BRAIDINGS AND QUANTIZATIONS OVER BIALGEBRAS

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Abstract. We describe braidings and quantizations in monoidal categories over bialgebras and group algebras of compact Lie groups. We introduce a relative variant of a braiding and a quantization more suitable in quantum problems. To describe quantizations we introduce nonlinear cohomologies and show their relations with Hochschild cohomologies and Poisson structures.

0.Introduction.

In this paper we continue the quantization program outlined in [L1,L2,L3]. Here we describe bradings and quantizations in monoidal categories over bialgebras. The main reason to start with categories over bialgebras is a Quantum Klein Program.

We enforce the Klein point of view in Quantum Universe by assumption that there is a *Quantum Symmetry* defined by some bialgebra or braided bialgebras [Mj].I would like to note that graded objects playing the main role in Quantum statistics are actually some type of a bialgebra action (cf. 2.7.) or a quantum symmetry.

In section 1 we briefly recall main definitions and main constructions in bialgebras.

In section 2 we describe braidings over group algebras of compact Lie group. We introduce also a notion of a relative braiding more suitable in problems of quantization.

In section 3 we introduce various notions of quantizations. More general one is a quantization of a covariant functor acting between monoidal categories. For a number of functors a quantization can be described in terms of *quantizers.* We show that the set of quantizers is a variant of non-linear cohomologies and after their linearization we get generalized Hochschild cohomologies and Poisson structures.

1.Bialgebras.

Here we fix our notations and recall main definitions.

1.1. Let k be a commutative ring with an unit. All modules are assumed to be unitary k-modules and algebras are assumed to be associative k-algebras with an unit. The tensor product is taken over k.

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Recall that a *coalgebra* C is a module together with a *coproduct* or a *diagonal* morphism

$$\Delta : C \otimes C \longrightarrow C$$

and a *counit* or an *augmentation* $\epsilon : C \longrightarrow k$ such that

$$(\Delta \otimes id_C) \circ \Delta = (id_C \otimes \Delta) \circ \Delta,$$

and

$$(id_C \otimes \epsilon) \circ \Delta = (\epsilon \otimes id_C) \circ \Delta = id_C.$$

An algebra A with a product

$$\mu : A \otimes A \longrightarrow A$$

and an unit $\eta : k \longrightarrow A$ is a *bialgebra* if there is a coalgebra structure

$$\Delta : A \longrightarrow A \otimes A, \quad \epsilon : A \longrightarrow k$$

such that Δ and ϵ are algebra morphisms.

A bialgebra A is a *Hopf algebra* if there is a k- algebra *anti - homomorphism* $S : A \longrightarrow A$, called *antipode,* such that

$$\mu \circ (S \otimes id_A) \circ \Delta = \mu \circ (id_A \otimes S) \circ \Delta = \epsilon.$$

1.2 Below we list main examples of bialgebras and Hopf algebras.

Example 1. Let G be a discrete group and $A = k(G)$ be a k *-algebra of functions* on G with values in k.

Define a coproduct, a counit and an antipode in the following way:

$$(\Delta f(x, y)) = f(xy), \quad \epsilon(f) = f(e), \quad S(f)(x) = f(x^{-1}),$$

where $f \in k(G); x, y \in G$, $\quad e$ is the unit of G and we identify $k(G) \otimes k(G)$ with $k(G \times G)$.

Example 2. Let G be a finite group and $A = k[G]$ be a *group algebra* of G.

Any element $f \in k[G]$ can be represented in the following way:

$$\theta = \sum_{x \in G} \theta(x) \cdot \delta_x,$$

where $\theta(x) \in k$ and $\{\delta_x, x \in G\}$ is a basis of $k[G]$ as k-module.

Define a Hopf algebra structure in terms of the basis in the following way:

$$\mu(\delta_x \otimes \delta_y) = \delta_{xy}, \quad \epsilon(1) = \delta_e,$$

$$\Delta(\delta_x) = \delta_x \otimes \delta_x, \epsilon(\delta_x) = 1, S(\delta_x) = \delta_{x^{-1}},$$

for all $x, y \in G$.

Example 3. Let G be an affine algebraic group (over $k = \mathbb{C}$) and $A = \mathbb{C}(G)$ be the *algebra of regular functions* on G with values in $\mathbb{C}$. Then A is converted into a Hopf algebra if we define (cf. ex.1):

$$(\Delta f)(x, y) = f(x \cdot y), \epsilon(f) = f(e), S(f)(x) = f(x^{-1}).$$

Example 4. Let $G = GL(n, \mathbb{C})$ and $A = \mathbb{C}[t_{ij}, d^{-1}]$ where t_{ij} are the coordinate functions on G and $d = \det ||t_{ij}||$.

Then morphisms

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \qquad \epsilon(t_{ij}) = \delta_{ij},$$

$$S(t_{ij}) = (-1)^{i+j} \cdot M_{ij} \cdot d^{-1}, \epsilon(d) = 1, S(d) = d^{-1}, \Delta(d) = d \otimes d,$$

where M_{ij} is the (i, j)- minor of the matrix $||t_{ij}||$, are defined a Hopf algebra structure.

Example 5. Let G be a Lie group. Recall that a continuous $\mathbb{C}$- valued function on G is called *representative* if the linear subspace of $C(G, \mathbb{C})$ generated by functions $x^*(f)$ for all $x \in G$ is a finite dimentional vector space. Denote by $\mathcal{R}(G) \subset C(G, \mathbb{C})$ the $\mathbb{C}$-algebra of all representative functions on G. Define a coproduct,a counit and an antipode analogously to what made in ex.1 in order to get a Hopf algebra structure.

If G is a compact group and $\lambda : G \longrightarrow Aut(E_\lambda)$ is a faithfull finite dimentional representation of G then $\mathcal{R}(G)$ is generated by functions $\lambda_{ij}(x), 1 \leq i, j \leq \dim E_\lambda$ and $(\det(\lambda(x))^{-1}$,where $\lambda_{ij}, x \in G$, are matrix coefficients of operators $\lambda(x)$ in some basis of E_λ.

In the particular case when G is a compact abelian group the algebra $\mathcal{R}(G)$ is the linear space spanned by the set of all characters of G.

Example 6. $\mathcal{R}(U(n)) = \mathbb{C}[t_{ij}, d^{-1}]$.

Example 7. The coordinate ring of the Manin quantum plane [M] is the algebra $\mathbb{C}_q^{2|0}$ generated by variables x, y and the following relation:

$$x \cdot y - q^{-1} \cdot y \cdot x = 0.$$

Actually the relation is nothing other than the relation of Quantum mechanics $[\hat{p}, \hat{q}] = -\imath h$ written in the exponential form:

$$x := \exp(\imath p), y := \exp(\imath q), q := -\imath h.$$

There is a bialgebra structure on $\mathbb{C}_q^{2|0}$ given by

$$\Delta(x) = x \otimes x, \quad \delta(y) = y \otimes 1 + x \otimes y, \quad \epsilon(x) = 1, \epsilon(y) = 0.$$

1.3 Let A be an algebra and let C be a coalgebra. Then k - module $Hom(C, A)$ carries of an algebra structure with respect to the *convolution product.* Recall that the convolution $\varphi * \psi : C \longrightarrow A$ of $\varphi : C \longrightarrow A$ with $\psi : C \longrightarrow A$ defines by the following commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\varphi*\psi} & A \\ {\scriptstyle\Delta}\downarrow & & \uparrow{\scriptstyle\mu} \\ C\otimes C & \xrightarrow{\varphi\otimes\psi} & A\otimes A \end{array}$$

and the unit of $Hom(C, A)$ is the composition:

$$\eta \circ \epsilon : C \xrightarrow{\epsilon} k \xrightarrow{\eta} A.$$

If we use the Sweedler notation:

$$\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$$

then the convolution product takes the form:

$$(\varphi * \psi)(x) = \sum_x \varphi(x_{(1)})\psi(x_{(2)}).$$

1.4. Let A be a bialgebra and A^* be the dual module. The adjoint Δ^* to the coproduct map defines an algebra stucture on A^*:

$$\mu_{A^*} = (\Delta_A)^* : A^* \otimes A^* \hookrightarrow (A \otimes A)^* \longrightarrow A^*,$$

and the unit $\epsilon \in A^*$.

