\mathcal{Q} = id$.

Theorem. *Any quantization in the monoidal category $\mathcal{G}Mod_k$ is trivial.*

3.7. Similarly to theorem 3.3. we get the following

Theorem. *Any quantization $\mathcal{Q}$ in the monoidal category $\mathcal{G}Mod_k$ is given by the formula*

$$\mathcal{Q}_{P,Q}(x_m \otimes y_n) = q(\hat{m}, \hat{n}) x_m \otimes y_n,$$

where $q : G \times G \longrightarrow U(k)$ is a **quantizer** *[L4], i.e. an $[G,G]$–invariant multiplicator on the group:*

(1) $q(\alpha\alpha_0, \beta) = q(\alpha, \alpha_0\beta)$,
(2) $q(e, \alpha) = q(\alpha, e) = 1$,
(3) $q(\alpha, \beta\gamma) q(\beta, \gamma) = q(\alpha\beta, \gamma) q(\alpha, \beta)$,

where $\alpha_0 \in [G,G]$, $\quad \alpha, \beta, \gamma \in G$.

Example. Let $G = \mathbb{Z}^n$.Then the theorems above show that quantizations act on the set of all symmetries in a transitive way. Therefore all modules of differential operators on quantum hyperplanes are isomorphic. The isomorphisms can be given by the quantizations [cf. O.].

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