If we denote the product $\mu_A^*(f \otimes g), \quad f, g \in A^*$, in the usual way $f \cdot g$ we get

$$\langle f \cdot g, x\rangle = \langle f \otimes g, \Delta x\rangle$$

$$\langle \epsilon \cdot f, x\rangle = \langle \epsilon \otimes f, \Delta x\rangle = \langle f, x\rangle.$$

To define a coalgebra structure on A^* by means of $(\mu_A)^*$ we should require that $Im(\mu_A)^* \subset A^* \otimes A^*$.

To this end we restrict oneself to some class of bialgebras.

Definition. *We say that an algebra A is a closed algebra if*

(1) $Im(\mu_A)^* \subset A^* \otimes A^*$,
(2) $(A^*)^* = A$.

A bialgebra A is called a closed bialgebra if A is a closed algebra.

1.5 Let A be a closed bialgebra. We shall consider two algebras linked with A: $A^* \otimes A^*$ and $Hom(A, A^*)$.

There is a natural imbedding:

$$\varkappa : A^* \otimes A^* \longrightarrow Hom(A, A^*),$$

$$\varkappa : \alpha \mapsto \hat{\alpha},$$

where

$$\langle \hat{\alpha}(x), y \rangle = \langle \alpha, y \otimes x \rangle,$$
$$\alpha \in A^* \otimes A^*, \quad x, y \in A.$$

We fix algebra structures: the convolution product in $Hom(A, A^*)$ and the usual product in $A^* \otimes A^*$. Note that the twist $\tau \colon A^* \otimes A^* \longrightarrow A^* \otimes A^*, \quad \tau(x \otimes y) = y \otimes x$, is an algebra morphism.

Respectively, with any morphism $\varphi : A \longrightarrow A^*$ we link the adjoint $\varphi^\dagger : A \longrightarrow A^*$ in the usual way:

$$\langle \varphi^\dagger(x), y \rangle = \langle \varphi(y), x \rangle,$$

where $x, y \in A$.

Proposition. *Let A be a closed bialgebra.Then $\varkappa : A^* \otimes A^* \longrightarrow Hom(A, A^*)$ is an algebra morphism such that*

$$\varkappa(\tau(\alpha)) = (\varkappa(\alpha))^\dagger.$$

1.6. Any element $h \in A^*$ defines by means of a coalgebra structure the element $\Delta(h) \in A^* \otimes A^*$ and therefore h defines the morphism

$$\lambda_h = \varkappa(\Delta h) : A \longrightarrow A^*.$$

By virtue of Δ is an algebra morphism we get $\lambda_{h_1 \cdot h_2} = \lambda_{h_1} * \lambda_{h_2}$. Hence, λ defines an A^*-module structure on $Hom(A, A^*)$.

2. Braidings.

In this chapter we recall the notion of a braiding in a monoidal category [JS] and describe all braidings in the category of G–modules over a compact Lie group G.

We introduce a relative variant of a braiding which is more suitable in quantum problems and describe relative braidings of the category G–modules too.

2.1 Let A be a bialgebra and $A - \mathcal{M}od$ be the category of left A-modules.The category becomes a monoidal [ML1] if one defines a tensor product bifunctor

$$\otimes : (A - \mathcal{M}od) \times (A - \mathcal{M}od) \longrightarrow A - \mathcal{M}od$$

in the following way:

$$X \otimes Y := X \underset{k}{\otimes} Y,$$

with an A-module structure:

$$a \cdot (x \otimes y) = \sum_a a_{(1)} x \otimes a_{(2)} y,$$

where $x \in X, y \in Y, a \in A$ and

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}.$$

2.2 Recall the following

Definition. *A* **braiding** *for monoidal category $\mathcal{C}$ is a natural isomorphism*

$$\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X,$$

such that the two natural hexagon conditions

$$\sigma_{X\otimes Y,Z} = (\sigma_{X,Z} \otimes id_Y) \circ (id_X \otimes \sigma_{Y,Z}), \tag{1}$$

$$\sigma_{X,Y\otimes Z} = (id_Y \otimes \sigma_{X,Z}) \circ (\sigma_{X,Y} \otimes id_Z), \tag{2}$$

hold[JS].

2.3. Actually the next theorem was used by Drinfeld [D1] in his definition of quasitriangular Hopf algebras.

Theorem.

(1) *Any braiding in the monoidal category $A - \mathcal{M}od$ is determineted by an invertible element $\sigma \in A \otimes A$ such that*

$$\begin{array}{ll} (i) & \sigma \cdot \tau(\Delta(a)) = \Delta(a) \cdot \sigma, \\ (ii) & (id_A \otimes \Delta)(\sigma) = (\sigma \otimes 1) \cdot \sigma_{13}, \\ (iii) & (\Delta \otimes id_A)(\sigma) = (1 \otimes \sigma) \cdot \sigma_{13}, \end{array}$$

where $a \in A$ and $\sigma_{13} = (id_A \otimes \tau)(\sigma \otimes 1)$.

(2) *The action*

$$\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$$

is given by the formula

$$\sigma_{X,Y}(x \otimes y) = \sigma \cdot (y \otimes x),$$

where

$$\sigma \cdot (y \otimes x) = \sum \sigma' y \otimes \sigma'' x,$$

if $\sigma = \sum \sigma' \otimes \sigma''$.

Proof. Let us identify elements $x \in X$ with morphisms $\varphi_x : A \longrightarrow X$ such that $\varphi_x(a) = ax$. By definition of a braiding one gets the following commutative diagram

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\sigma_{X,Y}} & Y \otimes X \\ {\scriptstyle \varphi_x \otimes \varphi_y}\uparrow & & \uparrow{\scriptstyle \varphi_y \otimes \varphi_x} \\ A \otimes A & \xrightarrow{\sigma_{A,A}} & A \otimes A. \end{array}$$

Therefore, if we denote by σ the element $\sigma_{A,A}(1 \otimes 1)$ we get

$$\sigma_{X,Y}(x \otimes y) = \sigma \cdot (y \otimes x).$$

Now the conditions (ii),(iii) are equivalent to definition (2.2) of a braiding and (i) means that $x \otimes y \mapsto \sigma \cdot (y \otimes x)$ is an A-algebra morphism.

Remark.

(1) The proof of the theorem is shown that the same result holds for arbitrary monoidal category over bialgebra A if we shall require that any morphism $X \otimes Y \longrightarrow Z$ is completly defined by images of decomposable elements $x \otimes y, x \in X, y \in Y$.

(2) Elements σ defining braidings in the category we will call *braiding elements.* Due to Drinfeld [D1] a bialgebra A with a given braiding element is called *quasitriangular.*

2.4. Here we reformulate theorem 2.3. by using the algebraic structure on $A^* \otimes A^*$.

Theorem. *Let A be a closed bialgebra. Then any braiding in the category $A^* - \mathcal{M}od$ is defined by an element $\sigma \in A^* \otimes A^*$ such that*

(1) $\lambda_h * \hat{\sigma} = \hat{\sigma} * \lambda_h^{\dagger}$,

(2) $\hat{\sigma} : A \longrightarrow A^*$ *and* $\hat{\sigma}^{\dagger} : A \longrightarrow A^*$ *are algebra homomorphisms.*

Proof. By using the condition(ii) of theorem 2.3. we get

$$\begin{aligned}\langle \hat{\sigma}(xy), z\rangle &= \langle \sigma, (id \otimes \mu_{A^*})(z \otimes x \otimes y)\rangle = \langle (id \otimes \Delta)(\sigma), z \otimes x \otimes y\rangle = \\ &= \langle (\sigma \otimes 1) \cdot (id \otimes \tau)(\sigma \otimes 1), z \otimes x \otimes y\rangle = \\ &= \langle (\sum \sigma' \otimes \sigma'' \otimes 1) \cdot (\sum \tilde{\sigma'} \otimes 1 \otimes \tilde{\sigma''}), z \ \otimes x \otimes y\rangle = \\ &= \langle \sum \sigma'\tilde{\sigma'} \otimes \sigma'' \otimes \tilde{\sigma}'', z \otimes x \otimes y\rangle = \langle \hat{\sigma}(x) * \hat{\sigma}(y), z\rangle.\end{aligned}$$

For the adjoint operator $\hat{\sigma}^{\dagger}$ we get

$$\begin{aligned}\langle \hat{\sigma}^{\dagger}(xy), z\rangle &= \langle \hat{\sigma}(z), xy\rangle = \langle \sigma, xy \otimes z\rangle = \\ &= \langle \sigma, (\mu_{A^*} \otimes id)(x \otimes y \otimes z\rangle = \langle (\Delta \otimes id)\sigma, x \otimes y \otimes z\rangle = \\ &= \langle (1 \otimes \sigma) \cdot (id \otimes \tau)(\sigma \otimes 1), x \otimes y \otimes z\rangle = \\ &= \langle \sum \tilde{\sigma}' \otimes \sigma' \otimes \sigma''\tilde{\sigma}'', x \otimes y \otimes z\rangle = \langle \hat{\sigma}^{\dagger}(x) * \hat{\sigma}^{\dagger}(y), z\rangle.\end{aligned}$$

Remark. Instead of condition (2) we shall use also the equivalent condition:

$$(2') \quad \hat{\sigma} : A \longrightarrow A^* \quad \text{is a bialgebra homomorphism.}$$

2.5. Now we outline some approach to a theory of representations in monoidal categories. More precisely we apply the Jacobson approach [J] to **ABC**–categories and introduce a notion of an algebra spectrum in the given **ABC**–category $\mathcal{C}$ (see [L2,L3] for the definition).

Let A be an algebra in category $\mathcal{C}$. A *representation* p of A in an object $E \in \mathcal{O}b(\mathcal{C})$ is a morphism of A into the algebra of *internal homomorphisms* $p : A \longrightarrow Hom(E, E)$.

There is a number of irreducibility notions. Here we shall use the simplest one: *algebraic irreducibility.*

We say that a subobject $E' \subset E$ is an *invariant* with respect to representation p if the image of the composition p and the *evaluation map*:

$$A \otimes E' \xrightarrow{p\otimes id} Hom(E, E') \xrightarrow{ev} E$$

is contained in E'.

Now we define *irreducible representations* in the usual way as representations with trivial invariant subobjects only.

Denote by $\mathcal{S}pec(A)$ (or to be more precise $\mathcal{S}pec_{\mathcal{C}}(A)$) the totality of all irreducible representations of an algebra A (in the given **ABC**–category $\mathcal{C}$). We shall call $\mathcal{S}pec(A)$ as a *spectrum* of the algebra A in the category $\mathcal{C}$.

The space $\mathcal{S}pec(A)$ topologized by making the closure $\overline{X}$ of $X \subset \mathcal{S}pec(A)$ the set of all irreducible representations $q \in \mathcal{S}pec(A)$ such that $q|_{\ker p} = 0$, for all $p \in X$.

The topology obtained in this way is called a *Jacobson topology.*

We produce the following interpretation of an algebra A. At every point $p \in \mathcal{S}pec(A)$ we have a *model space* E_p and a *model algebra* $A_p = Hom(E_p, E_p)$ and we may associate with any element $a \in A$ the "function" $\hat{a}$ on the space $\mathcal{S}pec(A)$ whose value at the point $p \in \mathcal{S}pec(A)$ is tne image $p(a) \in A_p$. We shall call a *Fourier image* or a *Fourier transformation* of $a \in A$ the function $\hat{a}$.

Let F be an algebra in the category $\mathcal{C}$. We shall call an algebra F *primitive* (cf. [J]) if F has a maximal left ideal $\mathcal{J} \subset F$ such that $\mathcal{J}$ contains trivial two–sided ideals only.

A two–sided ideal $\mathcal{I}$ of an algebra A is said to be *primitive* if $F = A/\mathcal{I}$ is a primitive algebra.

Actually the notion of primitive algebras gives a possibility to identify F with an irreducible subalgebra of $Hom(E_F, E_F)$, where $E_F = F/\mathcal{J}$. Hence, instead of the space $\mathcal{S}pec(A)$ one can consider the totality $\mathcal{P}rim(A)$ of all primitive ideals in A and define the *Jacobson topology* on $\mathcal{P}rim(A)$ as above.

Now fix some class $\mathcal{F}$ of non–isomorphic primitive algebras and consider primitive algebras from $\mathcal{F}$ only.

We shall call F*–point* an algebra morphism $p : A \longrightarrow F, \quad F \in \mathcal{F}$ such that

$$p : A \longrightarrow F \hookrightarrow Hom(E_F, E_F)$$

an irreducible representation .

Denote by $\mathcal{S}pec_F(A)$ the totality of all F–points. We shall call a $\mathcal{F}$*–spectrum* of A the union

$$\mathcal{S}pec_{\mathcal{F}}(A) = \bigcup_{F\in\mathcal{F}} \mathcal{S}pec_F(A)$$

. We shall call a F–part of the spectrumSubsets $\mathcal{S}pec_F(A) \subset \mathcal{S}pec_{\mathcal{F}}(A)$, where $F \in \mathcal{F}$,.

As above any element $a \in A$ defines the Fourier image $\hat{a}$ on $\mathcal{S}pec_{\mathcal{F}}(A)$ and the restriction of $\hat{a}$ on the F–part $\mathcal{S}pec_F(A)$ gives a F–valued function.

Example.

(1) Let $\mathcal{F}$ be the class of algebras of internal homomorphisms $Hom(E, E)$, where E are projective finite generated k– modules.Then $\mathcal{S}pec_{\mathcal{F}}(A)$ is a "finite dimensional" or "algebraic" part of the spectrum.

(2) Let $\mathcal{F}$ be the class of algebras *braided differential operators* $\mathcal{D}iff^{\sigma}_{*}(E,E)$ [cf.L2]. Then $\mathcal{S}pec_{\mathcal{F}}(A)$ is a "σ–differential" part of the spectrum.

2.6 Let G be a compact Lie group. We shall consider the following group algebras and bialgebras:

(1) $C(G,\mathbb{C})$–the algebra of all $\mathbb{C}$–valued continuous functions on G with respect to the convolution product:

$$(f*g)(x) = \int_G f(xy^{-1})g(y)\,dy,$$

and the counit

$$\epsilon(f) = f(e).$$

Here dx is the right Haar measure on G such that $\int_G dx = 1$, and $e \in G$ is the unit of group G.

(2) $\mathcal{D}(G,\mathbb{C})$–the algebra of all distributions on $C^\infty(G.\mathbb{C})$ with respect to the product:

$$\langle \alpha * \beta, f\rangle = \langle \alpha \otimes \beta, \Delta f\rangle,$$

and the unit

$$\delta_e \in \mathcal{D}(G,\mathbb{C}), \delta_e(f) = f(e),$$

where $\alpha,\beta \in \mathcal{D}(G,\mathbb{C})$ and

$$\Delta f \in C^\infty(G\times G,\mathbb{C}), \quad \Delta f(x,y) = f(xy).$$

We denote by $\hat{G}$ the finite dimensional spectrum of the group algebras i.e. the set of isomorphism classes $\{p : G \longrightarrow End(E_p)\}$ of all irreducible (finite dimentional) representations of G.

The *Fourier transform* $\hat{f} = \mathcal{F}(f)$ of a continuous function $f \in C(G,\mathbb{C})$ is the function

$$\hat{f} : p \in \hat{G} \mapsto \hat{f}(p) \in End(E_p),$$

on the spectrum $\hat{G}$ defined by the integral

$$\hat{f}(p) = \int_G f(x)p(x)\,dx.$$

Respectivly, the *Fourier transform* $\hat{\alpha} = \mathcal{F}(\alpha)$ of a distribution $\alpha \in \mathcal{D}(G,\mathbb{C})$ is the function on the spectrum is given by the formula

$$\hat{\alpha}(p) = \langle \alpha, p(x)\rangle.$$

Below we collected [cf.W] all main properties of the Fourier transformation.

(1)

$$\mathcal{F}(f*g) = \mathcal{F}(f)\circ\mathcal{F}(g),$$

(2)
$$\mathcal{F}(f^*) = \mathcal{F}(f)^*,$$
where $f^*(x) = \overline{f}(x^{-1})$, and the adjoint * in $End(E_p)$ is determinated with respect to an arbitrary positive G–invariant hermitian form,

(3)
$$\mathcal{F}(\ell_g(f)) = p(g) \circ \mathcal{F}(f),$$
$$\mathcal{F}(r_g(f)) = \mathcal{F}(f) \circ p(g),$$
where $\ell_g(f)(x) = f(g^{-1}x), \quad r_g(f)(x) = f(xg),$ are respectively left and right actions of G on $C(G, \mathbb{C}), \quad x, g \in G$

(4) Define the inner product on $C(G, \mathbb{C})$ and $End(E_p)$ in the following way:
$$(f, g) = \int_G \overline{f}(x) g(x)\, dx = \epsilon(f^* * g),$$
$$(A, B) = d_p \cdot Tr(A^* B).$$
Then
$$(\mathcal{F}(f), \mathcal{F}(g)) = (f, g), \qquad \textit{(Plancherel formula)}$$
where $f, g \in C(G, \mathbb{C}), \quad A, B \in End(E_p), \quad d_p = dim E_p$.

(5) The inverse $\mathcal{F}^{-1}$ (see [W] for complete description of the operator algebras) has the following form
$$\mathcal{F}^{-1}(h)(x) = \sum_{p \in \hat{G}} Tr(h(p) p^{-1}(x)) \cdot d_p,$$
where $x \in G$, and h is a function on $\hat{G}, \quad h : p \in \hat{G} \mapsto h(p) \in End(E_p)$.

Example. Let G be a finite commutative group and $\hat{G}$ be the group of characters.

The Fourier transformation defines an algebra isomorphism
$$\mathcal{F} : \mathbb{C}(G) \longrightarrow \mathbb{C}(\hat{G})^* = \mathbb{C}[\hat{G}].$$
In this case
$$\mathcal{F}(f)(\chi) = \frac{1}{|G|} \sum_{x \in G} f(x) \langle \chi, x \rangle,$$
and
$$\mathcal{F}^{-1}(g)(x) = \sum_{\chi \in \hat{G}} g(\chi) \overline{\langle \chi, x \rangle} \cdot d_\chi,$$
where $x \in G, \chi \in \hat{G}, f \in \mathbb{C}(G), g \in \mathbb{C}[\hat{G}]$.

2.7 Let G be a finite group and $A = k(G)$ be the algebra of k–valued functions on G.

The algebra is generated by "δ–functions" $\theta_g, g \in G$ such that $\theta_g(g) = 1$ and $\theta_g(x) = 0$, if $x \neq g$.

The bialgebra structure in terms of δ–functions takes the following form:

(1)
$$\theta_g \cdot \theta_g = \theta_g, \quad \theta_g \cdot \theta_h = 0, \quad \text{if} \quad g \neq h,$$

(2)
$$\Delta\theta_g = \sum_{h \in G} \theta_h \otimes \theta_{h^{-1}g},$$

(3)
$$\epsilon(\theta_g) = 0, \quad \text{if} \quad g \neq e, \quad \text{and} \quad \epsilon(\theta_e) = 1.$$

Hence the category $A-\mathcal{M}od$ coinsides with the category of G–graded k–modules.

Tensor products $X \otimes Y$ in the category are the usual G–graded products with the grading
$$(X \otimes Y)_g = \sum_{h \in G} X_h \otimes Y_{h^{-1}g}.$$

A braiding in the category is defined by an element
$$\sigma = \sum_{g,h \in G} \sigma(g,h)\, \theta_g \otimes \theta_h \in A \otimes A,$$

for some function
$$\sigma : G \times G \longrightarrow k, \qquad \sigma : h \times g \mapsto \sigma(h,g).$$

The condition 2.3(i) holds if and only if G is an abelian group.

The first hexagon condition 2.3.(ii) gives
$$\sigma(g_1 \cdot g_2, h) = \sigma(g_1, h) \cdot \sigma(g_2, h),$$

and from the second hexagon condition we get
$$\sigma(g, h_1 \cdot h_2) = \sigma(g, h_1) \cdot \sigma(g, h_2).$$

Therefore we obtain the following result.

Theorem. *Braidings in the category of G–graded k–modules over commutative group G are determined by group bihomomorphisms*
$$\sigma : G \times G \longrightarrow \mathcal{U}(k)$$

with values in the unit group $\mathcal{U}(k)$ of the ring k.

A braiding is a symmetry if and only if σ is a multiplicative skew–symmetry function:
$$\sigma(g,h) \cdot \sigma(h,g) = 1.$$

Remark. *The above theorem holds for an arbitrary discrete abelian group G.*

Example. We get a "quantum torus" if take $G = \mathbb{Z}^n$, and

$$\sigma(x, y) = \exp \pi\imath\langle Ax, y\rangle,$$

with skew symmetric $n \times n$–matrix A,$x, y \in \mathbb{Z}^n$.

For an arbitrary matrix A we get a "braided quantum torus".

2.8. Let G be a finite group, $A^* = \mathbb{C}[G]$ be a group algebra and $A^* - \mathcal{M}od$ be the category of G–modules.

One can consider any braiding σ in the category as a bialgebra homomorphism $\hat{\sigma} : A \longrightarrow A^*$.

Let $p \in \hat{G}$ be an irreducible representation. Then the composition

$$A \xrightarrow{\hat{\sigma}} A^* \xrightarrow{p} End(E_p)$$

is a homomorphism of commutative algebra A. Take $h = \delta_g \in A^*, g \in G$ in condition 2.4.(1) then we get the following commutative diagram:

$$\begin{array}{ccccc}
A & \xrightarrow{\hat{\sigma}} & A^* & \xrightarrow{p} & End(E_p) \\
{\scriptstyle Ad_g}\big\downarrow & & {\scriptstyle (Ad_g{}^*)^{-1}}\big\downarrow & & {\scriptstyle Ad_g}\big\downarrow \\
A & \xrightarrow{\hat{\sigma}} & A^* & \xrightarrow{p} & End(E_p)
\end{array}$$

where, for example, $Ad_g(f)(x) = f(g^{-1}xg)$, if $f \in A, x, g \in G$.

Since p is an irreducible representation the image $p \circ \hat{\sigma}$ belongs to the centre of $End(E_p)$.

Hence, $Im(p \circ \hat{\sigma}) \subset \mathbb{C}$ and we get an algebra homomorphism

$$p \circ \hat{\sigma} : A \longrightarrow \mathbb{C}.$$

Any algebra homomorphism $\varphi : A \longrightarrow \mathbb{C}$ can be represented in the following way:

$$\varphi(f) = f(x_\varphi),$$

for some point $x_\varphi \in G$.

Apply this remark to the morphisms $p \circ \hat{\sigma}, p \in \hat{G}$, we obtain a map:

$$x : \hat{G} \longrightarrow G,$$
$$x : p \mapsto x_p,$$

such that $(p \circ \hat{\sigma})(f) = f(x_p), \quad \text{for all} \quad p \in \hat{G}, f \in A$.

Since

$$\mathcal{F}(\hat{\sigma}(f))(p) = (p \circ \hat{\sigma})(f) = f(x_p),$$

we get

$$\langle \hat{\sigma}(f), g\rangle = \frac{1}{|G|} \sum_{y \in G} \sum_{p \in \hat{G}} f(x_p) g(y) \overline{\chi_p(y)},$$

where $f, g \in A$ and $\chi_p : G \longrightarrow \mathbb{C}^*$ are characters of representations $p \in \hat{G}$.

If
$$\sigma = \sum_{x,y \in G} \sigma(x,y)\, \delta_x \otimes \delta_y\,,$$
then
$$\sigma(x,y) = \langle \hat{\sigma}(\theta_y), \theta_x \rangle = \frac{1}{|G|} \sum_{p \in \hat{G}, x_p = y} \overline{\chi_p(x)},$$
and
$$\sigma = \frac{1}{|G|} \sum_{x,y \in G} \sum_{p \in \hat{G}, x_p = y} \overline{\chi}_p(x)\, \delta_x \otimes \delta_y\,. \tag{1}$$

By using the adjoint morphism $\hat{\sigma}^\dagger$ we get
$$\sigma = \frac{1}{|G|} \sum_{x,y \in G} \sum_{q \in \hat{G}, y_q = x} \overline{\chi}_q(y)\, \delta_x \otimes \delta_y\,, \tag{2}$$
for some map $y : \hat{G} \longrightarrow G$.

Let $H_1 = x(\hat{G})$ and $H_2 = y(\hat{G})$. Then from (1),(2) we also get the following:
$$\sigma = \sum_{x \in H_1, y \in H_2} \sigma(x,y)\, \delta_x \otimes \delta_y\,.$$

Condition 2.4.(i) gives imbeddings $H_1, H_2 \subset \mathcal{Z}(G)$, where $\mathcal{Z}(G)$ is the centre of group G.

Moreover, compare first terms in $(\Delta \otimes id)\sigma$ and $(1 \otimes \sigma)\sigma$ in condition 2.4.(ii) one gets that H_1 and H_2 are subgroups of $\mathcal{Z}(G)$.

For any subgroup $H \subset G$ we denote by $r : \mathbb{C}(G) \longrightarrow \mathbb{C}(H)$ the restriction map: $r(f) = f|_H$, and $r^* : \mathbb{C}[H] \longrightarrow \mathbb{C}[G]$– the adjoint of r. Remark that r and r^* are bialgebra homomorphisms.

Moreover, for any group homomorphism $\lambda : G_1 \longrightarrow G_2$ morphisms
$$\lambda^* : \mathbb{C}(G_2) \longrightarrow \mathbb{C}(G_1), \qquad \lambda^*(f) = f \circ \lambda,$$
and
$$\lambda_* : \mathbb{C}[G_1] \longrightarrow \mathbb{C}[G_2], \qquad \lambda_* = (\lambda^*)^*,$$
are bialgebra morphisms.

Summarise we obtain the following result.

Theorem. *Any braiding σ in the category of G–modules over a finite group G is determined by a group homomorphism $\lambda : \hat{H}_2 \longrightarrow H_1$, where H_1, H_2 are subgroups of the center $\mathcal{Z}(G)$. Namely, a morphism $\hat{\sigma} : \mathbb{C}(G) \longrightarrow \mathbb{C}[G]$ is obtained from the following commutative diagram:*

$$\begin{array}{ccc}
\mathbb{C}(G) & \xrightarrow{\hat{\sigma}} & \mathbb{C}[G] \\
{\scriptstyle r_1}\big\downarrow & & \big\uparrow{\scriptstyle r_2{}^*} \\
\mathbb{C}(H_1) & \xrightarrow{\tilde{\lambda}} & \mathbb{C}[H_2]
\end{array}$$

where $\tilde{\lambda}$ *is given by the composition:*

$$\tilde{\lambda} : \mathbb{C}(H_1) \xrightarrow{\lambda^*} \mathbb{C}(\hat{H}_2) \xrightarrow{\mathcal{F}^{-1}} \mathbb{C}[H_2].$$

2.9. Let G be a compact Lie group. Similarly to n.2.8. we get two maps $x : \hat{G} \longrightarrow G$ and $y : \hat{G} \longrightarrow G$, such that $Im(x) \subset \mathcal{Z}(G)$, $Im(y) \subset \mathcal{Z}(G)$.

Denote by H_1 and H_2 the closures of $Im(x)$ and $Im(y)$ respectively. Thus H_1 and H_2 are compact subgroups of $\mathcal{Z}(G)$ and Theorem 2.8. now take the following form.

Theorem. *Any braiding* $\hat{\sigma} : C^\infty(G, \mathbb{C}) \longrightarrow \mathcal{D}(G, \mathbb{C})$ *in the category of* G*–modules over a compact Lie group* G *is determined by a Lie group morphism* $\lambda : \hat{H}_2 \longrightarrow H_1$*, where* H_1 *and* H_2 *are closed subgroups of the centre* $\mathcal{Z}(G)$*, and* $\hat{\sigma}$ *is given by the following commutative diagram:*

$$\begin{array}{ccc} C^\infty(G, \mathbb{C}) & \xrightarrow{\hat{\sigma}} & \mathcal{D}(G, \mathbb{C}) \\ {\scriptstyle r_1}\big\downarrow & & \big\uparrow{\scriptstyle r_2{}^*} \\ C^\infty(H_1, \mathbb{C}) & \xrightarrow{\tilde{\lambda}} & \mathcal{D}(H_2, \mathbb{C}), \end{array}$$

where $\tilde{\lambda}$ *is given by composition:*

$$\tilde{\lambda} : C^\infty(H_1, \mathbb{C}) \xrightarrow{\lambda^*} \mathbb{C}(\hat{H}_2) \xrightarrow{\mathcal{F}^{-1}} \mathcal{D}(H_2, \mathbb{C}).$$

2.10. Examples.

(1) Let $G = \mathbb{Z}_n$. Take $H_1 = H_2 = G$,then any homomorphism

$$\lambda : \hat{\mathbb{Z}}_n \simeq \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$$

is determined by an element $k \in \mathbb{Z}_n$.

Fix the element $k \in \mathbb{Z}_n$ and denote by $|a| \subset \mathbb{Z}_n$, for any $a \in \mathbb{Z}_n$, the set of solutions the following equation : $kx = a$, $x \in \mathbb{Z}_n$.

Then the above Theorem gives the following description of braidings:

$$\sigma = \sum_{a,b \in \mathbb{Z}_n, s \in |a|} e^{-\frac{\pi\imath}{n} bs} \delta_a \otimes \delta_b. \tag{1}$$

(2) Let $G = SU(n)$, then $\mathcal{Z}(G) = \mathbb{Z}_n$ and all braidings can be described like the previuos example.

(3) Let $G = \mathbb{T}^n = H_1 = H_2$ is a n-dimentional torus. Then $\hat{G} = \mathbb{Z}^n$ and a homomorphism $\lambda : \mathbb{Z}^n \longrightarrow \mathbb{T}^n$ is given by an element $\theta \in \mathbb{T}^n$, where

$$\lambda(x) = \exp(\imath\pi\langle x, \theta\rangle).$$

For the morphism $\hat{\sigma}$ we get

$$\hat{\sigma}(f)(x) = \sum_{p \in \mathbb{Z}^n} f(\exp(\imath\pi\langle p, \theta\rangle)) \exp(-\imath\pi\langle x, p\rangle) \in \mathcal{D}(\mathbb{T}^n, \mathbb{C}). \tag{2}$$

(4) Let $G = U(n)$, then $\mathcal{Z}(G) = S^1$ and braidings is given by (2), with $n = 1$.

2.11. In this number we generalize the notion of a braiding on the relative case.

More precisely, let $\tilde{\mathcal{C}}$ be a monoidal subcategory of a monoidal category $\mathcal{C}$ [ML2]. This means that we have the imbedding functor $\tilde{\mathcal{C}} \hookrightarrow \mathcal{C}$, and the tensor product $X \otimes Y, \quad X, Y \in \mathcal{O}b(\tilde{\mathcal{C}})$ in the category $\tilde{\mathcal{C}}$ coinsides with the tensor product in the category $\mathcal{C}$.

Definition. *Say that σ is a braiding in the category $\tilde{\mathcal{C}}$ with values in the category $\mathcal{C}$ if σ is a natural isomorphism of the bifunctor:*

$$\otimes : \tilde{\mathcal{C}} \times \tilde{\mathcal{C}} \longrightarrow \mathcal{C}$$

with the two hexagon properties.

Thus, for arbitrary objects $X, Y \in \mathcal{O}b(\tilde{\mathcal{C}})$ we get $\mathcal{C}$–isomorphisms

$$\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X,$$

together with relations 2.4.

Recall that a natural isomorphism M of a monoidal category $\mathcal{C}$ is a *monoidal transformation* if M preserves the tensor product bifunctor:

$$M_{X \otimes Y} = M_X \otimes M_Y, \quad M_k = id.$$

The totality of all monoidal transformations is closed under the composition. Hence we get a group. We call this group the *Klein–Tannaka group* of the monoidal category $\mathcal{C}$ and denote by $KT(\mathcal{C})$.

The totality of all monoidal transformations of the given category $\mathcal{C}$ such that their restriction on $\tilde{\mathcal{C}}$ are monoidal transformations too generate a subgroup $KT(\mathcal{C}, \tilde{\mathcal{C}})$ of the Klein–Tannaka group.

Two braidings σ and σ' of the category $\tilde{\mathcal{C}}$ with values in the category $\mathcal{C}$ are *equivalent* if there exists a monoidal isomorphism $M \in KT(\mathcal{C}, \tilde{\mathcal{C}})$ such the following diagram

$$\begin{array}{ccc}
X \otimes Y & \xrightarrow{\sigma_{X,Y}} & Y \otimes X \\
{\scriptstyle M_{x \otimes Y}} \big\downarrow & & {\scriptstyle M_{Y \otimes X}} \big\downarrow \\
X \otimes Y & \xrightarrow{\sigma'_{X,Y}} & Y \otimes X
\end{array}$$

commute.

Theorem. *Let A be a k–algebra.*

(1) *Any braiding σ in the category $A{-}\mathcal{M}od$ with values in the category $k{-}\mathcal{M}od$ is determined by an invertible element $\sigma \in A \otimes A$ with relations 2.3.(i),(ii).*
(2) *Any braiging σ in the category $A^* - \mathcal{M}od$ with values in the category $k - \mathcal{M}od$ is given by a bialgebra homomorphism*

$$\hat{\sigma} : A \longrightarrow A^*.$$

3.Quantizations.

In this chapter we introduce our main notion of a (relative) quantization of a monoidal category with values in other monoidal category. For the case of $A-\mathcal{M}od$ category, where A is a bialgebra, we describe quantizations in terms of generating elements. This description leads to the notion of non–linear quantrum cohomology space.Their linearization is actually the Hochschild cohomology.

3.1. Let $\mathcal{C}$ be a monoidal category [ML1] with

(1) a bifunctor of a tensor product

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C},$$

$$\otimes : X \times Y \mapsto X \otimes Y,$$

where $X, Y \in \mathcal{O}b(\mathcal{C})$,

(2) an *associativity constraint*

$$\alpha = \alpha_{(X,Y,Z)} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z,$$

(3) an *unit object* k, with the natural isomorphisms

$$\eta^l_X : k \otimes X \longrightarrow X, \quad \text{and} \quad \eta^r_X : X \otimes k \longrightarrow X,$$

such that the following MacLane coherence conditions [McL]:

(1) the pentagon axiom

$$\alpha_{X\otimes Y,Z,T} \circ \alpha_{X,Y,Z\otimes T} = (\alpha_{X,Y,Z} \otimes id_T) \circ \alpha_{X,Y\otimes Z,T} \circ (id_X \otimes \alpha_{Y,Z,T})$$

and

(2) the unity axiom

$$(id_X \otimes \eta^l_Y) \circ \alpha_{X,k,Y} = \eta^l_X \otimes id_Y$$

hold.

3.2. One can consider the following definition as some type of the MacLane coherence conditions.

Definition. *A quantization Q of a monoidal category $\mathcal{C}$ is a natural isomorphism of the tensor product bifunctor*

$$Q_{X,Y} : X \otimes Y \longrightarrow X \otimes Y,$$

such that the following coherence conditions

$$\alpha_{X,Y,Z} \circ Q_{X,Y\otimes Z} \circ (id_X \otimes Q_{Y,Z}) = Q_{X\otimes Y,Z} \circ (Q_{X,Y} \otimes id_Z) \circ \alpha_{X,Y,Z}, \tag{1}$$

and

$$\eta^r_X \circ Q_{X,k} = \eta^r_X, \quad \eta^l_X \circ Q_{k,X} = \eta^l_X \tag{2}$$

hold.

3.3 A quantization preserves all natural algebraic notions related to the tensor product[L2]. Thus, if $\mu : A \otimes A \longrightarrow A$ is an algebra structure in the monoidal category $\mathcal{C}$ then one can define a quantization (A_q, μ^q) of the algebra as the object $A_q = A$ with the new multiplication $\mu^q = \mu \circ Q_{A,A}$. Similary, for any left (right or bi–) module X with a multiplication $\mu_X : A \otimes X \longrightarrow X$ we can define a quantization (X_q, μ^q_X), where $X_q = X, \mu^q_X = \mu_X \circ Q_{A,X}$. Moreover, (A_q, μ^q) and (X_q, μ^q_X) are an algebra and a A_q–module respectively.

If $\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ is a braiding in the category $\mathcal{C}$ then $\sigma^q_{X,Y}$ given by the diagram

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\sigma^q_{X,Y}} & Y \otimes X \\ {\scriptstyle Q_{X,Y}}\Big\downarrow & & {\scriptstyle Q_{Y,X}}\Big\downarrow \\ X \otimes Y & \xrightarrow{\sigma_{X,Y}} & Y \otimes X, \end{array}$$

is a braiding of the category too [L1].

3.4 Now we consider quantizations in the category $A - \mathcal{M}od$ over a bialgebra A.

Theorem.

(1) *Any quantization Q in the monoidal category $A - \mathcal{M}od$ over a bialgebra A is determined by an invertible element $q \in A \otimes A$, such that*

$$(i) \quad q \cdot \Delta(a) = \Delta(a) \cdot q, \qquad \forall a \in A,$$
$$(ii) \quad (\Delta \otimes id_A)(q) \cdot (q \otimes 1) = (id_A \otimes \Delta)(q) \cdot (1 \otimes q),$$
$$(iii) \quad (\epsilon \otimes id_A)(q) = (id_A \otimes \epsilon)(q) = 1.$$

(2) *The action*

$$Q_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$$

is given by the formula

$$Q_{X,Y}(x \otimes y) = q \cdot (x \otimes y).$$

Proof. In just the same way as in the proof of Theorem 2.3. we identify elements $x \in X, y \in Y$ with morphisms $\varphi_x : A \longrightarrow X$, and $\varphi_y : A \longrightarrow Y$. From the definition of a quantization we get the following commutative diagram

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{Q_{X,Y}} & X \otimes Y \\ \Big\uparrow{\scriptstyle \varphi_x \otimes \varphi_y} & & \Big\uparrow{\scriptstyle \varphi_x \otimes \varphi_y} \\ A \otimes A & \xrightarrow{Q_{A,A}} & A \otimes A. \end{array}$$

Denoting by q the element $Q_{A,A}(1 \otimes 1)$ one gets:

$$Q_{X,Y}(x \otimes y) = q \cdot (x \otimes y).$$

Now the conditions (ii),(iii) are equivalent to the conditions (1),(2) of definition 3.3., and the condition (i) means that the map $x \otimes y \mapsto q \cdot (y \otimes x)$ is an A–algebra morphism.

3.5. In this section we like n.2.11. generalize the notion of a quantization on the relative case.

Let $\tilde{\mathcal{C}}$ be a monoidal subcategory of a monoidal category $\mathcal{C}$.

Say that Q is a *relative quantization* of the category $\tilde{\mathcal{C}}$ with values in the category $\mathcal{C}$ if Q is a natural isomorphism of the bifunctor:

$$\otimes : \tilde{\mathcal{C}} \times \tilde{\mathcal{C}} \longrightarrow \tilde{\mathcal{C}} \hookrightarrow \mathcal{C},$$

such that relations n.3.1. hold.

Therefore, for arbitrary objects $X, Y \in \mathcal{O}b(\tilde{\mathcal{C}})$ we have $\mathcal{C}$*–morphisms*

$$Q_{X,Y} : X \otimes Y \longrightarrow X \otimes Y,$$

such that relations n.3.1. hold.

The relative quantization Q gives a quantization of algebras, modules and braidings in the category $\tilde{\mathcal{C}}$.

Thus, if $A, \mu \; : \; A \otimes A \; \longrightarrow \; A$ is an algebra in the category $\tilde{\mathcal{C}}$ then $(A_q = A, \mu^q = \mu \circ Q_{A,A})$ is an algebra in the category $\mathcal{C}$.

3.6. To make the notion of a relative quantization more clear we look at more general situation. To this end we consider monoidal categories $\mathcal{A}$ and $\mathcal{B}$ with tensor products $\otimes$ in $\mathcal{A}$ and $\hat{\otimes}$ in $\mathcal{B}$, and associativity constraints α in $\mathcal{A}$ and β in $\mathcal{B}$.

Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant functor preserving units.

Definition. *Say that a natural isomorphism*

$$Q_{X,Y} : \Phi(X)\hat{\otimes}\Phi(Y) \longrightarrow \Phi(X \otimes Y),$$

where $X, Y \in \mathcal{O}b(\mathcal{A})$, *is a quantization of the functor* Φ *if the following analog of relations 3.1.*

$$\Phi(\alpha_{X,Y,Z}) \circ Q_{X,Y\otimes Z} \circ (id_X \hat{\otimes} Q_{Y,Z}) = Q_{X\otimes Y,Z} \circ (QX, Y \hat{\otimes} id_Z) \circ \beta_{X,Y,Z}, \qquad (1)$$

and

$$Q_{X,k} \circ \Phi(\eta^r_{\mathcal{A}}) = \eta^r_{\mathcal{B}}, \qquad (2)$$

$$Q_{k,X} \circ \Phi(\eta^l_{\mathcal{A}}) = \eta^l_{\mathcal{B}}, \qquad (3)$$

hold.

Remark that relative quantization is a quantization of the imbedding functor $\tilde{\mathcal{C}} \hookrightarrow \mathcal{C}$.

Theorem. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be monoidal categories and* $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ *be a functor preserving units. Consider an algebra* (A, μ) *in the monoidal category* $\mathcal{A}$ *and* (X, μ_X) *a left (right, bi-)* A*–module in this category.*

Then for any quantization Q *of the functor* Φ, $(\Phi(A), \mu^q = \Phi(\mu) \circ Q_{A,A})$ *is an algebra in the category* $\mathcal{B}$ *and* $(\Phi(X), \mu^q_X = \Phi(\mu_X) \circ Q_{A,X})$ *is a left (right,bi-)* $\Phi(A)$*–module in the category* $\mathcal{B}$.

For the proof see Theorem 1.4. in [L2].

3.7. Quantizations of the forgetfull functor $A - \mathcal{M}od \longrightarrow k - \mathcal{M}od$ can be described analogously to Theorem 3.4.

Theorem. *Let A be a k–bialgebra. Then any quantization of the forgetfull functor $A-\mathcal{M}od \longrightarrow k-\mathcal{M}od$ is determined by an element $q \in A\otimes A$ such that*

$$(1)\quad (\Delta\otimes id_A)(q)(q\otimes 1) = (id_A\otimes\Delta)(q)(1\otimes q),$$
$$(2)\quad (\epsilon\otimes id_A)(q) = (id_A\otimes\epsilon)(q) = 1.$$

Definition. *We call a* **quantizer** *of a bialgebra A the element $q\in A\otimes A$ from the Theorem .*

Denote by $Q(A)\subset A\otimes A$ the set of all quantizers.

Let $Q_0(A)\subset Q(A)$ be a subset of all quantizers preserving the $A-\mathcal{M}od$ category:

$$Q_0(A) = \{q\in Q(A)|\Delta(h)\cdot q = q\cdot\Delta(h), \forall h\in A\}.$$

3.8. Define an A–module structure on A^* in the following way:

$$\langle a\cdot f, x\rangle = \langle f, xa\rangle,$$

where $a, x\in A, \quad f\in A^*$.

Hence, $A^*\in \mathcal{O}b(A-\mathcal{M}od)$ and one can quantize A^* as above.

Theorem. *An element $q\in A\otimes A$ is a quantizer if and only if the following morphism*

$$\mu^q{}_{A^*} = \mu_{A^*}\circ q : A^*\otimes A^* \longrightarrow A^*,$$

determines an associative algebra structure on A^*.

Proof. Denote by $f \underset{q}{*} g \quad f, g\in A^*$ the new multiplication:

$$\langle f\underset{q}{*} g, x\rangle = \langle f\otimes g, \Delta(x)\cdot q\rangle,$$

or

$$\langle f\underset{q}{*} g, x\rangle = \sum\langle f, x_(1)q'\rangle\langle g, x_(2)q''\rangle,$$

if $q=\sum q'\otimes q''$.

Then , for any elements $f, g, h\in A^*, \quad x\in A$ one has:

$$\begin{aligned}
&\langle (f\underset{q}{*} g)\underset{q}{*} h, x\rangle = \langle (f\underset{q}{*} g)\otimes h, \Delta(x)\cdot q\rangle = \\
&= \sum\langle f\underset{q}{*} g, x_{(1)}q'\rangle\langle h, x_{(2)}q''\rangle = \sum\langle f\otimes g, \Delta(x_{(1)}q')q\rangle\langle h, x_{(2)}q''\rangle = \\
&= \sum\langle f\otimes g\otimes h, \Delta(x_{(1)}q')q\otimes x_{(2)}q''\rangle = \\
&= \sum\langle f\otimes g\otimes h, (\Delta\otimes id_A)(x_{(1)}\otimes x_{(2)}\cdot(\Delta\otimes id_A)(q)(q\otimes 1)\rangle = \\
&= \langle f\otimes g\otimes h, ((\Delta\otimes id_A)\circ\Delta(x))\cdot((\Delta\otimes id_A)(q)\cdot(q\otimes 1))\rangle.
\end{aligned}$$

In the same way one gets

$$\langle f\underset{q}{*}(g\underset{q}{*} h), x\rangle = \langle f\otimes g\otimes h, ((id_A\otimes\Delta)\circ\Delta(x))\cdot((id_A\otimes\Delta)(q)\cdot(1\otimes q))\rangle.$$

Therefore, if

$$\gamma = (\Delta \otimes id_A)(q) \cdot (q \otimes 1) - (id_A \otimes \Delta)(q) \cdot (1 \otimes q),$$

then taking $x = 1$ one gets:

$$\langle f \otimes g \otimes h, \gamma \rangle = 0,$$

for all $f, g, h \in A^*$.

Hence, $\gamma = 0$.

3.9. For the case of $A^* - \mathcal{M}od$ category we have an additional property of a quantizer.

Theorem. *An element $q \in A^* \otimes A^*$ is a quantizer of the forgetfull functor $A^* - \mathcal{M}od \longrightarrow k - \mathcal{M}od$ if and only if*

$$({\lambda^\dagger}_{\hat{q}(x)} * \hat{q}^\dagger)(y) = (\lambda_{\hat{q}^\dagger(y)} * \hat{q})(x),$$

for all $x, y \in A^*$.

Proof. If $q = \sum q' \otimes q'' = \sum \tilde{q}' \otimes \tilde{q}''$, then

$$(\Delta \otimes id_A)(q) \cdot (q \otimes 1) = \sum q'_{(1)} \tilde{q}' \otimes q'_{(2)} \tilde{q}'' \otimes q''.$$

For all $x, y, z \in A$ one has:

$$\langle (\Delta \otimes id_A)(q) \cdot (q \otimes 1), x \otimes y \otimes z \rangle = \sum \langle q'_{(1)}, x_{(1)} \rangle \langle \tilde{q}, x_{(2)} \rangle \langle q'_{(2)}, y_{(1)} \rangle \langle \tilde{q}'', y_{(2)} \rangle \langle q'', z \rangle.$$

But

$$(\lambda_{\hat{q}(z)} * \hat{q})(a) = \sum \lambda_{\hat{q}(z)}(a_{(1)}) \langle \tilde{q}'', a_{(2)} \rangle \tilde{q}',$$

and

$$\lambda_{\hat{q}(z)}(b) = \sum \langle q'', z \rangle \langle q'_{(1)}, b \rangle q'_{(2)},$$

for all $a, b \in A$.

Hence,

$$\langle (\Delta \otimes id_A)(q) \cdot (q \otimes 1), x \otimes y \otimes z \rangle = \langle ({\lambda^\dagger}_{\hat{q}(z)} * \hat{q}^\dagger)(x), y \rangle.$$

In the same way one gets

$$\langle (id_A \otimes \Delta)(q) \cdot (q \otimes 1), x \otimes y \otimes z \rangle = \langle (\lambda_{\hat{q}^\dagger(x)} * \hat{q})(z), y \rangle.$$

Remarks.

(1) An element $q \in A^* \otimes A^*$ from the theorem gives a quantizer in the category $A - \mathcal{M}od$ if and only if

$$\hat{q} * \lambda_h = \lambda_h * \hat{q},$$

for all $h \in A^*$.

(2) If $\alpha, \beta \in Q(A)$ and $\alpha\beta = \beta\alpha$, then $\alpha\beta \in Q(A)$, because of $(\Delta \otimes id_A)$ is an algebra homomorphism. For example, if $k = \mathbb{C}$, $\quad q \in Q(A)$, and q^z, $\quad z \in \mathbb{C}$ make sence, then $q^z \in Q(A)$.

3.10. Consider the following action of the unit group $U(A)$ on the set $Q(A)$:

$$a \in U(A), \quad q \in Q(A) \mapsto a(q) = \Delta(a) \cdot q \cdot a^{-\otimes 2},$$

where $a^{-\otimes 2} = a^{-1} \otimes a^{-1} \in A \otimes A$.

Denote by $\mathcal{Q}H^2(A)$ the space of orbits of the action:

$$\mathcal{Q}H^2(A) = Q(A)/U(A).$$

One can consider this space as a non-linear analog of a (second) cohomology group.

For this reason we call $\mathcal{Q}H^2(A)$ a *cohomology space* of a bialgebra A.

To make the analogy more clear let's consider the linearization of the construction above.

Fix a quantizer $q \in Q(A)$. A "tangent plane" to $Q(A)$ at the point q we can identify with the kernel of the following operator

$$\partial_q : A^{\otimes 2} \longrightarrow A^{\otimes 3},$$

where

$$\begin{aligned}\partial_q(\alpha) =& (\Delta \otimes id_A)(q)(\alpha \otimes 1) - (id_A \otimes \Delta)(q)(1 \otimes \alpha) + \\ &+ (\Delta \otimes id_A)(\alpha)(q \otimes 1) - (id_A \otimes \Delta)(\alpha)(1 \otimes q),\end{aligned}$$

and $\alpha \in A^{\otimes 2}$.

Respectively, a "tangent plane" to the orbit $U(A)q \subset Q(A)$ is the image of the operator

$$\partial_q :: A \longrightarrow A^{\otimes 2},$$

where

$$\partial_q(a) = \Delta(a) \cdot q - q \cdot (a \otimes 1 + 1 \otimes a),$$

and $a \in A$.

It is easy to check that $\partial_q^2 = 0$. Hence, if we identify the tangent plane $T_q(\mathcal{Q}H^2(A))$ with the factor space

$$T_q(\mathcal{Q}H^2(A)) \simeq T_q(Q(A))/T_q(U(A)q),$$

then the second cohomology group $H^2(A, q)$ of the complex

$$A \xrightarrow{\partial_q} A^{\otimes 2} \xrightarrow{\partial_q} A^{\otimes 3}$$

can be identified with the tangent space $T_q(\mathcal{Q}H^2(A))$.

Define a bracket on $T_q(\mathcal{Q}H^2(A))$ in the following way:

$$\begin{aligned}[X, Y] =& ((\Delta \otimes id_A)(X)(Y \otimes 1) + (\Delta \otimes id_A)(Y)(X \otimes 1) - \\ &- (id_A \otimes \Delta)(X)(1 \otimes Y) - (id_A \otimes \Delta)(X)(1 \otimes Y)) \mod Im\, \partial_q.\end{aligned}$$

Then a vector $\nabla \in T_q(\mathcal{Q}H^2(A))$ is a tangent vector to some curve $q(t)$ on $\mathcal{Q}H^2(A)$, if

$$[\nabla, \nabla] = 0.$$

Defenition. *We say that an element* $\nabla \in T_q(\mathcal{Q}H^2(A))$ *defines a* q–**Poisson structure**, *if* $[\nabla, \nabla] = 0$.

Therefore, we get the following geometrical picture on the cohomology space: at any point $q \in \mathcal{Q}H^2(A)$ there is a cone K_q of Poisson structures,

$$K_q = \{\nabla \in T_q(\mathcal{Q}H^2(A)) | [\nabla, \nabla] = 0\}.$$

and curves in the space $\mathcal{Q}H^2(A)$ is an integral curves of the cone field.

3.11. Here we consider some examples of quantizers.

(1) Let A be a commutative and cocommutative bialgebra over $k = \mathbb{C}$. We will look for quantizers of the following form:

$$q = \exp(\alpha),$$

where $\alpha \in A \otimes A$.

One has:

$$(\Delta \otimes id_A)(q) = \exp((\Delta \otimes id_A)(\alpha), \quad (id_A \otimes \Delta)(q) = \exp((id_A \otimes \Delta)\alpha),$$

and hence the main equation for quantizers takes the form:

$$(\alpha \otimes 1) - (id_A \otimes \Delta)(\alpha) + (\Delta \otimes id_A)(\alpha) - (1 \otimes \alpha) = 0.$$

The last equation means that α is a 2-cocycle of the *coalgebra* A with values in k.

It is easy to check that the condition of q–Poisson structure does not depend on q.

For example, if $A = C^\infty(\mathbb{T}^n)$, then $\alpha \in \Lambda^2(D)(\mathbb{T}^n)$ is an usuall Poisson structure.

Note that, we get the *Moyal type quantizations* [V] of the category of $\mathbb{T}^n$–modules.

(2) Let G be an abelian finite group, $A = k(G)$. The category of $A - \mathcal{M}od$ is the category of G–graded k–modules.

Conditions 3.7. mean that q is a 2-cocycle on the group G with values in unit group $U(k)$:

$$q \in Z^2(G, U(k)).$$

Thus, in the given case,

$$\mathcal{Q}H^2(A) = H^2(G, U(k)).$$

(3) If we consider the category of modules over semi-simple Lie algebra. Then the Drienfeld construction [D2] of a quantization actually gives the (formal) quantizer.

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